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SEMANTICAL PROOF OF SUBFORMULA PROPERTY FOR THE MODAL LOGICS K4.3, KD4.3, AND S4.3

Abstract

The main purpose of this paper is to give alternative proofs of syntactical and semantical properties, i.e. the subformula property and the finite model property, of the sequent calculi for the modal logics **K4.3**, **KD4.3**, and **S4.3**. The application of the inference rules is said to be acceptable, if all the formulas in the upper sequents are subformula of the formulas in lower sequent. For some modal logics, Takano analyzed the relationships between the acceptable inference rules and semantical properties by constructing models. By using these relationships, he showed Kripke completeness and subformula property. However, his method is difficult to apply to inference rules for the sequent calculi for **K4.3**, **KD4.3**, and **S4.3**. Looking closely at Takano's proof, we find that his method can be modified to construct finite models based on the sequent calculus for **K4.3**, if the calculus has (cut) and all the applications of the inference rules are acceptable. Similarly, we can apply our results to the calculi for **KD4.3** and **S4.3**. This leads not only to Kripke completeness and subformula property, but also to finite model property of these logics simultaneously.

Keywords: modal logic, analytic cut, subformula property, finite model property.

1. Introduction

The sequent calculi for some modal logics possess subformula property and finite model property. Takano [2] proved that the sequent calculi for **K5** and **K5D** enjoy these properties through semantical method. Then, he generalized the method by introducing special unprovable sequent, *analytically saturated sequent*, in Takano [3].

In [3], Takano analyzed the relationships between acceptable inference rules and semantical properties by constructing Kripke models using the set of all analytically saturated sequents. (The application of the inference rules is said to be acceptable, if all formulas in the upper sequents are subformulas of the formulas in the lower sequent.) We call here this method as Takano's method. Then, he showed that the sequent calculi for modal logics which are obtained from **K** by adding axioms from **T**, **4**, **5**, **D**, and **B** enjoy subformula property and finite model property.

The main purpose of this paper is to give alternative proofs of subformula property and finite model property of the sequent calculi for the modal logics **K4.3**, **KD4.3**, and **S4.3**. For this purpose, we consider the relationships between the semantical properties and the inference rules ($\Box$4.3) and (S4.3) (introduced by Shimura [1]) based on Takano's method. However, the straightforward application of Takano's method does not work well for ($\Box$4.3) and (S4.3). Taking a close look at his proof, we find that Takano's method can be modified to construct finite models based on the sequent calculus for **K4.3**, if the calculus has ($\Box$4.3) and (*cut*), and all the applications of inference rules are acceptable. Similarly, we can apply this result to the inference rule (S4.3). This implies Kripke completeness of the sequent calculi for **K4.3** and **S4.3**, and these calculi enjoy not only subformula property, but also finite model property.

In Section 2, we introduce the definition and property of an analytically saturated sequent based on Takano [3]. In Section 3 and 4, we consider ($\Box$4.3) and (S4.3), respectively, and give the procedure for constructing finite models.

2. Preliminaries

In this paper, we use only $\neg$ (negation), $\supset$ (implication), and $\Box$ (necessity) as logical symbols, and other are considered as abbreviations. Propositional letters and formulas are denoted by p, q, $r, \cdots$ and A, B, $C, \cdots$, respectively. Finite sequences of formulas are denoted by Γ, Δ, Θ, $\Lambda, \cdots$, and a sequent is an expression of the form $\Gamma \to \Theta$. A $\Box$-formula is a formula whose outermost logical symbol is $\Box$. We mean by $\mathrm{Sf}(\Gamma)$ the set of all the subformulas of some formulas in Γ, and by $\Box\Gamma$ the set $\{\Box A \mid A \in \Gamma\}$.

Let us consider the following structural rules:

$$\frac{\Gamma \to \Theta}{A,\ \Gamma \to \Theta}\ (w \to) \qquad \frac{\Delta,\ B,\ A,\ \Gamma \to \Theta}{\Delta,\ A,\ B,\ \Gamma \to \Theta}\ (e \to) \qquad \frac{A,\ A,\ \Gamma \to \Theta}{A,\ \Gamma \to \Theta}\ (c \to)$$

$$\frac{\Gamma \to \Theta}{\Gamma \to \Theta,\ A}\ (\to w) \qquad \frac{\Gamma \to \Theta,\ B,\ A,\ \Lambda}{\Gamma \to \Theta,\ A,\ B,\ \Lambda}\ (\to e) \qquad \frac{\Gamma \to \Theta,\ A,\ A}{\Gamma \to \Theta,\ A}\ (\to c)$$

Every sequent calculus which we treat in this paper enjoys the following stipulation.

