Bulletin of the **S**ection of **L**ogic
Volume 49/1 (2020), pp. 13–51

http://dx.doi.org/10.18778/0138-0680.2020.02

Aldo V. Figallo, Inés Pascual, Gustavo Pelaitay

A TOPOLOGICAL APPROACH TO TENSE LM$_{n\times m}$-ALGEBRAS

Abstract

In 2015, tense $n \times m$-valued Łukasiewicz–Moisil algebras (or tense $LM_{n\times m}$-algebras) were introduced by A. V. Figallo and G. Pelaitay as an generalization of tense n-valued Łukasiewicz–Moisil algebras. In this paper we continue the study of tense $LM_{n\times m}$-algebras. More precisely, we determine a Priestley-style duality for these algebras. This duality enables us not only to describe the tense $LM_{n\times m}$-congruences on a tense $LM_{n\times m}$-algebra, but also to characterize the simple and subdirectly irreducible tense $LM_{n\times m}$-algebras.

Keywords: Tense $LM_{n\times m}$-algebras, Priestley-style topological duality, Priestley spaces, tense De Morgan algebras.

1. Introduction

In 1975, Suchoń ([36]) defined matrix Łukasiewicz algebras so generalizing n-valued Łukasiewicz algebras without negation ([29]). In 2000, A. V. Figallo and C. Sanza ([23]) introduced $n \times m$-valued Łukasiewicz algebras with negation which are both a particular case of matrix Łukasiewicz algebras and a generalization of n-valued Łukasiewicz–Moisil algebras ([1]). It is worth noting that unlike what happens in n-valued Łukasiewicz–Moisil algebras, generally the De Morgan reducts of $n \times m$-valued Łukasiewicz algebras with negation are not Kleene algebras. Furthermore, in [34] an important example which legitimated the study of this new class of algebras is provided. Following the terminology established in [1], these algebras were called $n \times m$-valued Łukasiewicz–Moisil algebras (or $\mathrm{LM}_{n\times m}$-algebras for short). $LM_{n\times m}$-algebras were studied in [24, 25, 15, 34] and [35].

Propositional logics usually do not incorporate the dimension of time; consequently, in order to obtain a tense logic, a propositional logic is enriched by the addition of new unary operators (or connectives) which are usually denoted by G, H, F and P. We can define F and P by means of G and H as follows: $F(x) = \neg G(\neg x)$ and $P(x) = \neg H(\neg x)$, where $\neg x$ denotes negation of the proposition x. Tense algebras (or tense Boolean algebras) are algebraic structures corresponding to the propositional tense logic (see [4, 19]). An algebra $\langle A, \vee, \wedge, \neg, G, H, 0, 1\rangle$ is a tense algebra if $\langle A, \vee, \wedge, \neg, 0, 1\rangle$ is a Boolean algebra and G, H are unary operators on A which satisfy the following axioms for all x, $y \in A$:

$$G(1) = 1,\ H(1) = 1,$$

$$G(x \wedge y) = G(x) \wedge G(y),\ H(x \wedge y) = H(x) \wedge H(y),$$

$$x \leq GP(x),\ x \leq HF(x),$$

where $P(x) = \neg H(\neg x)$ and $F(x) = \neg G(\neg x)$.

Taking into account that tense algebras constitute the algebraic basis for the bivalent tense logic, D. Diaconescu and G. Georgescu introduced in [12] the tense MV-algebras and the tense Łukasiewicz–Moisil algebras (or tense n-valued Łukasiewicz–Moisil algebras) as algebraic structures for some many-valued tense logics. In recent years, these two classes of algebras have become very interesting for several authors (see [2, 6, 8, 9, 15, 7, 17, 18]). In particular, in [8, 9], Chiriţă, introduced tense θ-valued Łukasiewicz–Moisil algebras and proved an important representation theorem which made it possible to show the completeness of the tense θ-valued Moisil logic (see [8]). In [12], the authors formulated an open problem about representation of tense MV-algebras, this problem was solved in [21, 3] for semisimple tense MV-algebras. Also, in [2], tense basic algebras which are an interesting generalization of tense MV-algebras, were studied.

The main purpose of this paper is to give a topological duality for tense $n \times m$-valued Łukasiewicz–Moisil algebras. In order to achieve this we will extend the topological duality given in [27], for $n \times m$-valued Łukasiewicz–Moisil algebras. In [35] another duality for $n \times m$-valued Łukasiewicz–Moisil algebras was developed, starting from De Morgan spaces and adding a family of continuous functions.

The paper is organized as follows: In Section 2, we briefly summarize the main definitions and results needed throughout this article. In Section 3, we developed a topological duality for tense $n \times m$-valued Łukasiewicz–Moisil algebras, extending the one obtained in [27] for $n \times m$-valued Łukasiewicz–Moisil algebras. In Section 4, the results of Section 3 are applied. Firstly, we characterize congruences on tense $n \times m$-valued Łukasiewicz–Moisil algebras by certain closed and increasing subsets of the space associated with them. This enables us to describe the subdirectly irreducible tense $n \times m$-valued Łukasiewicz–Moisil algebras and the simple tense $n \times m$-valued Łukasiewicz–Moisil algebras.

2. Preliminaries

2.1. Tense De Morgan algebras

In [16] A. V. Figallo and G. Pelaitay introduced the variety of algebras, which they call tense De Morgan algebras, and they also developed a representation theory for this class of algebras.

First, recall that an algebra $\langle A, \vee, \wedge, \sim, 0, 1\rangle$ is a De Morgan algebra if $\langle A, \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice and $\sim$ is a unary operation on A satisfying the following identities for all x, $y \in A$:

1. $\sim (x \vee y) = \sim x \wedge \sim y$,
2. $\sim\sim x = x$,
3. $\sim 0 = 1$.

In what follows a De Morgan algebra $\langle A, \vee, \wedge, \sim, 0, 1\rangle$ will be denoted briefly by $(A, \sim)$.

Definition 1. An algebra $(A, \sim, G, H)$ is a tense De Morgan algebra if $(A, \sim)$ is a De Morgan algebra and G and H are two unary operations on A such that for any $x, y \in A$:

1. $G(1) = 1$ and $H(1) = 1$,
2. $G(x \wedge y) = G(x) \wedge G(y)$ and $H(x \wedge y) = H(x) \wedge H(y)$,
3. $x \leq GP(x)$ and $x \leq HF(x)$, where $F(x) = \sim G(\sim x)$ and $P(x) = \sim H(\sim x)$,
4. $G(x \vee y) \leq G(x) \vee F(y)$ and $H(x \vee y) \leq H(x) \vee P(y)$.

In [16] a duality for tense De Morgan algebras is described taking into account the results established by W. Cornish and P. Fowler in [11]. To this purpose, the topological category **tmPS** of tmP-spaces and tmP-functions was considered, which we indicate below:

DEFINITION 2. A tense De Morgan space (or tmP-space) is a system (X, g, R, R^{-1}), where

(i) (X, g) is an mP-space ([11]). More precisely,
 (mP1) X is a Priestley space (or P-space),
 (mP2) $g : X \longrightarrow X$ is an involutive homeomorphism and an anti-isomorphism,

(ii) R is a binary relation on X and R^{-1} is the converse of R such that:
 (tS1) For each $U \in D(X)$ it holds that $G_R(U)$, $H_{R^{-1}}(U) \in D(X)$, where G_R and $H_{R^{-1}}$ are two operators on $\mathcal{P}(X)$ defined for any $U \subseteq X$ as follows:

$$G_R(U) = \{x \in X \mid R(x) \subseteq U\}, \tag{2.1}$$

$$H_{R^{-1}}(U) = \{x \in X \mid R^{-1}(x) \subseteq U\}, \tag{2.2}$$

 and $D(X)$ is the set of all increasing and clopen subsets of X,
 (tS2) $(x, y) \in R$ implies $(g(x), g(y)) \in R$ for any x, $y \in X$,
 (tS3) for each $x \in X$, $R(x)$ is a closed set in X,
 (tS4) for each $x \in X$, $R(x) = \downarrow R(x) \cap \uparrow R(x)$, where $\downarrow Y$ ($\uparrow Y$) denotes the set of all $x \in X$ such that $x \leq y$ ($y \leq x$) for some $y \in Y \subseteq X$.

DEFINITION 3. A tmP-function from a tmP-space $(X_1, g_1, R_1, R_1^{-1})$ into another one, $(X_2, g_2, R_2, R_2^{-1})$, is a continuous and increasing function (P-function) $f : X_1 \longrightarrow X_2$, which satisfies the following conditions:

(mf) $f \circ g_1 = g_2 \circ f$ (mP-function [11]),
(tf1) $(x, y) \in R_1$ implies $(f(x), f(y)) \in R_2$ for any $x, y \in X_1$,
(tf2) if $(f(x), y) \in R_2$, then there is an element $z \in X_1$ such that $(x, z) \in R_1$ and $f(z) \leq y$,
(tf3) if $(y, f(x)) \in R_2$, then there is an element $z \in X_1$ such that $(z, x) \in R_1$ and $f(z) \leq y$.

Next, A. V. Figallo and G. Pelaitay (see [16, Section 5]) showed that the category **tmPS** is dually equivalent to the category **TDMA** of tense De Morgan algebras and tense De Morgan homomorphisms. The following results are used to show the dual equivalence:

- Let (X, g, R, R^{-1}) be a tmP-space. Then, $(D(X), \sim_g, G_R, H_{R^{-1}})$ is a tense De Morgan algebra, where for all $U \in D(X)$, $\sim_g U$ is defined by
$$\sim_g U = X \setminus g(U), \tag{2.3}$$
and $G_R(U)$ and $H_{R^{-1}}(U)$ are defined as in (2.1) and (2.2), respectively.
- Let $(A, \sim, G, H)$ be a tense De Morgan algebra and $X(A)$ be the Priestley space associated with A, i.e. $X(A)$ is the set of all prime filters of A, ordered by inclusion and with the topology having as a sub-basis the following subsets of $X(A)$:
$$\sigma_A(a) = \{S \in X(A) : a \in S\} \text{ for each } a \in A, \tag{2.4}$$
and
$$X(A) \setminus \sigma_A(a) \text{ for each } a \in A.$$
Then, $(X(A), g_A, R^A_G, R^A_H)$ is a tmP-space, where $g_A(S)$ is defined by
$$g_A(S) = \{x \in A : \sim x \notin S\}, \text{ for all } S \in X(A), \tag{2.5}$$
and the relations R^A_G and R^A_H are defined for all $S, T \in X(A)$ as follows:
$$(S,T) \in R^A_G \Longleftrightarrow G^{-1}(S) \subseteq T \subseteq F^{-1}(S), \tag{2.6}$$
$$(S,T) \in R^A_H \Longleftrightarrow H^{-1}(S) \subseteq T \subseteq P^{-1}(S). \tag{2.7}$$
- Let $(A, \sim, G, H)$ be a tense De Morgan algebra; then, the function $\sigma_A : A \longrightarrow D(X(A))$ is a tense De Morgan isomorphism, where σ_A is defined as in (2.4).
- Let (X, g, R, R^{-1}) be a tmP-space; then, $\varepsilon_X : X \longrightarrow X(D(X))$ is an isomorphism of tmP-spaces, where ε_X is defined by
$$\varepsilon_X(x) = \{U \in D(X) : x \in U\}, \text{ for all } x \in X. \tag{2.8}$$

- Let $h : (A_1, \sim_1, G_1, H_1) \longrightarrow (A_2, \sim_2, G_2, H_2)$ be a tense De Morgan morphism. Then, the map $\Phi(h) : X(A_2) \longrightarrow X(A_1)$ is a morphism of tmP-spaces, where

$$\Phi(h)(S) = h^{-1}(S), \text{ for all } S \in X(A_2). \tag{2.9}$$

- Let $f : (X_1, g_1, R_1, R_1^{-1}) \longrightarrow (X_2, g_2, R_2, R_2^{-1})$ be a morphism of tmP-spaces. Then, $\Psi(f) : D(X_2) \longrightarrow D(X_1)$ is a tense De Morgan morphism, where

$$\Psi(f)(U) = f^{-1}(U), \text{ for all } U \in D(X_2). \tag{2.10}$$

In [16], the duality described above was used to characterize the congruence lattice $Con_{tM}(A)$ of a tense De Morgan algebra $(A, \sim, G, H)$. First the following notion was introduced:

Definition 4. Let $(X, \leq, g, R, R^{-1})$ be a tmP-space. An involutive (i.e. $Y = g(Y)$ [11]) closed subset Y of X is a tmP-subset if it satisfies the following conditions for $u, v \in X$:

(ts1) if $(v, u) \in R$ and $u \in Y$, then there exists, $w \in Y$ such that $(w, u) \in R$ and $w \leq v$.

(ts2) if $(u, v) \in R$ and $u \in Y$, then there exists, $z \in Y$ such that $(u, z) \in R$ and $z \leq v$.

The lattice of all tmP-subsets of the tmP-space associated with a tense De Morgan algebra was taken into account to characterize the congruence lattice of this algebra as it is indicated in the following theorem:

Theorem 1. ([16, Theorem 6.4]) *Let $(A, \sim, G, H)$ be a tense De Morgan algebra and $(X(A), \subseteq, g_A, R_G^A, R_H^A)$ be the tmP-space associated with A. Then, the lattice $\mathcal{C}_T(X(A))$ of all tmP-subsets of $X(A)$ is anti-isomorphic to the lattice $Con_{tM}(A)$ of the tense De Morgan congruences on A, and the anti-isomorphism is the function Θ_T defined by the prescription:*

$$\Theta_T(Y) = \{(a, b) \in A \times A : \sigma_A(a) \cap Y = \sigma_A(b) \cap Y\}, \text{ for all } Y \in \mathcal{C}_T(X(A)). \tag{2.11}$$

2.2. $n \times m$-valued Łukasiewicz–Moisil algebras

In the sequel n and m are positive integer numbers and we use the notation $[n] := \{1, \dots, n-1\}$ and so the cartesian product $\{1, \dots, n-1\} \times \{1, \dots, m-1\}$ is denoted by $[n] \times [m]$.

Definition 5. ([34, Definition 3.1.]) Let $n \geq 2$ and $m \geq 2$. An $n \times m$-valued Łukasiewicz–Moisil algebra (or $\mathrm{LM}_{n\times m}$-algebra) is an algebra $\langle A, \wedge, \vee, \sim, \{\sigma_{ij}\}_{(i,j)\in[n]\times[m]}, 0, 1\rangle$, such that:

(a) the reduct $\langle A, \wedge, \vee, \sim, 0, 1\rangle$ is a De Morgan algebra,

(b) $\{\sigma_{ij}\}_{(i,j)\in[n]\times[m]}$ is a family of unary operations on A which fulfills the following conditions for any $x, y \in A$ and any $(i, j), (r, s) \in [n] \times [m]$:

(C1) $\sigma_{ij}(x \vee y) = \sigma_{ij}x \vee \sigma_{ij}y$,
(C2) $\sigma_{ij}x \leq \sigma_{(i+1)j}x$,
(C3) $\sigma_{ij}x \leq \sigma_{i(j+1)}x$,
(C4) $\sigma_{ij}\sigma_{rs}x = \sigma_{rs}x$,
(C5) $\sigma_{ij}x = \sigma_{ij}y$ for all $(i, j) \in [n] \times [m]$ imply $x = y$,
(C6) $\sigma_{ij}x \vee \sim \sigma_{ij}x = 1$,
(C7) $\sigma_{ij}(\sim x) = \sim \sigma_{(n-i)(m-j)}x$.

