SEMANTICAL PROOF OF SUBFORMULA PROPERTY
FOR THE MODAL LOGICS K4.3, KD4.3, AND S4.3

Abstract

The main purpose of this paper is to give alternative proofs of syntactical and
semantical properties, i.e. the subformula property and the finite model prop-
erty, of the sequent calculi for the modal logics K4.3, KD4.3, and S4.3. The
application of the inference rules is said to be acceptable, if all the formulas in the
upper sequents are subformula of the formulas in lower sequent. For some modal
logics, Takano analyzed the relationships between the acceptable inference rules
and semantical properties by constructing models. By using these relationships,
he showed Kripke completeness and subformula property. However, his method
is difficult to apply to inference rules for the sequent calculi for K4.3, KD4.3,
and S4.3. Looking closely at Takano’s proof, we find that his method can be
modified to construct finite models based on the sequent calculus for K4.3, if the
calculus has (cut) and all the applications of the inference rules are acceptable.
Similarly, we can apply our results to the calculi for KD4.3 and S4.3. This
leads not only to Kripke completeness and subformula property, but also to finite
model property of these logics simultaneously.

Keywords: modal logic, analytic cut, subformula property, finite model
property.

1. Introduction

The sequent calculi for some modal logics possess subformula property and
finite model property. Takano [2] proved that the sequent calculi for K5
and K5D enjoy these properties through semantical method. Then, he
generalized the method by introducing special unprovable sequent, \textit{analytically saturated sequent}, in Takano [3].

In [3], Takano analyzed the relationships between acceptable inference rules and semantical properties by constructing Kripke models using the set of all analytically saturated sequents. (The application of the inference rules is said to be acceptable, if all formulas in the upper sequents are subformulas of the formulas in the lower sequent.) We call here this method as Takano’s method. Then, he showed that the sequent calculi for modal logics which are obtained from $K$ by adding axioms from $T$, $4$, $5$, $D$, and $B$ enjoy subformula property and finite model property.

The main purpose of this paper is to give alternative proofs of subformula property and finite model property of the sequent calculi for the modal logics $K4.3$, $KD4.3$, and $S4.3$. For this purpose, we consider the relationships between the semantical properties and the inference rules ($\Box4.3$) and ($S4.3$) (introduced by Shimura [1]) based on Takano’s method. However, the straightforward application of Takano’s method does not work well for ($\Box4.3$) and ($S4.3$). Taking a close look at his proof, we find that Takano’s method can be modified to construct finite models based on the sequent calculus for $K4.3$, if the calculus has ($\Box4.3$) and (cut), and all the applications of inference rules are acceptable. Similarly, we can apply this result to the inference rule ($S4.3$). This implies Kripke completeness of the sequent calculi for $K4.3$ and $S4.3$, and these calculi enjoy not only subformula property, but also finite model property.

In Section 2, we introduce the definition and property of an analytically saturated sequent based on Takano [3]. In Section 3 and 4, we consider ($\Box4.3$) and ($S4.3$), respectively, and give the procedure for constructing finite models.

2. Preliminaries

In this paper, we use only $\neg$ (negation), $\supset$ (implication), and $\Box$ (necessity) as logical symbols, and other are considered as abbreviations. Propositional letters and formulas are denoted by $p$, $q$, $r$, \ldots and $A$, $B$, $C$, \ldots, respectively. Finite sequences of formulas are denoted by $\Gamma$, $\Delta$, $\Theta$, $\Lambda$, \ldots, and a sequent is an expression of the form $\Gamma \rightarrow \Theta$. A $\Box$-formula is a formula whose outermost logical symbol is $\Box$. We mean by $Sf(\Gamma)$ the set of all the subformulas of some formulas in $\Gamma$, and by $\Box \Gamma$ the set \{ $\Box A \mid A \in \Gamma$ \}.
Let us consider the following structural rules:

\[
\frac{\Gamma \rightarrow \Theta}{A, \Gamma \rightarrow \Theta} \quad (w \rightarrow) \quad \frac{\Delta, B, A, \Gamma \rightarrow \Theta}{\Delta, A, B, \Gamma \rightarrow \Theta} \quad (\rightarrow w) \quad \frac{\Delta, A, \Gamma \rightarrow \Theta}{A, A, \Gamma \rightarrow \Theta} \quad (c \rightarrow) \\
\frac{\Gamma \rightarrow \Theta}{\Gamma \rightarrow \Theta, A} \quad (\rightarrow) \quad \frac{\Gamma \rightarrow \Theta, B, A, \Lambda}{\Gamma \rightarrow \Theta, A, B, \Lambda} \quad (\rightarrow c)
\]

Every sequent calculus which we treat in this paper enjoys the following stipulation.

