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Marek Nowak

DISJUNCTIVE MULTIPLE-CONCLUSION CONSEQUENCE RELATIONS

Abstract

The concept of multiple-conclusion consequence relation from [8] and [7] is considered. The closure operation C assigning to any binary relation r (defined on the power set of a set of all formulas of a given language) the least multiple-conclusion consequence relation containing r, is defined on the grounds of a natural Galois connection. It is shown that the very closure C is an isomorphism from the power set algebra of a simple binary relation to the Boolean algebra of all multiple-conclusion consequence relations.

Keywords: multiple-conclusion consequence relation, closure operation, Galois connection.

1. Preliminaries

Given a set A, any mapping $C : \wp(A) \longrightarrow \wp(A)$ such that for each $X, Y \subseteq A$, $X \subseteq C(X)$, $C(C(X)) \subseteq C(X)$ and C is monotone: $X \subseteq Y \Rightarrow C(X) \subseteq C(Y)$, is called a *closure operation* defined on the power set $\wp(A)$ of A. Any subset $\mathcal{B} \subseteq \wp(A)$ is said to be a *closure system over* A (or *of the complete lattice* $(\wp(A), \subseteq)$), if for each $\mathcal{X} \subseteq \mathcal{B}$, $\bigcap \mathcal{X} \in \mathcal{B}$. Given a closure operation C on $\wp(A)$, the set of all its fixed points called *closed elements*: $Cl(C) = \{X \subseteq A : X = C(X)\}$, is a closure system over A. Conversely, given a closure system $\mathcal{B}$ over A, the mapping $C : \wp(A) \longrightarrow \wp(A)$ defined by $C(X) = \bigcap\{Y \in \mathcal{B} : X \subseteq Y\}$, is a closure operation on $\wp(A)$. The closure system $\mathcal{B}$ is just the set of all its closed elements. On the other hand, the closure system $Cl(C)$ of all closed elements of a given closure operation C defines, in that way, just the operation C. Thus, there is a one to one correspondence between the class of all closure operations defined on $\wp(A)$ and of all closure systems of $(\wp(A), \subseteq)$, in fact, it is a dual isomorphism between the respective complete lattices of all closure operations and closure systems (the poset $(\mathcal{C}(A), \leq)$ of all closure operations defined on $\wp(A)$, where $C_1 \leq C_2$ iff $C_1(X) \subseteq C_2(X)$ for each $X \subseteq A$, forms a complete lattice such that for any class $\mathcal{E} \subseteq \mathcal{C}(A)$ its infimum, $\inf \mathcal{E}$, is a closure operation defined on $\wp(A)$ by $(\inf \mathcal{E})(X) = \bigcap\{C(X) : C \in \mathcal{E}\}$). Any closure system $\mathcal{B}$ of $(\wp(A), \subseteq)$ forms a complete lattice with respect to the order $\subseteq$ such that $\inf \mathcal{X} = \bigcap \mathcal{X}$ and $\sup \mathcal{X} = C(\bigcup \mathcal{X})$, for each $\mathcal{X} \subseteq \mathcal{B}$, where C is the closure operation corresponding to closure system $\mathcal{B}$. Given a family $\mathcal{X} \subseteq \wp(A)$, there exists the least closure system $\mathcal{B}$ of $(\wp(A), \subseteq)$ such that $\mathcal{X} \subseteq \mathcal{B}$. It is called a *closure system generated by* $\mathcal{X}$ and shall be denoted by $[\mathcal{X}]$. It is simply the intersection of all closure systems of $(\wp(A), \subseteq)$ containing $\mathcal{X}$ and is expressed by $[\mathcal{X}] = \{\bigcap \mathcal{Y} : \mathcal{Y} \subseteq \mathcal{X}\}$. The closure operation C corresponding to closure system $[\mathcal{X}]$ is defined by $C(X) = \bigcap\{Y \in \mathcal{X} : X \subseteq Y\}$, any $X \subseteq A$.

When A is a set of all formulae of a given formal language, a closure operation C defined on $\wp(A)$ is called a *consequence operation (in the sense of Tarski)*.

