Bulletin of the **S**ection of **L**ogic
Volume 50/2 (2021), pp. 177–204
https://doi.org/10.18778/0138-0680.2021.06

Mateusz Klonowski
Krzysztof Aleksander Krawczyk
Bożena Pięta

TABLEAU SYSTEMS FOR EPISTEMIC POSITIONAL LOGICS

Abstract

The goal of the article is twofold. The first one is to provide logics based on positional semantics which will be suitable for the analysis of epistemic modalities such as 'agent ... knows/beliefs that ...'. The second one is to define tableau systems for such logics. Firstly, we present the minimal positional logic MR. Then, we change the notion of formulas and semantics in order to consider iterations of the operator of realization and "free" classical formulas. After that, we move on to weaker logics in order to avoid the well known problem of logical omniscience. At the same time, we keep the positional counterparts of modal axioms (T), (4) and (5). For all of the considered logics we present sound and complete tableau systems.

Keywords: Epistemic logic, logical omniscience, positional logic, tableau system.

1. Introduction

Sentences like 'It is raining' can realize in certain points. Those points can be treated:

- temporally as moments or certain intervals: 'It is raining now', 'It has been raining since Monday',

Presented by: Tomasz Jarmużek, Fengkui Ju, Piotr Kulicki, Beishui Liao
Received: February 5, 2020
Published online: April 1, 2021

- spatially as points or certain parts of space: 'It is raining in Toruń',
- epistemically as (ir)rational agents: 'John knows that it is raining'.

Clearly, the amount of possible interpretations is much richer including alethic, deontic, etc. The goal of introducing positional logics is to enable one the expression of such relativized sentences. The difference between positional and modal logic, which in a sense is also about such relativization, is that the first one introduces points in the object language while the latter treats them implicitly as only semantic entities that are talked about in the metalanguage. On the other hand, the difference between hybrid and positional logics is that the points (worlds) can be treated as independent expressions in hybrid logic, whilst—in the case of positional logics—they can only be used to form more complex formulas.

The origin of positional logic is mainly associated with the emergence of temporal logic. The founder of positional logic, and at the same time of temporal logic, was Jerzy Łoś. His aim was to provide a logical tool for formalizing empirical sentences such as 'it is sunny in Warsaw on 26th July 2019'. The expression 'at ... it is the case that ... ' which Łoś analyzed may be called the connective of temporal realization. Nonetheless, the temporal interpretation of realization is not the only possible one.

In [11] Łoś used the realization operator, i.e. the sentence-forming connective from naming and sentential arguments, to express epistemic modality, while in [10] the temporal interpretation of such operator is considered. The letter used by Łoś for the realization operator in his investigations was U, but due to Rescher [15] it shall be denoted as $\mathcal{R}$ (cf. [17], [16]). Generally, the formula $\mathcal{R}_\alpha A$ can be read in the following manner: A is realised/realizes in α. In temporal understanding such formula would be read as: A takes place/happens in moment α. In epistemic context $\mathcal{R}_\alpha A$ means: agent α knows that A.

The work of Łoś was continued by Jarmużek and Pietruszczak in [6] where the minimal system of positional logic MR was introduced (cf. [1]). Logic MR is the minimal logic among positional logics that are closed under the law of distribution of $\mathcal{R}$ over standard connectives. In [9] Karczewska proved that MR is the maximal logic with respect to so-called single-index rules. Weaker logics than MR, for which the problem of distributivity of $\mathcal{R}$ is discussed, were considered by Tkaczyk in [20], [18], [19]. In [3, pp. 209–224] the discussion of the applications of the $\mathcal{R}$ operator for the analysis of the Master Argument was presented. In [8] an attempt to reduce unary modalities to $\mathcal{R}$-structures was considered. In [7] Jarmużek and Tkaczyk collected the main results and introduced some new ideas concerning positional logic. In fact, they considered normal logics, i.e. logics such that their connectives have the same standard meaning inside and outside the scope of the $\mathcal{R}$ operator.

In [12] the possible application of positional logic for the analysis of the reasoning concerning the social phenomena was presented. In this case the $\mathcal{R}$ operator was modified by replacing the individual constant by a tuple of individual constants. In [5] one can find investigations concerning the extended version of positional logic's language obtained by adding the predicate symbols.

In this paper we start with the presentation of logic MR. After that, we define logic MR^+ in which we can consider the iterations of $\mathcal{R}$. Such a system is the basis for the epistemic interpretation of positional operator. We consider three systems of epistemic positional logic. The first one is the minimal one that contains the counterpart of (T). Then we consider its extensions that contain the counterparts of schemata (4) and (5) respectively. In none of the epistemic logics the logical omniscience problem appears. For all of the considered logics we define sound and complete tableau systems.

