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A GENERAL MODEL OF NEUTROSOPHIC IDEALS IN BCK/BCI-ALGEBRAS BASED ON NEUTROSOPHIC POINTS

Abstract

More general form of $(\in, \in\vee q)$-neutrosophic ideal is introduced, and their properties are investigated. Relations between $(\in, \in)$-neutrosophic ideal and $(\in, \in\vee q_{(k_T, k_I, k_F)})$-neutrosophic ideal are discussed. Characterizations of $(\in, \in\vee q_{(k_T, k_I, k_F)})$-neutrosophic ideal are discussed, and conditions for a neutrosophic set to be an $(\in, \in\vee q_{(k_T, k_I, k_F)})$-neutrosophic ideal are displayed.

Keywords: Ideal, neutrosophic $\in$-subset, neutrosophic q_k-subset, neutrosophic $\in\vee q_k$-subset, $(\in, \in\vee q_{(k_T, k_I, k_F)})$-neutrosophic ideal.

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1. Introduction

Smarandache [23, 24] introduced the concept of neutrosophic sets which is a more general platform to extend the notions of the classical set and (intuitionistic, interval valued) fuzzy set. Neutrosophic set theory is applied to several parts which are referred to the site http://fs.gallup.unm.

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edu/neutrosophy.htm. Jun [10] introduced the notion of neutrosophic subalgebras in BCK/BCI-algebras based on neutrosophic points. Borumand and Jun [22] studied several properties of $(\in, \in \vee q)$-neutrosophic subalgebras and $(q, \in \vee q)$-neutrosophic subalgebras in BCK/BCI-algebras. Jun et al. [11] discussed neutrosophic $\mathcal{N}$-structures with an application in BCK/BCI-algebras, and in [13, 14] introduced neutrosophic quadruple numbers based on a set and construct neutrosophic quadruple BCK/BCI-algebras.

Song et al. [25] introduced the notion of commutative $\mathcal{N}$-ideal in BCK-algebras and investigated several properties. Bordbar, Jun and et al. [21] and [17] introduced the notion of $(q, \in \vee q)$-neutrosophic ideal, and $(\in, \in \vee q)$-neutrosophic ideal in BCK/BCI-algebras, and investigated related properties. Also in [7, 26], they discussed the notion of $BMBJ$-neutrosophic sets, subalgebra and ideals, as a generalisation of neutrosophic set, and investigated its application and related properties to BCI/BCK-algebras.

For more information about the mentioned topics, please refer to [3, 4, 8, 12, 16, 18, 19, 20].

In this paper, we introduce a more general form of $(\in, \in \vee q)$-neutrosophic ideal, and investigate their properties. We discuss relations between $(\in, \in)$-neutrosophic ideal and $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic ideal. We consider characterizations of $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic ideal. We investigate conditions for a neutrosophic set to be an $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic ideal. We find conditions for an $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic ideal to be an $(\in, \in)$-neutrosophic ideal.

2. Preliminaries

By a *BCI-algebra* we mean a set X with a binary operation $*$ and the special element 0 satisfying the axioms:

(a1) $((x * y) * (x * z)) * (z * y) = 0$,

(a2) $(x * (x * y)) * y = 0$,

(a3) $x * x = 0$,

(a4) $x * y = y * x = 0 \;\Rightarrow\; x = y$, for all $x, y, z \in X$. If a BCI-algebra X satisfies the axiom

(a5) $0 * x = 0$ for all $x \in X$,

then we say that X is a *BCK-algebra*. A subset I of a BCK/BCI-algebra X is called an *ideal* of X (see [9, 15]) if it satisfies:

$$0 \in I, \tag{2.1}$$

$$(\forall x, y \in X)\,(x * y \in I,\, y \in I \;\Rightarrow\; x \in I)\,. \tag{2.2}$$

The collection of all BCK-algebras and all BCI-algebras are denoted by $\mathcal{B}_K(X)$ and $\mathcal{B}_I(X)$, respectively. Also $\mathcal{B}(X) := \mathcal{B}_K(X) \cup \mathcal{B}_I(X)$.

We refer the reader to the books [9] and [15] for further information regarding BCK/BCI-algebras.

For any family $\{a_i \mid i \in \Lambda\}$ of real numbers, we define

$$\bigvee\{a_i \mid i \in \Lambda\} = \sup\{a_i \mid i \in \Lambda\}$$

and

$$\bigwedge\{a_i \mid i \in \Lambda\} = \inf\{a_i \mid i \in \Lambda\}.$$

If $\Lambda = \{1, 2\}$, we will also use $a_1 \vee a_2$ and $a_1 \wedge a_2$ instead of $\bigvee\{a_i \mid i \in \{1, 2\}\}$ and $\bigwedge\{a_i \mid i \in \{1, 2\}\}$, respectively.

Let X be a non-empty set. A *neutrosophic set* (NS) in X (see [23]) is a structure of the form:

$$A := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\}$$

where $A_T : X \to [0, 1]$ is a truth membership function, $A_I : X \to [0, 1]$ is an indeterminate membership function, and $A_F : X \to [0, 1]$ is a false membership function. For the sake of simplicity, we shall use the symbol $A = (A_T, A_I, A_F)$ for the neutrosophic set

$$A := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\}.$$

Given a neutrosophic set $A = (A_T, A_I, A_F)$ in a set X, $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$, we consider the following sets (see [10]):

$T_\in(A; \alpha) := \{x \in X \mid A_T(x) \geq \alpha\}$,

$I_\in(A; \beta) := \{x \in X \mid A_I(x) \geq \beta\}$,

$F_{\in}(A;\gamma) := \{x \in X \mid A_F(x) \leq \gamma\}$.

We say $T_{\in}(A;\alpha)$, $I_{\in}(A;\beta)$ and $F_{\in}(A;\gamma)$ are *neutrosophic $\in$-subsets.*

3. Generalizations of neutrosophic ideals based on neutrosophic points

In what follows, let k_T, k_I and k_F denote arbitrary elements of $[0,1)$ unless otherwise specified. If k_T, k_I and k_F are the same number in $[0,1)$, then it is denoted by k, i.e., $k = k_T = k_I = k_F$.

