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ROUGHNESS OF FILTERS IN EQUALITY ALGEBRAS

Abstract

Rough set theory is an excellent mathematical tool for the analysis of a vague description of actions in decision problems. Now, in this paper by considering the notion of an equality algebra, the notion of the lower and the upper approximations are introduced and some properties of them are given. Moreover, it is proved that the lower and the upper approximations define an interior operator and a closure operator, respectively. Also, using D-lower and D-upper approximation, conditions for a nonempty subset to be definable are provided and investigated that under which condition D-lower and D-upper approximation can be filter.

Keywords: equality algebra, approximation space, D-lower approximation, D-upper approximation, filter, D-lower filter, D-upper filter.

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1. Introduction

The rough sets theory introduced by Pawlak in [11] has often proved to be an excellent mathematical tool for the analysis of a vague description of objects called actions in decision problems. Many different problems

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can be addressed by rough sets theory. During the last few years some mathematicians studied about roughness theory in different fields of mathematics. For example an algebraic approach to rough sets has been given by Iwinski in [2]. Rough set theory is applied to semigroups and groups see [8, 9]. In 1994, Biswas and Nanda in [1] introduced and discussed the concept of rough groups and rough subgroups. Jun in [6] applied rough set theory to BCK-algebras. Recently, Rasouli in [12] introduced and studied the notion of roughness in MV-algebras. A new structure, called equality algebras, is introduced by Jenei in [4] and it is continued in [3, 5]. The study of equality algebras is motivated by EQ-algebras of Novák et al. in [10]. The equality algebra has two connectives, a meet operation and an equivalence, and a constant. Novák et al. in [10] introduced a closure operator in the class of equality algebras, and discussed relations between equality algebras and BCK-algebras.

Zebardast et al. in [13] have shown that there are relations among equality algebras and some of other logical algebras such as residuated lattice, MTL-algebra, BL-algebra, MV-algebra, Hertz-algebra, Heyting-algebra, Boolean-algebra, EQ-algebra and hoop-algebra. They found that under which conditions, equality algebras are equivalent to these logical algebras. Zebardast et al. in [13] also studied commutative equality algebras. They considered characterizations of commutative equality algebras.

In this paper we discuss the roughness of filter of an equality algebra. Using a filter D of an equality algebra E, we first define a congruence relation, so called a D-congruence relation, on E, and construct a D-lower and D-upper approximation and a D-approximation space. We investigate several properties of D-lower and D-upper approximation. We show that a D-lower (resp., D-upper) approximation is an interior (resp., closure) operator. In a D-approximation space, we define the notions of D-lower (resp. a D-upper) rough filter, and show that every filter containing D is both a D-lower and a D-upper rough filter. We provide a characterization of the definable subsets by using D-lower and D-upper approximation.

2. Preliminaries

In this section, we recollect some definitions and results which will be used in this paper.

Definition 2.1. [4] An algebraic structure $(E, \wedge, \sim, 1)$ is called an *equality algebra*, if for any $u, v, w \in E$ it satisfies in the following conditions.

(E1) $(E, \wedge, 1)$ is a commutative idempotent integral monoid,

(E2) the operation "$\sim$" is commutative,

(E3) $u \sim u = 1$,

(E4) $u \sim 1 = u$,

(E5) if $u \leq v \leq w$, then $u \sim w \leq v \sim w$ and $u \sim w \leq u \sim v$,

(E6) $u \sim v \leq (u \wedge w) \sim (v \wedge w)$,

(E7) $u \sim v \leq (u \sim w) \sim (v \sim w)$,

where $u \leq v$ if and only if $u \wedge v = u$.

In an equality algebra $(E, \wedge, \sim, 1)$, for any $u, v \in E$, we define an operation $\rightarrow$ (*implication*) on E by $u \rightarrow v := u \sim (u \wedge v)$.

PROPOSITION 2.2 ([4]). Let $(E, \wedge, \sim, 1)$ be an equality algebra. Then for any $u, v, w \in E$ the following assertions are valid.
(i) $u \rightarrow v = 1$ if and only if $u \leq v$,
(ii) $u \sim v = 1$ if and only if $u = v$,
(iii) $u \rightarrow (v \rightarrow w) = v \rightarrow (u \rightarrow w)$,
(iv) $1 \rightarrow u = u$, $u \rightarrow 1 = 1$ and $u \rightarrow u = 1$,
(v) $u \leq v \rightarrow w$ if and only if $v \leq u \rightarrow w$,
(vi) $u \leq v \rightarrow u$,
(vii) $u \leq (u \rightarrow v) \rightarrow v$,
$(viii)$ $u \rightarrow v \leq (v \rightarrow w) \rightarrow (u \rightarrow w)$,
(ix) if $v \leq u$, then $u \leftrightarrow v = u \rightarrow v = u \sim v$,
(x) if $u \leq v$, then $v \rightarrow w \leq u \rightarrow w$ and $w \rightarrow u \leq w \rightarrow v$,
(xi) $((u \rightarrow v) \rightarrow v) \rightarrow v = u \rightarrow v$.