2. Logic MR

2.1. Language and semantics of logic MR

The language of MR consists of propositional variables $p_0, p_1, p_2, \ldots$ (we will use letters p, q, r); standard connectives $\neg, \wedge, \vee, \to, \leftrightarrow$; the operator of realization $\mathcal{R}$; individual constants $a_1, a_2, a_3, \ldots$ (we will use letters a, b, c) and parentheses), (. Let VAR (resp. IC) be the set of propositional variables (resp. individual constants). By F we denote the set of formulas of Classical Propositional Logic (for short: CPL) defined in the standard way. The *set of* MR *formulas*, i.e. the set For, is the smallest set X meeting the following conditions:

- if $A \in \mathsf{F}$ than $\mathcal{R}_\alpha A \in X$, where $\alpha \in \mathsf{IC}$,
- if $A \in X$ than $\neg A \in X$,
- if $A, B \in X$ than $(A * B) \in X$, where $* \in \{\wedge, \vee, \to, \leftrightarrow\}$.

As we can see, in the language of MR there are no iterations of $\mathcal{R}$ operator and no "free" CPL formulas outside the scope of $\mathcal{R}$ operator.

By the *complexity of a formula* A we mean number $c(A)$, where $c\colon \mathsf{For} \longrightarrow \mathbb{N}$ is a function such that: $c(A) = 1$, if $A = \mathcal{R}_\alpha B$; $c(A) = c(B)+1$, if $A = \neg B$; $c(A) = c(B) + c(C) + 1$, if $A = B * C$, where $* \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$. Note that the complexity of the $\mathcal{R}_\alpha A$ formula equals 1, regardless of the formula A. In the proofs we present below, we will also use induction on the complexity of CPL formulas defined similarly to complexity of MR formulas as $o(A)$, where $o\colon \mathsf{F} \longrightarrow \mathbb{N}$ is a function defined as c except instead of $A = \mathcal{R}_\alpha B$ we have $A \in \mathsf{VAR}$ and we put 1.

A *model of* MR (a MR-*model*) is a triple $\langle W, f, v\rangle$ such that:

- W is the non-empty set,
- $f\colon \mathsf{IC} \longrightarrow W$ is the denotation function,
- $v\colon W \times \mathsf{F} \longrightarrow \{0,1\}$ is a valuation such that for any $w \in W$, for any $A, B \in \mathsf{F}$:

$$v(\langle w, \neg A\rangle) = 1 \quad \text{iff} \quad v(\langle w, A\rangle) = 0 \tag{v$_1$}$$
$$v(\langle w, A \wedge B\rangle) = 1 \quad \text{iff} \quad v(\langle w, A\rangle) = v(\langle w, B\rangle) = 1 \tag{v$_2$}$$
$$v(\langle w, A \vee B\rangle) = 1 \quad \text{iff} \quad v(\langle w, A\rangle) = 1 \text{ or } v(\langle w, B\rangle) = 1 \tag{v$_3$}$$
$$v(\langle w, A \rightarrow B\rangle) = 1 \quad \text{iff} \quad v(\langle w, A\rangle) = 0 \text{ or } v(\langle w, B\rangle) = 1 \tag{v$_4$}$$
$$v(\langle w, A \leftrightarrow B\rangle) = 1 \quad \text{iff} \quad v(\langle w, A\rangle) = v(\langle w, B\rangle). \tag{v$_5$}$$

We have the following truth-conditions for any $A \in \mathsf{F}$ and any $B, C \in \mathsf{For}$:

$$\mathfrak{M} \models \mathcal{R}_\alpha A \quad \text{iff} \quad v(\langle f(\alpha), A\rangle) = 1 \tag{m$_1$}$$
$$\mathfrak{M} \models \neg B \quad \text{iff} \quad \mathfrak{M} \not\models B \tag{m$_2$}$$
$$\mathfrak{M} \models B \wedge C \quad \text{iff} \quad \mathfrak{M} \models B \text{ and } \mathfrak{M} \models C \tag{m$_3$}$$
$$\mathfrak{M} \models B \vee C \quad \text{iff} \quad \mathfrak{M} \models B \text{ or } \mathfrak{M} \models C \tag{m$_4$}$$
$$\mathfrak{M} \models B \rightarrow C \quad \text{iff} \quad \mathfrak{M} \not\models B \text{ or } \mathfrak{M} \models C \tag{m$_5$}$$
$$\mathfrak{M} \models B \leftrightarrow C \quad \text{iff} \quad \mathfrak{M} \models B \text{ iff } \mathfrak{M} \models C. \tag{m$_6$}$$

The notions of the relation of semantic consequence $\models_{\mathsf{MR}}$ and the validity in MR are defined in the standard way. Logic MR might be identified with the relation $\models_{\mathsf{MR}}$. In the subsequent sections we will similarly define and denote other relations of semantic consequence and identify certain logics with them.