**Stipulation 1.** The sequent calculus has \( A \rightarrow A \) as an initial sequent for every \( A \), and contains the structural rules \((w \rightarrow), (\rightarrow w), (\rightarrow c), (\rightarrow e)\), and \((c \rightarrow)\).

Due to this in the rest of this paper, we recognize \( \Gamma, \Delta, \Theta, \Lambda, \cdots \) as finite sets.

**Definition 2.1.** (Takano [3, Definition 1.1]) Let \( GL \) be a sequent calculus with Stipulation 1. The sequent \( \Gamma \rightarrow \Theta \) is analytically saturated in \( GL \), iff the following properties hold.

(a) \( \Gamma \rightarrow \Theta \) is unprovable in \( GL \).
(b) Suppose \( A \in \text{Sf}(\Gamma \cup \Theta) \). If \( A, \Gamma \rightarrow \Theta \) is unprovable in \( GL \), then \( A \in \Gamma \); while if \( \Gamma \rightarrow \Theta, A \) is unprovable in \( GL \), then \( A \in \Theta \).

The set of all analytically saturated sequents is denoted by \( W_{GL} \).

**Lemma 2.2.** (Takano [3, Lemma 1.3]) For a sequent calculus \( GL \) with Stipulation 1, if the sequent \( \Gamma \rightarrow \Theta \) is unprovable in \( GL \), then there is an analytically saturated sequent \( u \) with the following properties:

(i) \( \Gamma \subseteq a(u) \) and \( \Theta \subseteq s(u) \)
(ii) \( a(u) \cup s(u) \subseteq \text{Sf}(\Gamma \cup \Theta) \)

**Definition 2.3.** (Takano [3, Definition 1.5]) An inference is admissible in a sequent calculus \( GL \), iff either some of the upper sequents of the inference is unprovable in \( GL \), or the lower one in provable in \( GL \).

For a sequent calculus \( GL \) with Stipulation 1, there are relationships between properties of analytically saturated sequents and inferences which are admissible in \( GL \). For example, we consider the following inferences.
Proposition 2.4. (Takano [3, Proposition 1.6]) For a sequent calculus $GL$ with Stipulation 1, the following equivalences hold for every $A$ and $B$.

1. The inference $(\neg \rightarrow)$ is admissible in $GL$ for every $\Gamma$ and $\Theta$, iff $\neg A \in a(u)$ implies $A \in s(u)$ for every $u$.
2. The inference $(\rightarrow \neg)$ is admissible in $GL$ for every $\Gamma$ and $\Theta$, iff $\neg A \in s(u)$ implies $A \in a(u)$ for every $u$.
3. The inference $(\supset \rightarrow)$ is admissible in $GL$ for every $\Gamma$ and $\Theta$, iff $A \supset B \in a(u)$ implies $A \in s(u)$ or $B \in a(u)$ for every $u$.
4. The inference $(\rightarrow \supset)$ is admissible in $GL$ for every $\Gamma$ and $\Theta$, iff $A \supset B \in s(u)$ implies $A \in a(u)$ and $B \in s(u)$ for every $u$.

Proposition 2.5. (Takano [3, Proposition 3.1]) For a sequent calculus $GL$ with Stipulation 1, the inference $(\text{cut})^a$ is admissible for every $\Gamma$, $\Theta$, $\Delta$, $\Lambda$, and $C$ with the restriction that $C \in Sf(\Gamma \cup \Theta \cup \Delta \cup \Lambda)$, iff $Sf(a(u) \cup s(u)) \subseteq a(u) \cup s(u)$.

We introduce Stipulation 2 as well.

Stipulation 2. The sequent calculus contains $(\neg \rightarrow)$, $(\rightarrow \neg)$, $(\supset \rightarrow)$, and $(\rightarrow \supset)$ as inference rules.