An equality algebra E is *bounded* if there exists an element $0 \in E$ such that $0 \leq u$, for all $u \in E$. In a bounded equality algebra E, we define the negation "$\,'\,$" on E by $u' = u \rightarrow 0 = u \sim 0$, for all $u \in E$.

A subset D of E is called a *deductive system* (or *filter*) of E if for any $u, v \in E$, it satisfies in the following statements:

$(F1)$ If $u \leq v$ such that $u \in D$, then $v \in D$,
$(F2)$ If $u \in D$ and $u \sim v \in D$, then $v \in D$.

Denote by $\mathcal{DS}(E)$ the set of all deductive systems of E (see [5]).

Lemma 2.3. [3] *Let $(E, \sim, \wedge, 1)$ be an equality algebra. A subset D of E is a deductive system of E if and only if $1 \in D$ and for any $u, v \in E$ if $u \in D$ and $u \to v \in D$, then $v \in D$.*

Definition 2.4. [13] An equality algebra $(E, \wedge, \sim, 1)$ is called *commutative*, if for any $u, v \in E$,

$$(u \to v) \to v = (v \to u) \to u.$$

Let ϱ be an equivalence relation on a set E and let $\mathcal{P}(E)$ denote the power set of E. For all $x \in E$, let $[x]_\varrho$ denote the equivalence class of x with respect to ϱ. Let ϱ_* and ϱ^* be mappings from $\mathcal{P}(E)$ to $\mathcal{P}(E)$ defined by

$$\varrho_* : \mathcal{P}(E) \to \mathcal{P}(E),\ D \mapsto \{x \in E \mid [x]_\varrho \subseteq D\}$$

and

$$\varrho^* : \mathcal{P}(E) \to \mathcal{P}(E),\ D \mapsto \{x \in E \mid [x]_\varrho \cap D \neq \emptyset\},$$

respectively. The pair (E, ϱ) is called an *approximation space* based on ϱ. A subset D of E is called *definable* if $\varrho_*(D) = \varrho^*(D)$, and *rough* otherwise. The set $\varrho_*(D)$ (resp., $\varrho^*(D)$) is called the *lower* (resp. *upper*) *approximation.*

Notation. In the following, we suppose $(E, \wedge, \sim, 1)$ is an equality algebra with the induced operation "$\to$" (or simply denoted by E) and D is a filter of E, unless otherwise stated.

3. Roughness of filters

In this section, we define the notion of the lower and the upper approximations on equality algebras and investigate some properties of them. Also, we show that the lower and the upper approximations form an interior operator and a closure operator, respectively.

Let $\cong_D$ be a relation on E which is defined by

$$x \cong_D y \text{ if and only if } x \sim y \in D.$$

By routine caculation, it is clear that $\cong_D$ is an equivalence relation on E related to D. Further, we know that $\cong_D$ satisfies the following condition:

$$\text{if } u \cong_D v \text{ and } x \cong_D y, \text{ then } (u \sim x) \cong_D (v \sim y) \text{ and } (u \wedge x) \cong_D (v \wedge y).$$

Thus $\cong_D$ is a congruence relation on E and we say $\cong_D$ is the *D-congruence relation* on E. Denote by E/D the collection of all equivalence classes, that is, $E/D = \{D[x] \mid x \in E\}$. Then $D[1] = D$. For any $D[x], D[y] \in E/D$, define two binary operations "$\sqcap$" and "$\approx$" on E/D as follows:

$$D[x] \sqcap D[y] = D[x \wedge y] \quad \text{and} \quad D[x] \approx D[y] = D[x \sim y].$$

It is routine to verify that $(E/D, \sqcap, \approx, D[1])$ is an equality algebra, and for any $D[x], D[y] \in E/D$, the implication "$\rightsquigarrow$" on E/D is given by,

$$D[x] \rightsquigarrow D[y] = D[x \to y].$$

For the D-congruence relation $\cong_D$ on E, consider the mappings

$$\underline{appr}_D : \mathcal{P}(E) \to \mathcal{P}(E),\ L \mapsto \{x \in E \mid D[x] \subseteq L\},$$
$$\overline{appr}_D : \mathcal{P}(E) \to \mathcal{P}(E),\ L \mapsto \{x \in E \mid D[x] \cap L \neq \emptyset\},$$

which are called the *D-lower approximation* and the *D-upper approximation* of L, respectively. Then $(E, \cong_D)$ is an approximation space based on the filter D of E (briefly, *D-approximation space*), and it is denoted by (E, D). A subset L of E is said to be *definable* with respect to D if $\underline{appr}_D(L) = \overline{appr}_D(L)$, and *rough* otherwise.