STIPULATION 1. *The sequent calculus has $A \to A$ as an initial sequent for every A, and contains the structural rules $(\to w)$, $(w \to)$, $(\to e)$, $(e \to)$, $(\to c)$, and $(c \to)$.*

Due to this in the rest of this paper, we recognize Γ, Δ, Θ, $\Lambda, \cdots$ as finite sets.

DEFINITION 2.1. (Takano [3, Definition 1.1]) Let GL be a sequent calculus with Stipulation 1. The sequent $\Gamma \to \Theta$ is *analytically saturated* in GL, iff the following properties hold.

(a) $\Gamma \to \Theta$ is unprovable in GL.
(b) Suppose $A \in \mathrm{Sf}(\Gamma \cup \Theta)$. If $A,\ \Gamma \to \Theta$ is unprovable in GL, then $A \in \Gamma$; while if $\Gamma \to \Theta,\ A$ is unprovable in GL, then $A \in \Theta$.

The set of all analytically saturated sequents is denoted by W_{GL}.

We denote the analytically saturated sequents by u, v, $w, \cdots$, and denote the antecedent and succedent of u by $a(u)$ and $s(u)$, respectively.

LEMMA 2.2. (Takano [3, Lemma 1.3]) *For a sequent calculus GL with Stipulation 1, if the sequent $\Gamma \to \Theta$ is unprovable in GL, then there is an analytically saturated sequent u with the following properties;*

(i) $\Gamma \subseteq a(u)$ *and* $\Theta \subseteq s(u)$
(ii) $a(u) \cup s(u) \subseteq \mathrm{Sf}(\Gamma \cup \Theta)$

DEFINITION 2.3. (Takano [3, Definition 1.5]) An inference is *admissible* in a sequent calculus GL, iff either some of the upper sequents of the inference is unprovable in GL, or the lower one in provable in GL.

For a sequent calculus GL with Stipulation 1, there are relationships between properties of analytically saturated sequents and inferences which are admissible in GL. For example, we consider the following inferences.

$$\frac{\Gamma \to \Theta,\ A}{\neg A,\ \Gamma \to \Theta}\ (\neg \to) \qquad \frac{A,\ \Gamma \to \Theta}{\Gamma \to \Theta,\ \neg A}\ (\to \neg)$$

$$\frac{\Gamma \to \Theta,\ A \qquad B,\ \Gamma \to \Theta}{A \supset B,\ \Gamma \to \Theta}\ (\supset\to) \quad \frac{A,\ \Gamma \to \Theta,\ B}{\Gamma \to \Theta,\ A \supset B}\ (\to\supset)$$

$$\frac{\Gamma \to \Theta,\ C \qquad C,\ \Delta \to \Lambda}{\Gamma, \Delta \to \Theta, \Lambda}\ (cut)^a \quad \textit{where } C \in \mathrm{Sf}(\Gamma \cup \Theta \cup \Delta \cup \Lambda)$$

Note that inference rule $(cut)^a$ is obtained from (cut) by applying appropriate restriction.

PROPOSITION 2.4. (Takano [3, Proposition 1.6]) *For a sequent calculus GL with Stipulation 1, the following equivalences hold for every A and B.*

(1) *The inference $(\neg \to)$ is admissible in GL for every Γ and Θ, iff $\neg A \in a(u)$ implies $A \in s(u)$ for every u.*
(2) *The inference $(\to \neg)$ is admissible in GL for every Γ and Θ, iff $\neg A \in s(u)$ implies $A \in a(u)$ for every u.*
(3) *The inference $(\supset\to)$ is admissible in GL for every Γ and Θ, iff $A \supset B \in a(u)$ implies $A \in s(u)$ or $B \in a(u)$ for every u.*
(4) *The inference $(\to\supset)$ is admissible in GL for every Γ and Θ, iff $A \supset B \in s(u)$ implies $A \in a(u)$ and $B \in s(u)$ for every u.*

PROPOSITION 2.5. (Takano [3, Proposition 3.1]) *For a sequent calculus GL with Stipulation 1, the inference $(cut)^a$ is admissible for every Γ, Θ, Δ, Λ, and C with the restriction that $C \in \mathrm{Sf}(\Gamma \cup \Theta \cup \Delta \cup \Lambda)$, iff $\mathrm{Sf}(a(u) \cup s(u)) \subseteq a(u) \cup s(u)$.*

We introduce Stipulation 2 as well.