In what follows and where no confusion might arise, we denote these algebras by A or $\big(A, \sim, \{\sigma_{ij}\}_{(i,j)\in[n]\times[m]}\big)$, in the case we need to specify unary operators.

In Lemma 1 we summarize the most important properties of these algebras necessary in what follows.

Lemma 1. ([34, Lemma 3.1.]) *Let $\big(A, \sim, \{\sigma_{ij}\}_{(i,j)\in[n]\times[m]}\big)$ be an $LM_{n\times m}$-algebra. Then, the following properties are satisfied for all $x, y \in A$ and for all $(i, j) \in [n] \times [m]$:*

(C8) $\sigma_{ij}(x \wedge y) = \sigma_{ij}x \wedge \sigma_{ij}y$,

(C9) $\sigma_{ij}x \wedge \sim \sigma_{ij}x = 0$,

(C10) $x \leq y$ *if and only if* $\sigma_{ij}x \leq \sigma_{ij}y$ *for all* $(i, j) \in [n] \times [m]$,

(C11) $x \leq \sigma_{(n-1)(m-1)}x$,

(C12) $\sigma_{ij}0 = 0,\ \sigma_{ij}1 = 1$,

(C13) $\sigma_{11}x \leq x$,

(C14) $\sim x \vee \sigma_{(n-1)(m-1)}x = 1$,

(C15) $x \vee \sim \sigma_{11}x = 1$.

Definition 6. ([28, Definition 2.1.]) Let $\big(A, \sim, \{\sigma_{ij}\}_{(i,j)\in[n]\times[m]}\big)$ and $(A', \sim', \{\sigma'_{ij}\}_{(i,j)\in[n]\times[m]})$ be two $LM_{n\times m}$-algebras. A function $h : A \longrightarrow A'$ is an $LM_{n\times m}$-homomorphism if it satisfies the following conditions for all $x, y \in A$ and for all $(i, j) \in [n] \times [m]$:

(a) h is a lattice homomorphism,
(b) $h(\sim x) = \sim' h(x)$,
(c) $h(\sigma_{ij} x) = \sigma'_{ij} h(x)$.

LEMMA 2. ([28, Remark 2.2.]) *Let* $(A, \sim, \{\sigma_{ij}\}_{(i,j)\in[n]\times[m]})$ *and* $(A', \sim',$ $\{\sigma'_{ij}\}_{(i,j)\in[n]\times[m]})$ *be two* $LM_{n\times m}$*-algebras and* $h : A \longrightarrow A'$ *be a lattice homomorphism. Then the following conditions are equivalent:*

(a) *h is an* $LM_{n\times m}$*-homomorphism,*
(b) $h(\sigma_{ij} x) = \sigma'_{ij} h(x)$ *for all* $x \in A$ *and for all* $(i, j) \in [n] \times [m]$.

The results announced here for $LM_{n\times m}$-algebras are used throughout the paper.

(LM1) $\sigma_{ij}(A) = B(A)$ for all $(i, j) \in [n] \times [m]$, where $B(A)$ is the set of all complemented elements of A ([33, Proposition 2.5]).

(LM2) Every $LM_{n\times 2}$-algebra is isomorphic to an n-valued Łukasiewicz–Moisil algebra. It is worth noting that $LM_{n\times m}$-algebras constitute a non-trivial generalization of the latter (see [34, Remark 2.1]).

(LM3) The class of $LM_{n\times m}$-algebras is a variety and two equational bases for it can be found in [33, Theorem 2.7] and [34, Theorem 4.6].

(LM4) Let X be a non-empty set and let A^X be the set of all functions from X into A. Then A^X is an $LM_{n\times m}$-algebra, where the operations are defined componentwise.

(LM5) Let $B(A)\uparrow^{[n]\times[m]} = \{f : [n] \times [m] \longrightarrow B(A)$ such that for arbitrary i, j, if $r \leq s$, then $f(r, j) \leq f(s, j)$ and $f(i, r) \leq f(i, s)\}$. Then $\langle B(A)\uparrow^{[n]\times[m]}, \wedge, \vee, \sim, \{\sigma_{ij}\}_{(i,j)\in[n]\times[m]}, 0, 1\rangle$ is an $LM_{n\times m}$-algebra, where for all $f \in B(A)\uparrow^{[n]\times[m]}$ and $(i, j) \in [n] \times [m]$ the operations $\sim$ and σ_{ij} are defined as follows:

$$(\sim f)(i, j) = \neg f(n - i, m - j), \tag{2.12}$$

where $\neg x$ is the Boolean complement of x,

$$(\sigma_{ij} f)(r, s) = f(i, j) \text{ for all } (r, s) \in [n] \times [m], \tag{2.13}$$

and the remaining operations are defined componentwise ([34, Proposition 3.2]). It is worth noting that this result can be generalized by replacing $B(A)$ by any Boolean algebra B. Furthermore, if B is a complete Boolean algebra, it is simple to check that $B\uparrow^{[n]\times[m]}$ is also a complete $LM_{n\times m}$-algebra.

(LM6) Every $LM_{n\times m}$-algebra $(A, \sim, \{\sigma_{ij}\}_{(i,j)\in[n]\times[m]})$ can be embedded into the $LM_{n\times m}$-algebra $B(A)\uparrow^{[n]\times[m]}$ ([34, Theorem 3.1]). Besides, A is isomorphic to $B(A)\uparrow^{[n]\times[m]}$ if and only if A is *centred* ([34, Corollary 3.1]), where A is centred if for each $(i,j)\in[n]\times[m]$ there exists $c_{ij}\in A$ such that

$$\sigma_{rs}c_{ij} = \begin{cases} 0 & \text{if } i > r \text{ or } j > s, \\ 1 & \text{if } i \leq r \text{ and } j \leq s. \end{cases}$$

(LM7) Let $\mathbf{2}\uparrow^{[\mathbf{n}]\times[\mathbf{m}]}$ be the set of all increasing functions from $[n]\times[m]$ to the Boolean algebra $\mathbf{2}$ with two elements. Then every simple $LM_{n\times m}$-algebra is a subalgebra of $\langle \mathbf{2}\uparrow^{[\mathbf{n}]\times[\mathbf{m}]}, \wedge, \vee, \sim, \{\sigma_{ij}\}_{(i,j)\in[n]\times[m]}, \mathbf{0}, \mathbf{1}\rangle$, where the operations of this $LM_{n\times m}$-algebra are defined as in statement (LM5) and $\mathbf{0}, \mathbf{1} \in \mathbf{2}\uparrow^{[\mathbf{n}]\times[\mathbf{m}]}$ are the functions $\mathbf{0}, \mathbf{1} : [n]\times[m] \longrightarrow \mathbf{2}$, defined for all $(i,j)\in[n]\times[m]$ by $\mathbf{0}((i,j)) = 0$ and $\mathbf{1}((i,j)) = 1$, respectively (see [34, Theorem 5.5]).

(LM8) Let A be an $LM_{n\times m}$-algebra. Then, the following conditions are equivalent:

(a) A is a subdirectly irreducible $LM_{n\times m}$-algebra,

(b) $B(A) = \{0,1\}$, where $B(A) = \{\sigma_{ij}a : a\in A,\ (i,j)\in[n]\times[m]\}$.

In [27], A. V. Figallo, I. Pascual and G. Pelaitay determined a topological duality for $LM_{n\times m}$-algebras. To this aim, these authors considered the topological category $\mathbf{LM_{n\times m}P}$ of $LM_{n\times m}$-spaces and $LM_{n\times m}$-functions. Specifically:

Definition 7. A system $\Big(X, g, \{f_{ij}\}_{(i,j)\in[n]\times[m]}\Big)$ is an $n\times m$-valued Łukasiewicz–Moisil space (or shortly $LM_{n\times m}$-space) if the following properties are fulfilled for all $x, y\in X$ and $(i,j), (r,s)\in[n]\times[m]$:

(LP1) (X, g) is an m-space,
(LP2) $f_{ij} : X \longrightarrow X$ is a continuous function,
(LP3) $f_{ij}(x) \leq f_{(i+1)j}(x)$,
(LP4) $f_{ij}(x) \leq f_{i(j+1)}(x)$,
(LP5) $x \leq y$ implies $f_{ij}(x) = f_{ij}(y)$ for all $(i,j)\in[n]\times[m]$,
(LP6) $f_{ij}\circ f_{rs} = f_{ij}$,
(LP7) $f_{ij}\circ g = f_{ij}$,

(LP8) $g \circ f_{ij} = f_{(n-i)(m-j)}$,

(LP9) $\bigcup\limits_{(i,j)\in[n]\times[m]} f_{ij}(X) = X$.

Remark 1. The axiom (LP5) is omitted in the Sanza's definition of $LM_{n\times m}$-space (see [35, Definition 2.1]). This axiom plays a fundamental role in the characterization of $LM_{n\times m}$-spaces and consequently in the characterization of congruences on $LM_{n\times m}$-algebras as we prove next.

DEFINITION 8. If $\left(X, g, \{f_{ij}\}_{(i,j)\in[n]\times[m]}\right)$ and $\left(X', g', \{f'_{ij}\}_{(i,j)\in[n]\times[m]}\right)$ are two $LM_{n\times m}$-spaces, then an $LM_{n\times m}$-function f from X to X' is a continuous and increasing function (P-function), which satisfies the following conditions:

(mPf) $f \circ g = g' \circ f$, (i.e., f is an m-function as in Defintion 6),

(LPf) $f'_{ij} \circ f = f \circ f_{ij}$ for all $(i,j) \in [n]\times[m]$.

Remark 2. The condition (mPf) in Definition 8 can be omitted.

PROPOSITION 1. ([27]) Let $\left(X, g, \{f_{ij}\}_{(i,j)\in[n]\times[m]}\right)$ be a system which satisfies the properties (LP1) to (LP8), and let $D(X)$ be the lattice of all increasing clopen (closed and open) of X. Then, the following conditions are equivalent:

(LP9) $\bigcup\limits_{(i,j)\in[n]\times[m]} f_{ij}(X) = X$,

(LP10) $\overline{\bigcup\limits_{(i,j)\in[n]\times[m]} f_{ij}(X)} = X$, where $\overline{Z}$ denotes the closure of $Z \subseteq X$,

(LP11) if $U, V \in D(X)$ and $f_{ij}^{-1}(U) = f_{ij}^{-1}(V)$ for all $(i,j) \in [n]\times[m]$, then $U = V$,

(LP12) for each $x \in X$, there is $(i_0, j_0) \in [n]\times[m]$ such that $f_{i_0 j_0}(x) = x$,

(LP13) if Y, $Z \subseteq X$ and $f_{ij}^{-1}(Y) = f_{ij}^{-1}(Z)$ for all $(i,j) \in [n]\times[m]$, then $Y = Z$.

DEFINITION 9. Let $(X, \leq)$ be a partial ordered set. For all $x, y \in X$ such that $x \leq y$, the subset $[x; y] := \{z \in X : x \leq z \leq y\}$ is said to be a segment or a closed interval in X.

It is worth mentioning the following properties of $LM_{n\times m}$-spaces because they are useful to describe these spaces:

LEMMA 3. *Let* $\left(X, g, \{f_{ij}\}_{(i,j)\in[n]\times[m]}\right)$ *be an* $LM_{n\times m}$*-space. Then, for any* $x \in X$,

(a) $[f_{11}(x); f_{(n-1),(m-1)}(x)] = \{f_{ij}(x) : (i,j) \in [n] \times [m]\}$,
(b) $x \in [f_{11}(x); f_{(n-1),(m-1)}(x)]$.

Proposition 2. Let $\Big(X, g, \{f_{ij}\}_{(i,j)\in[n]\times[m]}\Big)$ be an $LM_{n\times m}$-space. Then X is the cardinal sum of the sets $[f_{11}(x); f_{(n-1)(m-1)}(x)]$, $x \in X$.

Corollary 1. Let $\Big(X, g, \{f_{ij}\}_{(i,j)\in[n]\times[m]}\Big)$ be an $LM_{n\times m}$-space. Then it holds that

(LP14) $\min X = \{f_{11}(x) : x \in X\}$,
(LP15) $\max X = \{f_{(n-1)(m-1)}(x) : x \in X\}$.

Corollary 2. Let $\Big(X, g, \{f_{ij}\}_{(i,j)\in[n]\times[m]}\Big)$ be an $LM_{n\times m}$-space. Then for any $x \in X$ it holds that

(LP16) $f_{11}(x) \leq x$ and $f_{11}(x)$ is the unique minimal element in X that precedes x,
(LP17) $x \leq f_{(n-1)(m-1)}(x)$ and $f_{(n-1)(m-1)}(x)$ is the unique maximal element in X that follows x.

Corollary 3. Let $\Big(X, g, \{f_{ij}\}_{(i,j)\in[n]\times[m]}\Big)$ be an $LM_{n\times m}$-space. Then, for all interval $I \subseteq X$, the following conditions are equivalent:

(a) $I = [f_{11}(x); f_{(n-1)(m-1)}(x)]$ for some $x \in X$,
(b) I is a maximal interval in X.

In addition, in [27], the following results were established:

- If $(X, g, \{f_{ij}\}_{i\in[n]\times[m]})$ is an $LM_{n\times m}$-space. Then,
$$(D(X), \sim_g, \{\sigma^X_{ij}\}_{i\in[n]\times[m]})$$
is an $LM_{n\times m}$-algebra, where for every $U \in D(X)$, $\sim_g U$ is defined as in (2.3) and
$$\sigma^X_{ij}(U) = f^{-1}_{ij}(U) \text{ for all } (i,j) \in [n] \times [m]. \tag{2.14}$$
- If $(A, \sim, \{\sigma_{ij}\}_{i\in[n]\times[m]})$ is an $LM_{n\times m}$-algebra and $X(A)$ is the Priestley space associated with A, then $(X(A), g_A, \{f^A_{ij}\}_{i\in[n]\times[m]})$ is an $LM_{n\times m}$-space, where for every $S \in X(A)$, $g_A(S)$ is defined as (2.5) and
$$f^A_{ij}(S) = \sigma^{-1}_{ij}(S) \text{ for all } (i,j) \in [n] \times [m]. \tag{2.15}$$

- $(A, \sim, \{\sigma_{ij}\}_{(i,j)\in[n]\times[m]}) \cong (D(X(A)), \sim, \{\sigma_{ij}^{X(A)}\}_{(i,j)\in[n]\times[m]})$ and
- $(X, g, \{f_{ij}\}_{(i,j)\in[n]\times[m]}) \cong (X(D(X)), g_{D(X)}, \{f_{ij}^{D(X)}\}_{(i,j)\in[n]\times[m]})$, via the natural isomorphisms denoted by σ_A and ε_X respectively, which are defined as in (2.4) and (2.8), respectively.
- The correspondences between the morphisms of both categories are defined in the usual way as in (2.9) and (2.10).