We shall apply here the standard (called sometimes archetypal) antimonotone Galois connection (f, g) defined on the complete lattices $(\wp(A), \subseteq)$, $(\wp(B), \subseteq)$ of all subsets of given sets A, B by a binary relation $R \subseteq A \times B$ (cf. [3], a general theory is to be found for example in [1, 2, 4]). That is, $f : \wp(A) \longrightarrow \wp(B)$ and $g : \wp(B) \longrightarrow \wp(A)$ are the mappings defined for any $X \subseteq A$, $a \in A$, $Y \subseteq B$, $b \in B$ by

$$b \in f(X) \text{ iff for all } x \in X,\ (x, b) \in R,$$

$$a \in g(Y) \text{ iff for all } y \in Y,\ (a, y) \in R.$$

The following three facts are useful for our goals.

The compositions $f \circ g$, $g \circ f$ are closure operations on $\wp(A)$, $\wp(B)$, respectively.

The set $Cl(f \circ g)$ of all closed sets with respect to closure operation $f \circ g$ is the counterdomain of map g: $\{X \subseteq A : g(f(X)) = X\} = \{g(Y) : Y \subseteq B\}$ and similarly, $Cl(g \circ f) = \{Y \subseteq B : f(g(Y)) = Y\} = \{f(X) : X \subseteq A\}$.

The mapping f restricted to $Cl(f \circ g)$ is a dual isomorphism of the complete lattices $(Cl(f \circ g), \subseteq)$, $(Cl(g \circ f), \subseteq)$ as well as the map g restricted to $Cl(g \circ f)$ is the inverse dual isomorphism.

2. The concept of disjunctive multiple-conclusion consequence relation

This what will be called here a *disjunctive* consequence relation recalls the concept of multiple-conclusion entailment or multiple-conclusion consequence relation [7, 8]. In [8, p. 28] the following definition of multiple-conclusion consequence relation was introduced. Let V be a set of all formulae of a given language. For any $\mathcal{T} \subseteq \wp(V)$ a binary relation $\vdash_{\mathcal{T}}$ is defined on $\wp(V)$ by

$$(mc)\ X \vdash_{\mathcal{T}} Y \ \text{ iff } \ \forall\, T \in \mathcal{T}\ (X \subseteq T \ \Rightarrow \ Y \cap T \neq \emptyset).$$

We say that $\vdash \subseteq \wp(V) \times \wp(V)$ is a multiple-conclusion consequence relation iff $\vdash = \vdash_{\mathcal{T}}$ for some $\mathcal{T} \subseteq \wp(V)$. Next the authors of [8] prove the theorem (2.1, p. 30):

A relation $\vdash$ is a multiple-conclusion consequence relation iff it satisfies the following conditions for any $X, Y \subseteq V$:

$$(overlap)\ X \cap Y \neq \emptyset \ \Rightarrow \ X \vdash Y,$$

$$(dilution)\ X \vdash Y,\ X \subseteq X',\ Y \subseteq Y' \ \Rightarrow \ X' \vdash Y',$$

$$(cutforsets)\ \forall S \subseteq V\ \ ((\forall Z \subseteq S,\ X \cup Z \vdash Y \cup (S - Z)) \ \Rightarrow \ X \vdash Y).$$

Given $S \subseteq V$, the part $(\forall Z \subseteq S,\ X \cup Z \vdash Y \cup (S - Z)) \ \Rightarrow \ X \vdash Y$ of the condition $(cutforsets)$ is called $(cutforS)$. In turn, $(cutforformulae)$ denotes the family of all the conditions $(cutfor\{\alpha\})$, $\alpha \in V$:

$$(cutfor\{\alpha\})\ X \vdash Y \cup \{\alpha\}\ \&\ X \cup \{\alpha\} \vdash Y \ \Rightarrow \ X \vdash Y,$$

that is, stands to the cut rule of [5] from 1934. In general, granted $(dilution)$, the conditions $(cutforsets)$ and $(cutforV)$ are equivalent (Theorem 2.2 in [8], p. 31). Moreover, when a binary relation $\vdash \subseteq \wp(V) \times \wp(V)$ satisfies not only $(dilution)$ but also is compact, i.e fulfils the condition

$(compactness)$ $X \vdash Y \ \Rightarrow$ there exist finite subsets $X' \subseteq X,\ Y' \subseteq Y$ such that $X' \vdash Y'$, both conditions $(cut for sets)$, $(cut for formulae)$ are equivalent (Theorem 2.9 in [8], p. 37).