STIPULATION 2. *The sequent calculus contains $(\neg \to)$, $(\to \neg)$, $(\supset\to)$, and $(\to\supset)$ as inference rules.*

The aim of introducing analytically saturated sequents is obtaining the proof of Kripke completeness.

LEMMA 2.6. (Takano [3, Proposition 1.4]) *Let GL be a sequent calculus with Stipulation 1. Suppose that (W, R) is a Kripke frame such that $W \subseteq W_{GL}$, and the following properties hold for every A, B and every $u \in W$;*

$(\neg - a)$ *$\neg A \in a(u)$ implies $A \in s(u)$.*
$(\neg - s)$ *$\neg A \in s(u)$ implies $A \in a(u)$.*
$(\supset -a)$ *$A \supset B \in a(u)$ implies $A \in s(u)$ or $B \in a(u)$.*

$(\supset -s)$ *$A \supset B \in s(u)$ implies $A \in a(u)$ and $B \in s(u)$.*

$(\Box - a)$ *$\Box A \in a(u)$ implies $A \in a(v)$ for every $v \in W$ such that uRv.*

$(\Box - s)$ *$\Box A \in s(u)$ implies $A \in s(v)$ for some $v \in W$ such that uRv.*

Let $\models$ be the satisfaction relation on (W, R) such that $u \models p$ iff $p \in a(u)$ for every $u \in W$ and every p. Then for every C and every $u \in W$, if $C \in a(u)$ then $u \models C$; while if $C \in s(u)$ then $u \not\models C$.

The proof of this lemma is given by induction on the construction of C.

For a sequent calculus GL with Stipulation 1, assume that any $u \in W_{GL}$ has a Kripke frame (W, R) which satisfies following properties.

- $u \in W \subseteq W_{GL}$.
- (W, R) meets conditions of Lemma 2.6.
- the accessibility relation R meets the condition of Kripke frame for L.

Then, if $\Gamma \to \Theta$ is unprovable in GL, there is an analytically saturated sequent u such that $\Gamma \subseteq a(u)$ and $\Theta \subseteq s(u)$ by Lemma 2.2. And u has a Kripke frame (W, R) which satisfies the above properties. Adding satisfaction relation $\models$ introduced in Lemma 2.6, we obtain Kripke model $(W, R, \models)$ in which $C \in \Gamma$ implies $u \models C$ and $C \in \Theta$ implies $u \not\models C$. This leads to Kripke completeness of GL.

The key point is whether every $u \in W_{GL}$ has such Kripke frame or not. It depends on admissibility of inferences in GL. From Proposition 2.4, for any Kripke frame of sequent calculus GL with Stipulation 1 and 2 holds $(\neg - a)$, $(\neg - s)$, $(\supset -a)$, and $(\supset -s)$. The remaining conditions $(\Box - a)$ and $(\Box - s)$ depend not only on admissibility of inferences, but also on properties of accessibility relation. We will discuss them in the remaining sections.

3. The logics K4.3 and KD4.3

Modal logic **K4.3** is obtained from the least normal logic **K** by adding axioms $\Box p \supset \Box\Box p$ and $\Box((p \wedge \Box p) \supset q) \vee \Box((q \wedge \Box q) \supset p)$. Kripke frame (W, R) meets condition of **K4.3** iff the frame is transitive and *weakly connected*; the Kripke frame is said to be weakly connected, if a binary relation R enjoys the following condition.

$$\forall u, v, w (uRv \textit{ and } uRw \;\Rightarrow\; vRw \textit{ or } u = w \textit{ or } wRv)$$

Modal logic **KD4.3** is obtained from **K4.3** by adding axiom $\Box p \supset \neg\Box\neg p$. In this section, we consider the inference rule for **K4.3** introduced by Shimura [1].