Given a neutrosophic set $A = (A_T, A_I, A_F)$ in a set X, $\alpha, \beta \in (0,1]$ and $\gamma \in [0,1)$, we consider the following sets:

$$T_{q_{k_T}}(A;\alpha) := \{x \in X \mid A_T(x) + \alpha + k_T > 1\},$$

$$I_{q_{k_I}}(A;\beta) := \{x \in X \mid A_I(x) + \beta + k_I > 1\},$$

$$F_{q_{k_F}}(A;\gamma) := \{x \in X \mid A_F(x) + \gamma + k_F < 1\},$$

$$T_{\in\vee\, q_{k_T}}(A;\alpha) := \{x \in X \mid A_T(x) \geq \alpha \text{ or } A_T(x) + \alpha + k_T > 1\},$$

$$I_{\in\vee\, q_{k_I}}(A;\beta) := \{x \in X \mid A_I(x) \geq \beta \text{ or } A_I(x) + \beta + k_I > 1\},$$

$$F_{\in\vee\, q_{k_F}}(A;\gamma) := \{x \in X \mid A_F(x) \leq \gamma \text{ or } A_F(x) + \gamma + k_F < 1\}.$$

We say $T_{q_{k_T}}(A;\alpha)$, $I_{q_{k_I}}(A;\beta)$ and $F_{q_{k_F}}(A;\gamma)$ are *neutrosophic q_k-subsets*; and $T_{\in\vee\, q_{k_T}}(A;\alpha)$, $I_{\in\vee\, q_{k_I}}(A;\beta)$ and $F_{\in\vee\, q_{k_F}}(A;\gamma)$ are *neutrosophic $\in\vee\, q_k$-subsets.* For $\psi \in \{\in, q, q_k, q_{k_T}, q_{k_I}, q_{k_F}, \in\vee\, q, \in\vee\, q_k, \in\vee\, q_{k_T}, \in\vee\, q_{k_I}, \in\vee\, q_{k_F}\}$, the element of $T_\psi(A;\alpha)$ (resp., $I_\psi(A;\beta)$ and $F_\psi(A;\gamma)$) is called a *neutrosophic T_ψ-point* (resp., *neutrosophic I_ψ-point* and *neutrosophic F_ψ-point*) with value α (resp., β and γ).

It is clear that

$$T_{\in\vee\, q_{k_T}}(A;\alpha) = T_{\in}(A;\alpha) \cup T_{q_{k_T}}(A;\alpha), \tag{3.1}$$

$$I_{\in\vee\, q_{k_I}}(A;\beta) = I_{\in}(A;\beta) \cup I_{q_{k_I}}(A;\beta), \tag{3.2}$$

$$F_{\in\vee\, q_{k_F}}(A;\gamma) = F_{\in}(A;\gamma) \cup F_{q_{k_F}}(A;\gamma). \tag{3.3}$$

THEOREM 3.1. *Given a neutrosophic set $A = (A_T, A_I, A_F)$ in $X \in \mathcal{B}(X)$, the following assertions are equivalent.*

(1) *The nonempty neutrosophic* $\in$*-subsets* $T_\in(A;\alpha)$, $I_\in(A;\beta)$ *and* $F_\in(A;\gamma)$ *are ideals of* X *for all* $\alpha \in (\frac{1-k_T}{2}, 1]$, $\beta \in (\frac{1-k_I}{2}, 1]$ *and* $\gamma \in [0, \frac{1-k_F}{2})$.

(2) $A = (A_T, A_I, A_F)$ *satisfies the following assertion.*

$$(\forall x \in X)\left(\begin{array}{l} A_T(x) \leq A_T(0) \vee \frac{1-k_T}{2} \\ A_I(x) \leq A_I(0) \vee \frac{1-k_I}{2} \\ A_F(x) \geq A_F(0) \wedge \frac{1-k_F}{2} \end{array}\right) \tag{3.4}$$

and

$$(\forall x, y \in X)\left(\begin{array}{l} A_T(x) \vee \frac{1-k_T}{2} \geq A_T(x*y) \wedge A_T(y) \\ A_I(x) \vee \frac{1-k_I}{2} \geq A_I(x*y) \wedge A_I(y) \\ A_F(x) \wedge \frac{1-k_F}{2} \leq A_F(x*y) \vee A_F(y) \end{array}\right) \tag{3.5}$$

Proof: Assume that the nonempty neutrosophic $\in$-subsets $T_\in(A;\alpha)$, $I_\in(A;\beta)$ and $F_\in(A;\gamma)$ are ideals of X for all $\alpha \in (\frac{1-k_T}{2}, 1]$, $\beta \in (\frac{1-k_I}{2}, 1]$ and $\gamma \in [0, \frac{1-k_F}{2})$. If there are $a, b \in X$ such that $A_T(a) > A_T(0) \vee \frac{1-k_T}{2}$, then $a \in T_\in(A;\alpha_a)$ and $0 \notin T_\in(A;\alpha_a)$ for $\alpha_a := A_T(a) \in (\frac{1-k_T}{2}, 1]$. This is a contradiction, and so $A_T(x) \leq A_T(0) \vee \frac{1-k_T}{2}$ for all $x \in X$. We also know that $A_I(x) \leq A_I(0) \vee \frac{1-k_I}{2}$ for all $x \in X$ by the similar way. Now, let $x \in X$ be such that $A_F(x) < A_F(0) \wedge \frac{1-k_F}{2}$. If we take $\gamma_x := A_F(x)$, then $\gamma_x \in [0, \frac{1-k_F}{2})$ and so $0 \in F_\in(A;\gamma_x)$ since $F_\in(A;\gamma_x)$ is an ideal of X. Hence $A_F(0) \leq \gamma_x = A_F(x)$, which is a contradiction. Hence $A_F(x) \geq A_F(0) \wedge \frac{1-k_F}{2}$ for all $x \in X$. Suppose that $A_I(x) \vee \frac{1-k_I}{2} < A_I(x*y) \wedge A_I(y)$ for some $x, y \in X$ and take $\beta := A_I(x*y) \wedge A_I(y)$. Then $\beta \in (\frac{1-k_I}{2}, 1]$ and $x*y, y \in I_\in(A;\beta)$. But $x \notin I_\in(A;\beta)$ which is a contradiction. Thus $A_I(x) \vee \frac{1-k_I}{2} \geq A_I(x*y) \wedge A_I(y)$ for all $x, y \in X$. Similarly, we have $A_T(x) \vee \frac{1-k_T}{2} \geq A_T(x*y) \wedge A_T(y)$ for all $x, y \in X$. Suppose that there exist $x, y \in X$ such that $A_F(x) \wedge \frac{1-k_F}{2} > A_F(x*y) \vee A_F(y)$. Taking $\gamma := A_F(x*y) \vee A_F(y)$ implies that $\gamma \in [0, \frac{1-k_F}{2})$, $x*y \in F_\in(A;\gamma)$ and $y \in F_\in(A;\gamma)$, but $x \notin F_\in(A;\gamma)$. This is a contradiction, and so $A_F(x) \wedge \frac{1-k_F}{2} \leq A_F(x*y) \vee A_F(y)$ for all $x, y \in X$.