The conditions $(overlap), (dilution), (cut for formulae)$, under different names, were used to define on finite sets of formulas, the relation of multiple-conclusion entailment by D. Scott [7].

In [11] it was proved that when a family $\mathcal{T} \subseteq \wp(V)$ is a closure system over V, the consequence relation $\vdash_{\mathcal{T}}$ defined by (mc), may be expressed by

(dis) $X \vdash_{\mathcal{T}} Y$ iff $Y \cap C_{\mathcal{T}}(X) \neq \emptyset$,

where $C_{\mathcal{T}}$ is the closure operation determined by closure system $\mathcal{T}$. As it is seen, given a set of premises X some of conclusions of the consequence relation $\vdash_{\mathcal{T}}$ are conclusions of ordinary consequence operation $C_{\mathcal{T}}$ associated with the relation. So, one may say that the relation $\vdash_{\mathcal{T}}$ has a *disjunctive* character. It is worth to notice that in general, for arbitrary family $\mathcal{T} \subseteq \wp(V)$ only the implication $(\Leftarrow)$ from right to left holds true, where in case, $C_{\mathcal{T}}$ is the closure operation (consequence operation) determined by the family $\mathcal{T}$ (that is, by $[\mathcal{T}]$ – the least closure system over V containing $\mathcal{T}$): for a formula $\alpha \in V$, $\alpha \in C_{\mathcal{T}}(X)$ iff for any $T \in \mathcal{T}$, $X \subseteq T \Rightarrow \alpha \in T$.

Hereafter the consequence relations $\vdash_{\mathcal{T}}$, $\mathcal{T} \subseteq \wp(V)$ will be called *disjunctive*. Let $DR = \{\vdash_{\mathcal{T}} : \mathcal{T} \subseteq \wp(V)\}$.

3. Galois connection for disjunctive consequence relation

Taking into account the very definition of disjunctive consequence relation from the previous section (cf. (mc)), the following Galois connection (f, g) should be considered. Put $R \subseteq \wp(V)^2 \times \wp(V)$ of the form $((X, Y), T) \in R$ iff $X \subseteq T \Rightarrow Y \cap T \neq \emptyset$. So $f : (\wp(\wp(V) \times \wp(V)), \subseteq) \longrightarrow (\wp(\wp(V)), \subseteq)$, $g : (\wp(\wp(V)), \subseteq) \longrightarrow (\wp(\wp(V) \times \wp(V)), \subseteq)$ are defined for any relation $r \subseteq \wp(V) \times \wp(V)$ and any family $\mathcal{T} \subseteq \wp(V)$ by

$T \in f(r)$ iff for all $X, Y \subseteq V$ such that $(X, Y) \in r$, $X \subseteq T$ implies that $Y \cap T \neq \emptyset$, any $T \subseteq V$,

$(X, Y) \in g(\mathcal{T})$ iff for all $T \in \mathcal{T}$, $X \subseteq T$ implies that $Y \cap T \neq \emptyset$, any $X, Y \subseteq V$.

In more handy formulation,

(1) $T \in f(r)$ iff $\forall X, Y \subseteq V$ $(X \subseteq T \subseteq -Y \;\Rightarrow\; (X, Y) \notin r)$,

(2) $(X, Y) \in g(\mathcal{T})$ iff $\forall T \subseteq V$ $(X \subseteq T \subseteq -Y \;\Rightarrow\; T \notin \mathcal{T})$,

where "$-$" is the operation of complementation in the Boolean algebra of all subsets of V.