Definition 3.1. Suppose that $\Delta \neq \emptyset$. $P(\Delta)$ is defined as the set of all pairs (Σ, Λ) with following properties:

(1) $\Sigma \cup \Lambda = \Delta$ and $\Sigma \cap \Lambda = \emptyset$,
(2) $\Lambda \neq \emptyset$.

For example, if $\Delta = \{A,\ B\}$, then

$$P(\Delta) = \{(\{A\}, \{B\}),\ (\{B\}, \{A\}),\ (\emptyset, \{A, B\})\}$$

The inference rule ($\Box$4.3) is defined as follows:

$$\frac{\{\Gamma,\ \Box\Gamma \to \Box\Sigma,\ \Lambda |\ (\Sigma,\ \Lambda) \in P(\Delta)\}}{\Box\Gamma \to \Box\Delta}\ (\Box 4.3)$$

If $\Delta = \{A, B\}$, ($\Box$4.3) is of the form:

$$\frac{\Gamma,\ \Box\Gamma \to \Box A,\ B \qquad \Gamma,\ \Box\Gamma \to \Box B,\ A \qquad \Gamma,\ \Box\Gamma \to A,\ B}{\Box\Gamma \to \Box A,\ \Box B}\ (\Box 4.3)$$

Sequent calculus $G(\mathbf{K4.3})$ is obtained from Gentzen's original LK by adding inference rule ($\Box$4.3). Shimura proved that $G(\mathbf{K4.3})$ satisfies cut elimination using the syntactic method.

Restricted sequent calculi $G(\mathbf{K4.3})^-$ and $G(\mathbf{KD4.3})^-$ are obtained from LK by replacing (cut) with $(cut)^a$ and adding following rules.

	Rules	Condition on relations
$G(\mathbf{K4.3})^-$	LK with $(cut)^a$, ($\Box$4.3)	transitive and weakly connected
$G(\mathbf{KD4.3})^-$	LK with $(cut)^a$, ($\Box$4.3), $(4D)$	transitive, serial, and weakly connected

Where $(4D)$ is

$$\frac{\Gamma, \Box\Gamma \to}{\Box\Gamma \to}\ (4D).$$

We can prove their Kripke completeness by modifying Takano's method as follows. Note that the condition on Kripke frame is not equivalent to admissibility of ($\Box$4.3). (See Section 5.)

Definition 3.2. For a sequent calculus GL with Stipulation 1, the binary relation $R_{\mathbf{K4}}$ on W_{GL} is defined by: $uR_{\mathbf{K4}}v$, iff $\Box B \in a(u)$ implies B, $\Box B \in a(v)$ for every B.

From this definition, it follows that for every nonempty set $W \subseteq W_{GL}$, Kripke frame $(W, R_{\mathbf{K4}})$ is transitive and meets $(\Box - a)$.

Proposition 3.3. *Let GL be a sequent calculus with Stipulation 1 and the inference rule $(cut)^a$. If $(\Box 4.3)$ is admissible in GL for every Γ and Δ $(\Delta \neq \emptyset)$, then for every $u \in W_{GL}$, there is a finite set $W \subseteq W_{GL}$ with the following properties.*

(i) $u \in W$

(ii) *Kripke frame $(W,\ R_{\mathbf{K4}})$ enjoys the property $(\Box - a)$ and $(\Box - s)$, and meets condition for* **K4.3**.

Proof: Suppose $u \in W_{GL}$. We construct analytically saturated sequents $v_1, ..., v_n$ as follows.