For any $A, B \in \mathsf{For}$:

$$\mathrm{R}_{\wedge}: \frac{A \wedge B}{A, B} \qquad \mathrm{R}_{\vee}: \frac{A \vee B}{A|B} \qquad \mathrm{R}_{\to}: \frac{A \to B}{\neg A|B} \qquad \mathrm{R}_{\leftrightarrow}: \frac{A \leftrightarrow B}{A, B|\neg A, \neg B}$$

$$\mathrm{R}_{\neg\neg}: \frac{\neg\neg A}{A} \qquad \mathrm{R}_{\neg\wedge}: \frac{\neg(A \wedge B)}{\neg A|\neg B} \qquad \mathrm{R}_{\neg\vee}: \frac{\neg(A \vee B)}{\neg A, \neg B}$$

$$\mathrm{R}_{\neg\to}: \frac{\neg(A \to B)}{A, \neg B} \qquad \mathrm{R}_{\neg\leftrightarrow}: \frac{\neg(A \leftrightarrow B)}{A, \neg B|\neg A, B}$$

Figure 1. Elimination rules for standard connectives outside the scope of $\mathcal{R}$ operator

Let us notice that by [6, p. 150, p. 155] we have that, for any $A \in \mathsf{F}$, for any $\alpha \in \mathsf{IC}$:

$$\text{if } \vDash_{\mathsf{CPL}} A \text{ then } \vDash_{\mathsf{MR}} \mathcal{R}_\alpha A. \qquad (\dagger)$$

2.2. Tableau system for logic MR

In this and subsequent sections, in our analysis of tableau systems, we will adopt an approach described in [4] for relating logics that is based on the metatheoretical approach to tableau presented in [2] originally developed for modal logics. Let us start with a general definition of a t-inconsistent (tableau inconsistent) set of formulas. Let X be a set of formulas:

- X is *t-inconsistent* iff there is a formula A, such that $A, \neg A \in X$,
- X is *t-consistent* iff it is not t-inconsistent.

Let us present the tableau rules for logic MR. We are going to follow the index-free approach presented in [7, pp. 128–131]. Firstly, we assume classical rules for connectives outside the scope of $\mathcal{R}$ operator (see Figure 1). These are standard elimination rules for boolean connectives. Secondly, we have specific elimination rules for connectives within the range of $\mathcal{R}$ which are based on a kind of distribution of $\mathcal{R}$ operator over other connectives (see Figure 2).

The set of all tableau rules for logic MR will be denoted as **R**. For any rules from **R** formulas in numerator will be called input, while formulas from denominator will be called output. Let us take as an example the rule $\mathrm{R}_{\mathcal{R}\wedge}$. The input of $\mathrm{R}_{\mathcal{R}\wedge}$ is $\{\mathcal{R}_\alpha(A \wedge B)\}$ and the output is set $\{\mathcal{R}_\alpha A, \mathcal{R}_\alpha B\}$. Notice that this rule is a non-branching one, i.e. it has only one output (one set of formulas). On the other hand, $\mathrm{R}_{\neg\mathcal{R}\wedge}$ is a branching rule which means that we have two outputs: $\{\mathcal{R}_\alpha \neg A\}$ and $\{\mathcal{R}_\alpha \neg B\}$. Once we have a notion of input we can define the notion of applicability of a rule. Let $\mathrm{R} \in \mathbf{R}$ and $X \subseteq \mathsf{For}$. R is *applicable to* X iff for any A from the input of the R, $A \in X$.

We define the relation of tableau consequence by referring to the concept of closure under tableau rules, similarly as in [4]. Our general definition enables one to define a notion of tableau consequence for MR but also for logics considered in subsequent sections. Let $\mathbf{Q}$ be a set of tableau rules and X, Y be sets of formulas. X is a *closure of* Y *under tableau rules from* $\mathbf{Q}$ (for short: $\mathbf{Q}$-*closure of* Y) iff there exists such a subset of natural numbers K that:

- $K = \mathbb{N}$ or $K = \{1, \dots, n\}$ for some $n \in \mathbb{N}$,
- there exists such an injective string $f \colon K \longrightarrow \{\mathsf{Z} : \mathsf{Z}$ is a subset of formulas$\}$ that:
  - $\mathsf{Z}_1 = Y$,
  - for all $i, i+1 \in K$ there exists such a tableau rule $\mathrm{R} \in \mathbf{Q}$ that its input is included in Z_i, while one of its outputs is equal to $\mathsf{Z}_{i+1} \setminus \mathsf{Z}_i$,
  - for all $i, i+1 \in K$ for any tableau rule $\mathrm{R} \in \mathbf{Q}$ if the input of R is included in Z_i and one of outputs of R is equal to $\mathsf{Z}_{i+1} \setminus \mathsf{Z}_i$, then for no j such that $i < j, j+1 \in K$ one of the remaining outputs of R is equal to $\mathsf{Z}_{j+1} \setminus \mathsf{Z}_j$,
  - for any tableau rule $\mathrm{R} \in \mathbf{Q}$ if the input of R is included in $\bigcup_{i \in K} \mathsf{Z}_i$, then one of outputs of R is in $\bigcup_{i \in K} \mathsf{Z}_i$,
- $X = \bigcup_{i \in K} \mathsf{Z}_i$.