The aim of introducing analytically saturated sequents is obtaining the proof of Kripke completeness.

Lemma 2.6. (Takano [3, Proposition 1.4]) Let $GL$ be a sequent calculus with Stipulation 1. Suppose that $(W, R)$ is a Kripke frame such that $W \subseteq W_{GL}$, and the following properties hold for every $A$, $B$ and every $u \in W$:

- $(\neg - a)$ $\neg A \in a(u)$ implies $A \in s(u)$.
- $(\neg - s)$ $\neg A \in s(u)$ implies $A \in a(u)$.
- $(\supset - a)$ $A \supset B \in a(u)$ implies $A \in s(u)$ or $B \in a(u)$. 

Note that inference rule $(\text{cut})^a$ is obtained from $(\text{cut})$ by applying appropriate restriction.
Semantical Proof of Subformula Property for the Modal Logics...

Let $\models$ be the satisfaction relation on $(W, R)$ such that $u \models p$ iff $p \in a(u)$ for every $u \in W$ and every $p$. Then for every $C$ and every $u \in W$, if $C \in a(u)$ then $u \models C$; while if $C \in s(u)$ then $u \not\models C$.

The proof of this lemma is given by induction on the construction of $C$.

For a sequent calculus $GL$ with Stipulation 1, assume that any $u \in W_{GL}$ has a Kripke frame $(W, R)$ which satisfies following properties.

- $u \in W \subseteq W_{GL}$.
- $(W, R)$ meets conditions of Lemma 2.6.
- the accessibility relation $R$ meets the condition of Kripke frame for $L$.

Then, if $\Gamma \rightarrow \Theta$ is unprovable in $GL$, there is an analytically saturated sequent $u$ such that $\Gamma \subseteq a(u)$ and $\Theta \subseteq s(u)$ by Lemma 2.2. And $u$ has a Kripke frame $(W, R)$ which satisfies the above properties. Adding satisfaction relation $\models$ introduced in Lemma 2.6, we obtain Kripke model $(W, R, \models)$ in which $C \in \Gamma$ implies $u \models C$ and $C \in \Theta$ implies $u \not\models C$. This leads to Kripke completeness of $GL$.

The key point is whether every $u \in W_{GL}$ has such Kripke frame or not. It depends on admissibility of inferences in $GL$. From Proposition 2.4, for any Kripke frame of sequent calculus $GL$ with Stipulation 1 and 2 holds $(\neg - a)$, $(\neg - s)$, $(\top - a)$, and $(\top - s)$. The remaining conditions $(\square - a)$ and $(\square - s)$ depend not only on admissibility of inferences, but also on properties of accessibility relation. We will discuss them in the remaining sections.

3. The logics K4.3 and KD4.3

Modal logic K4.3 is obtained from the least normal logic K by adding axioms $\square p \supset \square \square p$ and $\square((p \land \square p) \supset q) \lor \square((q \land \square q) \supset p)$. Kripke frame $(W, R)$ meets condition of K4.3 if the frame is transitive and weakly connected; the Kripke frame is said to be weakly connected, if a binary relation $R$ enjoys the following condition.

$$\forall u, v, w (uRv \text{ and } uRw \Rightarrow vRw \text{ or } u = w \text{ or } wRv)$$
Modal logic $\textbf{KD4.3}$ is obtained from $\textbf{K4.3}$ by adding axiom $\Box p \supset \neg \Box \neg p$.

In this section, we consider the inference rule for $\textbf{K4.3}$ introduced by Shimura [1].

**Definition 3.1.** Suppose that $\Delta \neq \emptyset$. $P(\Delta)$ is defined as the set of all pairs $(\Sigma, \Lambda)$ with following properties:

1. $\Sigma \cup \Lambda = \Delta$ and $\Sigma \cap \Lambda = \emptyset$,
2. $\Lambda \neq \emptyset$.