- $v_1 := u$
- Suppose that $v_1, \cdots, v_k$ are constructed. Put Γ_k, Θ_k, L_k and Δ_k as follows:

$$\Gamma_k = \{B \mid \Box B \in a(v_k)\},\ \Theta_k = \{B \mid \Box B \in s(v_k)\},$$

$$L_k = \{B \in \Theta_k \mid \exists w \in \{v_1, ..., v_k\}\ s.t.\ v_k R_{\mathbf{K4}} w \ and\ B \in s(w)\},$$

$$\Delta_k = \Theta_k \setminus L_k.$$

We have two cases: $\Delta_k \neq \emptyset$ and $\Delta_k = \emptyset$.
Case (1): $\Delta_k \neq \emptyset$. By following procedure, we construct the analytically saturated v_{k+1} which satisfies $v_k R_{\mathbf{K4}} v_{k+1}$ and $B \in s(v_{k+1})$ for some $B \in \Delta_k$. Since $\Box\Gamma_k \to \Box\Delta_k$ is unprovable, $\Gamma_k, \Box\Gamma_k \to \Box\Sigma, \Lambda$ is unprovable for some $(\Sigma, \Lambda) \in P(\Delta_k)$. So, $\Gamma_k \cup \Box\Gamma_k \subseteq a(v)$, $\Box\Sigma \cup \Lambda \subseteq s(v)$ and $a(v) \cup s(v) \subseteq \mathrm{Sf}(\Gamma_k \cup \Box\Gamma_k \cup \Box\Sigma \cup \Lambda)$ for some v by Lemma 2.2. Put $v_{k+1} := v$, it is clear that $v_k R_{\mathbf{K4}} v_{k+1}$ and $B \in s(v_{k+1})$ for some $B \in \Delta_k$. Furthermore, $v_{k+1} \notin \{v_1, ..., v_k\}$. (Suppose $v_{k+1} \in \{v_1, ..., v_k\}$. Since $v_k R_{\mathbf{K4}} v_{k+1}$ and $\Lambda \subseteq s(v_{k+1})$, $\Lambda \subseteq L_k$ would follow, which is a contradiction.) Note that for all $\Box B \in s(v_k)$, $B \in s(v_{k+1})$, $\Box B \in s(v_{k+1})$, or $B \in L_k$ is satisfied.
Case (2): $\Delta_k = \emptyset$. Stop the construction.

To prove that this construction stops with finite steps, we will show that $\Delta_{k+1} \subsetneq \Delta_k$ or $\Theta_{k+1} \subsetneq \Theta_k$ holds. Since $a(v) \cup s(v) \subseteq \mathrm{Sf}(\Gamma_k \cup \Box\Gamma_k \cup \Box\Sigma \cup \Lambda) \subseteq \mathrm{Sf}(a(v_k) \cup s(v_k))$, it follows $a(v_{k+1}) \cup s(v_{k+1}) \subseteq a(v_k) \cup s(v_k)$ by $(cut)^a$. It is clear that $\Gamma_k \subseteq \Gamma_{k+1}$, so $\Theta_{k+1} \subseteq \Theta_k$. Suppose that $\Theta_{k+1} = \Theta_k$. We will derive $\Delta_{k+1} \subsetneq \Delta_k$ in this case. Since $a(v_{k+1}) \cup s(v_{k+1}) \subseteq a(v_k) \cup s(v_k)$ and $\Gamma_k \subseteq \Gamma_{k+1}$, it follows $\Gamma_k = \Gamma_{k+1}$. This implies that if $v_k R_{\mathbf{K4}} w$, then $v_{k+1} R_{\mathbf{K4}} w$ for any $w \in W_{GL}$, so $L_k \subseteq L_{k+1}$. Moreover, since $v_k R_{\mathbf{K4}} v_{k+1}$, it follows $v_{k+1} R_{\mathbf{K4}} v_{k+1}$ and $\Lambda \subseteq L_{k+1}$, although $\Lambda \not\subseteq L_k$. This implies $L_k \subsetneq L_{k+1}$; hence $\Delta_{k+1} \subsetneq \Delta_k$.

There is an analytically saturated sequent v_n with $\Delta_n = \emptyset$ by repeating this procedure. Put $W = \{v_1, \cdots, v_n\}$, it is clear that Kripke frame $(W, R_{\mathbf{K4}})$ is transitive and weakly connected frame, and enjoys $(\Box - a)$. If $\Box B \in s(v_n)$, then $B \in L_n$ since $\Delta_n = \emptyset$. So $(W, R_{\mathbf{K4}})$ enjoys $(\Box - s)$. $\Box$

From the above proposition, we can show Kripke completeness for $G(\mathbf{K4.3})^-$, and this leads to subformula property for $G(\mathbf{K4.3})$. Furthermore, this leads to finite model property simultaneously because the constructed model is finite.

