Bulletin of the **S**ection of **L**ogic
Volume 49/2 (2020), pp. 109–125
http://dx.doi.org/10.18778/0138-0680.2020.07

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COMPLETENESS, CATEGORICITY AND IMAGINARY NUMBERS: THE DEBATE ON HUSSERL[1]

Abstract

Husserl's two notions of "definiteness" enabled him to clarify the problem of imaginary numbers. The exact meaning of these notions is a topic of much controversy. A "definite" axiom system has been interpreted as a *syntactically complete* theory, and also as a *categorical* one. I discuss whether and how far these readings manage to capture Husserl's goal of elucidating the problem of imaginary numbers, raising objections to both positions. Then, I suggest an interpretation of "absolute definiteness" as *semantic completeness* and argue that this notion does not suffice to explain Husserl's solution to the problem of imaginary numbers.

Keywords: Husserl, completeness, categoricity, relative and absolute definiteness, imaginary numbers.

1. Introduction

Since the publication of Hill [10] and Majer [17], much attention has been devoted to Husserl's two notions of "definiteness" (*relative* and *absolute definiteness*), which were introduced in a Double Lecture (henceforth, *Doppelvortrag*) for the Göttingen Mathematical Society in 1901. These notions enabled him "to clarify the logical sense of the computational transition through the 'imaginary"' and, in connection with that, to bring

[1]This work was supported by the Spanish Ministry of Education under the Grant FPU15/00830. I thank the anonymous referee for his/her constructive comments and suggestions.

out the sound core of Hermann Hankel's[2] renowned, but logically unsubstantiated and unclear, "principle of the permanence of formal laws" (Husserl [13], p. 97).

What Husserl describes as the "computational transition through the 'imaginary'" is the extension of the number-concept[3]. The Principle of Permanence says that the progressive extension of the number-concept should preserve (to the greatest extent possible) the arithmetical laws of the positive whole numbers. Strictly speaking, it asserts that the laws governing the newly introduced numbers have to be *consistent* with the laws constraining the old ones.

Husserl's *Doppelvortrag* was an attempt to find a justification for this Principle. A consensus has emerged that, according to Husserl, if every level of the hierarchy of numbers has a *definite* axiom system, then the extension of the number-concept can never lead to contradictions. There is, however, disagreement in the literature as to the exact meaning of the word "definite". A passionate debate has opposed those like da Silva [4] and [5], who read "definiteness" as *syntactic completeness*, and those like Hartimo [8] and [9], who favor reading it in terms of *categoricity*. Centrone [3] pointed out that Husserl himself seems to oscillate between both characterizations.

In the present paper, I discuss the plausibility of the different interpretations of "definiteness" in the literature. Is a syntactically complete axiom system compatible with the extension of the number-concept? And a categorical one? I will provide a new interpretation of "absolute definiteness"[4] (as *semantic completeness*) which is, I think, conceptually stronger. I will also maintain that "definiteness" does not suffice to explain Husserl's justification of the transition through the imaginary: the hierarchy of numbers *must contain a copy* of the previous levels.

[2] Although the Principle of Permanence is discussed in Hankel [7], it was formulated by Peacock.

[3] "Here I of course take the term 'imaginary' in the widest possible sense, according to which also the negative, indeed even the fraction, the irrational number, and so forth, can be regarded as imaginary" (Husserl [14], p. 412).

[4] In my opinion, a *relatively definite theory* is not semantically complete. However, this would require a separate paper.

2. State of the art

In the *Doppelvortrag*, Husserl's notion of *definiteness* was introduced in a twofold manner and served a double purpose. First of all, it shows the perfect delimitation of a "domain" (or "sphere of existence")[5] by its axiom system:

> The further question: Would such a system be definite? It would be definite if, for the demarcated sphere of existence, for the given individuals, and for the individuals not given, no further new axiom were possible (Husserl [14], p. 424).

Secondly, it guarantees that every meaningful proposition of the language of the system *is decided* from the axioms:

> A formal axiom system which contains no extra-essential closure axiom is said to be a definite one if each proposition that has a sense at all through the axiom system *eo ipso* falls under the axiom system, be it as consequence or be it as contradiction (Husserl [14], p. 431).