Clearly, a set X is closed under applications of rules from $\mathbf{Q}$ (for short: $\mathbf{Q}$-closed) if X is a $\mathbf{Q}$-closure of some set Y. In practice, we can treat the closure in the presented sense as the so-called complete branch. In fact, it is a union of all elements that are on a complete branch.

For any $A \in \mathsf{F}$:

$$\mathrm{R}_{\mathcal{R}\neg}\colon \frac{\mathcal{R}_\alpha \neg A}{\neg \mathcal{R}_\alpha A} \qquad \mathrm{R}_{\mathcal{R}\wedge}\colon \frac{\mathcal{R}_\alpha (A \wedge B)}{\mathcal{R}_\alpha A, \mathcal{R}_\alpha B} \qquad \mathrm{R}_{\mathcal{R}\vee}\colon \frac{\mathcal{R}_\alpha (A \vee B)}{\mathcal{R}_\alpha A | \mathcal{R}_\alpha B}$$

$$\mathrm{R}_{\mathcal{R}\rightarrow}\colon \frac{\mathcal{R}_\alpha (A \rightarrow B)}{\mathcal{R}_\alpha \neg A | \mathcal{R}_\alpha B} \qquad \mathrm{R}_{\mathcal{R}\leftrightarrow}\colon \frac{\mathcal{R}_\alpha (A \leftrightarrow B)}{\mathcal{R}_\alpha A, \mathcal{R}_\alpha B | \mathcal{R}_\alpha \neg A, \mathcal{R}_\alpha \neg B}$$

$$\mathrm{R}_{\neg\mathcal{R}\neg}\colon \frac{\neg\mathcal{R}_\alpha \neg A}{\neg\neg \mathcal{R}_\alpha A} \qquad \mathrm{R}_{\neg\mathcal{R}\wedge}\colon \frac{\neg\mathcal{R}_\alpha (A \wedge B)}{\mathcal{R}_\alpha \neg A | \mathcal{R}_\alpha \neg B} \qquad \mathrm{R}_{\neg\mathcal{R}\vee}\colon \frac{\neg\mathcal{R}_\alpha (A \vee B)}{\mathcal{R}_\alpha \neg A, \mathcal{R}_\alpha \neg B}$$

$$\mathrm{R}_{\neg\mathcal{R}\rightarrow}\colon \frac{\neg\mathcal{R}_\alpha (A \rightarrow B)}{\mathcal{R}_\alpha A, \mathcal{R}_\alpha \neg B} \qquad \mathrm{R}_{\neg\mathcal{R}\leftrightarrow}\colon \frac{\neg\mathcal{R}_\alpha (A \leftrightarrow B)}{\mathcal{R}_\alpha A, \mathcal{R}_\alpha \neg B | \mathcal{R}_\alpha \neg A, \mathcal{R}_\alpha B}$$

Figure 2. Elimination rules for standard connectives inside the scope of $\mathcal{R}$ operator

A tableau consequence relation in logic MR is defined with respect to **R**-closed sets. A formula A is a *tableau consequence of* X *in* MR (in symb.: $X \rhd_{\mathsf{MR}} A$) iff there is a finite set $Y \subseteq X$ such that any **R**-closure of $Y \cup \{\neg A\}$ is t-inconsistent. And A is a *thesis in* MR (in symb.: $\rhd_{\mathsf{MR}} A$) iff $\emptyset \rhd_{\mathsf{MR}} A$.

2.3. Soundness and completeness of tableau system for MR

In order to prove the soundness and completeness of system MR and other system considered in the subsequent sections, we need to introduce some additional notions. Let $\mathfrak{M}$ be a model and X the set of formulas. We say that $\mathfrak{M}$ is *suitable for* X iff for any formula A, if $A \in X$ then $\mathfrak{M} \models A$.

The following lemma shows that by the applications of the rules from **R** from satisfiable formulas we receive some satisfiable formulas.

LEMMA 2.1. *Let* $X \subseteq \mathsf{For}$ *and* $\mathfrak{M} = \langle W, f, v \rangle$ *be a* MR*-model suitable for* X*. If any rule from* **R** *has been applied to* X*, then* $\mathfrak{M}$ *is suitable for the union of* X *and at least one output obtained by application of that rule.*

PROOF: For the cases of applications of the elimination rules for standard connectives outside the scope of $\mathcal{R}$ operator, i.e. rules R_*, $\mathrm{R}_{\neg *}$, where $*$ is a propositional connective, the proof is standard (cf. [14]).

Suppose $\mathrm{R}_{\mathcal{R}\neg}$ has been applied to X. Then $\mathcal{R}_\alpha\neg A \in X$. Since model $\mathfrak{M}$ is suitable for X, then $\mathfrak{M} \vDash \mathcal{R}_\alpha\neg A$. Thus, by the truth-condition (m_1), $v(\langle f(\alpha), \neg A\rangle) = 1$. Hence, by the condition (v_1), $v(\langle f(\alpha), A\rangle) \neq 1$. Thus, by the truth-conditions (m_1) and (m_2), $\mathfrak{M} \vDash \neg\mathcal{R}_\alpha A$.