For example, if $\Delta = \{A, B\}$, then

$$P(\Delta) = \{(\{A\}, \{B\}), (\{B\}, \{A\}), (\emptyset, \{A, B\})\}$$

The inference rule $(\Box 4.3)$ is defined as follows:

$$\frac{\Gamma, \Box \Gamma \rightarrow \Box \Sigma, A| (\Sigma, A) \in P(\Delta)}{\Box \Gamma \rightarrow \Box \Delta} (\Box 4.3)$$

If $\Delta = \{A, B\}$, $(\Box 4.3)$ is of the form:

$$\frac{\Gamma, \Box \Gamma \rightarrow \Box A, B \quad \Gamma, \Box \Gamma \rightarrow \Box B, A \quad \Gamma, \Box \Gamma \rightarrow A, B}{\Box \Gamma \rightarrow \Box A, \Box B} (\Box 4.3)$$

Sequent calculus $G(\textbf{K4.3})$ is obtained from Gentzen’s original $\textbf{LK}$ by adding inference rule $(\Box 4.3)$. Shimura proved that $G(\textbf{K4.3})$ satisfies cut elimination using the syntactic method.

Restricted sequent calculi $G(\textbf{K4.3})^-$ and $G(\textbf{KD4.3})^-$ are obtained from $\textbf{LK}$ by replacing $(\text{cut})$ with $(\text{cut})^a$ and adding following rules.

<table>
<thead>
<tr>
<th>Rules</th>
<th>Condition on relations</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G(\textbf{K4.3})^-$</td>
<td>$\textbf{LK}$ with $(\text{cut})^a$, transitive and weakly connected $(\Box 4.3)$</td>
</tr>
<tr>
<td>$G(\textbf{KD4.3})^-$</td>
<td>$\textbf{LK}$ with $(\text{cut})^a$, transitive, serial, $(\Box 4.3), (4D)$ and weakly connected</td>
</tr>
</tbody>
</table>

Where $(4D)$ is

$$\frac{\Gamma, \Box \Gamma \rightarrow (4D)}{\Box \Gamma \rightarrow (4D)}.$$

We can prove their Kripke completeness by modifying Takano’s method as follows. Note that the condition on Kripke frame is not equivalent to admissibility of $(\Box 4.3)$. (See Section 5.)
DEFINITION 3.2. For a sequent calculus $GL$ with Stipulation 1, the binary relation $R_{K4}$ on $W_{GL}$ is defined by: $u R_{K4} v$, iff $\Box B \in a(u)$ implies $B, \Box B \in a(v)$ for every $B$.

From this definition, it follows that for every nonempty set $W \subseteq W_{GL}$, Kripke frame $(W, R_{K4})$ is transitive and meets $(\Box - a)$.

PROPOSITION 3.3. Let $GL$ be a sequent calculus with Stipulation 1 and the inference rule (cut). If $(\Box - 3)$ is admissible in $GL$ for every $\Gamma$ and $\Delta$ ($\Delta \neq \emptyset$), then for every $u \in W_{GL}$, there is a finite set $W \subseteq W_{GL}$ with the following properties.

(i) $u \in W$

(ii) Kripke frame $(W, R_{K4})$ enjoys the property $(\Box - a)$ and $(\Box - s)$, and meets condition for $K4.3$.

PROOF: Suppose $u \in W_{GL}$. We construct analytically saturated sequents $v_1, \ldots, v_n$ as follows.

- $v_1 := u$
- Suppose that $v_1, \ldots, v_k$ are constructed. Put $\Gamma_k, \Theta_k, L_k$ and $\Delta_k$ as follows:

\[ \Gamma_k = \{ B \mid \Box B \in a(v_k) \}, \quad \Theta_k = \{ B \mid \Box B \in s(v_k) \}, \]

\[ L_k = \{ B \in \Theta_k \mid \exists w \in \{ v_1, \ldots, v_k \} \text{ s.t. } v_k R_{K4} w \text{ and } B \in s(w) \}, \]

\[ \Delta_k = \Theta_k \setminus L_k. \]

We have two cases: $\Delta_k \neq \emptyset$ and $\Delta_k = \emptyset$.