Almost thirty years after the *Doppelvortrag*, this duality between the "full description of the domain" and "maximality of the axiom system" remains invariable in Husserl's definition of his central notion. He [13] explicitly asserted that, if a domain is wholly captured by an axiom system (in modern terms, *if a theory axiomatizes a structure*), then every proposition constructed in the system has to be either a consequence of the axioms or an "analytic contradiction" (see p. 96).

Husserl also split the notion of definiteness in *relative* (the axiom system for "the whole and the fractional numbers") and *absolute definiteness*[6] (the axiom system for the "continuous number sequence" i.e., for the reals) in the context of the transition through the imaginary. The exact meaning of these notions, as well as their role in the extension of the number-concept, are matters of much controversy.

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[5] Husserl speaks of "domain" (or "sphere of existence") of a group of axioms in the sense that a system of objects satisfies certain general laws. I will use the term "domain" to refer to such a system of objects, because using the term "structure" seems quite anachronistic (see Hodges [11] and Husserl [14], pp. 437–38).

[6] "Therefore, absolutely definite = complete, in *Hilbert*'s sense" (Husserl [14], p. 127).

2.1. Centrone

Centrone [3] maintained an interpretation of “relative definiteness” as syntactic completeness and “absolute definiteness” as categoricity. Regarding the extension of the number-concept, she makes the following claim:

> The thesis that Husserl proposes in the *Doppelvortrag* is a conditional claim: if T is consistent and syntactically complete (definite) then every consistent extension of T is conservative, so that the transition through the imaginary is justified (Centrone [3], p. 178).

Syntactic completeness is a very unusual property of a set of sentences[7], because the set is so strong that, for every sentence φ of its language, either φ or $\neg\varphi$ has to be provable from the set. It follows that, if a sentence ψ formulated in the language of a complete set T is not provable from T, then $T \cup \{\psi\}$ will be inconsistent. This property is often known as the *maximality* of a consistent set.

But Centrone's solution does not function. Suppose for the sake of argument that T is a complete axiom system for the naturals. The sentence $\varphi :=$ “there exists an x such that when added to 1 gives 0” is not provable from T. Since T is complete, $\neg\varphi$ has to be provable from T. Let T' be an axiom system for the integers which is an extension of T. It is easy to see that φ is a theorem of T', which means that the extension T' of T is inconsistent. Thus, “definite” cannot be syntactically complete.

Furthermore, an extension T' of a theory T is *conservative* if T' is just a theory containing T. More precisely, every sentence of the language of T which is provable from T' is also a theorem of T. Is the extension of the number-concept a conservative extension?

Let T and T' be the axiom systems for the fields of real and complex numbers, respectively. While the reals can be ordered, there is no total ordering of the complexes that is compatible with the field operations. The sentence $\psi :=$ “there exists an x such that $x < 0$ and $-x < 0$” is provable from T' if we suppose that the complexes can be totally ordered. If Centrone were right, then ψ would be also provable from T, which contradicts the axioms of a total order. Consequently, the extension T' of T is not conservative.

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[7] A set of sentences is a *theory* (see Hodges [12], p. 33).

Contrary to Centrone's interpretation, Husserl did not believe that the extension of the number-concept had to be conservative. In the double lecture, he argued that the "expansion of the numbers series" leads to a new domain in which new relations and elements may be defined:

> The series of the positive whole numbers is a part of the series of numbers that is infinite at both ends. This in turn is part of the two-fold manifold of the complex numbers. The system of the positive whole numbers is defined by certain elementary relations. In these latter nothing is modified through expansion of the number series [...] In the new domain new relations as well as new elements may be defined. In the new domain there then will be such conceivable relations as include the old elements and old relations (Husserl [14], p. 457).

Husserl explicitly stated that a domain of numbers cannot be extended in a way that the same axiom system describes the broader domain (see [14], p. 427). If the same axiom system holds for both domains, then the narrower domain will not be extended at all. New propositions must be true in the broader domain (and hence the extension from T to T' cannot be conservative).