Suppose $\mathrm{R}_{\neg\mathcal{R}\neg}$ has been applied to X. Then $\neg\mathcal{R}_\alpha\neg A \in X$. Since model $\mathfrak{M}$ is suitable for X, then $\mathfrak{M} \nvDash \mathcal{R}_\alpha\neg A$. Thus, by the truth-condition (m_1), $v(\langle f(\alpha), \neg A\rangle) = 0$. Hence, by the condition (v_1), $v(\langle f(\alpha), A\rangle) = 1$. Thus, by the truth-conditions (m_1) and (m_2), $\mathfrak{M} \vDash \neg\neg\mathcal{R}_\alpha A$.

Suppose $\mathrm{R}_{\mathcal{R}\wedge}$ has been applied to X. Then $\mathcal{R}_\alpha(A \wedge B) \in X$. Since model $\mathfrak{M}$ is suitable for X, then $\mathfrak{M} \vDash \mathcal{R}_\alpha(A \wedge B)$. Thus, by the truth-condition (m_3), $v(\langle f(\alpha), A \wedge B\rangle) = 1$. Hence, by the condition (v_2), $v(\langle f(\alpha), A\rangle) = v(\langle f(\alpha), B\rangle) = 1$. Thus, by the truth-conditions (m_1), $\mathfrak{M} \vDash \mathcal{R}_\alpha A$ and $\mathfrak{M} \vDash \mathcal{R}_\alpha B$.

Suppose $\mathrm{R}_{\neg\mathcal{R}\wedge}$ has been applied to X. Then $\mathcal{R}_\alpha\neg(A \wedge B) \in X$. Since model $\mathfrak{M}$ is suitable for X, then $\mathfrak{M} \vDash \mathcal{R}_\alpha\neg(A \wedge B)$. Thus, by the truth-condition (m_1), $v(\langle f(\alpha), \neg(A \wedge B)\rangle) = 1$. Hence, by the conditions (v_1) and (v_2), either $v(\langle f(\alpha), \neg A\rangle) = 1$ or $v(\langle f(\alpha), \neg B\rangle) = 1$. Thus, by the truth-condition (m_1), either $\mathfrak{M} \vDash \mathcal{R}_\alpha\neg A$ or $\mathfrak{M} \vDash \mathcal{R}_\alpha\neg B$.

For the remaining cases, we reason in the similar way. $\square$

Let us now introduce the notion of a model generated by a t-consistent **R**-closed set. Let X be the t-consistent **R**-closed set and $\mathsf{IC}_X := \{\alpha \in \mathsf{IC} : \mathcal{R}_\alpha A \in X\}$. A MR-*model generated by* X (for short: MR-X-*model*) is a MR-model $\langle W, f, v\rangle$ such that:

- $W = \mathsf{IC}_X$,
- for any $\alpha \in \mathsf{IC}$ we put:

$$f(\alpha) = \begin{cases} \alpha, & \text{if } \alpha \in \mathsf{IC}_X \\ a_{\min\{n \in \mathbb{N} : a_n \in \mathsf{IC}_X\}}, & \text{if } \alpha \notin \mathsf{IC}_X \end{cases}$$

- for any $\alpha \in W$ and any $A \in X \cap \mathsf{VAR}$ we put:

$$v(\langle \alpha, A\rangle) = \begin{cases} 1, & \text{if } \mathcal{R}_\alpha A \in X \\ 0, & \text{if } \mathcal{R}_\alpha A \notin X \end{cases}$$

  we extend v on $W \times \mathsf{F}$ by means of conditions (v_1)–(v_5).

We have the following fact:

FACT 2.2. Let $X \subseteq \mathsf{For}$ be the t-consistent $\mathbf{R}$-closed set, $\mathfrak{M} = \langle W, f, v\rangle$ be a MR-X-model and $\alpha \in \mathsf{IC}$. Then, for any $A \in \mathsf{F}$:

- if $\mathcal{R}_\alpha A \in X$ then $v(\langle \alpha, A\rangle) = 1$,
- if $\neg\mathcal{R}_\alpha A \in X$ then $v(\langle \alpha, A\rangle) = 0$.

PROOF: *Base case.* We obtain the result by the definition of a MR-X-model and since X is t-consistent.

Inductive hypothesis. Let $n \in \mathbb{N}$. Suppose that for any $A \in \mathsf{F}$ such that $o(A) \leq n$:

- if $\mathcal{R}_\alpha A \in X$ then $v(\langle \alpha, A\rangle) = 1$,
- if $\neg\mathcal{R}_\alpha A \in X$ then $v(\langle \alpha, A\rangle) = 0$.

Inductive step. Let $A \in \mathsf{F}$ and $o(A) = n + 1$.