The next proposition is similar to the Proposition 3.3 in [7].

PROPOSITION 3.1. [7] Let (E, D) be a D-approximation space. For any $L, M \in \mathcal{P}(E)$, we have

(i) $\underline{appr}_D(L) \subseteq L \subseteq \overline{appr}_D(L)$,

(ii) $\underline{appr}_D(L \cap M) = \underline{appr}_D(L) \cap \underline{appr}_D(M)$,

(iii) $\underline{appr}_D(L) \cup \underline{appr}_D(M) \subseteq \underline{appr}_D(L \cup M)$,

(iv) $\overline{appr}_D(L \cap M) \subseteq \overline{appr}_D(L) \cap \overline{appr}_D(M)$,

(v) $\overline{appr}_D(L) \cup \overline{appr}_D(M) = \overline{appr}_D(L \cup M)$,

(vi) $\underline{appr}_D(\overline{appr}_D(L)) \subseteq \overline{appr}_D(\overline{appr}_D(L))$,

(vii) $\underline{appr}_D(\underline{appr}_D(L)) \subseteq \overline{appr}_D(\underline{appr}_D(L))$,

(viii) $\underline{appr}_D(L^c) = (\overline{appr}_D(L))^c$,

(ix) $\overline{appr}_D(L^c) = \left(\underline{appr}_D(L)\right)^c$,

(x) $\underline{appr}_D(L) = \emptyset$ for $L \neq E$,

(xi) $\overline{appr}_D(L) = L$ for $L \neq \emptyset$,

(xii) $\underline{appr}_D(L) = L$ if and only if $\overline{appr}_D(L^c) = L^c$.

Definition 3.2. Suppose S is a set. A function $C : \mathcal{P}(S) \to \mathcal{P}(S)$ is called a *closure operator* on a set S if for all subsets $X, Y \subseteq S$, the following conditions hold:

(C_1) $X \subseteq C(X)$,

(C_2) if $X \subseteq Y$, then $C(X) \subseteq C(Y)$,

(C_3) $C(C(X)) = C(X)$.

Definition 3.3. Suppose S is a set. A function $int : \mathcal{P}(S) \to \mathcal{P}(S)$ is said to be an *interior operator* on a set S if for all subsets $X, Y \subseteq S$, the following conditions hold:

(i) $int(X) \subseteq X$,

(ii) if $X \subseteq Y$, then $int(X) \subseteq int(Y)$,

(iii) $int(int(X)) = int(X)$.

Theorem 3.4. *Let (E, D) be a D-approximation space. Then $\underline{appr}_D$ and $\overline{appr}_D$ are an interior operator and a closure operator, respectively.*

Proof: The proof is clear. □

Proposition 3.5. Let (E, D) be a D-approximation space. Then $D[x]$ is definable with respective to D, for all $x \in E$.

Proof: By Proposition 3.1(i), it is clear that $\underline{appr}_D(D[x]) \subseteq \overline{appr}_D(D[x])$, for all $x \in E$. Let $y \in \overline{appr}_D(D[x])$. Then $D[y] \cap D[x] \neq \emptyset$, and so

$D[x] = D[y]$. Thus $y \in \underline{appr}_D(D[x])$. Therefore, $D[x]$ is definable with respective to D for all $x \in E$. □

PROPOSITION 3.6. Let (E, D) be a D-approximation space with $D = \{1\}$. Then every subset of E is definable.

PROOF: The proof is straightforward. □

COROLLARY 3.7. Every equality algebra is definable with respect to any filter.

PROOF: The proof is clear. □

PROPOSITION 3.8. Let $\cong_D$ and $\cong_B$ be equivalence relations on E related to filters D and B respectively. If $D \subseteq B$, then $\cong_D \subseteq \cong_B$.

PROOF: Let $x, y \in E$ such that $x \cong_D y$. Then $x \sim y \in D \subseteq B$, which implies that $x \cong_B y$. Hence $\cong_D \subseteq \cong_B$. □

For any subsets D and B of E, we define

$$D \wedge B = \{u \wedge v \mid u \in D, v \in B\}, \quad D \sim B = \{u \sim v \mid u \in D, v \in B\},$$

$$\text{and } D \to B = \{u \to v \mid u \in D, v \in B\}.$$

If either D or B is empty, then we define $D \wedge B = \emptyset$, $D \sim B = \emptyset$ and $D \to B = \emptyset$. It is clear that $D \to B = (D \wedge B) \sim D$.