2.2. Da Silva

Da Silva [4] and [5] read "relative definiteness" as syntactic completeness *relative to a particular set of expressions* and "absolute definiteness" as syntactic completeness. The former is the central notion for understanding Husserl's solution to the problem of imaginary numbers:

> Husserl's solution for the problem of imaginary elements has, I believe, the following form: given systems A and B such that A and B are consistent and B extends A, let $\mathcal{D}$ be the formal manifold determined by A [...] and suppose that A is complete *relative* to the assertions of $L_{\mathcal{D}}(A)$, i.e., the assertions of $L(A)$ with all variables restricted to $\mathcal{D}$. Now, if any of these assertions (i.e., assertions of $L_{\mathcal{D}}(A)$) is proved by B, it can also be accepted from the perspective of A (Da Silva [4], p. 423).

A theory is (syntactically) complete relative to a particular set of expressions Δ of its language if, for every sentence $\varphi \in \Delta$, either φ or $\neg\varphi$ has to be a theorem of the theory. Therefore, the set of expressions Δ is the collection of all statements that the theory can either prove or disprove, and it is called its *apophantic domain*[8]. This domain is obtained by restricting quantification to the domain of $\mathcal{D}$, so the sentences of Δ refer exclusively to the narrower domain (i.e., they do not contain terms denoting imaginary numbers).

The restriction of syntactic completeness to a particular set of sentences intends to avoid the difficulties of Centrone's approach. The sentence $\varphi :=$ "there exists an x such that when added to 1 gives 0" is now undecidable starting from the axioms of the natural numbers, because it refers to a number which belongs to the integers. Theorems of T are preserved in theories that extend T *provided that* they are about the narrower domain. For this reason, the provability of φ by means of the axioms of the integers does not imply a contradiction anymore. Does da Silva's restriction explain the transition through the imaginary?

Let T and T' be the axiom systems for the rationals and the reals, respectively. The sentence $\theta :=$ "$\sqrt{2}$ is an irrational number" does not belong to the apophantic domain of T, as it refers to a number which is imaginary from the point of view of T. If da Silva were right, then θ would be undecidable starting from the axioms of the rationals. However, the proof that shows the irrationality of $\sqrt{2}$ can be achieved by means of T and the rational root theorem. Hence, da Silva's restriction of syntactic completeness to a particular set of sentences does not account for the extension of the number-concept.

In the *Doppelvortrag*, Husserl claimed that the truth-value of an expression[9] that alludes to a broader domain is decided *on the basis* of the axioms for the narrower, for the reason that it is *false* in the old domain.

> Let us consider, for example, the axiom system of the whole numbers, positive and negative. Then $x^2 = -a$, $x = \pm\sqrt{-a}$ certainly has a sense. For square is defined, and $-a$, and $=$ also. But "in the domain" there exists no $\sqrt{-a}$. The equation

[8] "If an assertion belongs to the apophantic domain of a system, then it is either true on the basis of the axioms of the system, if they can prove it, or it is false on the basis of these axioms, if they can prove its negation" (Da Silva [4], p. 427).

[9] Since quantifiers had not been introduced in 1901, Husserl's "expressions" are probably just equations or operations among numbers. However, in the scholarly debate on Husserl these "expressions" are understood as "sentences" (in the modern sense). See, for instance, da Silva [4] and da Silva [5].

> is false in the domain, inasmuch as such an equation cannot hold at all in the domain (Husserl [14], pp. 438–39).

He also defended that an axiom system is definite if "it leaves open or undecided no question related to the domain and meaningful in terms of this system of axioms" ([14], pp. 438), which implies that no proposition will be undecidable from a definite set of axioms.

2.3. Hartimo

Hartimo [8] and [9] interpreted "relative" and "absolute definiteness" as categoricity. The usage of imaginary numbers in calculations is justified if both the narrower and the broader domain are fully described by a categorical set of axioms.

> Our suggestion is that Husserl's remarks in the *Doppelvortrag* are best understood if by the formal domain Husserl means something like a domain of a categorical theory [...] Each axiom system defines a unique formal domain that is included in the unique formal domain of the more extended axiom system (Hartimo [8], pp. 302–03).

A theory is *categorical* if for every pair $\mathcal{M}$ and $\mathcal{N}$ of its models there is an isomorphism between $\mathcal{M}$ and $\mathcal{N}$. In other words, a categorical theory has exactly only one model. It still remains to be explained how categoricity relates to justifying the extension of the number-concept. Hartimo [8] suggested that, according to Husserl, categoricity implies some kind of "maximality" which guarantees that the transition through the imaginary can never lead to contradictions. She also [9] argued that this maximality corresponds to syntactic completeness.