Then, from these results it was concluded that the category $\mathbf{LM_{n\times m}P}$ is dually equivalent to the category $\mathbf{LM_{n\times m}A}$ of $LM_{n\times m}$-algebras and $LM_{n\times m}$-homomorphisms. Moreover, this duality was taken into account to characterize the congruence lattice on an $LM_{n\times m}$-algebra as is indicated in Theorem 2. In order to obtain this characterization the modal subsets of the $LM_{n\times m}$-spaces were taken into account, which we mention below:

Definition 10. Let $(X, g, \{f_{ij}\}_{(i,j)\in[n]\times[m]})$ be an LM_n-space. A subset Y of X is modal if $Y = f_i^{-1}(Y)$ for all $(i,j) \in [n]\times[m]$.

Theorem 2. ([27]) *Let $(A, \sim, \{\sigma_{ij}\}_{(i,j)\in[n]\times[m]})$ be an $LM_{n\times m}$-algebra and $(X(A), g_A, \{f_{ij}^A\}_{(i,j)\in[n]\times[m]})$ be the $LM_{n\times m}$-space associated with A. Then, the lattice $C_M(X(A))$ of all modal and closed subsets of $X(A)$ is anti-isomorphic to the lattice $Con_{LM_{n\times m}}(A)$ of $LM_{n\times m}$-congruences on A, and the anti-isomorphism is the function $\Theta_M : C_M(X(A)) \longrightarrow Con_{LM_{n\times m}}(A)$ defined by the same prescription in* (2.11).

The previous results allow us to prove the following theorem.

Theorem 3. ([27]) *Let $\Big(X, g, \{f_{ij}\}_{(i,j)\in[n]\times[m]}\Big)$ be an $LM_{n\times m}$-space and let $\Big(D(X), \sim_g, \{\sigma_{ij}^X\}_{(i,j)\in[n]\times[m]}\Big)$ be the $LM_{n\times m}$-algebra associated with X. Then the following conditions are equivalent:*

(a) $X = [f_{11}(x), f_{(n-1)(m-1)}(x)]$ *for all* $x \in X$,

(b) $\Big(D(X), \sim_g, \{\sigma_{ij}^X\}_{(i,j)\in[n]\times[m]}\Big)$ *is a simple $LM_{n\times m}$-algebra,*

(c) $\Big(D(X), \sim_g, \{\sigma_{ij}^X\}_{(i,j)\in[n]\times[m]}\Big)$ *is a subdirectly irreducible $LM_{n\times m}$-algebra,*

(d) $D(X)$ *is finite and* $D(X)\setminus\{\emptyset, X\}$ *has least and greatest element.*

2.3. Tense $n \times m$-valued Łukasiewicz–Moisil algebras

In [17], A. V. Figallo and G. Pelaitay introduce the following notion:

Definition 11. An algebra $\langle A, \vee, \wedge, \sim, \{\sigma_{ij}\}_{(i,j)\in[n]\times[m]}, G, H, 0, 1\rangle$ is a tense $n \times m$-valued Łukasiewicz–Moisil algebra (or tense $LM_{n\times m}$-algebra) if $\langle A, \vee, \wedge, \sim, \{\sigma_{ij}\}_{(i,j)\in[n]\times[m]}, 0, 1\rangle$, is an $LM_{n\times m}$-algebra and G, H are two unary operators on A which satisfy the following properties:

(T1) $G(1) = 1$ and $H(1) = 1$,
(T2) $G(x \wedge y) = G(x) \wedge G(y)$ and $H(x \wedge y) = H(x) \wedge H(y)$,
(T3) $G\sigma_{ij}(x) = \sigma_{ij}G(x)$ and $H\sigma_{ij}(x) = \sigma_{ij}H(x)$,
(T4) $x \leq GP(x)$ and $x \leq HF(x)$, where $P(x) =\sim H(\sim x)$ and $F(x) =\sim G(\sim x)$, for any x, $y \in X$ and $(i,j) \in [n] \times [m]$.

A tense $LM_{n\times m}$-algebra $\langle A, \vee, \wedge, \sim, \{\sigma_{ij}\}_{(i,j)\in[n]\times[m]}, G, H, 0, 1\rangle$ will be denoted in the rest of this paper by (A, G, H) or by $(A, \sim, \{\sigma_{ij}\}_{(i,j)\in[n]\times[m]}, G, H)$.

The following lemma contains properties of tense LM_n-algebras that are useful in what follows.

Lemma 4. ([27]) *The following properties hold in every tense $LM_{n\times m}$-algebra (A, G, H):*

(T5) $x \leq y$ *implies* $G(x) \leq G(y)$ *and* $H(x) \leq H(y)$,
(T6) $x \leq y$ *implies* $F(x) \leq F(y)$ *and* $P(x) \leq P(y)$,
(T7) $F(0) = 0$ *and* $P(0) = 0$,
(T8) $F(x \vee y) = F(x) \vee F(y)$ *and* $P(x \vee y) = P(x) \vee P(y)$,
(T9) $PG(x) \leq x$ *and* $FH(x) \leq x$,
(T10) $GP(x) \wedge F(y) \leq F(P(x) \wedge y)$ *and* $HF(x) \wedge P(y) \leq P(F(x) \wedge y)$,
(T11) $G(x) \wedge F(y) \leq F(x \wedge y)$ *and* $H(x) \wedge P(y) \leq P(x \wedge y)$,
(T12) $G(x \vee y) \leq G(x) \vee F(y)$ *and* $H(x \vee y) \leq H(x) \vee P(y)$, *for any* $x, y \in X$.

Definition 12. ([27]) If (A, G, H) and (A', G', H') are two tense $LM_{n\times m}$-algebras, then a morphism of tense $LM_{n\times m}$-algebras $f : (A, G, H) \longrightarrow (A', G', H')$ is a morphism of $LM_{n\times m}$-algebras such that

(tf) $f(G(a)) = G'(f(a))$ and $f(H(a)) = H'(f(a))$, for any $a \in A$.

LEMMA 5. ([27]) *Let* (A, G, H) *be a tense* $LM_{n\times m}$*-algebra and let* $C(A) := \{a \in A : d(a) = a\}$*. Then,* $\langle C(A), \vee, \wedge, \sim, \{\sigma_{ij}\}_{(i,j)\in[n]\times[m]}, 0, 1\rangle$ *is an* $LM_{n\times m}$*-algebra.*

3. Topological duality for tense $LM_{n\times m}$-algebras

In this section, we will develop a topological duality for tense $n \times m$-valued Łukasiewicz–Moisil algebras, taking into account the results established by A. V. Figallo, I. Pascual and G. Pelaitay in [27] and the results obtained by A. V. Figallo and G. Pelaitay in [16]. In order to determine this duality, we introduce a topological category whose objects and their corresponding morphisms are described below.

DEFINITION 13. A system $(X, g, \{f_{ij}\}_{i\in[n]\times[m]}, R)$ is a tense $LM_{n\times m}$-space if the following conditions are satisfied:

(i) $(X, g, \{f_{ij}\}_{(i,j)\in[n]\times[m]})$ is an $LM_{n\times m}$-space (Definition 7),

(ii) R is a binary relation on X and R^{-1} is the converse of R such that:

(tS1) $(x, y) \in R$ implies $(g(x), g(y)) \in R$,

(tS2) for each $x \in X$, $R(x)$ and $R^{-1}(x)$ are closed subsets of X,

(tS3) for each $x \in X$, $R(x) =\downarrow R(x) \cap \uparrow R(x)$,

(tS4) $(x, y) \in R$ implies $(f_{ij}(x), f_{ij}(y)) \in R$ for any $(i, j) \in [n] \times [m]$,

(tS5) $(f_{ij}(x), y) \in R$, $(i, j) \in [n] \times [m]$, implies that there exists $z \in X$ such that $(x, z) \in R$ and $f_{ij}(z) \leq y$,

(tS6) $(y, f_{ij}(x)) \in R$, $(i, j) \in [n] \times [m]$, implies that there exists $z \in X$ such that $(z, x) \in R$ and $f_{ij}(z) \leq y$,

(tS7) for each $U \in D(X)$, $G_R(U), H_{R^{-1}}(U) \in D(X)$, where G_R and $H_{R^{-1}}$ are operators on $P(X)$ defined as in (2.1) and (2.2), respectively.

DEFINITION 14. A tense $LM_{n\times m}$-function f from a tense $LM_{n\times m}$-space $(X_1, g_1, \{f^1_{ij}\}_{(i,j)\in[n]\times[m]}, R_1)$ into another one, $(X_2, g_2, \{f^2_{ij}\}_{(i,j)\in[n]\times[m]}, R_2)$ is a function $f : X_1 \longrightarrow X_2$ such that:

(i) $f : X_1 \longrightarrow X_2$ is an $LM_{n\times m}$-function (Definition 8),

(ii) $f : X_1 \longrightarrow X_2$ satisfies the following conditions, for all $x \in X_1$:

(tf1) $f(R_1(x)) \subseteq R_2(f(x))$ and $f(R_1^{-1}(x)) \subseteq R_2^{-1}(f(x))$,

(tf2) $R_2(f(x)) \subseteq\uparrow f(R_1(x))$,

(tf3) ${R_2}^{-1}(f(x)) \subseteq\uparrow f({R_1}^{-1}(x))$.

The category that has tense $LM_{n\times m}$-spaces as objects and tense $LM_{n\times m}$-functions as morphisms will be denoted by $\mathbf{tLM_{n\times m}S}$, and $\mathbf{tLM_{n\times m}A}$ will denote the category of tense $LM_{n\times m}$-algebras and tense $LM_{n\times m}$-homomorphisms. Our next task will be to determine that the category $\mathbf{tLM_{n\times m}S}$ is naturally equivalent to the dual category of $\mathbf{tLM_{n\times m}A}$.

Now we will show a characterization of tense $LM_{n\times m}$-functions which will be useful later.

LEMMA 6. *Let $(X_1, g_1, \{f^1_{ij}\}_{i\in[n]\times[m]}, R_1)$ and $(X_2, g_2, \{f^2_{ij}\}_{i\in[n]\times[m]}, R_2)$ be two tense $LM_{n\times m}$-spaces and*
$f : X_1 \longrightarrow X_2$ be a tense $LM_{n\times m}$-function. Then, f satisfies the following conditions:

(tf4) $\uparrow f(R_1(x)) = \uparrow R_2(f(x))$,
(tf5) $\uparrow f({R_1}^{-1}(x)) = \uparrow {R_2}^{-1}(f(x))$, *for any $x \in X$.*

PROOF: It can be proved using a similar technique to that used in the proof of Lemma 3.4 in [14]. □

LEMMA 7. *Let $(X, g, \{f_{ij}\}_{(i,j)\in[n]\times[m]}, R)$ be a tense $LM_{n\times m}$-space. Then for all x, $y \in X$ such that $(x, y) \notin R$, the following conditions are satisfied:*

(i) *There is $U \in D(X)$ such that $y \notin U$ and $x \in G_R(U)$ or $y \in U$ and $x \notin F_R(U)$, where $F_R(U) := \{x \in X : R(x) \cap U \neq \emptyset\}$.*
(ii) *There is $V \in D(X)$ such that $y \notin V$ and $x \in H_{R^{-1}}(V)$ or $y \in V$ and $x \notin P_{R^{-1}}(V)$, where $P_{R^{-1}}(V) := \{x \in X : R^{-1}(x) \cap V \neq \emptyset\}$.*

PROOF: It can be proved in a similar way to Lemma 3.5 of [14]. □

LEMMA 8. *Let $(X_1, g_1, \{f^1_{ij}\}_{(i,j)\in[n]\times[m]}, R_1)$ and $(X_2, g_2, \{f^2_{ij}\}_{(i,j)\in[n]\times[m]}, R_2)$ be two tense $LM_{n\times m}$-spaces. Then, the following conditions are equivalent:*

(i) *$f : X_1 \longrightarrow X_2$ is a tense $LM_{n\times m}$-function,*
(ii) *$f : X_1 \longrightarrow X_2$ is an $LM_{n\times m}$-function such that, for any $U \in D(X_2)$:*
 (tf6) $f^{-1}(G_{R_2}(U)) = G_{R_1}(f^{-1}(U))$,
 (tf7) $f^{-1}(H_{R_2^{-1}}(U)) = H_{R_1^{-1}}(f^{-1}(U))$.

PROOF: The proof is similar in spirit to Lemma 3.6 of [14]. □

Lemma 9 and Corollary 4 can be proved in a similar way to Lemma 3.8 and Corollary 3.9, respectively of [14].

Lemma 9. *Let $(X, g, \{f_{ij}\}_{(i,j)\in[n]\times[m]}, R)$ be a tense $LM_{n\times m}$-space. Then, the following conditions are satisfied for any $x, y, \in X$ and $(i,j) \in [n]\times[m]$:*

(tS11) $R(g(x)) = g(R(x))$, $R^{-1}(g(x)) = g(R^{-1}(x))$,

(tS12) $R(f_{ij}(x)) \subseteq \bigcup\limits_{y\in R(f_{ij}(x))} \uparrow f_{ij}(y)$,

(tS13) $R^{-1}(f_{ij}(x)) \subseteq \bigcup\limits_{y\in R^{-1}(f_{ij}(x))} \uparrow f_{ij}(y)$,

(tS14) $\uparrow f_{ij}(R_1(x)) =\uparrow R_2(f(x))$,

(tS15) $\uparrow f_{ij}({R_1}^{-1}(x)) =\uparrow {R_2}^{-1}(f(x))$,

(tS16) $f_{ij}^{-1}(G_R(U)) = G_R(f_{ij}^{-1}(U))$,

(tS17) $f_{ij}^{-1}(H_{R^{-1}}(U)) = H_{R^{-1}}(f_{ij}^{-1}(U))$,

(tS18) $f_{ij}^{-1}(\sim_g U) =\sim_g (f^{-1}_{(n-i)(m-j)}(U))$,

(tS19) $f_{ij}^{-1}(F_R(U)) = F_R(f_{ij}^{-1}(U))$,

(tS20) $f_{ij}^{-1}(P_{R^{-1}}(U)) = P_{R^{-1}}(f_{ij}^{-1}(U))$.

Corollary 4. Let $(X, g, \{f_{ij}\}_{(i,j)\in[n]\times[m]})$ be a tense $LM_{n\times m}$-space. Then, the conditions (tS4), (tS5) and (tS6) can be replaced by the following conditions:

(tS16) $f_{ij}^{-1}(G_R(U)) = G_R(f_{ij}^{-1}(U))$ for any $U \in D(X)$

(tS17)$f_{ij}^{-1}(H_{R^{-1}}(U)) = H_{R^{-1}}(f_{ij}^{-1}(U))$ for any $U \in D(X)$.

Next, we will define a contravariant functor from $\mathbf{tLM_{n\times m}S}$ to $\mathbf{tLM_{n\times m}A}$.

Lemma 10. *Let $(X, g, \{f_{ij}\}_{(i,j)\in[n]\times[m]}, R)$ be a tense $LM_{n\times m}$-space. Then,*

$$\Psi(X) = \langle D(X), \sim_g, \{\sigma_{ij}^X\}_{(i,j)\in[n]\times[m]}, G_R, H_{R^{-1}}, \emptyset, X\rangle$$

is a tense $LM_{n\times m}$-algebra, where for all $U \in D(X)$, $\sim_g U$, $\sigma_{ij}^X(U)$, $(i,j) \in [n]\times[m]$, $G_R(U)$ and $H_{R^{-1}}(U)$ are defined as in (2.3), (2.14), (2.1) and (2.2), respectively.