PROPOSITION 3.9. Let (E, D) be a D-approximation space. Given a D-congruence relation $\cong_D$ on E, if $L, M \in \mathcal{P}(E)$, then

(i) $\overline{appr}_D(L) \to \overline{appr}_D(M) \subseteq \overline{appr}_D(L \to M)$,

(ii) $\overline{appr}_D(L) \wedge \overline{appr}_D(M) \subseteq \overline{appr}_D(L \wedge M)$,

(iii) $\overline{appr}_D(L) \sim \overline{appr}_D(M) \subseteq \overline{appr}_D(L \sim M)$.

PROOF: (i) Let $w \in \overline{appr}_D(L) \to \overline{appr}_D(M)$. Then $w = u \to v$ for some $u \in \overline{appr}_D(L)$ and $v \in \overline{appr}_D(M)$, and so $D[u] \cap L \neq \emptyset$ and $D[v] \cap M \neq \emptyset$. It follows that there are $x, y \in E$ such that $x \in D[u] \cap L$ and $y \in D[v] \cap M$. Since $\cong_D$ is a D-congruence relation on E, we have

$$x \to y \in D[u] \to D[v] = D[u \to v] = D[w].$$

Since $x \to y \in L \to M$, it follows that $x \to y \in D[w] \cap (L \to M)$, and so $w \in \overline{appr}_D(L \to M)$. Hence, (i) is valid.

(ii) Let $w \in \overline{appr}_D(L) \wedge \overline{appr}_D(M)$. Then $w = u \wedge v$ for some $u \in \overline{appr}_D(L)$ and $v \in \overline{appr}_D(M)$. Since $u \in \overline{appr}_D(L)$ and $v \in \overline{appr}_D(M)$, there exist $x \in D[u] \cap L$ and $y \in D[v] \cap M$. It follows that $x \cong_D u$ and $y \cong_D v$. Since $\cong_D$ is a congruence relation on E, we have $x \wedge y \cong_D u \wedge v = w$. Then $x \wedge y \in D[u \wedge v] = D[w]$ and $x \wedge y \in L \wedge M$. Hence $x \wedge y \in D[w] \cap (L \wedge M)$, that is, $D[w] \cap (L \wedge M) \neq \emptyset$, and so $w \in \overline{appr}_D(L \wedge M)$. Therefore

$$\overline{appr}_D(L) \wedge \overline{appr}_D(M) \subseteq \overline{appr}_D(L \wedge M).$$

(iii) The proof is similar to the proof of (ii). □

PROPOSITION 3.10. For a D-approximation space (E, D) and any $L, M \in \mathcal{P}(E)$, we have
(i) $\underline{appr}_D(L) \to \underline{appr}_D(M) \subseteq \underline{appr}_D(L \to M)$.
(ii) $\underline{appr}_D(L) \wedge \underline{appr}_D(M) \subseteq \underline{appr}_D(L \wedge M)$.
(iii) $\underline{appr}_D(L) \sim \underline{appr}_D(M) \subseteq \underline{appr}_D(L \sim M)$.

PROOF: (i) Let $w \in \underline{appr}_D(L) \to \underline{appr}_D(M)$. Then $w = u \to v$ for some $u \in \underline{appr}_D(L)$ and $v \in \underline{appr}_D(M)$. Hence $D[u] \subseteq L$ and $D[v] \subseteq M$. It follows that

$$D[u \to v] = D[u] \to D[v] \subseteq L \to M.$$

Then $w = u \to v \in \underline{appr}_D(L \to M)$.

(ii) If $x \in \underline{appr}_D(L) \wedge \underline{appr}_D(M)$, then there exist $u \in \underline{appr}_D(L)$ and $v \in \underline{appr}_D(M)$ such that $x = u \wedge v$, $D[u] \subseteq L$ and $D[v] \subseteq M$. It follows that

$$D[x] = D[u \wedge v] = D[u] \wedge D[v] \subseteq L \wedge M.$$

Hence $x \in \underline{appr}_D(L \wedge M)$, and therefore

$$\underline{appr}_D(L) \wedge \underline{appr}_D(M) \subseteq \underline{appr}_D(L \wedge M).$$

(iii) Let $x \in \underline{appr}_D(L) \sim \underline{appr}_D(M)$. Then $x = u \sim v$ for some $u \in \underline{appr}_D(L)$ and $v \in \underline{appr}_D(M)$. Thus $D[u] \subseteq L$ and $D[v] \subseteq M$, which imply that

$$D[x] = D[u \sim v] = D[u] \sim D[v] \subseteq L \sim M.$$

Hence $x \in \underline{appr}_D(L \sim M)$. □

PROPOSITION 3.11. Let (E, D) be a D-approximation space and $L, M \in \mathcal{P}(E)$. If $\underline{appr}_D(L \sim M) = \emptyset$ (resp., $\underline{appr}_D(L \wedge M) = \emptyset$ and $\underline{appr}_D(L \to M) = \emptyset$), then $\underline{appr}_D(L) = \emptyset$ or $\underline{appr}_D(M) = \emptyset$.