Let $A = \neg B$. Suppose $\mathcal{R}_\alpha\neg B \in X$. Since X is a $\mathbf{R}$-closed set, by the application of the rule $\mathrm{R}_{\mathcal{R}\neg}$ $\neg\mathcal{R}_\alpha B \in X$. By the inductive hypothesis $v(\langle \alpha, B\rangle) = 0$. Thus, by the condition (v_1), $v(\langle \alpha, \neg B\rangle) = 1$. Suppose $\neg\mathcal{R}_\alpha\neg B \in X$. Since X is a $\mathbf{R}$-closed set, by the application of the rule $\mathrm{R}_{\neg\mathcal{R}\neg}$ $\neg\neg\mathcal{R}_\alpha B \in X$. Hence, by the application of the rule $\mathrm{R}_{\neg\neg}$, $\mathcal{R}_\alpha B \in X$. By the inductive hypothesis $v(\langle \alpha, B\rangle) = 1$. Thus, by the condition (v_1), $v(\langle \alpha, \neg B\rangle) = 0$.

Let $A = B * C$, where $* \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$. Suppose $* = \wedge$, for other cases we reason in the similar way. Let us assume that $\mathcal{R}_\alpha(B \wedge C) \in X$. Since X is an $\mathbf{R}$-closed set, by the application of the rule $\mathrm{R}_{\mathcal{R}\wedge}$ $\mathcal{R}_\alpha B, \mathcal{R}_\alpha C \in X$. Hence, by the inductive hypothesis, $v(\langle \alpha, B\rangle) = 1$ and $v(\langle \alpha, C\rangle) = 1$. Thus, by the condition (v_2), $v(\langle \alpha, B \wedge C\rangle) = 1$. Suppose $\neg\mathcal{R}_\alpha(B \wedge C) \in X$. Since X is a $\mathbf{R}$-closed set, by the application of the rule $\mathrm{R}_{\neg\mathcal{R}\wedge}$ either $\mathcal{R}_\alpha\neg B$, or $\mathcal{R}_\alpha\neg C \in X$. Thus, by the application of the rule $\mathrm{R}_{\mathcal{R}\neg}$ either $\neg\mathcal{R}_\alpha B \in X$ or $\neg\mathcal{R}_\alpha C \in X$. Hence, by the inductive hypothesis, either $v(\langle \alpha, B\rangle) = 0$ or $v(\langle \alpha, C\rangle) = 0$. Therefore, by the condition (v_2), $v(\langle \alpha, B \wedge C\rangle) = 0$.

For the remaining cases we reason in the similar way. $\square$

By means of fact 2.2 we can prove the following lemma:

LEMMA 2.3. *Let* $X \subseteq \mathsf{For}$ *be a t-consistent* $\mathbf{R}$*-closed set and* $\mathfrak{M} = \langle W, f, v \rangle$ *be an* MR-X*-model. Then, for any* $A \in \mathsf{For}$*:*

- *if* $A \in X$ *then* $\mathfrak{M} \models A$*,*
- *if* $\neg A \in X$ *then* $\mathfrak{M} \not\models A$*.*

PROOF: *Base case.* Let $A \in \mathsf{For}$ and $c(A) = 1$. Thus $A = \mathcal{R}_\alpha B$, where $B \in \mathsf{F}$. Suppose $\mathcal{R}_\alpha B \in X$. Then, by fact 2.2 (1), $v(\langle \alpha, B \rangle) = 1$. Therefore, by the truth-condition (m_1), $\mathfrak{M} \models \mathcal{R}_\alpha B$. Suppose $\neg \mathcal{R}_\alpha B \in X$. Hence, by fact 2.2 (2), $v(\langle \alpha, B \rangle) = 0$. Thus, by the truth-conditions (m_1), $\mathfrak{M} \not\models \mathcal{R}_\alpha B$.

Inductive hypothesis. Let $n \in \mathbb{N}$. Suppose that for any $A \in \mathsf{For}$ such that $c(A) \leq n$:

- if $A \in X$ then $\mathfrak{M} \models A$,
- if $\neg A \in X$ then $\mathfrak{M} \not\models A$.

Inductive step. Let $A \in \mathsf{For}$ and $c(A) = n + 1$.

Let $A = \neg\neg B$. Suppose $\neg\neg B \in X$. Hence, by the application of the rule $\mathrm{R}_{\neg\neg}$, $B \in X$. Thus, by the inductive hypothesis, $\mathfrak{M} \models B$. Therefore, by the truth-condition (m_2), $\mathfrak{M} \models \neg\neg B$.

Let $A = B * C$, where $* \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$. Suppose $* = \wedge$, for other cases we reason in the similar way. Let us assume that $B \wedge C \in X$. Since X is a $\mathbf{R}$-closed set, by the application of the rule $\mathrm{R}_\wedge$ $B, C \in X$. Hence, by the inductive hypothesis, $\mathfrak{M} \models B$ and $\mathfrak{M} \models C$. Therefore, by the truth-condition (m_3), $\mathfrak{M} \models B \wedge C$.