Conversely, suppose that $A = (A_T, A_I, A_F)$ satisfies two conditions (3.4) and (3.5). Let $\alpha \in (\frac{1-k_T}{2}, 1]$, $\beta \in (\frac{1-k_I}{2}, 1]$ and $\gamma \in [0, \frac{1-k_F}{2})$ be such that $T_\in(A;\alpha)$, $I_\in(A;\beta)$ and $F_\in(A;\gamma)$ are nonempty. For any $x \in T_\in(A;\alpha)$, $y \in I_\in(A;\beta)$ and $z \in F_\in(A;\gamma)$, we get

$$\begin{aligned}
&A_T(0) \vee \tfrac{1-k_T}{2} \geq A_T(x) \geq \alpha > \tfrac{1-k_T}{2},\\
&A_I(0) \vee \tfrac{1-k_I}{2} \geq A_I(y) \geq \beta > \tfrac{1-k_I}{2},\\
&A_F(0) \wedge \tfrac{1-k_F}{2} \leq A_F(z) \leq \gamma < \tfrac{1-k_F}{2},
\end{aligned}$$

and so $A_T(0) \geq \alpha$, $A_I(0) \geq \beta$ and $A_F(0) \leq \gamma$. Hence $0 \in T_\in(A;\alpha)$, $0 \in I_\in(A;\beta)$ and $0 \in F_\in(A;\gamma)$. Let $a,b,x,y,u,v \in X$ be such that $a*b \in T_\in(A;\alpha)$, $b \in T_\in(A;\alpha)$, $x*y \in I_\in(A;\beta)$, $y \in I_\in(A;\beta)$, $u*v \in F_\in(A;\gamma)$, and $v \in F_\in(A;\gamma)$. It follows from (3.5) that

$$\begin{aligned}
&A_T(a) \vee \tfrac{1-k_T}{2} \geq A_T(a*b) \wedge A_T(b) \geq \alpha > \tfrac{1-k_T}{2},\\
&A_I(x) \vee \tfrac{1-k_I}{2} \geq A_I(x*y) \wedge A_I(y) \geq \beta > \tfrac{1-k_I}{2},\\
&A_F(u) \wedge \tfrac{1-k_F}{2} \leq A_F(u*v) \vee A_F(v) \leq \gamma < \tfrac{1-k_F}{2}.
\end{aligned}$$

Hence $A_T(a) \geq \alpha$, $A_I(x) \geq \beta$ and $A_F(u) \leq \gamma$, that is, $a \in T_\in(A;\alpha)$, $x \in I_\in(A;\beta)$ and $u \in F_\in(A;\gamma)$. Therefore $T_\in(A;\alpha)$, $I_\in(A;\beta)$ and $F_\in(A;\gamma)$ are ideals of X for all $\alpha \in (\frac{1-k_T}{2}, 1]$, $\beta \in (\frac{1-k_I}{2}, 1]$ and $\gamma \in [0, \frac{1-k_F}{2})$. □

Corollary 3.2 ([21]). Given a neutrosophic set $A = (A_T, A_I, A_F)$ in $X \in \mathcal{B}(X)$, the following assertions are equivalent.

(1) The nonempty neutrosophic $\in$-subsets $T_\in(A;\alpha)$, $I_\in(A;\beta)$ and $F_\in(A;\gamma)$ are ideals of X for all $\alpha, \beta \in (0.5, 1]$ and $\gamma \in [0, 0.5)$.

(2) $A = (A_T, A_I, A_F)$ satisfies the following assertion.

$$(\forall x \in X)\left(\begin{array}{l} A_T(x) \leq A_T(0) \vee 0.5 \\ A_I(x) \leq A_I(0) \vee 0.5 \\ A_F(x) \geq A_F(0) \wedge 0.5 \end{array}\right)$$

and

$$(\forall x, y \in X)\left(\begin{array}{l} A_T(x) \vee 0.5 \geq A_T(x*y) \wedge A_T(y) \\ A_I(x) \vee 0.5 \geq A_I(x*y) \wedge A_I(y) \\ A_F(x) \wedge 0.5 \leq A_F(x*y) \vee A_F(y) \end{array}\right)$$

Definition 3.3. A neutrosophic set $A = (A_T, A_I, A_F)$ in $X \in \mathcal{B}(X)$ is called an $(\in, \in \vee\, q_{(k_T,k_I,k_F)})$-*neutrosophic ideal* of X if the following assertions are valid.

$$(\forall x \in X)\left(\begin{array}{l} x \in T_{\in}(A;\alpha_x) \;\Rightarrow\; 0 \in T_{\in\vee\, q_{k_T}}(A;\alpha_x) \\ x \in I_{\in}(A;\beta_x) \;\Rightarrow\; 0 \in I_{\in\vee\, q_{k_I}}(A;\beta_x) \\ x \in F_{\in}(A;\gamma_x) \;\Rightarrow\; 0 \in F_{\in\vee\, q_{k_F}}(A;\gamma_x) \end{array}\right), \tag{3.6}$$

$$(\forall x, y \in X)\left(\begin{array}{l} x*y \in T_{\in}(A;\alpha_x),\ y \in T_{\in}(A;\alpha_y) \Rightarrow x \in T_{\in\vee\, q_{k_T}}(A;\alpha_x \wedge \alpha_y) \\ x*y \in I_{\in}(A;\beta_x),\ y \in I_{\in}(A;\beta_y) \Rightarrow x \in I_{\in\vee\, q_{k_I}}(A;\beta_x \wedge \beta_y) \\ x*y \in F_{\in}(A;\gamma_x),\ y \in F_{\in}(A;\gamma_y) \Rightarrow x \in F_{\in\vee\, q_{k_F}}(A;\gamma_x \vee \gamma_y) \end{array}\right) \tag{3.7}$$

for all $\alpha_x, \alpha_y, \beta_x, \beta_y \in (0,1]$ and $\gamma_x, \gamma_y \in [0,1)$.