Proof: From [27] and [16, Lemma 4.3] it follows that the reduct $\langle D(X), \sim_g, \{\sigma_{ij}^X\}_{(i,j)\in[n]\times[m]}, \emptyset, X\rangle$ is an $LM_{n\times m}$-algebra and the structure $\langle D(X), \sim_g, G_R, H_{R^{-1}}, \emptyset, X\rangle$ is a tense De Morgan algebra, respectively. Therefore, the properties (T1), (T2) and (T4) of tense $LM_{n\times m}$-algebras (Definition 11) hold. In addition, since any $U \in D(X)$ satisfies properties (tS16) and (tS17) in Lemma 9, then we can assert that property (T3) holds too, and so the proof is complete. □

LEMMA 11. *Let $f : (X_1, g_1, \{f^1_{ij}\}_{(i,j)\in[n]\times[m]}) \longrightarrow (X_2, g_2, \{f^2_i\}_{(i,j)\in[n]\times[m]})$ be a morphism of tense $LM_{n\times m}$-spaces. Then, the map $\Psi(f) : D(X_2) \longrightarrow D(X_1)$ defined by $\Psi(f)(U) = f^{-1}(U)$ for all $U \in D(X_2)$, is a tense $LM_{n\times m}$-homomorphism.*

PROOF: It follows from the results established in [27] and Lemma 8. □

The previous two lemmas show that Ψ is a contravariant functor from $\mathbf{tLM_nS}$ to $\mathbf{tLM_nA}$. To achieve our goal we need to define a contravariant functor from $\mathbf{tLM_nA}$ to $\mathbf{tLM_nS}$.

LEMMA 12. *Let (A, G, H) be a tense $LM_{n\times m}$-algebra and let $S, T \in X(A)$. Then the following conditions are equivalent:*

(i) $G^{-1}(S) \subseteq T \subseteq F^{-1}(S)$,

(ii) $H^{-1}(T) \subseteq S \subseteq P^{-1}(T)$.

PROOF: In a similar way to [18, Lemma 3.8]. □

DEFINITION 15. Let (A, G, H) be a tense $LM_{n\times m}$-algebra and let R^A be the relation defined on $X(A)$ by the prescription:

$$(S, T) \in R^A \Longleftrightarrow G^{-1}(S) \subseteq T \subseteq F^{-1}(S). \tag{3.1}$$

Remark 3. Lemma 12 means that we have two ways to define the relation R^A, either by using G and F, or by using H and P.

The following lemma, whose proof can be obtained as in [18, Lemma 3.11], will be essential for the proof of Lemma 14.

LEMMA 13. *Let (A, G, H) be a tense $LM_{n\times m}$-algebra and let $S \in X(A)$ and $a \in A$. Then,*

(i) *$G(a) \notin S$ if and only if there exists $T \in X(A)$ such that $(S, T) \in R^A$ and $a \notin T$,*

(ii) *$H(a) \notin S$ if and only if there exists $T \in X(A)$ such that $(S, T) \in {R^A}^{-1}$ and $a \notin T$.*

LEMMA 14. *Let (A, G, H) be an $LM_{n\times m}$-algebra and $X(A)$ be the Priestley space associated with A. Then, $\Phi(A) = (X(A), g_A, \{f^A_{ij}\}_{(i,j)\in[n]\times[m]}, R^A)$ is a tense $LM_{n\times m}$-space, where for every $S \in X(A)$, $g_A(S)$ and $f^A_{ij}(S)$ are defined as in (2.5) and (2.15), respectively and R^A is the relation defined on $X(A)$ as in (3.1). Besides, $\sigma_A : A \longrightarrow D(X(A))$, defined by the prescription (2.4), is a tense $LM_{n\times m}$-isomorphism.*

Proof: From [27] and [16, Lemma 5.6] it follows that the system $(X(A), g_A, \{f_{ij}^A\}_{(i,j)\in[n]\times[m]})$ is an $LM_{n\times m}$-space and $(X(A), g_A, R^A, R^{A^{-1}})$ is a tense mP-space, and so properties (tS1), (tS2) (tS3) and (tS7) of tense $LM_{n\times m}$-spaces hold (Definition 13). Also, from Corollary 4 we have that the conditions (tS4), (tS5) and (tS6) are satisfied. Therefore, we have that $(X(A), g_A, \{f_{ij}^A\}_{(i,j)\in[n]\times[m]}, R^A)$ is a tense $LM_{n\times m}$-space. In addition, from [27] we have that σ_A is an $LM_{n\times m}$-isomorphism. Also for all $a \in A$, $G_{R^A}(\sigma_A(a)) = \sigma_A(G(a))$ and $H_{R^{A^{-1}}}(\sigma_A(a)) = \sigma_A(H(a))$. Indeed, let us take a prime filter S such that $G(a) \notin S$. By Lemma 13, there exists $T \in X(A)$ such that $(S, T) \in R^A$ and $a \notin T$. Then, $R^A(S) \not\subseteq \sigma_A(a)$. So, $S \notin G_{R^A}(\sigma_A(a))$ and, therefore, $G_{R^A}(\sigma(a)) \subseteq \sigma_A(G(a))$. Moreover, it is immediate that $\sigma_A(G(a)) \subseteq G_{R_A}(\sigma_A(a))$. Similarly we obtain that $H_{R^{A^{-1}}}(\sigma_A(a)) = \sigma_A(H(a))$ and so σ_A is a tense $LM_{n\times m}$-isomorphism. □

Lemma 15. *Let (A_1, G_1, H_1) and (A_2, G_2, H_2) be two $LM_{n\times m}$-algebras and $h : A_1 \longrightarrow A_2$ be a tense $LM_{n\times m}$-homomorphism. Then, the map $\Phi(h) : X(A_2) \longrightarrow X(A_1)$, defined by $\Phi(h)(S) = h^{-1}(S)$ for all $S \in X(A_2)$, is a tense $LM_{n\times m}$-function.*

Proof: It follows from the results established in [27] and [16, Lemma 5.7]. □

Lemmas 14 and 15 show that Φ is a contravariant functor from $\mathbf{tLM_{n\times m}A}$ to $\mathbf{tLM_{n\times m}S}$.

The following characterization of isomorphisms in the category $\mathbf{tLM_{n\times m}S}$ will be used to determine the duality that we set out to prove.

Proposition 3. Let $(X_1, g_1, \{f_{ij}^1\}_{(i,j)\in[n]\times[m]}, R_1)$ and $(X_2, g_2, \{f_{ij}^2\}_{(i,j)\in[n]\times[m]}, R_2)$ be two tense $LM_{n\times m}$-spaces. Then, the following conditions are equivalent, for every function $f : X_1 \longrightarrow X_2$:

(i) f is an isomorphism in the category $\mathbf{tLM_nS}$,

(ii) f is a bijective $LM_{n\times m}$-function such that for all $x, y \in X_1$:

(itf) $(x, y) \in R_1 \Longleftrightarrow (f(x), f(y)) \in R_2$.

Proof: It is routine. □

The map $\varepsilon_X : X \longrightarrow X(D(X))$, defined as in (2.8), leads to another characterization of tense $LM_{n\times m}$-spaces, which also allow us to assert that this map is an isomorphism in the category $\mathbf{tLM_{n\times m}S}$, as we will describe below:

Lemma 16. *Let $(X, g, \{f_{ij}\}_{(i,j)\in[n]\times[m]}, R)$ be a tense $LM_{n\times m}$-space, $\varepsilon_X : X \longrightarrow X(D(X))$ be the map defined by the prescription (2.8) and let $R^{D(X)}$ be the relation defined on $X(D(X))$ by means of the operators G_R and F_R as follows:*

$$(\varepsilon_X(x), \varepsilon_X(y)) \in R^{D(X)} \Longleftrightarrow G_R^{-1}(\varepsilon_X(x)) \subseteq \varepsilon_X(y) \subseteq F_R^{-1}(\varepsilon_X(x)). \quad (3.2)$$

Then, the following property holds:

(tS5) $(x, y) \in R$ *implies* $(\varepsilon_X(x), \varepsilon_X(y)) \in R^{D(X)}$.

Proof: It is routine. □

Proposition 4. Let $(X, g, \{f_{ij}\}_{(i,j)\in[n]\times[m]}, R)$ be a tense $LM_{n\times m}$-space, $\varepsilon_X : X \longrightarrow X(D(X))$ be the function defined by the prescription (2.8) and let $R^{D(X)}$ be the relation defined on $X(D(X))$ by the prescription (3.2). Then, the condition (tS3) can be replaced by the following one:

(tS18) $(\varepsilon_X(x), \varepsilon_X(y)) \in R^{D(X)} \Longleftrightarrow (x, y) \in R$.

Proof: It can be proved in a similar way to [16, Proposition 5.5]. □

Corollary 5. Let $(X, g, \{f_{ij}\}_{(i,j)\in[n]\times[m]}, R)$ be a tense $LM_{n\times m}$-space. Then, the map $\varepsilon_X : X \longrightarrow X(D(X))$ is an isomorphism in the category $\mathbf{tLM_{n\times m}S}$.

Proof: It follows from the results established in [27], Lemma 16, Propositions 3 and 4. □

Then, from the above results and using the usual procedures we can prove that the functors $\Phi \circ \Psi$ and $\Psi \circ \Phi$ are naturally equivalent to the identity functors on $\mathbf{tLM_{n\times m}S}$ and $\mathbf{tLM_{n\times m}A}$, respectively, from which we conclude:

Theorem 4. *The category $\mathbf{tLM_{n\times m}S}$ is naturally equivalent to the dual of the category $\mathbf{tLM_{n}A}$.*

4. Subdirectly irreducible tense $\mathbf{LM_{n\times m}}$-algebras

In this section, our first objective is the characterization of the congruence lattice on a tense $LM_{n\times m}$-algebra by means of certain closed and modal subsets of its associated tense $LM_{n\times m}$-space. Later, this result will be taken into account to characterize simple and subdirectly irreducible tense

$LM_{n\times m}$-algebras. With this purpose, we will start by introducing the following notion.

DEFINITION 16. Let $(X, g, \{f_{ij}\}_{(i,j)\in[n]\times[m]}, R)$ be a tense $LM_{n\times m}$-space. A subset Y of X is a tense subset if it satisfies the following conditions for all $y, z \in X$:

(ts1) if $y \in Y$ and $z \in R(y)$, then there is $w \in Y$ such that $w \in R(y) \cap \downarrow z$,
(ts2) if $y \in Y$ and $z \in R^{-1}(y)$, then there is $v \in Y$ such that $v \in R^{-1}(y) \cap \downarrow z$.

In [27] the following characterizations of a modal subset of an $LM_{n\times m}$-space were obtained.

PROPOSITION 5. ([27]) Let $(X, g, \{f_{ij}\}_{(i,j)\in[n]\times[m]})$ be an $LM_{n\times m}$-space and Y be a nonempty subset of X. Then, the following conditions are equivalent:

(a) Y is modal,
(b) Y is involutive and increasing,
(c) $Y = \bigcup\limits_{y\in Y} [f_{11}(y), f_{(n-1)(m-1)}(y)]$ (i.e. Y is the cardinal sum of certain maximal intervals of X).

COROLLARY 6. Let $(X, g, \{f_{ij}\}_{(i,j)\in[n]\times[m]})$ be an $LM_{n\times m}$-space. If $\{Y_i\}_{i\in I}$ is a family of modal subsets of X, then $\bigcap\limits_{i\in I} Y_i$ is a modal subset of X.

PROOF: It is a direct consequence of Proposition 5. □

The notion of a modal and tense subset of a tense $LM_{n\times m}$-space has several equivalent formulations, which will be useful later:

PROPOSITION 6. Let $(X, g, \{f_{ij}\}_{(i,j)\in[n]\times[m]}, R)$ be a tense $LM_{n\times m}$-space. If Y is a modal subset of X, then the following conditions are equivalent:

(i) Y is a tense subset,
(ii) for all $y \in Y$, the following conditions are satisfied:
  (ts3) $R(y) \subseteq Y$,
  (ts4) $R^{-1}(y) \subseteq Y$,
(iii) $Y = G_R(Y) \cap Y \cap H_{R^{-1}}(Y)$, where $G_R(Y) := \{x \in X : R(x) \subseteq Y\}$ and $H_{R^{-1}}(Y) := \{x \in X : R^{-1}(x) \subseteq Y\}$.

PROOF: (i) $\Rightarrow$ (ii): Let $y \in Y$ and $z \in R(y)$, then by (i) and (ts1), there is $w \in Y$ such that $w \in R(y)$ and $w \leq z$. Since Y is modal, from Proposition 5 it follows that $z \in Y$ and therefore $R(y) \subseteq Y$. Using an analogous reasoning we get that $R^{-1}(y) \subseteq Y$.

(ii) $\Rightarrow$ (i): It is immediate.

(ii) $\Leftrightarrow$ (iii): It is immediate. $\square$

The closed, modal and tense subsets of the tense $LM_{n\times m}$-space associated with a tense $LM_{n\times m}$-algebra perform a fundamental roll in the characterization of the tense $LM_{n\times m}$-congruences on these algebras as we will show next.

THEOREM 5. *Let (A, G, H) be a tense $LM_{n\times m}$-algebra, and $(X(A), g_A, \{f_{ij}^A\}_{(i,j)\in[n]\times[m]}, R^A)$ be the tense $LM_{n\times m}$-space associated with A. Then, the lattice $C_{MT}(X(A))$ of all closed, modal and tense subsets of $X(A)$ is anti-isomorphic to the lattice $Con_{tLM_{n\times m}}(A)$ of tense $LM_{n\times m}$-congruences on A, and the isomorphism is the function Θ_{MT} defined by the same prescription as in* (2.11).

PROOF: It immediately follows from Theorems 1 and 2 and the fact that $C_{MT}(X(A)) = C_M(X(A)) \cap C_T(X(A))$ and for all $\varphi \subseteq A \times A$, $\varphi \in Con_{tLM_{n\times m}}(A)$ iff φ is both an $LM_{n\times m}$-congruence on A and a tense De Morgan congruence on A. $\square$

Next, we will use the results already obtained in order to determine the simple and subdirectly irreducible tense $LM_{n\times m}$-algebras.

COROLLARY 7. Let (A, G, H) be a tense $LM_{n\times m}$-algebra, and $(X(A), g_A, \{f_i^A\}_{(i,j)\in[n]\times[m]}, R^A)$ be the tense $LM_{n\times m}$-space associated with A. Then, the following conditions are equivalent:

(i) (A, G, H) is a simple tense $LM_{n\times m}$-algebra,

(ii) $C_{MT}(X(A)) = \{\emptyset, X(A)\}$.

PROOF: It is a direct consequence of Theorem 5. $\square$

COROLLARY 8. Let (A, G, H) be a tense $LM_{n\times m}$-algebra, and $(X(A), g_A, \{f_{ij}^A\}_{(i,j)\in[n]\times[m]}, R^A)$ be the tense $LM_{n\times m}$-space associated with A. Then, the following conditions are equivalent:

(i) (A, G, H) is a subdirectly irreducible tene $LM_{n\times m}$-algebra,

(ii) there is $Y \in C_{MT}(X(A)) \setminus \{X(A)\}$ such that $Z \subseteq Y$ for all $Z \in C_{MT}(X(A)) \setminus \{X(A)\}$.