PROOF: Let $L, M \in \mathcal{P}(E)$ such that $\underline{appr}_D(L) \neq \emptyset$ and $\underline{appr}_D(M) \neq \emptyset$. Then there exist $u \in \underline{appr}_D(L)$ and $v \in \underline{appr}_D(M)$, such that $D[u] \subseteq L$ and $D[v] \subseteq M$. Since $u \in D[u]$ and $v \in D[v]$, we have $u \in L$ and $v \in M$. Then $u \sim v \in L \sim M$, and so

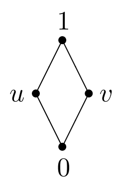
$$u \sim v \in D[u \sim v] = D[u] \sim D[v] \subseteq L \sim M.$$

Hence $\underline{appr}_D(L \sim M) \neq \emptyset$, which is a contradiction. Therefore, $\underline{appr}_D(L) = \emptyset$ or $\underline{appr}_D(M) = \emptyset$.

The proof of other cases is similar. □

DEFINITION 3.12. Let (E, D) be a D-approximation space. A subset L of E is called a *D-lower* (resp. a *D-upper*) *rough filter* of E if $\underline{appr}_D(L)$ (resp., $\overline{appr}_D(L)$) is a filter of E. If L is both a D-lower and a D-upper filters of E, then L is called a *D-rough filter* of E.

Example 3.13. Let $E = \{0, u, v, 1\}$ be a set with the following Hasse diagram.



Then $(E, \wedge, 1)$ is a commutative idempotent integral monoid. We define a binary operation "$\sim$" on E by the following table.

$\sim$	0	u	v	1
0	1	v	u	0
u	v	1	0	u
v	u	0	1	v
1	0	u	v	1

Then $\mathcal{E} = (E, \wedge, \sim, 1)$ is an equality algebra, and the implication "$\rightarrow$" is given by the following Cayley table.

$\rightarrow$	0	u	v	1
0	1	1	1	1
u	v	1	v	1
v	u	u	1	1
1	0	u	v	1

Consider a D-approximation space (E, D) where $D = \{u, 1\}$ is a filter of E. Then $D[u] = D[1] = \{u, 1\}$ and $D[v] = D[0] = \{v, 0\}$. For a subset $L = \{0, u, 1\}$ of E, we have

$$\underline{appr}_D(L) = \{x \in E \mid D[x] \subseteq \{0, u, 1\}\} = \{u, 1\},$$

and

$$\overline{appr}_D(L) = \{x \in E \mid D[x] \cap \{0, u, 1\} \neq \emptyset\} = \{0, u, v, 1\},$$

are filters of E. Hence D is a D-rough filter of E. If we take a subset $M = \{v\}$ of E, then $\underline{appr}_D(M) = \emptyset$ and $\overline{appr}_D(M) = \{0, v\}$ are not filters of E. Hence D is not a D-rough filter of E. Also, if we take a subset $K = \{u, 1\}$ of E, then $\underline{appr}_D(K) = \emptyset$ that is not a filter of E and $\overline{appr}_D(K) = \{u, 1\}$ is a filter of E. Hence D is a D-upper rough filter of E.

The extension theorem of D-upper rough filter of E is obtained from the following theorem.

Theorem 3.14. *Let (E, D) be a D-approximation space. Then every filter L of E which contains D is a D-upper rough filter of E.*

Proof: Let L be a filter of E such that $D \subseteq L$. Then $D[1] \cap L \neq \emptyset$, and so $1 \in \overline{appr}_D(L)$. Suppose $x, y \in E$ such that $x \in \overline{appr}_D(L)$ and $x \sim y \in \overline{appr}_D(L)$. Then $D[x] \cap L \neq \emptyset$ and $D[x \sim y] \cap L \neq \emptyset$, which imply that there exist $u, v \in L$ such that $u \in D[x]$ and $v \in D[x \sim y]$. Hence $u \cong_D x$ and $v \cong_D (x \sim y)$. It follows that $u \sim x \in D \subseteq L$ and $v \sim (x \sim y) \in D \subseteq L$. Since $u, v \in L$ and L is a filter of E, we have $x \in L$ and $x \sim y \in L$, and so $y \in L$. Note that $y \in D[y]$, and so $y \in D[y] \cap L$. Hence $y \in \overline{appr}_D(L)$, and therefore $\overline{appr}_D(L)$ is a filter of E, that is, L is a D-upper rough filter of E. □

Corollary 3.15. Let (E, D) be a D-approximation space with $D = \{1\}$. Then every filter L of E is a D-upper rough filter of E.

In the following example we show that the converse of Theorem 3.14 is not true, in general.