Case (1): $\Delta_k \neq \emptyset$. By following procedure, we construct the analytically saturated $v_{k+1}$ which satisfies $v_k R_{K4} v_{k+1}$ and $B \in s(v_{k+1})$ for some $B \in \Delta_k$. Since $\Box \Gamma_k \rightarrow \Box \Delta_k$ is unprovable, $\Gamma_k, \Box \Gamma_k \rightarrow \Box \Sigma, \Lambda$ is unprovable for some $(\Sigma, \Lambda) \in P(\Delta_k)$. So, $\Gamma_k \cup \Box \Gamma_k \subseteq a(v)$, $\Box \Sigma \cup \Lambda \subseteq s(v)$ and $a(v) \cup s(v) \subseteq Sf(\Gamma_k \cup \Box \Gamma_k \cup \Box \Sigma \cup \Lambda)$ for some $v$ by Lemma 2.2. Put $v_{k+1} := v$, it is clear that $v_k R_{K4} v_{k+1}$ and $B \in s(v_{k+1})$ for some $B \in \Delta_k$. Furthermore, $v_{k+1} \not\in \{ v_1, \ldots, v_k \}$. (Suppose $v_{k+1} \in \{ v_1, \ldots, v_k \}$. Since $v_k R_{K4} v_{k+1}$ and $\Lambda \subseteq \Delta_k$, $\Delta_k \subseteq L_k$ would follow, which is a contradiction.) Note that for all $\Box B \in s(v_k), B \in s(v_{k+1}), \Box B \in s(v_{k+1})$, or $B \in L_k$ is satisfied.

Case (2): $\Delta_k = \emptyset$. Stop the construction.
To prove that this construction stops with finite steps, we will show that \( \Delta_{k+1} \subseteq \Delta_k \) or \( \Theta_{k+1} \subseteq \Theta_k \) holds. Since \( a(v) \cup s(v) \subseteq \text{Sf}(\Gamma_k \sqcup \square \Gamma_k \sqcup \square \Sigma \cup A) \subseteq \text{Sf}(a(v_k) \cup s(v_k)) \), it follows \( a(v_{k+1}) \cup s(v_{k+1}) \subseteq a(v_k) \cup s(v_k) \) by \((\text{cut})^a\).

It is clear that \( \Gamma_k \subseteq \Gamma_{k+1} \), so \( \Theta_{k+1} \subseteq \Theta_k \). Suppose that \( \Theta_{k+1} = \Theta_k \). We will derive \( \Delta_{k+1} \subseteq \Delta_k \) in this case. Since \( a(v_{k+1}) \cup s(v_{k+1}) \subseteq a(v_k) \cup s(v_k) \) and \( \Gamma_k \subseteq \Gamma_{k+1} \), it follows \( \Gamma_k = \Gamma_{k+1} \). This implies that if \( v_kR_{K4}w \), then \( v_{k+1}R_{K4}w \) for any \( w \in W_{GL} \), so \( L_k \subseteq L_{k+1} \). Moreover, since \( v_kR_{K4}v_{k+1} \), it follows \( v_kR_{K4}v_{k+1} \) and \( \Lambda \subseteq L_{k+1} \), although \( \Lambda \not\subseteq L_k \). This implies \( L_k \subseteq L_{k+1} \); hence \( \Delta_{k+1} \subseteq \Delta_k \).

There is an analytically saturated sequent \( v_n \) with \( \Delta_n = \emptyset \) by repeating this procedure. Put \( W = \{ v_1, \ldots, v_n \} \), it is clear that Kripke frame \((W, R_{K4})\) is transitive and weakly connected frame, and enjoys \((\Box - a)\). If \( \Box B \in s(v_n) \), then \( B \in L_n \) since \( \Delta_n = \emptyset \). So \((W, R_{K4})\) enjoys \((\Box - s)\). \( \square \)

From the above proposition, we can show Kripke completeness for \( G(K4.3)^- \), and this leads to subformula property for \( G(K4.3) \). Furthermore, this leads to finite model property simultaneously because the constructed model is finite.

Similarly, \( G(KD4.3) \) has subformula property and finite model property.

**Lemma 3.4.** Let \( GL \) be a sequent calculus with Stipulation 1 and the inference rule \((\text{cut})^a\). If \((\Box 4.3)\) and \((AD)\) are admissible in \( GL \) for every \( \Gamma \) and \( \Delta (\Delta \neq \emptyset) \), then for every \( u \in W_{GL} \), there is a finite set \( W \subseteq W_{GL} \) with the following properties.