Let us put $C = f \circ g$ and $C' = g \circ f$, that is, C is a closure operation defined on $\wp(\wp(V) \times \wp(V))$ assigning to each binary relation r defined on $\wp(V)$ the least relation from DR containing r (the operation C is the counterpart of closure introduced in [6, p. 1006, definition 3.1] for Scott's multiple-conclusion relations from [7]); in turn C' is a closure operation whose closed sets correspond via dual isomorphism f restricted to DR to disjunctive consequence relations. Using (1) and (2) we obtain that for any binary relation $r \subseteq \wp(V) \times \wp(V)$, $(X, Y) \in C(r)$ iff $(X, Y) \in g(f(r))$ iff $\forall T \subseteq V$ $(X \subseteq T \subseteq -Y \;\Rightarrow\; T \notin f(r))$ iff $\forall T \subseteq V$ $(X \subseteq T \subseteq -Y \;\Rightarrow\; \exists U, Z \subseteq V$ $(U \subseteq T \subseteq -Z \;\&\; (U, Z) \in r))$. Finally,

(3) $(X, Y) \in C(r)$ iff $[X, -Y] \subseteq \bigcup\{[U, -Z] : (U, Z) \in r\}$,

where for any $X, Y \subseteq V$, $[X, Y] = \{U \subseteq V : X \subseteq U \subseteq Y\}$. However, the equivalence:

(4) $(X, Y) \in C(r)$ iff $\forall T \subseteq V$ $(X \subseteq T \subseteq -Y \;\Rightarrow\; T \notin f(r))$,

is also interesting since from it one may derive that for any set $T \subseteq V$ and any binary relation $r \subseteq \wp(V) \times \wp(V)$,

(5) $T \in f(r)$ iff $(T, -T) \notin C(r)$.

Similarly, for any family $\mathcal{T} \subseteq \wp(V)$: $T \in C'(\mathcal{T})$ iff $T \in f(g(\mathcal{T}))$ iff $\forall X, Y \subseteq V$ $(X \subseteq T \subseteq -Y \;\Rightarrow \exists T' \subseteq V (X \subseteq T' \subseteq -Y \;\&\; T' \in \mathcal{T}))$ iff $T \in \mathcal{T}$. In this way, C' is the identity mapping on $\wp(\wp(V))$ so $Cl(C') = Cl(g \circ f) = \wp(\wp(V))$. On the other hand, $Cl(C) = Cl(f \circ g) = \{g(\mathcal{T}) : \mathcal{T} \subseteq \wp(V)\} = \{\vdash_{\mathcal{T}} : \mathcal{T} \subseteq \wp(V)\} = DR$. Thus we have the following corollary.

Corollary. *The mapping f restricted to DR (that is f defined for each $r \in DR$ by $f(r) = \{T \subseteq V : (T, -T) \notin r\}$ due to (5)) is a dual isomorphism of the complete lattices $(DR, \subseteq)$, $(\wp(\wp(V)), \subseteq)$ and the mapping g is the inverse dual isomorphism.*

This result, obtained first in [11] without application of Galois connection, can be strengthened (cf. also [11]) to a dual isomorphism of complete and atomic Boolean algebras $(DR, \cap, \vee, -, \vdash_0, \wp(V)^2)$, $(\wp(\wp(V)), \cap, \cup, -, \emptyset, \wp(V))$, by equipping the family DR of disjunctive relations with the operation of Boolean complementation $-$ in such a way that the dual isomorphism of complete lattices preserves it : $-r = -g(f(r)) = g(\wp(V) - f(r)) = g(\{T \subseteq V : (T, -T) \in r\})$. Here for any $r_1, r_2 \in DR$, $r_1 \vee r_2 = C(r_1 \cup r_2)$ and $\vdash_0 = g(\wp(V)) = \{(X, Y) : X \cap Y \neq \emptyset\}$ is the least disjunctive relation.