Example 3.16. Let E be the equality algebra as in Example 3.13 and (E, D) be a D-approximation space of E. Suppose $D = \{u, 1\}$ is a filter of E and $\cong_D$ is an equivalence relation on E related to D. Then $D[0] = \{0, v\} = D[v]$ and $D[u] = D = D[1]$. Let $L = \{v, 1\}$ be a subset of E. Then L does not contain D and

$$\overline{appr}_D(L) = \{x \in E \mid D[x] \cap L \neq \emptyset\} = E.$$

Thus L is a D-upper rough filter of E.

THEOREM 3.17. *Let (E, D) be a D-approximation space. Then every filter L of E which contains D is a D-lower rough filter of E.*

PROOF: Let L be a filter of E such that $D \subseteq L$. Since $D = D[1]$, if $x \in D[1]$, then $x \in D \subseteq L$, and so $D[1] \subseteq L$. Hence $1 \in \underline{appr}_D(L)$. Let $x, y \in E$ such that $x \in \underline{appr}_D(L)$ and $x \sim y \in \underline{appr}_D(L)$. Then $D[x] \subseteq L$ and $D[x] \sim D[y] = D[x \sim y] \subseteq L$. Let $u \in D[x]$ and $v \in D[y]$. Then $u \cong_D x$ and $v \cong_D y$, which imply that $(u \sim v) \cong_D (x \sim y)$, that is, $u \sim v \in D[x \sim y] \subseteq L$. Since $u \in L$ and L is a filter of E, we get $v \in L$ and $D[y] \subseteq L$. Thus $y \in \underline{appr}_D(L)$, and therefore $\underline{appr}_D(L)$ is a filter of E. Consequently, L is a D-lower rough filter of E. □

COROLLARY 3.18. Let (E, D) be a D-approximation space such that $D = \{1\}$. Then every filter L of E is a D-lower rough filter of E.

PROPOSITION 3.19. Let (E, D) be a D-approximation space. For any subset L of E, we have

(i) $D \subseteq L$ if and only if $D \subseteq \underline{appr}_D(L)$.

(ii) $L \subseteq D$ if and only if $\overline{appr}_D(L) = D$.

PROOF: (i) Assume that $D \subseteq L$. If $x \in D$, then $D[x] = D \subseteq L$. Hence $x \in \underline{appr}_D(L)$, and so $D \subseteq \underline{appr}_D(L)$. By Proposition 3.1(i), the proof of converse is clear.

(ii) Suppose $L \subseteq D$ and $x \in \overline{appr}_D(L)$. Then $D[x] \cap L \neq \emptyset$, and thus there exists $y \in D[x] \cap L$ which implies that $D[x] = D[y]$ and $y \in L$. Hence $D[y] = D$, and so $x \in D$. This shows that $\overline{appr}_D(L) \subseteq D$. Let $z \in D$.

Then $D[z] = D$ and so $D[z] \cap L = D \cap L \neq \emptyset$. Thus $z \in \overline{appr}_D(L)$, that is, $D \subseteq \overline{appr}_D(L)$. By Proposition 3.1(i), the proof of converse is clear. □

Corollary 3.20. Let (E, D) be a D-approximation space. If L is a filter of E such that $L \subseteq D$, then L is a D-upper rough filter of E.

Theorem 3.21. *If L is a filter in a D-approximation space (E, D), then*

(i) $D \subseteq \overline{appr}_D(L)$.

(ii) $D \subseteq L$ *if and only if* $\underline{appr}_D(L) \subseteq L = \overline{appr}_D(L)$.

Proof: (i) Let $x \in D$. Since $x \in D[x]$, it is clear that $1 \in D[x]$. Moreover, since L is a filter in a D-approximation space (E, D), we have $1 \in L$ and so $1 \in D[x] \cap L$. Hence $x \in \overline{appr}_D(L)$, and therefore $D \subseteq \overline{appr}_D(L)$.

(ii) Assume that $D \subseteq L$. Then by Proposition 3.1(i), $\underline{appr}_D(L) \subseteq L \subseteq \overline{appr}_D(L)$. Let $x \in \overline{appr}_D(L)$. Then $D[x] \cap L \neq \emptyset$ and thus there exists $u \in L$ such that $u \in D[x]$. Since $D \subseteq L$, it follows that $u \sim x \in D \subseteq L$. Hence $x \in L$ and so $\overline{appr}_D(L) \subseteq L$.

Conversely, suppose $\underline{appr}_D(L) \subseteq L = \overline{appr}_D(L)$ and $x \in D$. Since D and L are filters, we get $1 \in D \cap L = D[x] \cap L$. Hence $x \in \overline{appr}_D(L) = L$. Therefore $D \subseteq L$. □

Corollary 3.22. If L is a filter of a D-approximation space (E, D), then

$$\underline{appr}_D(L) = L = \overline{appr}_D(L),$$

and L is a D-rough filter of E.