(i) \( u \in W \)

(ii) Kripke frame \((W, R_{K4})\) enjoys the property \((\Box - a)\) and \((\Box - s)\), and meets condition for \( KD4.3 \).

**Proof:** Suppose \( u \in W_{GL} \). From Proposition 3.3, there is a finite set \( \{ v_1, \ldots, v_n \} \) which meets the condition of Proposition 3.3 with \( v_1 = u \). If \( v_n \) has \( R_{K4} \) successor in \( \{ v_1, \ldots, v_n \} \), then the set is the desired one. If not so, we construct the analytically saturated \( v_{n+1} \) by following procedure.

Put \( \Gamma_n \) and \( \Theta_n \) same as Proposition 3.3. It is clear that \( \Theta_n = \emptyset \) and \( \Gamma_n \neq \emptyset \). (Suppose otherwise \( \Gamma_n = \emptyset \). Then, all analytically saturated sequents of \( W_{GL} \) are \( R_{K4} \) successors of \( v_n \). This is a contradiction.) Since \( \Box \Gamma_n \rightarrow \Box \Gamma_n \) is unprovable, \( \Gamma_n, \Box \Gamma_n \rightarrow \Box \Gamma_n \) is unprovable. Then, by Lemma 2.2, \( v_n \neq a(v) \) and \( a(v) \cup s(v) \subseteq \text{Sf}(\Box \Gamma_n) \) for some \( v \). Put \( v = v_{n+1} \), it is clear \( v_nR_{K4}v_{n+1} \). Furthermore, since \( GL \) has inference rule \((\text{cut})^a\), \( a(v_{n+1}) \cup s(v_{n+1}) \subseteq a(v_n) \cup s(v_n) \). So, \( s(v_{n+1}) \) has no \( \Box \)-formulas. (Suppose
$\square B \in s(v_{n+1})$, this implies $\square B \in a(v_n)$ or $\square B \in s(v_n)$. Since $\Theta_n = \emptyset$, $\square B \in a(v_n)$. It follows $\square B \in a(v_n) \subseteq a(v_{n+1})$, which is a contradiction.) Similarly, suppose $\square B \in a(v_{n+1})$, it follows $B \in a(v_{n+1})$, and this implies $v_{n+1} R_{K4} v_{n+1}$. Thus, $W = \{v_1, \cdots, v_n, v_{n+1}\}$ meets the conditions. $\square$

4. The logic S4.3

Modal logic $\mathbf{S4.3}$ is obtained form $\mathbf{K4.3}$ by adding axiom $\square p \supset p$. Kripke frame $(W, R)$ meets condition of $\mathbf{S4.3}$ iff the frame is transitive, weakly connected, and reflexive. Shimura [1] also introduced inference rule for $\mathbf{S4.3}$.

\[
\frac{\square \Gamma \rightarrow \square (\Delta \setminus \{A\}), \ A \mid A \in \Delta}{\square \Gamma \rightarrow \square \Delta} \quad (S4.3)
\]

If $\Delta = \{A, B\}$, $(\square 4.3)$ is of the form:

\[
\frac{\square \Gamma \rightarrow \square A, \ B}{\square \Gamma \rightarrow \square A, \ \square B} \quad (S4.3)
\]

Sequent calculus $G(\mathbf{S4.3})$ is obtained from $LK$ by adding inference rule $(S4.3)$ and $(T)$.

\[
\frac{A, \ \Gamma \rightarrow \Theta}{\square A, \ \Gamma \rightarrow \Theta} \quad (T)
\]

Shimura proved that $G(\mathbf{S4.3})$ satisfies cut elimination using the syntactic method.

Restricted sequent calculus $G(\mathbf{S4.3})^-$ is obtained from $LK$ by replacing $(cut)$ with $(cut)^a$ and adding the following rules.

\[
\begin{array}{c|c|c}
\text{Rules} & \text{Condition on relations} \\
\hline
G(\mathbf{S4.3})^- & LK with (cut)$^a$, & transitive, reflexive \\
& (S4.3), (T) & and weakly connected \\
\end{array}
\]

We can prove its Kripke completeness by modifying Takano’s method as follows.