PROOF: It is a direct consequence of Theorem 5. $\square$

Proposition 7. Let $(X, g, \{f_{ij}\}_{(i,j)\in[n]\times[m]}, R)$ be a tense $LM_{n\times m}$-space. If Y is a modal subset of X, then $G_R(Y)$ and $H_{R^{-1}}(Y)$ are also modal.

Proof: Let Y be a modal subset of X. From Proposition 2 it follows immediately that (1) $G_R(Y) \subseteq \bigcup_{z\in G_R(Y)} [f_{11}(z), f_{(n-1)(m-1)}(z)]$. Let (2) $z \in G_R(Y)$ and let (3) $w \in [f_{11}(z), f_{(n-1)(m-1)}(z)]$, then from (3) and properties (LP5) and (LP6), we obtain that (4) $f_{ij}(w) = f_{rs}(z)$ for all $(i,j),(r,s) \in [n]\times[m]$. Let (5) $t \in R(w)$, then by (4), (5) and property (tS3), we infer that $f_{11}(t) \in R(f_{11}(z))$ and therefore, from properties (tS4), (LP5) and (LP6), we can assert that there exists $y \in X$ such that (5) $y \in R(z)$ and (6) $f_{ij}(y) = f_{rs}(t)$ for all $(i,j),(r,s) \in [n]\times[m]$. From (2) and (5) we get that $y \in Y$. Since Y is modal, then from this last assertion and (6) it results that $f_{ij}(t) \in Y$ for all $(i,j) \in [n]\times[m]$. Then, since Y is modal, we have that $t \in Y$, from which we deduce by (5) that $R(w) \subseteq Y$, which allows to assert that $w \in G_R(Y)$. Therefore, from (3) we can set that $\bigcup_{z\in G_R(Y)} [f_{11}(z), f_{(n-1)(m-1)}(z)] \subseteq G_R(Y)$. Then, from (1) it follows that $G_R(Y) = \bigcup_{z\in G_R(Y)} [f_{11}(z), f_{(n-1)(m-1)}(z)]$, and so from Proposition 5, we conclude that $G_R(Y)$ is modal. The proof that $H_{R^{-1}}(Y)$ is modal is similar. □

The characterization of modal and tense subsets of a tense $LM_{n\times m}$-space, given in Proposition 6, prompts us to introduce the following definition:

Definition 17. Let $(X, g, \{f_{ij}\}_{(i,j)\in[n]\times[m]}, R)$ be a tense $LM_{n\times m}$-space and let $d_X : P(X) \longrightarrow P(X)$ defined by:

$$d_X(Z) = G_R(Z) \cap Z \cap H_{R^{-1}}(Z), \text{ for all } Z \in P(X). \tag{4.1}$$

For each $n \in \omega$, let $d_X^n : P(X) \longrightarrow P(X)$, defined by:

$$d_X^0(Z) = Z,\ d_X^{n+1}(Z) = d_X(d_X^n(Z)), \text{ for all } Z \in P(X). \tag{4.2}$$

By using the above functions d_X, d_X^n, $n \in \omega$, we obtain another equivalent formulation of the notion of modal and tense subset of a tense $LM_{n\times m}$-space.

Lemma 17. *Let $(X, g, \{f_{ij}\}_{(i,j)\in[n]\times[m]}, R)$ be a tense $LM_{n\times m}$-space. If Y is modal subset of X, then the following conditions are equivalent:*

(i) *Y is a tense subset,*

(ii) $Y = d_X^n(Y)$, *for all* $n \in \omega$,
(iii) $Y = \bigcap\limits_{n\in\omega} d_X^n(Y)$.

PROOF: It is an immediate consequence of Proposition 6 and Definition 17. $\square$

PROPOSITION 8. Let $(X, g, \{f_{ij}\}_{(i,j)\in[n]\times[m]}, R)$ be a tense $LM_{n\times m}$-space and $(D(X), G_R, H_{R^{-1}})$ be the tense $LM_{n\times m}$-algebra associated with X. Then, for all $n \in \omega$, for all $U, V \in D(X)$ and for all $(i,j) \in [n]\times[m]$, the following conditions are satisfied:

(d0) $d_X^n(U) \in D(X)$,
(d1) $d_X^n(X) = X$ and $d_X^n(\emptyset) = \emptyset$,
(d2) $d_X^{n+1}(U) \subseteq d_X^n(U)$,
(d3) $d_X^n(U \cap V) = d_X^n(U) \cap d_X^n(V)$,
(d4) $U \subseteq V$ implies $d_X^n(U) \subseteq d_X^n(V)$,
(d5) $d_X^n(U) \subseteq U$,
(d6) $d_X^{n+1}(U) \subseteq G_R(d_X^n(U))$ and $d_X^{n+1}(U) \subseteq H_{R^{-1}}(d_X^n(U))$,
(d7) $d_X^n(f_{ij}^{-1}(U)) = f_{ij}^{-1}(d_X^n(U))$ for any $n \in \omega$ and $(i,j) \in [n]\times[m]$,
(d8) if U is modal, then $d_X^n(U)$ is modal,
(d9) $\bigcap\limits_{n\in\omega} d_X^n(f_{ij}^{-1}(U))$ is a closed, modal and tense subset of X and therefore $d_X(\bigcap\limits_{n\in\omega} d_X^n(f_{ij}^{-1}(U))) = \bigcap\limits_{n\in\omega} d_X^n(f_{ij}^{-1}(U))$.

PROOF: From Definition 17, Lemma 14 and the fact that G_R, $H_{R^{-1}}$ and d_X^n, $n \in \omega$, are monotonic operations it immediately follows that properties (d0), (d1), (d2), (d3), (d4), (d5) and (d6) hold.

(d7): Let $U \in D(X)$ and $(i,j) \in [n]\times[m]$, then $d_X(f_{ij}^{-1}(U)) = f_{ij}^{-1}(U) \cap G_R(f_{ij}^{-1}(U)) \cap H_{R^{-1}}(f_{ij}^{-1}(U))$. From the last assertion and properties (tS17) and (tS18) in Lemma 9, we infer that (1) $d_X(f_{ij}^{-1}(U)) = f_{ij}^{-1}(U \cap G_R(U) \cap H_{R^{-1}}(U)) = f_{ij}^{-1}(d_X(U))$ for any $U \in D(X)$ and $(i,j) \in [n]\times[m]$. Suppose that $d_X^{n-1}(f_{ij}^{-1}(U)) = f_{ij}^{-1}(d_X^{n-1}(U))$, for any $n \in \omega$ and $(i,j) \in [n]\times[m]$, then (2) $d_X^n(f_{ij}^{-1}(U)) = d_X(d_X^{n-1}(f_{ij}^{-1}(U))) = d_X(f_{ij}^{-1}(d_X^{n-1}(U)))$. Taking into account that $d_X^{n-1}(U) \in D(X)$ and (1), we get that $d_X(f_{ij}^{-1}(d_X^{n-1}(U))) = f_{ij}^{-1}(d_X(d_X^{n-1}(U))) = f_{ij}^{-1}(d_X^n(U))$, and so from (2) the proof is complete.

(d8): It is a direct consequence of Corollary 6 and Proposition 7.

(d9): Let $U \in D(X)$. Then, from Lemma 14 and the prescription (2.14), we have that $f_{ij}^{-1}(U) \in D(X)$. Also, from (LP5), $f_{ij}^{-1}(U)$ is a modal subset of X for all $(i,j) \in [n] \times [m]$, from which it follows by (d7) that for $n \in \omega$ and $(i,j) \in [n] \times [m]$, $d_X^n(f_{ij}^{-1}(U))$ is a modal and closed subset of X, and so by Corollary 6 and the fact that the arbitrary intersection of closed subsets of X is closed, we get that $\bigcap\limits_{n\in\omega} d_X^n(f_{ij}^{-1}(U))$ is a modal and closed subset of X. If $\bigcap\limits_{n\in\omega} d_X^n(f_{ij}^{-1}(U)) = \emptyset$, then it is verified that $\bigcap\limits_{n\in\omega} d_X^n(f_{ij}^{-1}(U))$ is a closed, modal and tense subset of X. Suppose now that there exists $y \in \bigcap\limits_{n\in\omega} d_X^n(f_{ij}^{-1}(U))$. Since, $f_{ij}^{-1}(U) \in D(X)$ for any $(i,j) \in [n] \times [m]$, then from (d6) it follows that $y \in G_R(d_X^{n-1}(f_{ij}^{-1}(U)))$ and $y \in H_{R^{-1}}(d_X^{n-1}(f_{ij}^{-1}(U)))$ for all $n \in \omega$. Therefore, $R(y) \subseteq d_X^{n-1}(f_{ij}^{-1}(U))$ and $R^{-1}(y) \subseteq d_X^{n-1}(f_{ij}^{-1}(U))$ for all $n \in \omega$ and consequently $R(y) \subseteq \bigcap\limits_{n\in\omega} d_X^n(f_{ij}^{-1}(U))$ and $R^{-1}(y) \subseteq \bigcap\limits_{n\in\omega} d_X^n(f_{ij}^{-1}(U))$ for all $(i,j) \in [n] \times [m]$. From these last assertions, the fact that $\bigcap\limits_{n\in\omega} d_X^n(f_{ij}^{-1}(U))$ is a modal and closed subset of X and Proposition 8, we have that $\bigcap\limits_{n\in\omega} d_X^n(f_{ij}^{-1}(U))$ is a tense subset, from which we conclude, by Lemma 17, that $d_X(\bigcap\limits_{n\in\omega} d_X^n(f_{ij}^{-1}(U))) = \bigcap\limits_{n\in\omega} d_X^n(f_{ij}^{-1}(U))$. □

As consequences of Proposition 8 and the above duality for tense $LM_{n\times m}$-algebras (Lemma 14) we obtain the following corollaries.

Corollary 9. Let $(A, \sim, \{\sigma_{ij}\}_{(i,j)\in[n]\times[m]}, G, H)$ be a tense $LM_{n\times m}$-algebra and consider the function $d : A \longrightarrow A$, defined by $d(a) = G(a) \wedge a \wedge H(a)$, for all $a \in A$. For all $n \in \omega$, let $d^n : A \longrightarrow A$ be a function, defined by $d^0(a) = a$ and $d^{n+1}(a) = d(d^n(a))$, for all $a \in A$. Then, for all $n \in \omega$ and $a, b \in A$, the following conditions are satisfied:

(d1) $d^n(1) = 1$ and $d^n(0) = 0$,

(d2) $d^{n+1}(a) \leq d^n(a)$,

(d3) $d^n(a \wedge b) = d^n(a) \wedge d^n(b)$,

(d4) $a \leq b$ implies $d^n(a) \leq d^n(b)$,

(d5) $d^n(a) \leq a$,

(d6) $d^{n+1}(a) \leq G(d^n(a))$ and $d^{n+1}(a) \leq H(d^n(a))$, (d7) for all $(i,j) \in [n] \times [m]$ and $n \in \omega$, $d^n(\sigma_{ij}(a)) = \sigma_{ij}(d^n(a))$.

Corollary 10. Let (A, G, H) be a tense $LM_{n\times m}$-algebra, $(X(A), g_A, \{f_{ij}^A\}_{(i,j)\in[n]\times[m]}, R^A)$ be the tense $LM_{n\times m}$-space associated with A and let $\sigma_A : A \longrightarrow D(X(A))$ be the map defined by the prescription (2.4). Then, $\sigma_A(d^n(a)) = d^n_{X(A)}(\sigma_A(a))$ for all $a \in A$ and $n \in \omega$.

Proof: It is a direct consequence of Lemma 14. □

It seems worth mentioning that the operator d defined in Corollary 9 was previously defined in [19] for tense algebras, in [12] for tense MV-algebras, and in [8, 9] for tense θ-valued Łukasiewicz–Moisil algebras, respectively.

Lemma 18. *Let (A, G, H) be a tense $LM_{n\times m}$-algebra. If $\bigwedge\limits_{i\in I} a_i$ exists, then the following conditions hold:*

(i) $\bigwedge\limits_{i\in I} G(a_i)$ *exists and* $\bigwedge\limits_{i\in I} G(a_i) = G(\bigwedge\limits_{i\in I} a_i)$,

(ii) $\bigwedge\limits_{i\in I} H(a_i)$ *exists and* $\bigwedge\limits_{i\in I} H(a_i) = H(\bigwedge\limits_{i\in I} a_i)$,

(iii) $\bigwedge\limits_{i\in I} d(a_i)$ *exists and* $\bigwedge\limits_{i\in I} d^n(a_i) = d^n(\bigwedge\limits_{i\in I} a_i)$ *for all* $n \in \omega$.

Proof:

(i): Assume that $a_i \in A$ for all $i \in I$ and $\bigwedge\limits_{i\in I} a_i$ exists. Since $\bigwedge\limits_{i\in I} a_i \leq a_i$, we have by (T2) that $G(\bigwedge\limits_{i\in I} a_i) \leq G(a_i)$ for each $i \in I$. Thus, $G(\bigwedge\limits_{i\in I} a_i)$ is a lower bound of the set $\{G(a_i) : i \in I\}$. Assume now that b is a lower bound of the set $\{G(a_i) : i \in I\}$. From (T5) and (T6) we have that $P(b) \leq PG(a_i) \leq a_i$ for each $i \in I$. So, $P(b) \leq \bigwedge\limits_{i\in I} a_i$. Besides, the pair (G, P) is a Galois connection, this means that $x \leq G(y) \Longleftrightarrow P(x) \leq y$, for all $x, y \in A$. So, we can infer that $b \leq G(\bigwedge\limits_{i\in I} a_i)$. This proves that $\bigwedge\limits_{i\in I} G(a_i)$ exists and $\bigwedge\limits_{i\in I} G(a_i) = G(\bigwedge\limits_{i\in I} a_i)$.

(ii): The proof for the operator H is analogous to the proof for G.

(iii): It is a direct consequence of (i) and (ii). □

For invariance properties we have:

Lemma 19. *Let $(X, g, \{f_{ij}\}_{(i,j)\in[n]\times[m]}, R)$ be a tense $LM_{n\times m}$-space and $(D(X), G_R, H_{R^{-1}})$ be the tense $LM_{n\times m}$-algebra associated with X. Then,* *for all U, $V, W \in D(X)$ such that $U = d_X(U)$, $V = d_X(V)$ and for some $(i_0, j_0) \in [n] \times [m]$, $d_X(f^{-1}_{i_0 j_0}(W)) = f^{-1}_{i_0 j_0}(W)$, the following properties are satisfied:*

(i) $U \cap V = d_X(U \cap V)$,
(ii) $U \cup V = d_X(U \cup V)$,
(iii) $\sim_g U = d_X(\sim_g U)$,
(iv) $d_X(f^{-1}_{ij}(W)) = f^{-1}_{ij}(W)$ *for all* $(i, j) \in [n] \times [m]$.

Proof:
(i): It immediately follows from the definition of the function d_X and property (T2) of tense $LM_{n\times m}$-algebras.

(ii): Taking into account that $U = d_X(U)$ and $V = d_X(V)$ and the fact that the operations G_R and $H_{R^{-1}}$ are increasing, we infer that $U \cup V \subseteq G_R(U \cup V)$ and $U \cup V \subseteq H_{R^{-1}}(U \cup V)$, which imply that $U \cup V = d_X(U \cup V)$.