For any nonempty subset L of E, we let $L' = \{x' \mid x \in L\}$. It is clear that if L and M are nonempty subsets of E, then $L \subseteq M$ staisfies $L' \subseteq M'$.

Proposition 3.23. In a D-approximation space (E, D), for any $L \in \mathcal{P}(E) \setminus \{\emptyset\}$, we have $(\overline{appr}_D(L))' \subseteq \overline{appr}_D(L')$.

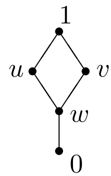
Proof: Let $u \in (\overline{appr}_D(L))'$ for any nonempty subset L of E. Then $u = x'$ for some $x \in \overline{appr}_D(L)$ and so $D[x] \cap L \neq \emptyset$. It follows that there exists $v \in L$ such that $v \in D[x]$, which implies that $v' \in L'$ and $v \sim x \in D$. By (E2) and (E7) we have

$$v \sim x = x \sim v \leq (x \sim 0) \sim (v \sim 0) = x' \sim v'.$$

Since D is a filter of E and $u = x'$, it follows that $u \sim v' = x' \sim v' \in D$. Hence $v' \in D[u] \cap L'$, that is, $D[u] \cap L' \neq \emptyset$. Therefore $u \in \overline{appr}_D(L')$ which shows that $(\overline{appr}_D(L))' \subseteq \overline{appr}_D(L')$. □

The next example shows that the converse of Proposition 3.23 is not true in general.

Example 3.24. Let $E = \{0, u, v, w, 1\}$ be a set with the following Hasse diagram.



Then $(E, \wedge, 1)$ is a commutative idempotent integral monoid. We define a binary operation "$\sim$" on E by Table 1.

Table 1. Table of the implication "$\sim$"

$\sim$	0	u	v	w	1
0	1	0	0	0	0
u	0	1	w	v	u
v	0	w	1	u	v
w	0	v	u	1	w
1	0	u	v	w	1

Then $\mathcal{E} = (E, \wedge, \sim, 1)$ is an equality algebra, and the implication "$\to$" is given by Table 2.

Table 2. Table of the implication "$\to$"

$\to$	0	u	v	w	1
0	1	1	1	1	1
u	0	1	v	v	1
v	0	u	1	u	1
w	0	1	1	1	1
1	0	u	v	w	1

Let $D = \{u, 1\}$. It is clear that D is a filter of E. Let $\cong_D$ be an equivalence relation on E related to D. Then $D[1] = D[u] = \{u, 1\}$, $D[w] = D[v] = \{v, w\}$ and $D[0] = \{0\}$. If $L = \{0, u\}$, then $L' = \{0, 1\}$. Thus

$$\overline{appr}_D(L') = \{0, u, 1\} \quad , \quad \overline{appr}_D(L) = \{0, u, 1\}.$$

But $(\overline{appr}_D(L))' = (\{0, u, 1\})' = \{0, 1\}$. Hence $\overline{appr}_D(L') \nsubseteq (\overline{appr}_D(L))'$.

In the following example, we show that there exists a nonempty subset L of E such that $\underline{appr}_D(L') \nsubseteq (\underline{appr}_D(L))'$.

Example 3.25. Let (E, D) be a D-approximation space where E be the equality algebra as in Example 3.13 and $D = \{u, 1\}$ be a filter of E. If $L = \{u, 0\}$, then $L' = \{0, 1\}$. Thus $\underline{appr}_D(L') = \{0\}$ and $\underline{appr}_D(L) = \{0\}$, and so $(\underline{appr}_D(L))' = \{1\}$. Hence $\underline{appr}_D(L') \nsubseteq (\underline{appr}_D(L))'$.

Proposition 3.26. Let (E, D) be a D-approximation space and L be a nonempty subset of E. Then
(i) $\mathcal{R}(E) \cap \overline{appr}_D(L') \subseteq (\overline{appr}_D(L''))'$,
(ii) $\mathcal{R}(E) \cap \overline{appr}_D((L \cap \mathcal{R}(E))') \subseteq (\overline{appr}_D(L))'$,
where $\mathcal{R}(E) := \{x \in E \mid x'' = x\}$.