4. Isomorphism theorem for disjunctive consequence relations

Let us put $\mathcal{R}_0 = \{(T, -T) : T \subseteq V\}$. Consider the mapping $p : \wp(\mathcal{R}_0) \longrightarrow \wp(\wp(V))$ defined by $p(\rho) = \{T \subseteq V : (T, -T) \in \rho\}$. It is obvious that p is a Boolean and complete isomorphism of Boolean algebras $(\wp(\mathcal{R}_0), \cap, \cup, -, \emptyset, \mathcal{R}_0)$, $(\wp(\wp(V)), \cap, \cup, -, \emptyset, \wp(V))$. Consider the following composition of mappings:

$$\wp(\mathcal{R}_0) \ni \rho \longmapsto p(\rho) \longmapsto \wp(V) - p(\rho) \longmapsto g(\wp(V) - p(\rho)) \in DR.$$

The correspondence $\wp(\wp(V)) \ni \mathcal{T} \longmapsto \wp(V) - \mathcal{T}$ is obviously a dual Boolean complete isomorphism from $(\wp(\wp(V)), \cap, \cup, -, \emptyset, \wp(V))$ onto itself. So the composition $\wp(\mathcal{R}_0) \ni \rho \longmapsto g(\wp(V) - p(\rho)) \in DR$ (one isomorphism and two dual isomorphisms are here composed) is a complete Boolean isomorphism from $(\wp(\mathcal{R}_0), \cap, \cup, -, \emptyset, \mathcal{R}_0)$ onto $(DR, \cap, \vee, -, \vdash_0, \wp(V)^2)$.

Using (2) one may calculate the value of that isomorphism on a $\rho \subseteq \mathcal{R}_0$: for any $X, Y \subseteq V$, $(X, Y) \in g(\wp(V) - p(\rho))$ iff $[X, -Y] \subseteq p(\rho)$. Moreover, from (3) we have

(6) $(X, Y) \in C(\rho)$ iff $[X, -Y] \subseteq \bigcup\{[T, T] : (T, -T) \in \rho\}$ iff $[X, -Y] \subseteq p(\rho)$.

Therefore, for any $\rho \subseteq \mathcal{R}_0$, $C(\rho) = g(\wp(V) - p(\rho))$. Furthermore, one may consider the inverse isomorphism as the following composition:

$DR \ni r \longmapsto f(r) \longmapsto \wp(V) - f(r) = \{T \subseteq V : (T, -T) \in r\}$ (by (5)) $\longmapsto r \cap \mathcal{R}_0$.

In this way the following result is proved.

PROPOSITION. *The closure operation C (assigning to each binary relation r defined on $\wp(V)$ the least disjunctive relation containing r) restricted to the power set of $\mathcal{R}_0 = \{(T, -T) : T \subseteq V\}$ is a Boolean and complete isomorphism from the power set algebra $(\wp(\mathcal{R}_0), \cap, \cup, -, \emptyset, \mathcal{R}_0)$ onto atomic and complete Boolean algebra $(DR, \cap, \vee, -, \vdash_0, \wp(V)^2)$ of all disjunctive relations defined on the language V. The inverse isomorphism, say $h : DR \longrightarrow \wp(\mathcal{R}_0)$ is defined by $h(r) = r \cap \mathcal{R}_0$. In this way, for any $r \in DR$ and $\rho \subseteq \mathcal{R}_0$, $r = C(r \cap \mathcal{R}_0)$ and $\rho = C(\rho) \cap \mathcal{R}_0$.*

5. Some applications

Applying (6) one may show that for any $T_1, T_2 \subseteq V$ such that $T_1 \subseteq T_2$ and for any $X, Y \subseteq V$,

(7) $(X, Y) \in C(\{(T, -T) : T \in [T_1, T_2]\})$ iff either $X \vdash_0 Y$ or $T_1 \subseteq X \subseteq -Y \subseteq T_2$.

In particular, using (7) and Proposition, one may find a form of atoms in the Boolean algebra $(DR, \cap, \vee, -, \vdash_0, \wp(V)^2)$ of all disjunctive relations. Let us take any atom $\{(T, -T)\}$, $T \subseteq V$, of $(\wp(\mathcal{R}_0), \cap, \cup, -, \emptyset, \mathcal{R}_0)$. Then the corresponding atom in the Boolean algebra of all disjunctive relations is of the form:

(8) $C(\{(T, -T)\}) = \vdash_0 \cup \{(T, -T)\}$.