Example 3.4. Let $X = \{0,1,2,3,4\}$ be a set with the binary operation $*$ which is given in Table 1.

Table 1. Cayley table for the binary operation "$*$"

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	2	0
3	3	1	3	0	3
4	4	4	4	4	0

Then $(X,*,0)$ is a BCK-algebra (see [15]). Consider a neutrosophic set $A = (A_T, A_I, A_F)$ in X which is given by Table 2.

Table 2. Tabular representation of $A = (A_T, A_I, A_F)$

X	$A_T(x)$	$A_I(x)$	$A_F(x)$
0	0.6	0.5	0.45
1	0.5	0.3	0.93
2	0.3	0.7	0.67
3	0.4	0.3	0.93
4	0.1	0.2	0.74

Routine calculations show that $A = (A_T, A_I, A_F)$ is an $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic ideal of X for $k_T = 0.24$, $k_I = 0.08$ and $k_F = 0.16$.

THEOREM 3.5. *A neutrosophic set $A = (A_T, A_I, A_F)$ in $X \in \mathcal{B}(X)$ is an $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic ideal of $X \in \mathcal{B}(X)$ if and only if $A = (A_T, A_I, A_F)$ satisfies the following assertions.*

$$(\forall x \in X) \begin{pmatrix} A_T(0) \geq A_T(x) \wedge \frac{1-k_T}{2} \\ A_I(0) \geq A_I(x) \wedge \frac{1-k_I}{2} \\ A_F(0) \leq A_F(x) \vee \frac{1-k_F}{2} \end{pmatrix}, \tag{3.8}$$

$$(\forall x, y \in X) \begin{pmatrix} A_T(x) \geq \bigwedge\{A_T(x * y), A_T(y), \frac{1-k_T}{2}\} \\ A_I(x) \geq \bigwedge\{A_I(x * y), A_I(y), \frac{1-k_I}{2}\} \\ A_F(x) \leq \bigvee\{A_F(x * y), A_F(y), \frac{1-k_F}{2}\} \end{pmatrix}. \tag{3.9}$$

PROOF: Assume that $A = (A_T, A_I, A_F)$ in $X \in \mathcal{B}(X)$ is an $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic ideal of $X \in \mathcal{B}(X)$. If $A_T(0) < A_T(a) \wedge \frac{1-k_T}{2}$ for some $a \in X$, then there exists $\alpha_a \in (0, 1]$ such that $A_T(0) < \alpha_a \leq A_T(a) \wedge \frac{1-k_T}{2}$. It follows that $\alpha_a \in (0, \frac{1-k_T}{2}]$, $a \in T_\in(A; \alpha_a)$ and $0 \notin T_\in(A; \alpha_a)$. Also, $A_T(0) + \alpha_a + k_T < 2\alpha_a + k_T \leq 1$, i.e., $0 \notin T_{q_{k_T}}(A; \alpha_a)$. Hence $0 \notin T_{\in \vee\, q_{k_T}}(A; \alpha_a)$, a contradiction. Thus $A_T(0) \geq A_T(x) \wedge \frac{1-k_T}{2}$ for all $x \in X$. Similarly, we have $A_I(0) \geq A_I(x) \wedge \frac{1-k_I}{2}$ for all $x \in X$. Suppose that $A_F(0) > A_F(z) \vee \frac{1-k_F}{2}$ for some $z \in X$ and take $\gamma_z := A_F(z) \vee \frac{1-k_F}{2}$. Then $\gamma_z \geq \frac{1-k_F}{2}$, $z \in F_\in(A; \gamma_z)$ and $0 \notin F_\in(A; \gamma_z)$. Also $A_F(0) + \gamma_z + k_F \geq 1$, that is, $0 \notin F_{q_{k_F}}(A; \gamma_z)$. This is a contradiction, and thus $A_F(0) \leq A_F(x) \vee \frac{1-k_F}{2}$ for all $x \in X$. Suppose that $A_I(a) < \bigwedge\{A_I(a*b), A_I(b), \frac{1-k_I}{2}\}$ for some $a, b \in X$ and take $\beta := \bigwedge\{A_I(a * b), A_I(b), \frac{1-k_I}{2}\}$. Then $\beta \leq \frac{1-k_I}{2}$, $a * b \in I_\in(A; \beta)$, $b \in I_\in(A; \beta)$ and $a \notin I_\in(A; \beta)$. Also, we have $A_I(a) + \beta + k_I \leq 1$, i.e., $a \notin I_{q_{k_F}}(A; \beta)$. This is impossible, and therefore $A_I(x) \geq \bigwedge\{A_I(x*y), A_I(y), \frac{1-k_I}{2}\}$ for all $x, y \in X$. By the similar way, we can verify that $A_T(x) \geq \bigwedge\{A_T(x * y), A_T(y), \frac{1-k_T}{2}\}$ for all $x, y \in X$. Now assume that $A_F(a) > \bigvee\{A_F(a * b), A_F(b), \frac{1-k_F}{2}\}$ for some $a, b \in X$. Then there exists $\gamma \in [0, 1)$ such that $A_F(a) > \gamma \geq \bigvee\{A_F(a * b), A_F(b), \frac{1-k_F}{2}\}$. Then $\gamma \geq \frac{1-k_F}{2}$, $a * b \in F_\in(A; \gamma)$, $b \in F_\in(A; \gamma)$ and $a \notin F_\in(A; \gamma)$. Also, $A_F(a)+\gamma+k_F \geq 1$, i.e., $a \notin F_{q_{k_F}}(A; \gamma)$. Thus $a \notin F_{\in \vee\, q_{k_F}}(A; \gamma)$, which is a contradiction. Hence $A_F(x) \leq \bigvee\{A_F(x*y), A_F(y), \frac{1-k_F}{2}\}$ for all $x, y \in X$.