Proof: (i) Let $z \in \mathcal{R}(E) \cap \overline{appr}_D(L')$. Then $z'' = z$ and $D[z] \cap L' \neq \emptyset$, which imply that there exists $x \in L$ such that $D[x'] = D[z]$. Hence

$$D[z'] \cap L'' = D[x''] \cap L'' \neq \emptyset,$$

i.e., $z' \in \overline{appr}_D(L'')$. Therefore $z \in (\overline{appr}_D(L''))'$.
(ii) Let $u \in \mathcal{R}(E) \cap \overline{appr}_D((L \cap \mathcal{R}(E))')$. Then $u'' = u$ and $D[u] \cap (L \cap \mathcal{R}(E))' \neq \emptyset$. It follows that there exists $x \in L \cap \mathcal{R}(E)$ such that $D[u] = D[x']$ and $x'' = x$. Hence

$$D[u'] \cap L = D[x''] \cap L = D[x] \cap L \neq \emptyset,$$

and so $u' \in \overline{appr}_D(L)$, i.e., $u \in (\overline{appr}_D(L))'$. Therefore

$$\mathcal{R}(E) \cap \overline{appr}_D((L \cap \mathcal{R}(E))') \subseteq (\overline{appr}_D(L))'. \qquad \square$$

Lemma 3.27. *If E is a bounded equality algebra, then the set*

$$\mathcal{E}(E) := \{x \in E \mid x' = 0\},$$

is a filter of E.

PROOF: Obviously $1 \in \mathcal{E}(E)$. Let $x, y \in E$ such that $x \in \mathcal{E}(E)$ and $x \to y \in \mathcal{E}(E)$. Then $x' = 0$ and $(x \to y)' = 0$. Since $y \leq y''$, by Proposition 2.2(x), we get $x \to y \leq x \to y'' = y' \to x'$. Hence

$$y' = y''' = (y' \to 0)' = (y' \to x')' \leq (x \to y)' = 0$$

and so $y' = 0$, that is, $y \in \mathcal{E}(E)$. Therefore $\mathcal{E}(E)$ is a filter of E. □

PROPOSITION 3.28. Let (E, D) be a D-approximation space and L be a nonempty subset of E. Then

$$D \subseteq \overline{appr}_D(\mathcal{E}(E)) \subseteq \{y \in E \mid y'' \in D\}. \tag{3.1}$$

PROOF: Using Lemma 3.27 and Theorem 3.21(i), we get $D \subseteq \overline{appr}_D(\mathcal{E}(E))$. Let $x \in \overline{appr}_D(\mathcal{E}(E))$. Then $D[x] \cap \mathcal{E}(E) \neq \emptyset$ and so there exists $u \in D[x]$ such that $u' = 0$. Thus $u \sim x \in D$. By (E2) and (E7), $u \sim x \leq (x \sim 0) \sim (u \sim 0) = x' \sim u'$ and D is a filter of E, we have $x' \sim u' \in D$. Thus by (E2), $x'' = 0 \sim x' = u' \sim x' \in D$. Therefore $\overline{appr}_D(\mathcal{E}(E)) \subseteq \{y \in E \mid y'' \in D\}$. □

We provide conditions for a nonempty subset to be definable.

THEOREM 3.29. *Let (E, D) be a D-approximation space. Then a nonempty subset L of E is definable with respect to D if and only if $\underline{appr}_D(L) = L$ or $\overline{appr}_D(L) = L$.*

PROOF: Assume that L is definable with respect to D. Then $L \subseteq \overline{appr}_D(L) = \underline{appr}_D(L) \subseteq L$ and so

$$\overline{appr}_D(L) = \underline{appr}_D(L) = L.$$

Conversely, suppose that $\underline{appr}_D(L) = L$ or $\overline{appr}_D(L) = L$. For the case $\underline{appr}_D(L) = L$, let $x \in \overline{appr}_D(L)$. Then $D[x] \cap L \neq \emptyset$ which implies that $D[x] = D[z]$ for some $z \in L$. It follows from $\underline{appr}_D(L) = L$ that $D[x] = D[z] \subseteq L$. Hence $x \in L$, and therefore $\overline{appr}_D(L) \subseteq L$. Consequently, $\overline{appr}_D(L) = L$. Suppose that $\overline{appr}_D(L) = L$. For any $x \in L$ let $z \in D[x]$. Then $D[z] \cap L = D[x] \cap L \neq \emptyset$ and so $z \in \overline{appr}_D(L) = L$. This shows that $D[x] \subseteq L$, that is, $x \in \underline{appr}_D(L)$. Hence $L \subseteq \underline{appr}_D(L)$, and so $\underline{appr}_D(L) = L$. Therefore L is definable with respect to D. □

4. Conclusions and future works

In this paper the notion of the lower and the upper approximations are introduced on equality algebras and some properties of them are investigated. Moreover, the relation among the lower and the upper approximations with an interior operator and a closure operator are investigated. Also, the conditions for a nonempty subset to be definable are provided. Also, due to the importance of this subject in the field of decision making, we decided to introduce these concepts on equality algebras in order to introduce concepts related to rough soft and soft rough equality algebras and fuzzification of them in the future. Moreover, in the future further study is possible in the direction of roughness with different types of filters and ideals in equality algebras.

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