The coatoms of $(DR, \cap, \vee, -, \vdash_0, \wp(V)^2)$ are much more interesting. Take any $T \subseteq V$. Then the corresponding coatom in this Boolean algebra to the coatom $\mathcal{R}_0 - \{(T, -T)\}$ of $(\wp(\mathcal{R}_0), \cap, \cup, -, \emptyset, \mathcal{R}_0)$ is, due to (6) and (mc), of the form

(9) $(X, Y) \in C(\mathcal{R}_0 - \{(T, -T)\})$ iff $[X, -Y] \subseteq \wp(V) - \{T\}$ iff either $X \not\subseteq T$ or $Y \cap T \neq \emptyset$ iff $X \vdash_{\{T\}} Y$.

More figuratively,

(10) $C(\mathcal{R}_0 - \{(T, -T)\}) = \vdash_{\{T\}} = \bigcup\{[(\{\alpha\}, \emptyset)) : \alpha \notin T\} \cup \bigcup\{[(\emptyset, \{\alpha\})) : \alpha \in T\}$,

where for any $X, Y \subseteq V$, $[(X, Y)) = \{(X', Y') \in \wp(V)^2 : X \subseteq X'$ & $Y \subseteq Y'\}$.

The following lemma provides a useful characteristics of coatoms.

LEMMA. *For any* $\vdash \in DR$ *and* $T \subseteq V$, $\vdash = \vdash_{\{T\}}$ *iff for each* $\alpha \in V$, $(\emptyset \vdash \{\alpha\}$ *iff* $\alpha \in T)$ *and* $(\{\alpha\} \vdash \emptyset$ *iff* $\alpha \notin T)$.

PROOF. Consider any disjunctive relation $\vdash$ and $T \subseteq V$.

($\Rightarrow$): By (10).

($\Leftarrow$): Assume that for each $\alpha \in V$, $(\emptyset \vdash \{\alpha\}$ iff $\alpha \in T)$ and $(\{\alpha\} \vdash \emptyset$ iff $\alpha \notin T)$. First we show that $\vdash_{\{T\}} \subseteq \vdash$. So suppose that $X \vdash_{\{T\}} Y$, that is, either $X \not\subseteq T$ or $Y \cap T \neq \emptyset$. In the first case, from the assumption it follows that $\{\alpha\} \vdash \emptyset$ for some $\alpha \in X$ so $X \vdash Y$ by (*dilution*). In the second case, analogously, $\emptyset \vdash \{\alpha\}$ for some $\alpha \in Y$ so $X \vdash Y$. Now notice that $\vdash_{\{T\}}$ is a coatom in the Boolean algebra of all disjunctive relations, therefore the inclusion $\vdash_{\{T\}} \subseteq \vdash$ implies that $\vdash_{\{T\}} = \vdash$ or $\vdash = \wp(V)^2$. Since the relation $\wp(V)^2$ does not satisfy the assumption we obtain $\vdash_{\{T\}} = \vdash$. □

The coatoms in the Boolean algebra of all disjunctive consequence relations are easily expressible in terms of [7]. In order to show this let us apply the definition from [7, p. 416], for any disjunctive relation. A relation $\vdash \in DR$ is said to be *consistent* (*complete*) iff for any $\alpha \in V$, either $\emptyset \nvdash \{\alpha\}$ or $\{\alpha\} \nvdash \emptyset$ (for any $\alpha \in V$, either $\emptyset \vdash \{\alpha\}$ or $\{\alpha\} \vdash \emptyset$). In this way, for any $\vdash \in DR$,

(11) $\vdash$ is consistent and complete iff for any $\alpha \in V$, $\emptyset \vdash \{\alpha\}$ iff $\{\alpha\} \nvdash \emptyset$.

FACT. *For any* $\vdash \in DR$, $\vdash$ *is consistent and complete iff for some* $T \subseteq V$, $\vdash = \vdash_{\{T\}}$.

PROOF. Consider any disjunctive relation $\vdash$.

($\Rightarrow$): Assume that $\vdash$ is consistent and complete. Put $T = \{\alpha \in V : \emptyset \vdash \{\alpha\}\}$. Then from the assumption and (11) it follows that $-T = \{\alpha \in V : \{\alpha\} \vdash \emptyset\}$. In this way, $\vdash = \vdash_{\{T\}}$ due to Lemma.