Similarly, $G(\mathbf{KD4.3})$ has subformula property and finite model property.

Lemma 3.4. *Let GL be a sequent calculus with Stipulation 1 and the inference rule $(cut)^a$. If $(\Box 4.3)$ and $(4D)$ are admissible in GL for every Γ and Δ $(\Delta \neq \emptyset)$, then for every $u \in W_{GL}$, there is a finite set $W \subseteq W_{GL}$ with the following properties.*

- *(i) $u \in W$*
- *(ii) Kripke frame $(W,\ R_{\mathbf{K4}})$ enjoys the property $(\Box - a)$ and $(\Box - s)$, and meets condition for* **KD4.3**.

Proof: Suppose $u \in W_{GL}$. From Proposition 3.3, there is a finite set $\{v_1, \cdots, v_n\}$ which meets the condition of Proposition 3.3 with $v_1 = u$. If v_n has $R_{\mathbf{K4}}$ successor in $\{v_1, \cdots, v_n\}$, then the set is the desired one. If not so, we construct the analytically saturated v_{n+1} by following procedure.

Put Γ_n and Θ_n same as Proposition 3.3. It is clear that $\Theta_n = \emptyset$ and $\Gamma_n \neq \emptyset$. (Suppose otherwise $\Gamma_n = \emptyset$. Then, all analytically saturated sequents of W_{GL} are $R_{\mathbf{K4}}$ successors of v_n. This is a contradiction.) Since $\Box\Gamma_n \to$ is unprovable, $\Gamma_n, \Box\Gamma_n \to$ is unprovable. Then, by Lemma 2.2, $\Gamma_n,\ \Box\Gamma_n \subseteq a(v)$ and $a(v) \cup s(v) \subseteq \mathrm{Sf}(\Box\Gamma_n)$ for some v. Put $v = v_{n+1}$, it is clear $v_n R_{\mathbf{K4}} v_{n+1}$. Furthermore, since GL has inference rule $(cut)^a$, $a(v_{n+1}) \cup s(v_{n+1}) \subseteq a(v_n) \cup s(v_n)$. So, $s(v_{n+1})$ has no $\Box$-formulas. (Suppose

$\Box B \in s(v_{n+1})$, this implies $\Box B \in a(v_n)$ or $\Box B \in s(v_n)$. Since $\Theta_n = \emptyset$, $\Box B \in a(v_n)$. It follows $\Box B \in \Box\Gamma_n \subseteq a(v_{n+1})$, which is a contradiction.) Similarly, suppose $\Box B \in a(v_{n+1})$, it follows $B \in a(v_{n+1})$, and this implies $v_{n+1} R_{\mathbf{K4}} v_{n+1}$. Thus, $W = \{v_1, \cdots, v_n, v_{n+1}\}$ meets the conditions. $\Box$

4. The logic S4.3

Modal logic **S4.3** is obtained form **K4.3** by adding axiom $\Box p \supset p$. Kripke frame (W, R) meets condition of **S4.3** iff the frame is transitive, weakly connected, and reflexive. Shimura [1] also introduced inference rule for **S4.3**.

$$\frac{\{\Box\Gamma \to \Box(\Delta \setminus \{A\}),\ A|\ A \in \Delta\}}{\Box\Gamma \to \Box\Delta}\ (S4.3)$$

If $\Delta = \{A, B\}$, ($\Box$4.3) is of the form:

$$\frac{\Box\Gamma \to \Box A,\ B \qquad \Box\Gamma \to \Box B,\ A}{\Box\Gamma \to \Box A,\ \Box B}\ (S4.3)$$

Sequent calculus $G(\mathbf{S4.3})$ is obtained from LK by adding inference rule $(S4.3)$ and (T).

$$\frac{A,\ \Gamma \to \Theta}{\Box A, \Gamma \to \Theta}\ (T)$$

Shimura proved that $G(\mathbf{S4.3})$ satisfies cut elimination using the syntactic method.

Restricted sequent calculus $G(\mathbf{S4.3})^-$ is obtained from LK by replacing (cut) with $(cut)^a$ and adding the following rules.

	Rules	Condition on relations
$G(\mathbf{S4.3})^-$	LK with $(cut)^a$, $(S4.3)$, (T)	transitive, reflexive and weakly connected

We can prove its Kripke completeness by modifying Takano's method as follows.