Conversely, suppose that $A = (A_T, A_I, A_F)$ satisfies two conditions (3.8) and (3.9). For any $x, y, z \in X$, let $\alpha_x, \beta_y \in (0,1]$ and $\gamma_z \in [0,1)$ be such that $x \in T_{\in}(A;\alpha_x)$, $y \in I_{\in}(A;\beta_y)$ and $z \in F_{\in}(A;\gamma_z)$. Then $A_T(x) \geq \alpha_x$, $A_I(y) \geq \beta_y$ and $A_F(z) \leq \gamma_z$. Assume that $A_T(0) < \alpha_x$, $A_I(0) < \beta_y$ and $A_F(0) > \gamma_z$. If $A_T(x) < \frac{1-k_T}{2}$, then

$$A_T(0) \geq A_T(x) \wedge \tfrac{1-k_T}{2} = A_T(x) \geq \alpha_x,$$

a contradiction. Hence $A_T(x) \geq \frac{1-k_T}{2}$, and so

$$A_T(0) + \alpha_x + k_T > 2A_T(0) + k_T \geq 2\left(A_T(x) \wedge \tfrac{1-k_T}{2}\right) + k_T = 1.$$

Hence $0 \in T_{q_{k_T}}(A;\alpha_x) \subseteq T_{\in\vee\, q_{k_T}}(A;\alpha_x)$. Similarly, we get $0 \in I_{q_{k_I}}(A;\beta_y) \subseteq I_{\in\vee\, q_{k_I}}(A;\beta_y)$. If $A_F(z) > \frac{1-k_F}{2}$, then $A_F(0) \leq A_F(z) \vee \frac{1-k_F}{2} = A_F(z) \leq \gamma_z$ which is a contradiction. Hence $A_F(z) \leq \frac{1-k_F}{2}$, and thus

$$A_F(0) + \gamma_z + k_F < 2A_F(0) + k_F \leq 2\left(A_F(z) \vee \tfrac{1-k_F}{2}\right) + k_F = 1.$$

Hence $0 \in F_{q_{k_F}}(A;\gamma_z) \subseteq F_{\in\vee\, q_{k_F}}(A;\gamma_z)$. For any $a, b, p, q, x, y \in X$, let $\alpha_a, \alpha_b, \beta_p, \beta_q \in (0,1]$ and $\gamma_x, \gamma_y \in [0,1)$ be such that $a * b \in T_{\in}(A;\alpha_a)$, $b \in T_{\in}(A;\alpha_b)$, $p * q \in I_{\in}(A;\beta_p)$, $q \in I_{\in}(A;\beta_q)$, $x * y \in F_{\in}(A;\gamma_x)$, and $y \in F_{\in}(A;\gamma_y)$. Then $A_T(a*b) \geq \alpha_a$, $A_T(b) \geq \alpha_b$, $A_I(p*q) \geq \beta_p$, $A_I(q) \geq \beta_q$, $A_F(x*y) \leq \gamma_x$, and $A_F(y) \leq \gamma_y$. Suppose that $a \notin T_{\in}(A;\alpha_a \wedge \alpha_b)$. Then $A_T(a) < \alpha_a \wedge \alpha_b$. If $A_T(a*b) \wedge A_T(b) < \frac{1-k_T}{2}$, then

$$A_T(a) \geq \bigwedge\{A_T(a*b), A_T(b), \tfrac{1-k_T}{2}\} = A_T(a*b) \wedge A_T(b) \geq \alpha_a \wedge \alpha_b.$$

This is a contradiction, and so $A_T(a*b) \wedge A_T(b) \geq \frac{1-k_T}{2}$. Thus

$$\begin{aligned} A_T(a) + (\alpha_a \wedge \alpha_b) + k_T &> 2A_T(a) + k_T \\ &\geq 2\left(\bigwedge\{A_T(a*b), A_T(b), \tfrac{1-k_T}{2}\}\right) + k_T = 1, \end{aligned}$$

which induces $a \in T_{q_{k_T}}(A;\alpha_a \wedge \alpha_b) \subseteq T_{\in\vee\, q_{k_T}}(A;\alpha_a \wedge \alpha_b)$. By the similarly way, we get $p \in I_{\in\vee\, q_{k_I}}(A;\beta_p \wedge \beta_q)$. Suppose that $x \notin F_{\in}(A;\gamma_x \vee \gamma_y)$, that is, $A_F(x) > \gamma_x \vee \gamma_y$. If $A_F(x*y) \vee A_F(y) > \frac{1-k_F}{2}$, then

$$A_F(x) \leq \bigvee\{A_F(x*y), A_F(y), \tfrac{1-k_F}{2}\} = A_F(x*y) \vee A_F(y) \leq \gamma_x \vee \gamma_y,$$

which is impossible. Thus $A_F(x*y) \vee A_F(y) \leq \frac{1-k_F}{2}$, and so

$$\begin{aligned} A_F(x) + (\gamma_x \vee \gamma_y) + k_F &< 2A_F(x) \\ &\leq 2\left(\bigvee\{A_F(x*y), A_F(y), \tfrac{1-k_F}{2}\}\right) + k_F = 1. \end{aligned}$$

This implies that $x \in F_{q_{k_F}}(A; \gamma_x \vee \gamma_y) \subseteq F_{\in\vee\, q_{k_F}}(A; \gamma_x \vee \gamma_y)$. Consequently, $A = (A_T, A_I, A_F)$ is an $(\in, \in \vee\, q_{(k_T,k_I,k_F)})$-neutrosophic ideal of $X \in \mathcal{B}(X)$. □

Corollary 3.6 ([21]). For a neutrosophic set $A = (A_T, A_I, A_F)$ in $X \in \mathcal{B}(X)$, the following are equivalent.

(1) $A = (A_T, A_I, A_F)$ is an $(\in, \in \vee\, q)$-neutrosophic ideal of $X \in \mathcal{B}(X)$.

(2) $A = (A_T, A_I, A_F)$ satisfies the following assertions.

$$(\forall x \in X)\begin{pmatrix} A_T(0) \geq A_T(x) \wedge 0.5 \\ A_I(0) \geq A_I(x) \wedge 0.5 \\ A_F(0) \leq A_F(x) \vee 0.5 \end{pmatrix},$$

$$(\forall x, y \in X)\begin{pmatrix} A_T(x) \geq \bigwedge\{A_T(x*y), A_T(y), 0.5\} \\ A_I(x) \geq \bigwedge\{A_I(x*y), A_I(y), 0.5\} \\ A_F(x) \leq \bigvee\{A_F(x*y), A_F(y), 0.5\} \end{pmatrix}.$$

Theorem 3.7. *A neutrosophic set $A = (A_T, A_I, A_F)$ in $X \in \mathcal{B}(X)$ is an $(\in, \in \vee\, q_{(k_T,k_I,k_F)})$-neutrosophic ideal of $X \in \mathcal{B}(X)$ if and only if the nonempty neutrosophic $\in$-subsets $T_\in(A;\alpha)$, $I_\in(A;\beta)$ and $F_\in(A;\gamma)$ are ideals of X for all $\alpha \in (0, \frac{1-k_T}{2}]$, $\beta \in (0, \frac{1-k_I}{2}]$ and $\gamma \in [\frac{1-k_F}{2}, 1)$.*