\textbf{Definition 4.1}. For a sequent calculus $GL$ with Stipulation 1, the binary relation $R_{S4}$ on $W_{GL}$ is defined by: $u R_{S4} v$, iff $\square B \in a(u)$ implies $\square B \in a(v)$ for every $B$.

By this definition, for every nonempty set $W \subseteq W_{GL}$, Kripke frame $(W, R_{S4})$ is transitive and reflexive.
Proposition 4.2. Let $GL$ be a sequent calculus with Stipulation 1 and the inference rule (cut)\(^a\). If (S4.3) is admissible in $GL$ for every $\Gamma$ and $\Delta$ ($\Delta \neq \emptyset$), then for every $u \in W_{GL}$, there is a finite set $W \subseteq W_{GL}$ with the following properties.

(i) $u \in W$

(ii) Kripke frame $(W, R_{S4})$ enjoys the property $(\Box - s)$, and meets condition for $S4.3$.

The proof is similar to Proposition 3.3. Note that the Kripke frame constructed by the above proposition does not enjoy $(\Box - a)$. If $GL$ has $(T)$ as inference rule, then the constructed model enjoys $(\Box - a)$ by following lemma.

Lemma 4.3. Let $GL$ be a sequent calculus with Stipulation 1. If the inference $(T)$ is admissible in $GL$ for every $A$, $\Gamma$, and $\Theta$, then Kripke frame $(W, R_{S4})$ holds the property $(\Box - a)$ for every $W \subseteq W_{GL}$.

Proof: Suppose that $u \in W$. If $\Box B \in a(u)$, then $uR_{K4}v$ implies $\Box B \in a(v)$ for every $v \in W$. Since $\Box B, a(v) \rightarrow s(v)$ is unprovable, we have that $B, a(v) \rightarrow s(v)$ is unprovable by applying rule $(T)$. Hence, $B \in a(v)$. $\Box$

We can show Kripke completeness of $G(S4.3)$ by Propositions 4.2 and 4.3. This implies not only subformula property, but also finite model property of $G(S4.3)$.

5. Concluding remark

In this paper, we gave alternative proofs of Kripke completeness, subformula property and finite model property for $K4.3$, $KD4.3$ and $S4.3$ by modifying Takano’s method in [3].

Takano’s method in [3] was developed originally to analyze relationships between admissibility of acceptable inference rules and semantical properties. Then, by using these relationships, he showed Kripke completeness of some modal logics as well. But, the straightforward application of Takano’s method does not work well for $(\Box 4.3)$ and (S4.3). Takano’s method is useful to prove Kripke completeness, but has limitations. Let us explain this with examples. We consider the following inference.

\[
\frac{\Gamma, \Box \Gamma \rightarrow A}{\Box \Gamma \rightarrow A} \quad (4) \quad \frac{\Box \Gamma \rightarrow A}{\Box \Gamma \rightarrow A} \quad (S4)
\]
Proposition 5.1. (Takano [3, Proposition 2.2]) For a sequent calculus GL with Stipulation 1, the following equivalences hold for every A.

(i) The inference \((4)\) is admissible in GL for every \(\Gamma\), iff the Kripke frame \((W_{GL}, R_{K4})\) enjoys the property \((\Box - s)\).

(ii) The inference \((S4)\) is admissible in GL for every \(\Gamma\), iff the Kripke frame \((W_{GL}, R_{S4})\) enjoys the property \((\Box - s)\).

Sequent calculi \(G(K4)\) and \(G(S4)\) are obtained on the basis sequent calculus with Stipulation 1 and 2 by adding the following inference rules, respectively.

<table>
<thead>
<tr>
<th>Additional rules</th>
<th>Condition on relations</th>
</tr>
</thead>
<tbody>
<tr>
<td>(G(K4))</td>
<td>((4)) transitive</td>
</tr>
<tr>
<td>(G(S4))</td>
<td>((S4), (T)) transitive and reflexive</td>
</tr>
</tbody>
</table>

From Proposition 5.1 and Lemma 4.3, it follows that \((W_{G(K4)}, R_{K4})\) and \((W_{G(S4)}, R_{S4})\) meet conditions in Lemma 2.6. Hence, we have Kripke completeness for \(G(K4)\) and \(G(S4)\).