Let $A = \neg(B * C)$, where $* \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$. Suppose $* = \wedge$, for the other cases we reason in a similar way. Let us assume that $\neg(B \wedge C) \in X$. Hence, by the application of the rule $\mathrm{R}_{\neg\wedge}$, either $\neg B \in X$ or $\neg C \in X$. Thus, by the inductive hypothesis, either $\mathfrak{M} \models \neg B$ or $\mathfrak{M} \models \neg C$. Therefore, by the truth-conditions (m_2) and (m_3), $\mathfrak{M} \models \neg(B \wedge C)$. □

Having proven the introduced facts, we can easily receive the soundness and completeness of our tableau system.

THEOREM 2.4. *Let* $X \cup \{A\} \subseteq \mathsf{For}$*. Then,* $X \rhd_{\mathsf{MR}} A$ *iff* $X \models_{\mathsf{MR}} A$*.*

PROOF: Suppose there is finite $Y \subseteq X$ such that any closure of $Y \cup \{\neg A\}$ on rules from $\mathbf{R}$ is t-inconsistent. Let us assume that there is a MR-model

$\mathfrak{M}$ such that $\mathfrak{M} \models X \cup \{\neg A\}$. Hence $\mathfrak{M}$ is suitable to $X \cup \{\neg A\}$, so also to $Y \cup \{\neg A\}$. By lemma 2.1 there is an **R**-closure of $X \cup \{\neg A\}$ to which $\mathfrak{M}$ is suitable. But such closure is t-inconsistent. Hence, there is $A \in \mathsf{For}$ such that $\mathfrak{M} \models A$ and $\mathfrak{M} \not\models A$. Therefore, for any MR-model $\mathfrak{M}$, if $\mathfrak{M} \models X$ then $\mathfrak{M} \models A$, and so $X \models_{\mathsf{MR}} A$.

Suppose $X \models_{\mathsf{MR}} A$. Let us assume that for any finite $Y \subseteq X$ there is t-consistent **R**-closure of $Y \cup \{\neg A\}$. Hence, there is a t-consistent **R**-closure Z such that $X \cup \{\neg A\} \subseteq Z$. Otherwise, any of such a closure would consist some t-inconsistency. But by the definition of a **R**-closure of a set, this would mean that for some finite $Y \subseteq X$ no **R**-closure of $Y \cup \{\neg A\}$ is t-consistent. As a consequence, by lemma 2.3, $\mathfrak{M} \models X \cup \{\neg A\}$, where $\mathfrak{M}$ is a MR-Z-model. Therefore $X \not\models_{\mathsf{MR}} A$. $\square$

3. Logic MR^+

As we noticed, in the language of MR there are no "free" CPL formulas outside the scope of the $\mathcal{R}$ operator. Whereas on the ground of epistemic logic, it is important to be able to refer both to sentences stating that a given agent knows a given thing and to sentences simply expressing states of affairs not propositional attitudes. Furthermore, in the language of MR there are no iterations of the $\mathcal{R}$ operator. But iterations matter in epistemic contexts, especially if we want to consider so-called positive and negative introspection. For this reason, we introduce a modification of the language and semantics of MR.

3.1. Language and semantics of logic MR^+

The language of MR^+ is an extension of the language of MR. The *set of* MR^+ *formulas*, i.e. the set For^+, is defined in the usual way as the smallest set X meeting the following conditions:

- $\mathsf{VAR} \subseteq X$,
- if $A \in X$ than $\mathcal{R}_\alpha A \in X$, where $\alpha \in \mathsf{IC}$,
- if $A \in X$ than $\neg A \in X$,
- if $A, B \in X$ than $(A * B) \in X$, where $* \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$.

Obviously $\mathsf{F}, \mathsf{For} \subset \mathsf{For}^+$.

Let us modify the notion of the complexity of a formula. We define function $c^+\colon \mathsf{For}^+ \longrightarrow \mathbb{N}$ in the standard way, i.e.: $c^+(A) = 1$, if $A \in \mathsf{VAR}$; $c^+(A) = c^+(B) + 1$, if $A = \neg B$ or $A = \mathcal{R}_\alpha B$; $c^+(A) = c^+(B) + c^+(C) + 1$, if $A = B * C$, where $* \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$.

In this section we also employ the function assigning to a formula its subformulas, i.e. a function $s\colon \mathsf{For}^+ \longrightarrow \mathcal{P}(\mathsf{For}^+)$ such that: $s(A) = \{A\}$, if $A \in \mathsf{VAR}$; $s(A) = \{A\} \cup s(B)$, if $A = \neg B$ or $A = \mathcal{R}_\alpha B$; $s(A) = \{B * C\} \cup s(B) \cup s(C)$, if $A = B * C$, where $* \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$. Let $s(X) := \{s(A) : A \in X\}$.