Proof: Suppose that $A = (A_T, A_I, A_F)$ is an $(\in, \in \vee\, q_{(k_T,k_I,k_F)})$-neutrosophic ideal of $X \in \mathcal{B}(X)$ and let $\alpha \in (0, \frac{1-k_T}{2}]$, $\beta \in (0, \frac{1-k_I}{2}]$ and $\gamma \in [\frac{1-k_F}{2}, 1)$ be such that $T_\in(A;\alpha)$, $I_\in(A;\beta)$ and $F_\in(A;\gamma)$ are nonempty. Using (3.8), we get $A_T(0) \geq A_T(x) \wedge \frac{1-k_T}{2}$, $A_I(0) \geq A_I(y) \wedge \frac{1-k_I}{2}$, and $A_F(0) \leq A_F(z) \vee \frac{1-k_F}{2}$ for all $x \in T_\in(A;\alpha)$, $y \in I_\in(A;\beta)$ and $z \in F_\in(A;\gamma)$. It follows that $A_T(0) \geq \alpha \wedge \frac{1-k_T}{2} = \alpha$, $A_I(0) \geq \beta \wedge \frac{1-k_I}{2} = \beta$, and $A_F(0) \leq \gamma \vee \frac{1-k_F}{2} = \gamma$, that is, $0 \in T_\in(A;\alpha)$, $0 \in I_\in(A;\beta)$ and $0 \in F_\in(A;\gamma)$. Let $x, y, a, b, u, v \in X$ be such that $x*y \in T_\in(A;\alpha)$, $y \in T_\in(A;\alpha)$, $a*b \in I_\in(A;\beta)$, $b \in I_\in(A;\beta)$, $u*v \in F_\in(A;\gamma)$, and $v \in F_\in(A;\gamma)$ for $\alpha \in (0, \frac{1-k_T}{2}]$, $\beta \in (0, \frac{1-k_I}{2}]$ and $\gamma \in [\frac{1-k_F}{2}, 1)$. Then $A_T(x*y) \geq \alpha$, $A_T(y) \geq \alpha$, $A_I(a*b) \geq \beta$, $A_I(b) \geq \beta$, $A_F(u*v) \leq \gamma$, and $A_F(v) \leq \gamma$. It follows from (3.9) that

$A_T(x) \geq \bigwedge\{A_T(x * y), A_T(y), \frac{1-k_T}{2}\} \geq \alpha \wedge \frac{1-k_T}{2} = \alpha,$

$A_I(a) \geq \bigwedge\{A_I(a * b), A_I(b), \frac{1-k_I}{2}\} \geq \beta \wedge \frac{1-k_I}{2} = \beta,$

$A_F(u) \leq \bigvee\{A_F(u * v), A_F(v), \frac{1-k_F}{2}\} \leq \gamma \vee \frac{1-k_F}{2} = \gamma$

and so that $x \in T_\in(A;\alpha)$, $a \in I_\in(A;\beta)$ and $u \in F_\in(A;\gamma)$. Therefore $T_\in(A;\alpha)$, $I_\in(A;\beta)$ and $F_\in(A;\gamma)$ are ideals of X for all $\alpha \in (0, \frac{1-k_T}{2}]$, $\beta \in (0, \frac{1-k_I}{2}]$ and $\gamma \in [\frac{1-k_F}{2}, 1)$.

Conversely, let $A = (A_T, A_I, A_F)$ be a neutrosophic set in $X \in \mathcal{B}(X)$ such that the nonempty neutrosophic $\in$-subsets $T_\in(A;\alpha)$, $I_\in(A;\beta)$ and $F_\in(A;\gamma)$ are ideals of X for all $\alpha \in (0, \frac{1-k_T}{2}]$, $\beta \in (0, \frac{1-k_I}{2}]$ and $\gamma \in [\frac{1-k_F}{2}, 1)$. If there exist $x, y, z \in X$ such that $A_T(0) < A_T(x) \wedge \frac{1-k_T}{2}$, $A_I(0) < A_I(y) \wedge \frac{1-k_I}{2}$, and $A_F(0) > A_F(z) \vee \frac{1-k_F}{2}$, then $0 \notin T_\in(A;\alpha_x)$, $0 \notin I_\in(A;\beta_y)$ and $0 \notin F_\in(A;\gamma_z)$ by taking $\alpha_x := A_T(x) \wedge \frac{1-k_T}{2}$, $\beta_y := A_I(y) \wedge \frac{1-k_I}{2}$, and $\gamma_z := A_F(z) \vee \frac{1-k_F}{2}$. This is a contradiction, and so $A_T(0) \geq A_T(x) \wedge \frac{1-k_T}{2}$, $A_I(0) \geq A_I(x) \wedge \frac{1-k_I}{2}$, and $A_F(0) \leq A_F(x) \vee \frac{1-k_F}{2}$ for all $x \in X$. Now, suppose that there $x, y, a, b, u, v \in X$ be such that $A_T(x) < \bigwedge\{A_T(x * y), A_T(y), \frac{1-k_T}{2}\}$, $A_I(a) < \bigwedge\{A_I(a * b), A_I(b), \frac{1-k_I}{2}\}$, and $A_F(u) > \bigvee\{A_F(u * v), A_F(v), \frac{1-k_F}{2}\}$. If we take $\alpha := \bigwedge\{A_T(x * y), A_T(y), \frac{1-k_T}{2}\}$, $\beta := \bigwedge\{A_I(a * b), A_I(b), \frac{1-k_I}{2}\}$, and $\gamma := \bigvee\{A_F(u * v), A_F(v), \frac{1-k_F}{2}\}$, then $\alpha \leq \frac{1-k_T}{2}$, $\beta \leq \frac{1-k_I}{2}$, $\gamma \geq \frac{1-k_F}{2}$, $x * y \in T_\in(A;\alpha)$, $y \in T_\in(A;\alpha)$, $a * b \in I_\in(A;\beta)$, $b \in I_\in(A;\beta)$, $u * v \in F_\in(A;\gamma)$, and $v \in F_\in(A;\gamma)$. But $x \notin T_\in(A;\alpha)$, $a \notin I_\in(A;\beta)$ and $u \notin F_\in(A;\gamma)$, which induces a contradiction. Therefore $A_T(x) \geq \bigwedge\{A_T(x * y), A_T(y), \frac{1-k_T}{2}\}$, $A_I(x) \geq \bigwedge\{A_I(x * y), A_I(y), \frac{1-k_I}{2}\}$, and $A_F(x) \leq \bigvee\{A_F(x * y), A_F(y), \frac{1-k_F}{2}\}$ for all $x, y \in X$. Using Theorem 3.5, we conclude that $A = (A_T, A_I, A_F)$ is an $(\in, \in \vee\, q_{(k_T,k_I,k_F)})$-neutrosophic ideal of $X \in \mathcal{B}(X)$. □