(iii): Let $U \in D(X)$ such that (1) $U = d_X(U)$. Then, it is verified that $\sim_g U \subseteq G_R(\sim_g U)$. Indeed, let $x \in \sim_g U$ and (2) $y \in R(x)$. Then, $x \in X \setminus g(U)$ and hence (3) $x \notin g(U)$. Suppose that $y \in g(U)$, then there is $z \in U$ such that $y = g(z)$, and by (tS11) in Lemma 9, we get that $R^{-1}(y) = R^{-1}(g(z)) = g(R^{-1}(z))$. Since $z \in U$, from (1) it follows that $R^{-1}(z) \subseteq U$ and so $g(R^{-1}(z))) \subseteq g(U)$. Thus, $R^{-1}(y) \subseteq g(U)$. From the last statement and (2), we infer that $x \in g(U)$, which contradicts (3). Consequently, $y \in \sim_g U$, which allows us to assert that $R(x) \subseteq \sim_g U$ and therefore $\sim_g U \subseteq G_R(\sim U)$. In a similar way, we can prove that $\sim_g U \subseteq H_{R^{-1}}(\sim_g U)$. From the two last assertions we conclude that $\sim_g U = d_X(\sim_g U)$.

(iv): If $W \in D(X)$ and $d_X(f^{-1}_{i_0 j_0}(W)) = f^{-1}_{i_0 j_0}(W)$ for some $(i_0, j_0) \in [n] \times [m]$, then from (d7) it follows that $f^{-1}_{i_0 j_0}(d_X(W)) = f^{-1}_{i_0 j_0}(W)$. From the last assertion and (LP5) we infer that $f^{-1}_{ij}(d_X(W)) = f^{-1}_{ij}(W)$ for all $(i, j) \in [n] \times [m]$, and so from (d7), we get that $d_X(f^{-1}_{ij}(W)) = f^{-1}_{ij}(W)$ for all $(i, j) \in [n] \times [m]$. □

Corollary 11. Let $(A, \sim, \{\sigma_{ij}\}_{(i,j)\in[n]\times[m]}, G, H)$ be a tense $LM_{n\times m}$-algebra. Then, for all $a, b, c \in A$, such that $a = d(a)$, $b = d(b)$ and $\varphi_{i_0 j_0}(c) = d\,(\varphi_{i_0 j_0}(c))$ for some $(i_0, j_0) \in [n] \times [m]$, the following properties are satisfied:

(i) $d(a \wedge b) = a \wedge b$,
(ii) $d(a \vee b) = a \vee b$,
(iii) $d(\sim a) =\sim a$,
(iv) $\sigma_{ij}(c) = d\,(\sigma_{ij}(c))$ for all all $(i,j) \in [n] \times [m]$.

Proof: It is a direct consequence of Lemmas 14 and 19. $\square$

Lemma 20. *Let (A, G, H) be a tense $LM_{n\times m}$-algebra. Then, for all $a \in A$, the following conditions are equivalent:*

(i) $a = d(a)$,
(ii) $a = d^n(a)$ *for all* $n \in \omega$.

Proof: It immediately follows from Corollary 9. $\square$

Lemma 21. *Let $(A, \sim, \{\sigma_{ij}\}_{(i,j)\in[n]\times[m]}, G, H)$ be a tense $LM_{n\times m}$-algebra and $C(A) := \{a \in A : d(a) = a\}$. Then, $\langle C(A), \vee, \wedge, \sim, \{\sigma_{ij}\}_{(i,j)\in[n]\times[m]}, 0, 1\rangle$ is an $LM_{n\times m}$-algebra.*

Proof: From Corollary 11 and property (d1) in Corollary 9, we have that $\langle C(A), \vee, \wedge, \sim, 0, 1\rangle$ is a De Morgan algebra. Taking into account that $a = d(a)$ for all $a \in C(A)$, and the property (iv) in Corollary 11 it follows that $\sigma_{ij}(a) = \sigma_{ij}(d(a)) = d(\sigma_{ij}(a))$ for all $a \in C(A)$ and $(i,j) \in [n] \times [m]$. Therefore, $\sigma_{ij}(a) \in C(A)$ for all $a \in C(A)$ and $(i,j) \in [n] \times [m]$, from which we conclude that $\langle C(A), \vee, \wedge, \sim, \{\sigma_{ij}\}_{(i,j)\in[n]\times[m]}, 0, 1\rangle$ is an $LM_{n\times m}$-algebra. $\square$

Corollary 12. Let (A, G, H) be a tense $LM_{n\times m}$-algebra. Then, the structure $(B(C(A)), G, H)$ is a tense Boolean algebra, where $B(C(A))$ is the Boolean algebra of all complemented elements of $C(A)$.

Proof: It is a direct consequence of Lemmas 5 and 21 and property (iv) in Corollary 11. $\square$

Remark 4. Let us recall that under the Priestley duality, the lattice of all filters of a bounded distributive lattice is dually isomorphic to the lattice of all increasing closed subsets of the dual space. Under that isomorphism, any filter T of a bounded distributive lattice A corresponds to the increasing closed set

$$Y_T = \{S \in X(A) : T \subseteq S\} = \bigcap \{\sigma_A(a) : a \in T\} \tag{4.3}$$

and $\Theta_C(Y_T) = \Theta(T)$, where Θ_C is defined as in (2.11) and $\Theta(T)$ is the lattice congruence associated with T.

Conversely any increasing closed subset Y of $X(A)$ corresponds to the filter

$$T_Y = \{a \in A : Y \subseteq \sigma_A(a)\}, \tag{4.4}$$

and $\Theta(T_Y) = \Theta_C(Y)$, where Θ_C is defined as in (2.11), and $\Theta(T_Y)$ is the lattice congruence associated with T_Y.

Taking into account these last remarks on Priestley duality, Theorem 5 and Proposition 5, we can say that the congruences on a tense $LM_{n\times m}$-algebra are the lattice congruences associated with certain filters of this algebra. So our next goal is to determine the conditions that a filter of a tense $LM_{n\times m}$-algebra must fulfill for the associated lattice congruence to be a tense $LM_{n\times m}$-congruence.

THEOREM 6. *Let $(A, \sim, \{\sigma_{ij}\}_{(i,j)\in[n]\times[m]}, G, H)$ be a tense $LM_{n\times m}$-algebra. If S is a filter of A, then, the following conditions are equivalent:*

(i) $\Theta(S) \in Con_{tLM_{n\times m}}(A)$,
(ii) $d(\sigma_{ij}(a)) \in S$ *for any* $a \in S$ *and* $(i,j) \in [n]\times[m]$,
(iii) $d^n(\sigma_{ij}(a)) \in S$ *for any* $a \in S$, $n \in \omega$ *and* $(i,j) \in [n]\times[m]$.

PROOF: (i) $\Rightarrow$ (ii): Let S be a filter of A such that $\Theta(S) \in Con_{tLM_{n\times m}}(A)$. Then, from Priestley duality and Theorem 5 it follows that $\Theta(S) = \Theta_{MT}(Y_S)$, where $\Theta(S)$ is the lattice congruence associated with S, and $Y_S = \{x \in X(A) : S \subseteq x\} = \bigcap\limits_{a\in S} \sigma_A(a)$ is a closed, modal and tense subset of the tense $LM_{n\times m}$-space $X(A)$, associated with A. Since Y_S is modal and σ_A is an $LM_{n\times m}$-isomorphism, then $Y_S = f_{ij}^{A^{-1}}(Y_S) = f_{ij}^{A^{-1}}\Big(\bigcap\limits_{a\in S} \sigma_A(a)\Big) = \bigcap\limits_{a\in S} \sigma_A(\sigma_{ij}(a))$ for any $(i,j) \in [n]\times[m]$. From the last assertion, and taking into account that Y is a tense subset, Lemmas 17 and 9, Corollary 10 and the fact that the function $d_{X(A)} : X(A) \longrightarrow X(A)$ is monotone, we infer that $Y_S = d_{X(A)}\Big(\bigcap\limits_{a\in S} \sigma_A(\sigma_{ij}(a))\Big) \subseteq$

$\bigcap\limits_{a\in S} d_{X(A)}\left(\sigma_A(\sigma_{ij}(a))\right) = \bigcap\limits_{a\in S} \sigma_A(d(\sigma_{ij}(a))) \subseteq \bigcap\limits_{a\in S} \sigma_A(\sigma_{ij}(a)) = Y_S$, for any $(i,j) \in [n]\times[m]$. Hence $Y_S = \bigcap\limits_{a\in S} \sigma_A(d(\sigma_{ij}(a))$ for any $(i,j) \in [n]\times[m]$, from which we conclude that $d(\sigma_{ij}(a)) \in S$ for any $a \in S$ and $(i,j) \in [n]\times[m]$. Indeed, assume that $a \in S$, then $a \in x$ for all $x \in Y_S$, from which it follows that $x \in \bigcap\limits_{a\in S} \sigma_A(d(\sigma_{ij}(a))$ for any $(i,j) \in [n]\times[m]$, and thus $d(\sigma_{ij}(a)) \in x$ for all $x \in Y_S$ and $(i,j) \in [n]\times[m]$. Therefore, $d(\sigma_{ij}(a)) \in \bigcap\limits_{x\in Y_S} x$ for any $(i,j) \in [n]\times[m]$, and taking into account that $S = \bigcap\limits_{x\in Y_S} x$, we obtain that $d(\sigma_{ij}(a)) \in S$ for any $(i,j) \in [n]\times[m]$.
(ii) $\Rightarrow$ (i): From Priestley duality and (4.3), we have that $\bigcap\limits_{a\in S} \sigma_A(a) = Y_S = \{x \in X(A) : S \subseteq x\}$ is an increasing and closed subset of $X(A)$ and $\Theta(S) = \Theta(Y_S)$. By Theorem 5, it remains to show that Y_S is a modal and tense subset of $X(A)$. From the hypothesis (ii), it follows that for all $a \in S$, $(i,j) \in [n]\times[m]$ and $x \in Y_S$, $d(\sigma_{ij}(a)) \in x$. Therefore, from this last fact and Corollary 11, it results that $\sigma_{ij}(d(a)) \in x$ for all $(i,j) \in [n]\times[m]$ and all $x \in Y_S$, and hence (1) $Y_S \subseteq \bigcap\limits_{a\in S} \sigma_A(\sigma_{ij}(d(a)))$ for all $(i,j) \in [n]\times[m]$. Consequently, by Corollary 9, $Y_S \subseteq \bigcap\limits_{a\in S} \sigma_A(\sigma_{ij}(a))$ for all $(i,j) \in [n]\times[m]$, and from this assertion it follows that $Y_S \subseteq \bigcap\limits_{a\in S} \sigma_A\left(\varphi_1(a)\right) \subseteq \bigcap\limits_{a\in S} \sigma_A(a) = Y_s$. Since σ_A is an $LM_{n\times m}$-isomorphism, then we get that (2) $Y_s = \bigcap\limits_{a\in S} \sigma_A\Big(\sigma_{11}(a)\Big) = \bigcap\limits_{a\in S} f_{11}^{A^{-1}}(\sigma_A(a)) = f_{11}^{A^{-1}}\left(\bigcap\limits_{a\in S} \sigma_A(a)\right) = f_{11}^{A^{-1}}(Y_S)$. Therefore from the last statement and (LP6) we conclude that $Y_S = f_{ij}^{A}(Y_S)$ for all $(i,j) \in [n]\times[m]$ and so Y_S is modal. In addition, from (1), (2) and Corollary 9 we infer that $Y_S \subseteq \bigcap\limits_{a\in S} \sigma_A(d(\sigma_{11}(a)) \subseteq \bigcap\limits_{a\in S} \sigma_A(\sigma_{11}(a)) = Y_S$ and hence, $Y_S = \bigcap\limits_{a\in S} \sigma_A(d(\sigma_{11}(a))$. Then, taking into account Corollary 10 and that $\bigcap\limits_{a\in S} d_{X(A)}(\sigma_A(\sigma_{11}(a))) = d_{X(A)}\left(\bigcap\limits_{a\in S} \sigma_A(\sigma_{11}(a))\right)$, we obtain that $Y_S = d_{X(A)}(Y_S)$, and thus, from Lemma 17 and the fact that Y_S is modal, we infer that Y_S is a tense subset of $X(A)$. Finally, since Y_S is a closed, modal and tense subset of $X(A)$ and $\Theta(S) = \Theta_{MT}(Y_S)$, we conclude, from Theorem 5, that $\Theta(S) \in Con_{tLM_{n\times m}}(A)$.
(ii) $\Leftrightarrow$ (iii): It is trivial. $\square$

Theorem 6 leads us to introduce the following definition:

DEFINITION 18. Let (A, G, H) be a tense $LM_{n\times m}$-algebra. A filter S of A is a tense filter iff

(tf) $d(a) \in S$ for all $a \in S$ or equivalently $d^n(a) \in S$ for all $a \in S$ and $n \in \omega$.

Now, we remember the notion of Stone filter of an $LM_{n\times m}$-algebra.

DEFINITION 19. Let $(A, \sim, \{\sigma_{ij}\}_{(i,j)\in[n]\times[m]})$ be an $LM_{n\times m}$-algebra. A filter S of A is a Stone filter iff

(sf) $\sigma_{ij}(a) \in S$ for all $a \in S$ and $(i, j) \in [n] \times [m]$, or equivalently $\sigma_{11}(a) \in S$ for all $a \in S$.

LEMMA 22. *Let $(A, \sim, \{\sigma_{ij}\}_{(i,j)\in[n]\times[m]}, G, H)$ be a tense $LM_{n\times m}$-algebra. If S is a Stone filter of A, then the following conditions are equivalent:*

(i) *S is a tense filter of A,*

(ii) *$d^n(\sigma_{ij}(a)) \in S$ for all $a \in S$, $n \in \omega$ and $(i, j) \in [n] \times [m]$.*

PROOF:
(i) $\Rightarrow$ (ii): Let S be a Stone filter of A, $a \in S$, $n \in \omega$ and $(i, j) \in [n] \times [m]$. Since S is an Stone filter of A, we have that $\sigma_{ij}(a) \in S$. From this last assertion and the fact that S is a tense filter we conclude that $d^n(\sigma_{ij}(a)) \in S$.
(ii) $\Rightarrow$ (i): Let $a \in S$. Then, from the hypothesis (ii) we obtain that $d^n(\sigma_{11}(a)) \in S$. From the last assertion, properties (C13) and (d5) and the fact that S is a filter of A we infer that $d^n(a) \in S$ for all $n \in \omega$, and therefore S is a tense filter of A. $\square$

We will denote by $F_{TS}(A)$ the set of all tense Stone filters of a tense $LM_{n\times m}$-algebra (A, G, H).

PROPOSITION 9. Let $(A, \sim, \{\sigma_{ij}\}_{(i,j)\in[n]\times[m]}, G, H)$ be a tense $LM_{n\times m}$-algebra. Then, the following conditions are equivalent for all $\theta \subseteq A \times A$:

(i) $\theta \in Con_{tLM_{n\times m}}(A)$,

(ii) there is $S \in F_{TS}(A)$ such that $\theta = \Theta(S)$, where $\Theta(S)$ is the lattice congruence associated with the filter S.