($\Leftarrow$): Immediately from Lemma and (11). □

In the light of this fact, the result of [7] that any multiple-conclusion consequence relation is an intersection of all consistent and complete relations containing it, becomes absolutely clear. Since for every $\vdash \in DR$, the identity $\vdash = \bigcap\{\vdash_{\{T\}} : \vdash \subseteq \vdash_{\{T\}}\}$ holds. In turn, the latter connection is an obvious consequence of the following one: $\rho = \bigcap\{\mathcal{R}_0 - \{(T, -T)\} : (T, -T) \notin \rho\}$, any $\rho \subseteq \mathcal{R}_0$ (implying together with Proposition and (9) that $C(\rho) = \bigcap\{C(\mathcal{R}_0 - \{(T, -T)\}) : \rho \subseteq \mathcal{R}_0 - \{(T, -T)\}\} = \bigcap\{\vdash_{\{T\}} : C(\rho) \subseteq \vdash_{\{T\}}\}$).

Notice that the power set $\wp(\mathcal{R}_0)$ is closed on the operation $^\sim$ of taking the converse relation. Applying (6) for a given $\rho \subseteq \mathcal{R}_0$ we have $(X, Y) \in C(\rho^\sim)$ iff $[X, -Y] \subseteq p(\rho^\sim)$ iff $[X, -Y] \subseteq \{-T : T \in p(\rho)\}$ iff $[Y, -X] \subseteq p(\rho)$ iff $(Y, X) \in C(\rho)$ iff $(X, Y) \in C(\rho)^\sim$. Hence, $C(\rho^\sim) = C(\rho)^\sim$ so the operation $^\sim$ is preserved under the isomorphism C and the set DR is closed on this operation. Denoting for a given family $\mathcal{T} \subseteq \wp(V)$, $\mathcal{T}^\sim = \{-T : T \in \mathcal{T}\}$ we have $g(\mathcal{T}^\sim) = g(\mathcal{T})^\sim$ due to (2), that is, in terms of (mc):

(12) $\vdash_{\mathcal{T}^\sim} = \vdash_{\mathcal{T}}^\sim$.

Given $\vdash \in DR$ the relation $\vdash^\sim$ could be called *dual with respect to* $\vdash$. For example, assume that V is the set of all formulas of propositional language equipped with the standard connectives $\neg, \wedge, \vee, \rightarrow$ and let Val be the set of all Boolean valuations of the language into $\{0, 1\}$. Consider the disjunctive relation $\vdash_{\mathcal{T}_{Max}}$ determined (according to (mc)) by the family of all maximal theories of classical propositional logics $\mathcal{T}_{Max} = \{T_v : v \in Val\}$, where for each $v \in Val$, $T_v = \{\alpha \in V : v(\alpha) = 1\}$ (cf. also [9, p. 242, definition 1]):

$X \vdash_{\mathcal{T}_{Max}} Y$ iff $\forall v \in Val(X \subseteq T_v \Rightarrow Y \cap T_v \neq \emptyset)$ iff $\forall v \in Val(v[X] \subseteq \{1\} \Rightarrow \exists \alpha \in Y,\ v(\alpha) = 1)$.

The dual relation with respect to $\vdash_{\mathcal{T}_{Max}}$ is, according to (12), determined by the family $\mathcal{T}_{Max}^\sim = \{\{\alpha \in V : v(\alpha) = 0\} : v \in Val\}$ (notice that the consequence operation corresponding to the closure system $[\mathcal{T}_{Max}^\sim]$ over V is dual in the sense of Wójcicki [10] with respect to the consequence operation of classical propositional logic, that is, corresponding to the closure system $[\mathcal{T}_{Max}]$). One may consider the dual disjunctive relation with respect to a coatom $\vdash_{\{T\}}$, $T \subseteq V$ which is the coatom $\vdash_{\{-T\}}$ (cf. also (10)). In particular $\vdash_{\{-T_v\}}$, $v \in Val$ is considered in [9, p. 245, definition 3].

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University of Łódź
Department of Logic
Lindleya 3/5, 90-131 Łódź
e-mail: marek.nowak@filozof.uni.lodz.pl