In favor of Hartimo's reading, it has to be said that the axiomatically constructed second-order arithmetic of natural numbers is categorical. But it is also incomplete by Gödel's theorems. Hartimo alleged that Husserl's view of "definiteness" combines *expressive* power (categoricity) and *deductive* power (syntactic completeness). Both ideals combined, which are not simultaneously attainable in the interesting cases, were called "monomathematics" by Tennant [23].

From these ideals, we can draw some important conclusions regarding the problem of imaginary numbers. If a definite axiom system is categorical *and* complete, then Hartimo's proposal is open to the same objections as

Centrone's. If it is categorical *and* complete relative to a particular set of expressions, then Hartimo is forced to address the objections against da Silva.

In short, it seems that the interpretation of "definiteness" as syntactic completeness (or implying syntactic completeness) does not make plausible Husserl's idea of how the number-concept should be extended.

3. Semantic completeness

In a lecture probably delivered in 1939, which Tarski never published and entitled "On the Completeness and Categoricity of Deductive Systems"[10], he introduced the notion of "semantic completeness". After remarking that every theory affected by Gödel's first incompleteness theorem is essentially incomplete (i.e., it always contains undecidable propositions), Tarski aimed to present semantic analogues of syntactic completeness (he called "absolute completeness" to syntactic completeness):

> On the basis of the foregoing we see that absolute completeness occurs rather as an exception in the domain of the deductive sciences, and by no means can it be treated as a universal methodological demand. In this connection, I want to call your attention to certain concepts very closely related to the concept of absolute completeness, which are the result of a weakening of this concept and whose occurrence is not such an exceptional phenomenon. (Tarski, [22], p. 488).

Tarski believed that the notion of *provability* developed in modern logic was not the formal counterpart of the intuitive concept of consequence[11]. For this reason, the notion of semantic completeness is obtained by replacing "provability" with "logical consequence" in the definition of syntactic completeness. A (consistent) theory is *semantically complete* if, for every sentence φ of its language, φ or $\neg\varphi$ is a logical consequence of the axioms. Awodey and Reck [1] stated the following four equivalent conditions for semantic completeness:

1. For all sentences φ and all models $\mathcal{M}$ and $\mathcal{N}$ of T, if $\models_{\mathcal{M}} \varphi$ then $\models_{\mathcal{N}} \varphi$.

[10] It is published in Mancosu [18].

[11] See Tarski [21], p. 409 and Tarski [22], p. 489.

2. For all sentences φ, either $T \models \varphi$ or $T \models \neg\varphi$.
3. For all sentences φ, either $T \models \varphi$ or $T \cup \{\varphi\}$ is not satisfiable.
4. There is no sentence φ such that both $T \cup \{\varphi\}$ and $T \cup \{\neg\varphi\}$ are satisfiable (p. 3).

As Centrone [3] rightly noticed, one cannot seriously defend that Husserl already distinguished between the syntactic notion of provability and the semantic concept of logical consequence. However, the interpretation of "definiteness" as semantic completeness, instead of syntactic completeness, certainly makes more plausible Husserl's attempts to link the full description of a domain with the maximality of its axiom system.

To begin with, it is clear that a semantically complete theory is maximal in some general sense. Consider, for instance, the fourth condition above. It corresponds to Carnap's notion of "non-forkability"[12], which was identified by Fraenkel [6] and states that there is no sentence φ (of the language of T) such that $T \cup \{\varphi\}$ and $T \cup \{\neg\varphi\}$ have a model. In other words, a theory is *non-forkable* if it does not branch out to other sets of sentences containing both φ and $\neg\varphi$. A proof of the implication from semantic completeness to relative completeness, which Tarski considered equivalent to non-forkability, is given in Mancosu [18] (see pp. 457–58).

The implication from categoricity to semantic completeness also holds[13]. We saw that reading (absolute and relative) definiteness as categoricity implying syntactic completeness weakened Husserl's position, because categoricity and syntactic completeness are not both simultaneously attainable for the interesting cases. In contrast, Tarski [22] showed that every categorical theory – semantically categorical, in Tarski's terminology – is semantically complete.