DEFINITION 4.1. For a sequent calculus GL with Stipulation 1, the binary relation $R_{\mathbf{S4}}$ on W_{GL} is defined by: $uR_{\mathbf{S4}}v$, iff $\Box B \in a(u)$ implies $\Box B \in a(v)$ for every B.

By this definition, for every nonempty set $W \subseteq W_{GL}$, Kripke frame $(W, R_{\mathbf{S4}})$ is transitive and reflexive.

PROPOSITION 4.2. *Let GL be a sequent calculus with Stipulation 1 and the inference rule $(cut)^a$. If $(S4.3)$ is admissible in GL for every Γ and Δ $(\Delta \neq \emptyset)$, then for every $u \in W_{GL}$, there is a finite set $W \subseteq W_{GL}$ with the following properties.*

(i) $u \in W$

(ii) *Kripke frame $(W,\ R_{\mathbf{S4}})$ enjoys the property $(\Box - s)$, and meets condition for* **S4.3***.*

The proof is similar to Proposition 3.3. Note that the Kripke frame constructed by the above proposition does not enjoy $(\Box - a)$. If GL has (T) as inference rule, then the constructed model enjoys $(\Box - a)$ by following lemma.

LEMMA 4.3. *Let GL be a sequent calculus with Stipulation 1. If the inference (T) is admissible in GL for every A, Γ, and Θ, then Kripke frame $(W, R_{\mathbf{S4}})$ holds the property $(\Box - a)$ for every $W \subseteq W_{GL}$.*

PROOF: Suppose that $u \in W$. If $\Box B \in a(u)$, then $uR_{\mathbf{K4}}v$ implies $\Box B \in a(v)$ for every $v \in W$. Since $\Box B, a(v) \to s(v)$ is unprovable, we have that $B, a(v) \to s(v)$ is unprovable by applying rule (T). Hence, $B \in a(v)$. $\Box$

We can show Kripke completeness of $G(\mathbf{S4.3})^-$ by Propositions 4.2 and 4.3. This implies not only subformula property, but also finite model property of $G(\mathbf{S4.3})$.

5. Concluding remark

In this paper, we gave alternative proofs of Kripke completeness, subformula property and finite model property for **K4.3**, **KD4.3** and **S4.3** by modifying Takano's method in [3].

Takano's method in [3] was developed originally to analyze relationships between admissibility of acceptable inference rules and semantical properties. Then, by using these relationships, he showed Kripke completeness of some modal logics as well. But, the straightforward application of Takano's method does not work well for $(\Box 4.3)$ and $(S4.3)$. Takano's method is useful to prove Kripke completeness, but has limitations. Let us explain this with examples. We consider the following inference.

$$\frac{\Gamma,\ \Box\Gamma \to A}{\Box\Gamma \to \Box A}\,(4) \qquad \frac{\Box\Gamma \to A}{\Box\Gamma \to \Box A}\,(S4)$$

Proposition 5.1. (Takano [3, Proposition 2.2]) *For a sequent calculus GL with Stipulation 1, the following equivalences hold for every A.*

(i) *The inference* (4) *is admissible in GL for every Γ, iff the Kripke frame $(W_{GL},\ R_{\mathbf{K4}})$ enjoys the property $(\Box - s)$.*

(ii) *The inference* ($S4$) *is admissible in GL for every Γ, iff the Kripke frame $(W_{GL},\ R_{\mathbf{S4}})$ enjoys the property $(\Box - s)$.*

Sequent calculi $G(\mathbf{K4})$ and $G(\mathbf{S4})$ are obtained on the basis sequent calculus with Stipulation 1 and 2 by adding the following inference rules, respectively.

	Additional rules	Condition on relations
$G(\mathbf{K4})$	(4)	transitive
$G(\mathbf{S4})$	$(S4)$, (T)	transitive and reflexive

From Proposition 5.1 and Lemma 4.3, it follows that $(W_{G(\mathbf{K4})}, R_{\mathbf{K4}})$ and $(W_{G(\mathbf{S4})}, R_{\mathbf{S4}})$ meet conditions in Lemma 2.6. Hence, we have Kripke completeness for $G(\mathbf{K4})$ and $G(\mathbf{S4})$.