In this way, we can show Kripke completeness of sequent calculi for some modal logics by using the conditions of Kripke frame which are equivalent to admissibility of their inferences.

On the other hands, we cannot deal with \((\Box 4.3)\) and \((S4.3)\) in a similar way, although the following Propositions 5.2 and 5.3 give conditions of Kripke frames which are equivalent to admissibility of \((\Box 4.3)\) and \((S4.3)\).

Proposition 5.2. For a sequent calculus GL with Stipulation 1, the following conditions are equivalent for every nonempty set \(\Delta\).

(i) The inference \((\Box 4.3)\) is admissible in GL for every \(\Gamma\).

(ii) For every \(u \in W_{GL}\), if \(\Box \Delta \subseteq s(u)\), then there is an analytically saturated sequent \(v\) with the following properties.

\[
* \ uR_{K4}v \\
* \ \forall B \in \Delta \ B \in s(v) \text{ or } \Box B \in s(v) \\
* \ \exists B \in \Delta \text{ s.t. } B \in s(v)
\]

Proof: \((\Rightarrow)\) Suppose that \(\Box \Delta \subseteq s(u)\). Put \(\Gamma = \{B \mid \Box B \in a(u)\}\). \(\Box \Gamma \rightarrow \Box \Delta\) is unprovable in GL. Since \((\Box 4.3)\) is admissible in GL, \(\Gamma, \Box \Gamma \rightarrow \Box \Sigma, \Lambda\) is unprovable for some \((\Sigma, \Lambda) \in P(\Delta)\). By Lemma 2.2, we have \(\Gamma, \Box \Gamma \subseteq a(v)\) and \(\Box \Sigma, \Lambda \subseteq s(v)\) for some \(v\). It is clear that \(uR_{K4}v\). Since \(\Lambda \neq \emptyset\), \(v\) satisfies remaining properties.
(⇐) Take a finite set $\Gamma$ such that $\square \Gamma \rightarrow \square \Delta$ is unprovable. By Lemma 2.2, $\square \Gamma \subseteq a(u)$, $\square \Delta \subseteq s(u)$ and $a(u) \cup s(u) \subseteq \text{Sf}(\square \Gamma \cup \square \Delta)$ for some $u$. Since $\square \Delta \subseteq s(u)$, there is an analytically saturated $v$ which satisfies properties. Note that $uR_{K4}v$ leads to $\Gamma$, $\square \Gamma \subseteq a(v)$. Put $\Lambda$ and $\Sigma$ as follows:

$$\Lambda = \{B \in \Delta \mid B \in s(v)\},$$

$$\Sigma = \Delta \setminus \Lambda.$$

It is clear that $(\Sigma, \Lambda) \in P(\Delta)$ since $\Lambda \neq \emptyset$ by the third condition. Note that $\Gamma$, $\square \Gamma \rightarrow \square \Sigma$, $\Lambda$ is one of upper sequents of $\square \Gamma \rightarrow \square \Delta$. Therefore $(\square 4.3)$ is admissible for this $\Gamma$.

**Proposition 5.3.** For a sequent calculus $GL$ with Stipulation 1, the following conditions are equivalent for every nonempty set $\Delta$.

(i) The inference $(S4.3)$ is admissible in $GL$ for every $\Gamma$.

(ii) For every $u \in W_{GL}$, if $\square \Delta \subseteq s(u)$, then there is an analytically saturated sequent $v$ with the following properties.

* $uR_{S4}v$
* $\exists B \in \Delta$ s.t. $B \in s(v)$ and $\square(\Delta \setminus \{B\}) \subseteq s(v)$

By the above propositions, we can show that if $GL$ with Stipulation 1 has $(\square 4.3)$ or $(S4.3)$, then $(W_{GL}, R_{K4})$ or $(W_{GL}, R_{S4})$ enjoys $(\square - s)$ respectively. But these Kripke frames are not weakly connected. Thus, we cannot use these conditions to the proof of Kripke completeness of the calculi with $(\square 4.3)$ and $(S4.3)$. So, we extended Takano's method and established our results in this paper.

As of now, we do not have the Kripke frame condition suitable for the proof of Kripke completeness and for the weak connectedness. In order to obtain the condition, author expects that it is necessary to improve the definitions of analytically saturated or binary relation.

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References


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