Finally, a few words about the transition through the imaginary. If according to Husserl the axiom systems for the naturals, integers, rationals, reals, and complexes must be *definite*, then these systems must be semantically complete. Otherwise, my interpretation will be flawed. Fortunately:

> We know many systems of sentences that are categorical; we know, for instance, categorical systems of axioms for the arithmetic of natural, integral, rational, real, and complex numbers,

[12] See Carnap [2], pp. 130–33.

[13] See Carnap [2], p. 138, and Lindenbaum and Tarski [15], pp. 390–92.

> for the metric, affine, projective geometry of any number of dimensions etc [...] From theorems I and II, we see that all mentioned systems are at the same time semantically or relatively complete. Thus, in opposition to absolute completeness, relative or semantical completeness occurs as a common phenomenon (Tarski [22], p. 492).

For instance, from the categoricity of second-order Peano arithmetic we conclude that this theory is semantically complete (see Manzano [19], p. 128). One could be tempted to infer that the extension of the number-concept will be justified if every logical consequence of T (the axiom system for the narrower domain) is likewise a logical consequence of T' (the axiom system for the broader one), but this is clearly not true. Instead, I will argue that such an extension is permitted iff every sentence which is true in the narrower domain $\mathcal{M}$ is also true in the copy of $\mathcal{M}$ contained in the broader domain $\mathcal{N}$. In model theory, we say that there is an *embedding* of $\mathcal{M}$ in $\mathcal{N}$.

4. Not a sufficient condition

The debate on Husserl's notions of definiteness presupposes that a *definite* axiom system is a sufficient condition for the transition through the imaginary. But there is another necessary condition that has not been emphasized as deserved in the literature. Let me quote the entire relevant passage.

> According to this the following general law seems to result: A transition through the imaginary is permitted 1) if the imaginary can be formally defined in a consistent and comprehensive system of deduction, and 2) if the original domain of deduction when formalized has the property that every proposition falling within that domain is either true on the basis of the axioms of that domain or else is false on the same basis (i.e., is contradictory to the axioms).
> However, it is easily seen that this formulation does not suffice, although it already brings to expression the most essential part of the truth [...]
> But there is still the question whether the derived propositions of the broader domain fall in this sense within the narrower

> domain. If that is not determined in advance, we can say absolutely nothing about it (Husserl [14], pp. 428–29).

There are two points that are important here. First, Husserl highlights the role of consistency and definiteness in the extension of the number-concept. The axiom system for the original domain has to be consistent and definite. Second, he claims that both requirements do not suffice. Propositions about the narrower domain *but obtained from the axioms of the broader* are permitted if they are true propositions in the narrower domain[14]. The question is: How can such a result be established?

In the passages following the above, Husserl argues that this result can be proved if the extension of the number-concept does not induce new determinations on the old domain. For instance, the sentence $\chi :=$ "there exists an x whose square is -1", which extends the number-concept when added to the axiom system for the reals, does not define any arithmetical law of the real numbers. Husserl believed that, "if I expand an $\mathcal{M}_0$ to $\mathcal{M}$, then the $\mathcal{M}_0$ remains in $\mathcal{M}$ thus as structure still an $\mathcal{M}_0$. It is not thereby modified in species" (Husserl [14], p. 456).

Notice that for Husserl, the broader domain must contain a copy of the narrower one. The textual evidence for this is given in the first appendix of his *Doppelvortrag*:

> $\mathcal{M}_E$ is to be an expansion of $\mathcal{M}_0$. Thus $\mathcal{M}_E$ consists of the elements of $\mathcal{M}_0$ plus other elements. But that does not suffice. The $\mathcal{M}_0$ must be a part of $\mathcal{M}_E$. $\mathcal{M}_E$ has a part that falls under the concept $\mathcal{M}_0$. But that too is not sufficient. The expansion to $\mathcal{M}_E$ must not disturb $\mathcal{M}_0$ as that which it is, and above all must not specialize it (Husserl [14], p. 454).

> If a manifold is given to me as an $\mathcal{M}_0$, then $\mathcal{M}$ is an expansion of $\mathcal{M}_0$ if $\mathcal{M}_0$ undergoes no further "specialization" within $\mathcal{M}$ (Husserl [14], p. 456).