In this way, we can show Kripke completeness of sequent calculi for some modal logics by using the conditions of Kripke frame which are equivalent to admissibility of their inferences.

On the other hands, we cannot deal with $(\Box 4.3)$ and $(S4.3)$ in a similar way, although the following Propositions 5.2 and 5.3 give conditions of Kripke frames which are equivalent to admissibility of $(\Box 4.3)$ and $(S4.3)$.

Proposition 5.2. *For a sequent calculus GL with Stipulation 1, the following conditions are equivalent for every nonempty set Δ.*

(i) *The inference $(\Box 4.3)$ is admissible in GL for every Γ.*

(ii) *For every $u \in W_{GL}$, if $\Box\Delta \subseteq s(u)$, then there is an analytically saturated sequent v with the following properties.*

- $uR_{\mathbf{K4}}v$
- $\forall B \in \Delta,\ B \in s(v)$ *or* $\Box B \in s(v)$
- $\exists B \in \Delta$ *s.t.* $B \in s(v)$

Proof: ($\Rightarrow$) Suppose that $\Box\Delta \subseteq s(u)$. Put $\Gamma = \{B \mid \Box B \in a(u)\}$. $\Box\Gamma \to \Box\Delta$ is unprovable in GL. Since $(\Box 4.3)$ is admissible in GL, $\Gamma, \Box\Gamma \to \Box\Sigma, \Lambda$ is unprovable for some $(\Sigma, \Lambda) \in P(\Delta)$. By Lemma 2.2, we have $\Gamma, \Box\Gamma \subseteq a(v)$ and $\Box\Sigma, \Lambda \subseteq s(v)$ for some v. It is clear that $uR_{\mathbf{K4}}v$. Since $\Lambda \neq \emptyset$, v satisfies remaining properties.

($\Leftarrow$) Take a finite set Γ such that $\Box\Gamma \to \Box\Delta$ is unprovable. By Lemma 2.2, $\Box\Gamma \subseteq a(u)$, $\Box\Delta \subseteq s(u)$ and $a(u) \cup s(u) \subseteq \mathrm{Sf}(\Box\Gamma \cup \Box\Delta)$ for some u. Since $\Box\Delta \subseteq s(u)$, there is an analytically saturated v which satisfies properties. Note that $uR_{\mathbf{K4}}v$ leads to $\Gamma, \Box\Gamma \subseteq a(v)$. Put Λ and Σ as follows:

$$\Lambda = \{B \in \Delta \mid B \in s(v)\},$$

$$\Sigma = \Delta \setminus \Lambda.$$

It is clear that $(\Sigma, \Lambda) \in P(\Delta)$ since $\Lambda \neq \emptyset$ by the third condition. Note that $\Gamma,\ \Box\Gamma \to \Box\Sigma,\ \Lambda$ is one of upper sequents of $\Box\Gamma \to \Box\Delta$. Therefore ($\Box$4.3) is admissible for this Γ. $\Box$

Proposition 5.3. *For a sequent calculus GL with Stipulation 1, the following conditions are equivalent for every nonempty set Δ.*

(*i*) *The inference (S4.3) is admissible in GL for every Γ.*

(*ii*) *For every $u \in W_{GL}$, if $\Box\Delta \subseteq s(u)$, then there is an analytically saturated sequent v with the following properties.*

- $uR_{\mathbf{S4}}v$
- $\exists B \in \Delta$ *s.t.* $B \in s(v)$ *and* $\Box(\Delta \setminus \{B\}) \subseteq s(v)$

By the above propositions, we can show that if GL with Stipulation 1 has ($\Box$4.3) or (S4.3), then $(W_{GL},\ R_{\mathbf{K4}})$ or $(W_{GL},\ R_{\mathbf{S4}})$ enjoys $(\Box - s)$ respectively. But these Kripke frames are not weakly connected. Thus, we cannot use these conditions to the proof of Kripke completeness of the calculi with ($\Box$4.3) and (S4.3). So, we extended Takano's method and established our results in this paper.

As of now, we do not have the Kripke frame condition suitable for the proof of Kripke completeness and for the weak connectedness. In order to obtain the condition, author expects that it is necessary to improve the definitions of analytically saturated or binary relation.

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