PROOF:
(i) $\Rightarrow$ (ii): From (i) and Theorem 5, it follows that there exists $Y \in C_{MT}(X(A))$ such that (1) $\Theta_{MT}(Y) = \theta$. Then, from Remark 4, we infer that $T_Y = \{a \in A : Y \subseteq \sigma_A(a)\}$ is a filter on A and (2) $\Theta(T_Y) =$

$\Theta(Y) = \Theta_{MT}(Y)$. Therefore $\Theta(T_Y) \in Con_{tLM_{n\times m}}(A)$, and so from Theorem 6, we obtain that $Y \in F_{TS}(A)$. This last assertion, (1) and (2) enable us to conclude the proof.

(ii) $\Rightarrow$ (i): It immediatly follows from Theorem 6. □

Corollary 13. Let (A, G, H) be a tense $LM_{n\times m}$-algebra. Then,

(i) (A, G, H) is a simple tense $LM_{n\times m}$-algebra if and only if $F_{TS}(A) = \{A, \{1\}\}$.

(ii) (A, G, H) is a subdirectly irreducible tense $LM_{n\times m}$-algebra if and only if there is $T \in F_{TS}(A)$, $T \neq \{1\}$ such that $T \subseteq S$ for all $S \in F_{TS}(A)$, $S \neq \{1\}$.

Proof: It is a direct consequence of Corollaries 7 and 8, Remark 4 and Proposition 9. □

Finally, we will describe the simple and subdirectly irreducible tense $LM_{n\times m}$-algebras.

In the proof of the following proposition we will use the finite intersection property of compact spaces, which establishes that if X is a compact topological space, then for each family $\{M_i\}_{i\in I}$ of closed subsets of X satisfying $\bigcap_{i\in I} M_i = \emptyset$, there is a finite subfamily $\{M_{i_1}, \ldots, M_{i_n}\}$ such that $\bigcap_{j=1}^{n} M_{i_j} = \emptyset$.

Proposition 10. Let (A, G, H) be a tense $LM_{n\times m}$-algebra and $(X(A), g_A, \{f_{ij}^A\}_{(i,j)\in[n]\times[m]}, R^A)$ be the tense $LM_{n\times m}$-space associated with A. Then, the following conditions are equivalent:

(i) (A, G, H) is a simple tense $LM_{n\times m}$-algebra,

(ii) for every $U \in D(X(A)) \setminus \{X(A)\}$ and for every $(i, j) \in [n] \times [m]$ such that ${f_{ij}^A}^{-1}(U) \neq X(A)$, $\bigcap_{n\in\omega} d^n_{X(A)}({f_{ij}^A}^{-1}(U)) = \emptyset$,

(iii) for every $U \in D(X(A)) \setminus \{X(A)\}$ and for every $(i, j) \in [n] \times [m]$ such that ${f_{ij}^A}^{-1}(U) \neq X(A)$, $d^{n^U_{ij}}_{X(A)}({f_{ij}^A}^{-1}(U)) = \emptyset$ for some $n^U_{ij} \in \omega$,

(iv) for every $U \in B(D(X(A))) \setminus \{X(A)\}$, there is $n_U \in \omega$ such that $d^{n_U}_{X(A)}(U) = \emptyset$,

(v) $F_{TS}(D(X(A))) = \{D(X(A)), \{X(A)\}\}$.

Proof:
(i) $\Rightarrow$ (ii): Let $U \in D(X(A)) \setminus \{X(A)\}$. Now, let $(i,j) \in [n] \times [m]$ such that $f_{ij}^{A^{-1}}(U) \neq X(A)$, then from (d5) in Proposition 8 we have that $d^n_{X(A)}(f_{ij}^{A^{-1}}(U)) \neq X(A)$. From this last assertion and (d9) in Proposition 8, we obtain that $\bigcap\limits_{n\in\omega} d^n_{X(A)}(f_{ij}^{A^{-1}}(U)) \in C_{MT}(X(A)) \setminus \{X(A)\}$. From this last assertion, the hypothesis (i) and Corollary 7, we conclude that $\bigcap\limits_{n\in\omega} d^n_{X(A)}(f_{ij}^{A^{-1}}(U)) = \emptyset$.
(ii) $\Rightarrow$ (iii): Let $U \in D(X(A)) \setminus \{X(A)\}$ and $(i,j) \in [n] \times [m]$ such that $f_{ij}^{A^{-1}}(U) \neq X(A)$. Then, from the hypothesis (ii), we have that

$$(1)\ \bigcap_{n\in\omega} d^n_{X(A)}(f_{ij}^{A^{-1}}(U)) = \emptyset.$$

Besides, for all $n \in \omega$, $d^n_{X(A)}(f_{ij}^{A^{-1}}(U))$ is a closed subset of $X(A)$ and $d^n_{X(A)}(f_{ij}^{A^{-1}}(U)) = \bigcap\limits_{k=1}^{n} d^k_{X(A)}(f_{ij}^{A^{-1}}(U))$. Then, from (1), the last statement, the fact that $X(A)$ is compact and the finite intersection property of compact spaces, we conclude that there is $n_{ij}^U \in \omega$ such that $d^{n_{ij}^U}_{X(A)}(f_{ij}^{A^{-1}}(U)) = \emptyset$.
(iii) $\Rightarrow$ (iv): From Lemma 5, we have that $U \in B(D(X(A)))$ if and only if $U = f_{ij}^{A^{-1}}(U)$ for all $(i,j) \in [n] \times [m]$, and so from property (LP10) of $LM_{n\times m}$-spaces, we infer that $U \in B(D(X(A))) \setminus \{X(A)\}$ iff $f_{ij}^{A^{-1}}(U) \neq X(A)$ for all $(i,j) \in [n] \times [m]$. Therefore, from the previous assertion and the hypothesis (iii), we obtain that for each $U \in B(D(X(A)))$ and each $(i,j) \in [n] \times [m]$, there is $n_{ij}^U \in \omega$ such that $d^{n_{ij}^U}_{X(A)}(U) = \emptyset$. Since, from (1) it follows that for all $(i,j),(r,s) \in [n] \times [m]$, $n_{ij}^U = n_{rs}^U = n^U$, then the proof is complete.
(iv) $\Rightarrow$ (v): Assume that $S \in F_{TS}(D(X(A)))$, $S \neq \{X(A)\}$. Then there is (1) $U \in S$, $U \neq X(A)$ and so from property (LP10) of $LM_{n\times m}$-spaces, we infer that there is $(i,j) \in [n]\times[m]$ such that $f_{ij}^{A^{-1}}(U) \neq X(A)$. Considering (2) $V = f_{ij}^{A^{-1}}(U)$, then from Lemma 5, we obtain that $V \in B(D(X(A)))$, $V \neq X(A)$. Hence, from the hypothesis (iv), we can assert that there is $n_V \in \omega$ such that $d^{n_V}_{X(A)}(V) = \emptyset$. From (1), (2), the preceding assertion and Definitions 18 and 19, we deduce that $\emptyset \in S$, which implies that $S = D(X(A))$.

(v) $\Rightarrow$ (i): It immediately follows from Corollary 13 and the fact that (A, G, H) is isomorphic to the tense $LM_{n\times m}$-algebra $(D(X(A)), G_{R^A}, H_{R^{A^{-1}}})$. □

Corollary 14. Let $(A, \sim, \{\sigma_{ij}\}_{(i,j)\in[n]\times[m]}, G, H)$ be a tense $LM_{n\times m}$-algebra. Then, the following conditions are equivalent:

(i) $(A, \sim, \{\sigma_{ij}\}_{(i,j)\in[n]\times[m]}, G, H)$ is a simple tense $LM_{n\times m}$-algebra,
(ii) for every $a \in A \setminus \{1\}$ and for every $(i,j) \in [n]\times[m]$ such that $\sigma_{ij}(a) \neq 1$, $d^{n^a_{ij}}(\sigma_{ij}(a)) = 0$ for some $n^a_{ij} \in \omega$,
(iii) for each $a \in B(A) \setminus \{1\}$, there is $n_a \in \omega$ such that $d^{n_a}(a) = 0$,
(iv) $F_{TS}(A) = \{A, \{1\}\}$.

Proof: It is a direct consequence of Proposition 10 and the fact that $\sigma_A : A \longrightarrow D(X(A))$ is a tense $LM_{n\times m}$-isomorphism. □

Corollary 15. If $(A, \sim, \{\sigma_{ij}\}_{(i,j)\in[n]\times[m]}, G, H)$ is a simple tense $LM_{n\times m}$-algebra, then $B(C(A)) = \{0, 1\}$ and therefore $(C(A), \sim, \{\sigma_{ij}\}_{(i,j)\in[n]\times[m]})$ is a simple $LM_{n\times m}$-algebra.

Proof: From Lemmas 5 and 20, property (iv) in Corollary 11 and property (ii) in Corollary 14 it follows that $B(C(A)) = \{0, 1\}$. From this last assertion, (LM8) and Lemma 21 the proof is complete. □

Next, we will recall two concepts which will play a fundamental role in this paper. Let Y be a topological space and $y_0 \in Y$. A net in a space Y is a map $\varphi : D \longrightarrow Y$ of some directed set $(D, \prec)$ (i.e. $D \neq \emptyset$ and $\prec$ is a preorder on D and for all d_1, $d_2 \in D$ there is $d_3 \in D$ such that $d_1 \prec d_3$ and $d_2 \prec d_3$). Besides, we say that φ converges to y_0 (written $\varphi \to y_0$) if for all neighborhoods $U(y_0)$ of y_0 there is $d_0 \in D$ such that for all $d \in D$, $d_0 \prec d$, $\varphi(d) \in U(y_0)$. We also say that φ accumulates at y_0 (written $\varphi \succ y_0$) if for all neighborhoods $U(y_0)$ of y_0 and for all $d \in D$, there is $d_c \in D$ such that $d \prec d_c$ and $\varphi(d_c) \in U(y_0)$. If $\varphi : D \longrightarrow Y$ is a net and $y_d = \varphi(d)$ for all $d \in D$, then the net φ it will be denoted by $(y_d)_{d\in D}$. If $\varphi \to y_0$, it will be denoted by $(y_d) \underset{d\in D}{\longrightarrow} y_0$. If $\varphi \succ y_0$, it will be denoted $(y_d)_{d\in D} \succ y_0$.

Proposition 11. Let (A, G, H) be a tense $LM_{n\times m}$-algebra and $(X(A), g_A, \{f^A_{ij}\}_{(i,j)\in[n]\times[m]}, R^A)$ be the tense $LM_{n\times m}$-space associated with A. Then, the following conditions are equivalent:

(i) (A, G, H) is a subdirectly irreducible tense $LM_{n\times m}$-algebra,

(ii) there is $V \in B(D(X(A)))$, $V \neq X(A)$, such that for each $U \in D(X(A))$, $U \neq X(A)$ and for each $(i,j) \in [n]\times[m]$ such that $f_{ij}^{A^{-1}}(U) \neq X(A)$, $\bigcap\limits_{n\in\omega} d^n_{X(A)}(f_{ij}^{A^{-1}}(U)) \subseteq V$,

(iii) there is $V \in B(D(X(A)))$, $V \neq X(A)$, such that for each $U \in D(X(A))$, $U \neq X(A)$ and for each $(i,j) \in [n]\times[m]$ such that $f_{ij}^{A^{-1}}(U) \neq X(A)$, $d^{n_i^U}_{X(A)}(f_{ij}^{A^{-1}}(U)) \subseteq V$ for some $n^U_{ij} \in \omega$,

(iv) there is $V \in B(D(X(A)))$, $V \neq X(A)$, such that for all $U \in B(D(X(A)))$, $U \neq X(A)$, $d^{n^U}_{X(A)}(U) \subseteq V$, for some $n^U \in \omega$,

(v) there is $T \in F_{TS}(D(X(A))$, $T \neq \{X(A)\}$, such that $T \subseteq S$ for all $S \in F_{TS}(D(X(A)))$, $S \neq \{X(A)\}$.

PROOF:
(i) $\Rightarrow$ (ii): From (i) and Corollary 8 we infer that there exists $Y \in C_{MT}(X(A)) \setminus \{X(A)\}$ such that (1) $Z \subseteq Y$ for all $Z \in C_{MT}(X(A)) \setminus \{X(A)\}$. Since Y is modal, then by Proposition 5, there is (2) $x \in \max X(A) \setminus Y$. Taking into account that Y is a closed subset of $X(A)$ and hence it is compact, we can assert that there is $W \in D(X(A))$, such that (3) $Y \subseteq W$ and (4) $x \notin W$. In addition from (2) and (LP15) in Corollary 1, we have that $x = f^A_{(n-1)(m-1)}(x)$ and so by (4) we infer that $x \notin f^{A^{-1}}_{(n-1)(m-1)}(W)$. If $V = f^{A^{-1}}_{(n-1)(m-1)}(W)$, then $V \in B(D(X(A))) \setminus \{X(A)\}$. Besides, from (3) and the fact that $Y = f^{A^{-1}}_{(n-1)(m-1)}(Y)$, we get that (5) $Y \subseteq f^{A^{-1}}_{(n-1)(m-1)}(W) = V$. On the other hand, if $U \in D(X(A)) \setminus \{X(A)\}$, then from Lemma 14 and property (LP10) of $LM_{n\times m}$-spaces, we infer that there is at least $(i_0, j_0) \in [n]\times[m]$ such that $f^{A^{-1}}_{i_0j_0}(U) \neq X(A)$. Now, let $(i,j) \in [n]\times[m]$ such that $f^{A^{-1}}_{ij}(U) \neq X(A)$, then from Proposition 8 we obtain that $\bigcap\limits_{n\in\omega} d^n_{X(A)}(f^{A^{-1}}_{ij}(U)) \in C_{MT}(X(A)) \setminus \{X(A)\}$, from which we conclude, by the assertions (1) and (5), that $\bigcap\limits_{n\in\omega} d^n_{X(A)}(f^{A^{-1}}_{ij}(U)) \subseteq V$.