Furthermore, in the *Doppelvortrag* he stated that every domain of numbers of a lower level is completely contained in the higher levels. When a domain is contained in another one, Husserl explicitly speaks about "expansion" of the narrower domain or "contraction" of the broader one (see

[14] "The inference from the imaginary is permitted in the singular case or for a class, if we can know in advance and can see that for this case or for this class the inference is decided by the narrower system" (Husserl [14], p. 437).

[14], p. 421). If every object a of a domain $\mathcal{M}$ must occur in $\mathcal{N}$, and if every operation f defined on $\mathcal{M}$ must be defined on $\mathcal{N}$, then, Husserl says, $\mathcal{M}$ is contained in $\mathcal{N}$. The inclusion of the narrower domain *as a part* of the broader one "is the presupposition for the possibility of the transition through the Imaginary" (Husserl [14], p. 451).

This "presupposition" is also coherent with the construction of numbers. As it is well-known, the hierarchy of numbers is formally expressed as $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$. However, these inclusions are an abuse of notation because the set of integers is a quotient set of $\mathbb{N} \times \mathbb{N}$; the set of rationals is a quotient set of $\mathbb{Z} \times \mathbb{Z}^*$; the set of real numbers is the set of *all* the Dedekind cuts; and the set of complex numbers is the set $\mathbb{R} \times \mathbb{R}$. Let me briefly explain why the construction of numbers *speaks in favor of Husserl's presupposition*.

The inclusion $\mathbb{N} \subset \mathbb{Z}$ really expresses the identification of $\mathbb{N}$ with $\mathbb{Z}_+$, which means that there is an isomorphism between $\mathbb{N}$ and the subset $\mathbb{Z}_+$ of $\mathbb{Z}$. Therefore, we can put into one-one correspondence every number n of the naturals with every number $[(n, 0)]$ of $\mathbb{Z}_+$. Likewise, $\mathbb{Q}$ contains an ordered ring *isomorphic* to the ordered ring of the integers, and so on. Every level of the hierarchy of numbers contains a copy of the previous levels, which is mathematically indistinguishable from them. Hence, the extension of the number-concept does not introduce new determinations on the narrower domains, just as Husserl required.

5. Isomorphism and elementary equivalence

At the beginning of the *Doppelvortrag*, Husserl faces the problem of calculating with those numbers which are "absurd" or "imaginary" from the point of view of the original domain[15]. The main challenge, related to the Principle of Permanence, was introduced next.

> How is it to be explained that one can operate with the absurd according to rules, and that, if the absurd is then eliminated from the propositions, the propositions obtained are correct? (Husserl [14], p. 433).

[15] "Imaginary objects = objects which do not occur in A, are not defined there, are not established by means of the axioms and existential definitions of A, so that, therefore, if we regard A as the axiom system of a domain which has no other axioms – and thus also no other objects – those objects are in fact 'impossible"' (Husserl [14], p. 433).

Before looking at this passage in detail, I want to call your attention to Husserl's concept of "proposition'." He argued that the equation "$7 + 5 = 12$" *is a proposition*, which is correct iff its truth necessarily follows from the definitions of the numbers "7'," ' '5," and "12," and from the definition of addition (see Husserl [14], p. 194). If we extend the number-concept to solve the equation "$7 + 5 + x = 0$'," then our domain of numbers must include the number "-12,", which is "absurd" from the point of view of the naturals. But we still can single out propositions about the old domain (i.e., equations without imaginary numbers). The question is: Why correct propositions about natural numbers, such as "$7+5 = 12$", are still correct if we *restrict* a broader domain of numbers to the copy of the naturals contained in such a domain?

Consider, for instance, the truth of the proposition "$7 + 5 = 12$" in the domain of the positive integers. This proposition is true in both the natural numbers and the positive integers because the result of adding "$7_{\mathbb{Z}}$" (the equivalence class representing the number "7") to "$5_{\mathbb{Z}}$" (the equivalence class representing the number "5") is "$12_{\mathbb{Z}}$" (the equivalence class representing the number "12"). Let h be the isomorphism between $\mathbb{N}$ and $\mathbb{Z}_+$. More formally, we would say that "$7 + 5 = 12$" is true in the positive integers, for the reason that $h(7) + h(5) = h(7 + 5)$.