Corollary 3.8 ([21]). A neutrosophic set $A = (A_T, A_I, A_F)$ in $X \in \mathcal{B}(X)$ is an $(\in, \in \vee\, q)$-neutrosophic ideal of $X \in \mathcal{B}(X)$ if and only if the nonempty neutrosophic $\in$-subsets $T_\in(A;\alpha)$, $I_\in(A;\beta)$ and $F_\in(A;\gamma)$ are ideals of X for all $\alpha, \beta \in (0, 0.5]$ and $\gamma \in [0.5, 1)$.

It is clear that every $(\in, \in)$-neutrosophic ideal is an $(\in, \in \vee\, q_{(k_T,k_I,k_F)})$-neutrosophic ideal. But the converse is not true in general. For example, the $(\in, \in \vee\, q_{(k_T,k_I,k_F)})$-neutrosophic ideal $A = (A_T, A_I, A_F)$ with $k_T = 0.24$, $k_I = 0.08$ and $k_F = 0.16$ in Example 3.4 is not an $(\in, \in)$-neutrosophic ideal since $2 \in I_\in(A; 0.56)$ and $0 \notin I_\in(A; 0.56)$.

We now consider conditions for an $(\in, \in\vee q_{(k_T,k_I,k_F)})$-neutrosophic ideal to be an $(\in,\in)$-neutrosophic ideal.

THEOREM 3.9. *Let $A = (A_T, A_I, A_F)$ be an $(\in, \in\vee q_{(k_T,k_I,k_F)})$-neutrosophic ideal of $X \in \mathcal{B}(X)$ such that*

$$(\forall x \in X)\left(A_T(x) < \tfrac{1-k_T}{2},\ A_I(x) < \tfrac{1-k_I}{2},\ A_F(x) > \tfrac{1-k_F}{2}\right).$$

Then $A = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic ideal of $X \in \mathcal{B}(X)$.

PROOF: Let $x, y, z \in X$, $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$ be such that $x \in T_\in(A;\alpha)$, $y \in I_\in(A;\beta)$ and $z \in F_\in(A;\gamma)$. Then $A_T(x) \geq \alpha$, $A_I(y) \geq \beta$ and $A_F(z) \leq \gamma$. It follows from (3.8) that

$$A_T(0) \geq A_T(x) \wedge \tfrac{1-k_T}{2} = A_T(x) \geq \alpha,$$

$$A_I(0) \geq A_I(y) \wedge \tfrac{1-k_I}{2} = A_I(y) \geq \beta,$$

$$A_F(0) \leq A_F(z) \vee \tfrac{1-k_F}{2} = A_F(z) \leq \gamma.$$

Hence $0 \in T_\in(A;\alpha)$, $0 \in I_\in(A;\beta)$ and $0 \in F_\in(A;\gamma)$. For any $x, y, a, b, u, v \in X$, let α_x, α_y, β_a, $\beta_b \in (0, 1]$ and γ_u, $\gamma_v \in [0, 1)$ be such that $x * y \in T_\in(A;\alpha_x)$, $y \in T_\in(A;\alpha_y)$, $a*b \in I_\in(A;\beta_a)$, $b \in I_\in(A;\beta_b)$, $u*v \in F_\in(A;\gamma_u)$, and $v \in F_\in(A;\gamma_v)$. Then $A_T(x * y) \geq \alpha_x$, $A_T(y) \geq \alpha_y$, $A_I(a * b) \geq \beta_a$, $A_I(b) \geq \beta_b$, $A_F(u * v) \leq \gamma_u$, and $A_F(v) \leq \gamma_v$. It follows from (3.9) that

$$A_T(x) \geq \bigwedge\{A_T(x * y), A_T(y), \tfrac{1-k_T}{2}\} = A_T(x * y) \wedge A_T(y) \geq \alpha_x \wedge \alpha_y,$$

$$A_I(a) \geq \bigwedge\{A_I(a * b), A_I(b), \tfrac{1-k_I}{2}\} = A_I(a * b) \wedge A_I(b) \geq \beta_a \wedge \beta_b,$$

$$A_F(u) \leq \bigvee\{A_F(u * v), A_F(v), \tfrac{1-k_F}{2}\} = A_F(u * v) \vee A_F(v) \leq \gamma_u \vee \gamma_v.$$

Thus $x \in T_\in(A;\alpha_x \wedge \alpha_y)$, $a \in I_\in(A;\beta_a \wedge \beta_b)$ and $u \in F_\in(A;\gamma_u \vee \gamma_v)$. Therefore $A = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic ideal of $X \in \mathcal{B}(X)$. □

COROLLARY 3.10 ([21]). Let $A = (A_T, A_I, A_F)$ be an $(\in, \in\vee q)$-neutrosophic ideal of $X \in \mathcal{B}(X)$ such that

$$(\forall x \in X)\,(A_T(x) < 0.5,\ A_I(x) < 0.5,\ A_F(x) > 0.5)\,.$$

Then $A = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic ideal of $X \in \mathcal{B}(X)$.

THEOREM 3.11. *Given a neutrosophic set $A = (A_T, A_I, A_F)$ in $X \in \mathcal{B}(X)$, if the nonempty neutrosophic $\in\vee q_k$-subsets $T_{\in\vee q_{k_T}}(A;\alpha)$, $I_{\in\vee q_{k_I}}(A;\beta)$*

and $F_{\in\vee\, q_{k_F}}(A;\gamma)$ are ideals of X for all $\alpha \in (0, \frac{1-k_T}{2}]$, $\beta \in (0, \frac{1-k_I}{2}]$ and $\gamma \in [\frac{1-k_F}{2}, 1)$, then $A = (A_T, A_I, A_F)$ is an $(\in, \in \vee\, q_{(k_T,k_I,k_F)})$-neutrosophic ideal of X.