(ii) $\Rightarrow$ (iii): From the hypothesis (ii), we have that there is $V \in B(D(X(A))) \setminus \{X(A)\}$, such that (1) $\bigcap\limits_{n\in\omega} d^n_{X(A)}(f^{A^{-1}}_{ij}(U)) \subseteq V$ for each $U \in D(X(A)) \setminus \{X(A)\}$ and each $i \in [n]\times[m]$ such that $f^{A^{-1}}_{ij}(U) \neq X(A)$. Suppose that there is $U \in D(X(A)) \setminus \{X(A)\}$ and there is $i_0 \in [n]\times[m]$, which satisfy (1) and $d^n_{X(A)}({f^A_{i_0j_0}}^{-1}(U)) \not\subseteq V$ for all $n \in \omega$. Then for each $n \in \omega$, there exists (2) $x_n \in d^n_{X(A)}({f^A_{i_0j_0}}^{-1}(U))$ and $x_n \notin V$. Hence $(x_n)_{n\in\omega}$ is a sequence in $X(A) \setminus V$ and since $X(A) \setminus V$ is compact, we can assert that there exists (3) $x \in X(A) \setminus V$ such that $(x_n)_{n\in\omega}$ accumulates at x. In addition, by (1) and (3), we have that $x \notin \bigcap\limits_{n\in\omega} d^n_{X(A)}({f^A_{i_0j_0}}^{-1}(U))$, and thus $x \in X(A) \setminus d^{n_0}_{X(A)}({f^A_{i_0j_0}}^{-1}(U))$ for some $n_0 \in \omega$. Since x is an accumulation point of $(x_n)_{n\in\omega}$, then the preceding assertion and the fact that $X(A) \setminus d^{n_0}_{X(A)}({f^A_{i_0j_0}}^{-1}(U))$ is an open subset of $X(A)$ allows us to infer that for all $n \in \omega$ there is $m_n \in \omega$ such that $n \leq m_n$ and $x_{m_n} \in X(A) \setminus d^{n_0}_{X(A)}({f^A_{i_0j_0}}^{-1}(U))$. Thus $x_{m_{n_0}} \in X(A) \setminus d^{n_0}_{X(A)}({f^A_{i_0j_0}}^{-1}(U))$ and $n_0 \leq m_{n_0}$. As a consequence of Proposition 8 we have that $X(A) \setminus d^{n_0}_{X(A)}({f^A_{i_0j_0}}^{-1}(U)) \subseteq X(A) \setminus d^{m_{n_0}}_{X(A)}({f^A_{i_0j_0}}^{-1}(U))$ and so $x_{m_{n_0}} \notin d^{m_{n_0}}_{X(A)}({f^A_{i_0j_0}}^{-1}(U)))$, which contradicts (2). Therefore, for every $U \in D(X(A)) \setminus \{X(A)\}$ and $(i,j) \in [n] \times [m]$ such that ${f^A_{ij}}^{-1}(U) \neq X(A)$, $d^{n^U_i}_{X(A)}({f^A_{ij}}^{-1}(U)) \subseteq V$ for some $n^U_i \in \omega$.

(iii) $\Rightarrow$ (iv): From Lemma 5 and the property (LP10) of $LM_{n\times m}$-spaces, we infer that for all $U \in B(D(X(A)))$, $U \neq X(A)$ if and only if ${f^A_{ij}}^{-1}(U) \neq X(A)$ for all $(i,j) \in [n] \times [m]$. Therefore, from the last statement and the hypothesis (iii), we obtain that for each $U \in B(D(X(A)))$, $U \neq X(A)$ and each $(i,j) \in [n] \times [m]$, there is $n^U_{ij} \in \omega$ such that $d^{n^U_{ij}}_{X(A)}(U) \subseteq V$. Then, considering $n_U = \max\{n^U_{ij} : (i,j) \in [n] \times [m]\}$, from (d2) in Proposition 8 we conclude that $d^{n_U}_{X(A)}(U) \subseteq V$.

(iv) $\Rightarrow$ (v): Let $S \in F_{TS}(D(X(A)))$, $S \neq \{X(A)\}$. Then there exists (1) $U \in S \setminus \{X(A)\}$ and so from property (LP10) we infer that there is $(i,j) \in [n] \times [m]$ such that ${f^A_{ij}}^{-1}(U) \neq X(A)$. Let (2) $W = {f^A_{ij}}^{-1}(U)$. Then, from Lemma 5 we have that $W \in B(D(X(A)))$, $W \neq X(A)$ and thus by the hypothesis (iv), we can assert that there is $n_W \in \omega$ such that (3) $d^{n_W}_{X(A)}(W) \subseteq V$. Besides, from the assertions (1) and (2) and Lemma 22, we obtain that $d^{n_W}_{X(A)}(W) \in S$. From the last statement, (3) and the fact that S is a filter of $D(X(A))$, we get that $V \in S$, and so $V \in \bigcap\limits_{S\in\Omega} S$, where $\Omega = \{S \in F_{TS}(D(X(A))) : S \neq \{X(A)\}\}$. Therefore, considering $T = \bigcap\limits_{S\in\Omega} S$ and taking into account that $V \neq X(A)$, we conclude that $T \in \Omega$ and $T \subseteq S$, for all $S \in \Omega$.

(v) $\Rightarrow$ (i): It follows from the fact that (A, G, H) and $(D(X(A)), G_{R^A}, H_{R^{A^{-1}}})$ are isomorphic tense $LM_{n\times m}$-algebras. $\square$

COROLLARY 16. Let $(A, \sim, \{\sigma_{ij}\}_{(i,j)\in[n]\times[m]}, G, H)$ be a tense $LM_{n\times m}$-algebra. Then, the following conditions are equivalent:

(i) $(A, \sim, \{\sigma_{ij}\}_{(i,j)\in[n]\times[m]}, G, H)$ is a subdirectly irreducible tense $LM_{n\times m}$-algebra,

(ii) there is $b \in B(A)\setminus\{1\}$ such that for every $a \in A\setminus\{1\}$ and for every $(i,j) \in [n]\times[m]$ such that $\sigma_{ij}(a) \neq 1$, $d^{n^a_{ij}}(\sigma_{ij}(a)) \leq b$ for some $n^a_{ij} \in \omega$,

(iii) there is $b \in B(A)\setminus\{1\}$ such that for every $a \in B(A)\setminus\{1\}$, there is $n_a \in \omega$ such that $d^{n_a}(a) \leq b$,

(iv) there is $T \in F_{TS}(A)$, $T \neq \{1\}$ such that $T \subseteq S$ for all $S \in F_{TS}(A)$, $S \neq \{1\}$.

PROOF: It is a direct consequence of Proposition 11 and the fact that $\sigma_A : A \longrightarrow D(X(A))$ is a tense $LM_{n\times m}$-isomorphism. $\square$

COROLLARY 17. Let $(A, \sim, \{\sigma_{ij}\}_{(i,j)\in[n]\times[m]}, G, H)$ be a subdirectly irreducible tense $LM_{n\times m}$-algebra such that for every $a \in B(A)\setminus\{1\}$, $d^n(a) = d^{n_a}(a)$ for some $n_a \in \omega$ and for all $n \in \omega$, $n_a \leq n$. Then, $(C(A) \sim\ , \{\sigma_{ij}\}_{(i,j)\in[n]\times[m]})$ is a simple $LM_{n\times m}$-algebra.

PROOF: From Corollary 16, we can assert that there exists $b \in B(A)\setminus\{1\}$ such that (1) for every $a \in B(A)\setminus\{1\}$, $d^{n_a}(a) \leq b$ for some $n_a \in \omega$. Also, from hypothesis we have that there is $n_b \in \omega$ such that $d^n(b) = d^{n_b}(b)$ for all $n \in \omega$, $n_b \leq n$. Considering $u = d^{n_b}(b)$, then from the last assertion, properties (d5) and (d7) in Corollary 9 and the fact that $b \in B(A)\setminus\{1\}$, we obtain that $u \in B(C(A))$, $u \neq 1$. In addition, let $c \in B(C(A))$, $c \neq 1$, then by Lemma 20, $c = d^n(c)$ for all $n \in \omega$, and thus from (1) we get that $c = d^{n_c}(c) \leq b$. Then from property (d4) in Corollary 9, we infer that $c = d^{n_b}(c) \leq d^{n_b}(b) = u$. Consequently, from Corollary 12, $B(C(A))$ is a totally ordered Boolean algebra and so $B(C(A)) = \{0, 1\}$. Therefore, from (LM8) and Lemma 21, we conclude that $(C(A) \sim, \{\sigma_{ij}\}_{(i,j)\in[n]\times[m]})$ is a simple $LM_{n\times m}$-algebra. $\square$

5. Conclusion and future research

Priestley spaces arise more naturally in relation with logics, as Priestley spaces incorporate the now widely used Kripke semantics in them. As a result, Priestley's duality became rather popular among logicians, and most dualities for distributive lattices with operators have been performed in terms of Priestley spaces. In particular, in this paper we have determined a topological duality for tense $n \times m$-valued Łukasiewicz–Moisil algebras, extending the one obtained for $n \times m$-valued Łukasiewicz–Moisil algebras in [27]. By means of the above duality we have characterized simple and subdirectly irreducible tense $n \times m$-valued Łukasiewicz–Moisil algebras. We expect that our method can be easily applied to modal operators or monadic operators on $n \times m$-valued Łukasiewicz–Moisil algebras (see, [25], [27]).

References

[1] V. Boicescu, A. Filipoiu, G. Georgescu and S. Rudeanu, *Łukasiewicz–Moisil Algebras*, **Annals of Discrete Mathematics**, Vol. 49 (1991), North-Holland.

[2] M. Botur, I. Chajda, R. Halaš and M. Kolařík, *Tense operators on Basic Algebras*, **International Journal of Theoretical Physics**, Vol. 50, No. 12 (2011), pp. 3737–3749.

[3] M. Botur, J. Paseka, *On tense MV-algebras*, **Fuzzy Sets and Systems**, Vol. 259 (2015), pp. 111–125.

[4] J. Burges, *Basic tense logic*, [in:] D. M. Gabbay, F. Günter (eds.), **Handbook of Philosophical Logic**, Vol. II, Reidel, Dordrecht (1984), pp. 89–139.

[5] I. Chajda, *Algebraic axiomatization of tense intuitionistic logic*, **Central European Journal of Mathematics**, Vol. 9, No. 5 (2011), pp. 1185–1191.

[6] I. Chajda and J. Paseka, *Dynamic effect algebras and their representations*, **Soft Computing**, Vol. 16, No. 10 (2012), pp. 1733–1741.

[7] I. Chajda and M. Kolařík, *Dynamic Effect Algebras*, **Mathematica Slovaca**, Vol. 62, No. 3 (2012), pp. 379–388.

[8] C. Chiriţă, *Tense θ-valued Moisil propositional logic*, **International Journal of Computers Communications and Control**, Vol. 5 (2010), pp. 642–653.

[9] C. Chiriţă, *Tense θ-valued Łukasiewicz–Moisil algebras*, **Journal of Multiple-Valued Logic and Soft Computing**, Vol. 17, No. 1 (2011), pp. 1–24.

[10] R. Cignoli, *Moisil Algebras*, **Notas de Lógica Matemática**, Vol. 27 (1970), Instituto de Matemática, Universidad Nacional del Sur, Bahía Blanca.

[11] W. Cornish and P. Fowler, *Coproducts of De Morgan algebras*, **Bulletin of the Australian Mathematical Society**, Vol. 16 (1977), pp. 1–13.

[12] D. Diaconescu and G. Georgescu, *Tense operators on MV-algebras and Łukasiewicz–Moisil algebras*, **Fundamenta Informaticae**, Vol. 81, No. 4 (2007), pp. 379–408.

[13] A. V. Figallo, G. Pelaitay, *$n \times m$-valued Łukasiewicz–Moisil algebras with two modal operators*, **South American Journal of Logic**, Vol. 1, No. 1 (2015), pp. 267–281.

[14] A. V. Figallo, I. Pascual, G. Pelaitay, *A topological duality for tense LM_n-algebras and applications*, **Logic Journal of the IGPL**, Vol. 26, No. 4 (2018), pp. 339–380.

[15] A. V. Figallo and G. Pelaitay, *Note on tense SHn-algebras*, **Analele Universitatii din Craiova. Seria Matematica-Informatica**, Vol. 38, No. 4 (2011), pp. 24–32.

[16] A. V. Figallo and G. Pelaitay, *Tense operators on De Morgan algebras*, **Logic Journal of the IGPL**, Vol. 22, No. 2, (2014), pp. 255–267.

[17] A. V. Figallo and G. Pelaitay, *A representation theorem for tense $n \times m$-valued Łukasiewicz–Moisil algebras*, **Mathematica Bohemica**, Vol. 140, No. 3 (2015), pp. 345–360.

[18] A. V. Figallo and G. Pelaitay, *Discrete duality for tense Łukasiewicz–Moisil algebras*, **Fundamenta Informaticae**, Vol. 136, No. 4 (2015), pp. 317–329.

[19] T. Kowalski, *Varieties of tense algebras*, **Reports on Mathematical Logic**, Vol. 32 (1998), pp. 53–95.

[20] Gr. C. Moisil, *Recherches sur les logiques non-chrysippiennes*, **Annales Scientifiques de l'Université de Jassy**, Vol. 26 (1940), pp. 431–466.

[21] J. Paseka, *Operators on MV-algebras and their representations*, **Fuzzy Sets and Systems**, Vol. 232 (2013), pp. 62–73.

[22] H. A. Priestley, *Representation of distributive lattices by means of ordered Stone spaces*, **Bulletin of the London Mathematical Society**, Vol. 2 (1970), pp. 186–190.

[23] A. V. Figallo and C. Sanza, *Álgebras de Łukasiewicz $n \times m$-valuadas con negación*, **Noticiero de la Unión Matemática Argentina**, Vol. 93 (2000).

[24] A. V. Figallo and C. Sanza, *The $NS_{n\times m}$-propositional calculus*, **Bulletin of the Section of Logic**, Vol. 35, No. 2 (2008), pp. 67–79.

[25] A. V. Figallo and C. Sanza, *Monadic $n \times m$-valued Łukasiewicz–Mosil algebras*, **Mathematica Bohemica**, Vol. 137, No. 4 (2012), pp. 425–447.

[26] A. V. Figallo and G. Pelaitay, *A representation theorem for tense $n \times m$-valued Łukasiewicz–Moisil algebras*, **Mathematica Bohemica**, Vol. 140, No. 3 (2015), pp. 345–360.

[27] A. V. Figallo, I. Pascual, G. Pelaitay, *A new topological duality for $n \times m$-valued Łukasiewicz–Moisil algebras*, **Asian–European Journal of Mathematics** (2019).

[28] C. Gallardo, C. Sanza and A. Ziliani, *F-multipliers and the localization of $LM_{n\times m}$-algebras*, **Analele Stiintifice ale Universitatii Ovidius Constanta**, Vol. 21, No. 1 (2013), pp. 285–304.

[29] Gr. C. Moisil, **Essais sur les logiques non Chrysippiennes**, Ed. Academiei, Bucarest, 1972.

[30] H. Priestley, *Representation of distributive lattices by means of ordered Stone spaces*, **Bulletin of the London Mathematical Society**, Vol. 2 (1970), pp. 186–190.

[31] H. Priestley, *Ordered topological spaces and the representation of distributive lattices*, **Proceedings of the London Mathematical Society**, Vol. 3 (1972), pp. 507–530.

[32] H. Priestley, *Ordered sets duality for distributive lattices*, **Annals of Discrete Mathematics**, Vol. 23 (1984), pp. 39–60.

[33] C. Sanza, *Notes on $n \times m$-valued Łukasiewicz algebras with negation*, **Logic Journal of the IGPL**, Vol. 6, No. 12 (2004), pp. 499–507.

[34] C. Sanza, *$n \times m$-valued Łukasiewicz algebras with negation*, **Reports on Mathematical Logic**, Vol. 40 (2006), pp. 83–106.

[35] C. Sanza, *On $n\times m$-valued Łukasiewicz–Moisil algebras*, **Central European Journal of Mathematics**, Vol. 6, No. 3 (2008), pp. 372–383.

[36] W. Suchoń, *Matrix Łukasiewicz Algebras*, **Reports on Mathematical Logic**, Vol. 4 (1975), pp. 91–104.

Universidad Nacional de San Juan
Instituto de Ciencias Básicas
5400 San Juan, Argentina
e-mail: avfigallo@gmail.com
ipascual756@hotmail.com
gpelaitay@gmail.com