We only need to generalize these reflections on the preservation of truth to arrive at the solution of the problem quoted above. True propositions of a certain domain must also be true propositions of every isomorphic domain. In contemporary model theory, the isomorphism theorem[16] states that, if there is an isomorphism between $\mathcal{M}$ and $\mathcal{N}$, then every formula φ satisfied by $\mathcal{M}$ will be satisfied by $\mathcal{N}$. Thus, every n-tuple $a_1, ..., a_n$ of $\mathcal{M}$ satisfies φ if $h(a_1, ..., a_n)$ satisfies φ. It also establishes that, if a term t denotes an individual a in $\mathcal{M}$, then its denotation in $\mathcal{N}$ will be $h(a)$, where h is the isomorphism from $\mathcal{M}$ to $\mathcal{N}$.

Although many commentators have read Husserl's *Doppelvortrag* through the glasses of modern logic (see, for instance, da Silva [5], p. 1928), he never proved an isomorphism theorem. Since the oldest theorem of model theory is probably due to Löwenheim [16], it would be anachronistic to look for such a proof in the *Doppelvortrag*. However, Husserl felt that this kind of result could be actually achieved.

[16]See Manzano [20], pp. 68–69.

> The utilization of a broader system in order to bring forth propositions of the narrower one can only be permitted *if we possess some characterizing mark* by which we recognize that every proposition that has a sense in the narrower domain also is decided in the broader one, thus must be its consequence or its contradictory (Husserl [14], p. 437; my emphasis).

I claim that this "characterizing mark" is the fact that the isomorphism between $\mathcal{M}$ and $\mathcal{N}$ implies elementary equivalence. For instance, the ordered ring $\mathbb{Q}$ contains an ordered ring isomorphic (and hence elementarily equivalent) to $\mathbb{Z}$. It follows that every sentence that is true in the narrower ring *is also true in its copy* contained in the broader ring. "The laws of the expanded domain include those of the narrower one, but in such a way, however, that for the old domain no new laws are established" (Husserl [14], p. 457).

Let me conclude by pointing out the main difference between the other readings of Husserl's *Doppelvortrag* and my own approach. Whereas the justification of the transition through the imaginary has usually been associated with the preservation of the theorems of a (syntactically) complete theory, I have argued that it is better understood as the preservation of the true sentences of certain isomorphic domains ($\mathbb{N}$ and $\mathbb{Z}_+$, and so on).

6. Conclusions

This paper began with a discussion of the recent contributions to the debate on Husserl's two notions of "definiteness". We saw that the interpretation of (relative) definiteness as syntactic completeness seems unsatisfactory, because it presupposes that every extension from T (the axiom system for the narrower domain) to T' (the axiom system for the broader) must be *conservative*. Furthermore, if T is complete, then *proper extensions* of T will be inconsistent. On the other hand, the interpretation of (relative) definiteness as syntactic completeness is *relative to a set of sentences* flaws, for the reason that certain provable sentences (from the axioms of the old domain) are considered to be *undecidable*. Finally, the reading of definiteness as categoricity implying syntactic completeness (due to a pre-gödelian predicament which is called "monomathematics") is open to the same conceptual difficulties.

I claimed that the interpretation of absolute definiteness as *semantic completeness* makes Husserl's position more plausible. There are categorical axiom systems for the natural numbers, the integers, and so on, which are also semantically complete, as categoricity implies semantic completeness. Semantic completeness is not such an uncommon phenomenon. However, this implication does not suffice to explain Husserl's justification of the "transition through the imaginary". He remarks that the extension of the number-concept must not induce any new determinations on the narrower domains. This necessary condition has not been fairly emphasized in the literature.

I offered textual evidence in favor of understanding this requirement as the fact that the highest domains of the hierarchy of numbers contain a copy of the previous levels. For instance, the set of the integers includes a subset that is mathematically indistinguishable from the natural numbers ($\mathbb{N} \cong \mathbb{Z}_+$). There is also an isomorphism from the integers to a certain subset of the rationals, and so on. Every true sentence of $\mathbb{N}$ is a true sentence of $\mathbb{Z}_+$ by the isomorphism theorem, which explains that true formulas about the naturals are preserved *if we restrict the integers* to the positive ones. Husserl never proved such a result, but the fact that isomorphism implies elementary equivalence enabled me to explain his solution (definiteness + a hierarchy of numbers containing the lowest levels) to the problem of imaginary numbers.

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