Proof: Let $A = (A_T, A_I, A_F)$ be a neutrosophic set in $X \in \mathcal{B}(X)$ such that the nonempty neutrosophic $\in \vee\, q_k$-subsets $T_{\in\vee\, q_{k_T}}(A;\alpha)$, $I_{\in\vee\, q_{k_I}}(A;\beta)$ and $F_{\in\vee\, q_{k_F}}(A;\gamma)$ are ideals of X for all $\alpha \in (0, \frac{1-k_T}{2}]$, $\beta \in (0, \frac{1-k_I}{2}]$ and $\gamma \in [\frac{1-k_F}{2}, 1)$. If $A_T(0) < A_T(x) \wedge \frac{1-k_T}{2} := \alpha_x$, $A_I(0) < A_I(y) \wedge \frac{1-k_I}{2} := \beta_y$ and $A_F(0) > A_F(z) \vee \frac{1-k_F}{2} := \gamma_z$ for some $x, y, z \in X$, then $x \in T_\in(A;\alpha_x) \subseteq T_{\in\vee\, q_{k_T}}(A;\alpha_x)$, $y \in I_\in(A;\beta_y) \subseteq I_{\in\vee\, q_{k_I}}(A;\beta_y)$, $z \in F_\in(A;\gamma_z) \subseteq F_{\in\vee\, q_{k_F}}(A;\gamma_z)$, $0 \notin T_\in(A;\alpha_x)$, $0 \notin I_\in(A;\beta_y)$, and $0 \notin F_\in(A;\gamma_z)$. Also, since $A_T(0) + \alpha_x + k_T < 2\alpha_x + k_T \leq 1$, i.e., $0 \notin T_{q_{k_T}}(A;\alpha_x)$, $A_I(0) + \beta_y + k_I < 2\beta_y + k_I \leq 1$, i.e., $0 \notin I_{q_{k_I}}(A;\beta_Y)$, $A_F(0) + \gamma_z + k_F > 2\gamma_z + k_F \geq 1$, i.e., $0 \notin F_{q_{k_F}}(A;\gamma_z)$, we get $0 \notin T_{\in\vee\, q_{k_T}}(A;\alpha_x)$, $0 \notin I_{\in\vee\, q_{k_I}}(A;\beta_y)$, and $0 \notin F_{\in\vee\, q_{k_F}}(A;\gamma_z)$. This is a contradiction, and thus (3.8) is valid. Suppose that there exist $a, b \in X$ such that $A_I(a) < \bigwedge\{A_I(a*b), A_I(b), \frac{1-k_I}{2}\}$. Taking $\beta := \bigwedge\{A_I(a*b), A_I(b), \frac{1-k_I}{2}\}$ implies that $a*b \in I_\in(A;\beta) \subseteq I_{\in\vee\, q_{k_I}}(A;\beta)$, $b \in I_\in(A;\beta) \subseteq I_{\in\vee\, q_{k_I}}(A;\beta)$. Since $I_{\in\vee\, q_{k_I}}(A;\beta)$ is an ideal of X, it follows that $a \in I_{\in\vee\, q_{k_I}}(A;\beta)$, i.e., $a \in I_\in(A;\beta)$ or $a \in I_{q_{k_I}}(A;\beta)$, and so that $a \in I_{q_{k_I}}(A;\beta)$, i.e., $A_I(a) + \beta + k_I > 1$, since $a \notin I_\in(A;\beta)$. But $A_I(a) + \beta + k_I < 2\beta + k_I \leq 1$, a contradiction. Hence $A_I(x) \geq \bigwedge\{A_I(x*y), A_I(y), \frac{1-k_I}{2}\}$ for all $x, y \in X$. Similarly, we can verify that $A_T(x) \geq \bigwedge\{A_T(x*y), A_T(y), \frac{1-k_T}{2}\}$ for all $x, y \in X$. Assume that $A_F(a) > \bigvee\{A_F(a*b), A_F(b), \frac{1-k_F}{2}\} := \gamma$ for some $a, b \in X$. Then $a \notin F_\in(A;\gamma)$, $a*b \in F_\in(A;\gamma) \subseteq F_{\in\vee\, q_{k_F}}(A;\gamma)$, $b \in F_\in(A;\gamma) \subseteq F_{\in\vee\, q_{k_F}}(A;\gamma)$. Since $F_{\in\vee\, q_{k_F}}(A;\gamma)$ is an ideal of X, we have $a \in F_{\in\vee\, q_{k_F}}(A;\gamma)$. On the other hand, $A_F(a) + \gamma + k_F > 2\gamma + k_F \geq 1$, that is, $a \notin F_{q_{k_F}}(A;\gamma)$. Hence $a \notin F_{\in\vee\, q_{k_F}}(A;\gamma)$, a contradiction. Thus $A_F(x) \leq \bigvee\{A_F(x*y), A_F(y), \frac{1-k_F}{2}\}$ for all $x, y \in X$. Therefore (3.9) is valid, and consequently $A = (A_T, A_I, A_F)$ is an $(\in, \in \vee\, q_{(k_T,k_I,k_F)})$-neutrosophic ideal of X by Theorem 3.5. □

Corollary 3.12 ([21]). Given a neutrosophic set $A = (A_T, A_I, A_F)$ in $X \in \mathcal{B}(X)$, if the nonempty neutrosophic $\in \vee\, q$-subsets $T_{\in\vee\, q}(A;\alpha)$, $I_{\in\vee\, q}(A;\beta)$ and $F_{\in\vee\, q}(A;\gamma)$ are ideals of X for all $\alpha, \beta \in (0, 0.5]$ and $\gamma \in [0.5, 1)$, then $A = (A_T, A_I, A_F)$ is an $(\in, \in \vee\, q)$-neutrosophic ideal of X.

4. Conclusions

More general form of $(\in, \in \vee q)$-neutrosophic ideal was introduced, and their properties were investigated. Relations between $(\in, \in)$-neutrosophic ideal and $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic ideal were discussed. Characterizations of $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic ideal were discussed, and conditions for a neutrosophic set to be an $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic ideal were displayed.

These results can be applied to characterize the neutrosophic ideals in a *BCK*/*BCI*-algebra. In our future research, we will focus on some properties of ideal such as intersections, unions, maximality, primeness and height, and try to find the relations between these properties of ideals and the results of this paper. For instance, how we can define the prime and maximal neutrosophic ideals? Whatis the meaning of height of these types of ideals? For information about the maximality, primeness and height of ideals, please refer to [1, 2, 6, 5].

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