Let W be a non-empty set. By $\overrightarrow{W}$ we denote the set of all finite strings of elements from W. We have $(w_1, \ldots, w_n) \in \overrightarrow{W}$ iff $n \in \mathbb{N}$ and $w_i \in W$, for any i such that $1 \leqslant i \leqslant n$. If a string has one element w we write w instead of (w). A *model of* MR^+ (a MR^+-*model*) is an ordered triple $\langle W, f, v\rangle$ such that:

- W, f are the same as in the case of MR-model,
- $v\colon (\overrightarrow{W} \times \mathsf{For}^+) \cup \mathsf{F} \longrightarrow \{0, 1\}$ is such that:
  - $v\restriction_{\overrightarrow{W}\times\mathsf{For}^+}$ is a valuation such that for any $\overrightarrow{w} = (w_1, \ldots, w_n) \in \overrightarrow{W}$ and any $A, B \in \mathsf{For}^+$:

$$v(\langle \overrightarrow{w}, \neg A\rangle) = 1 \ \text{ iff } \ v(\langle \overrightarrow{w}, A\rangle) = 0 \tag{v_1^+}$$

$$v(\langle \overrightarrow{w}, A \wedge B\rangle) = 1 \ \text{ iff } \ v(\langle \overrightarrow{w}, A\rangle) = v(\langle \overrightarrow{w}, B\rangle) = 1 \tag{v_2^+}$$

$$v(\langle \overrightarrow{w}, A \vee B\rangle) = 1 \ \text{ iff } \ v(\langle \overrightarrow{w}, A\rangle) = 1 \text{ or } v(\langle \overrightarrow{w}, B\rangle) = 1 \tag{v_3^+}$$

$$v(\langle \overrightarrow{w}, A \rightarrow B\rangle) = 1 \ \text{ iff } \ v(\langle \overrightarrow{w}, A\rangle) = 0 \text{ or } v(\langle w, B\rangle) = 1 \tag{v_4^+}$$

$$v(\langle \overrightarrow{w}, A \leftrightarrow B\rangle) = 1 \ \text{ iff } \ v(\langle \overrightarrow{w}, A\rangle) = v(\langle \overrightarrow{w}, B\rangle) \tag{v_5^+}$$

$$v(\langle (w_1, \ldots, w_n),\ \mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_m} A\rangle) = 1 \ \text{ iff } \ v(\langle (w_1, \ldots, w_n, f(\alpha_1), \ldots, f(\alpha_m)), A\rangle) = 1 \tag{v_6^+}$$

  - $v\restriction_{\mathsf{F}}$ is the classical CPL valuation.

The truth-conditions (m_1)–(m_6) are now determined for formulas from For^+. Notice that in (m_1) we now have a one element string $(f(\alpha))$ not a point $f(\alpha)$. Moreover, we add the following truth-condition, for any $A \in \mathsf{VAR}$:

$$\mathfrak{M} \models A \quad \text{iff} \quad v(A) = 1. \tag{m_7}$$

Thus we get that for any $A \in \mathsf{F}$, $\mathfrak{M} \vDash A$ iff $v(A) = 1$. And so CPL is a proper sublogic of MR^+, i.e. $\vDash_{\mathsf{CPL}} \subset \vDash_{\mathsf{MR}^+}$. Let us also state that MR must be the proper sublogic of MR^+.

Let us notice that for MR^+ we have the counterpart of the property (†). For any $\alpha \in \mathsf{IC}$, for any $A \in \mathsf{For}^+$:

$$\text{if } \vDash_{\mathsf{MR}^+} A \text{ then } \vDash_{\mathsf{MR}^+} \mathcal{R}_\alpha A. \tag{‡}$$

In order to prove that we define a notion of α-model.

Let $\mathfrak{M} = \langle W, f, v\rangle$ be a MR^+- model and $\alpha \in \mathsf{IC}$. An *α-model received from* $\mathfrak{M}$ (for short: *α-model*) is a MR^+-model $\mathfrak{N} = \langle W, f, u\rangle$ where $u\colon (\overrightarrow{W} \times \mathsf{For}^+) \cup \mathsf{F} \longrightarrow \{1, 0\}$ is such that, for any $A \in \mathsf{VAR}$, for any $(w_1, \ldots, w_n) \in \overrightarrow{W}$ we put:

$$u(\langle (w_1, \ldots, w_n), A\rangle) = \begin{cases} 1, & \text{if } v(\langle (f(\alpha), w_1, \ldots, w_n), A\rangle) = 1 \\ 0, & \text{if } v(\langle (f(\alpha), w_1, \ldots, w_n), A\rangle) = 0 \end{cases}$$

$$u(A) = \begin{cases} 1, & \text{if } v(\langle f(\alpha), A\rangle) = 1 \\ 0, & \text{if } v(\langle f(\alpha), A\rangle) = 0 \end{cases}$$

we extend u on $(\overrightarrow{W} \times \mathsf{For}^+) \cup \mathsf{F}$ by means of standard conditions for CPL formulas and conditions (v_1^+)–(v_6^+).

We have the following fact:

Fact 3.1. Let $\mathfrak{M} = \langle W, f, v\rangle$ be a MR^+- model, $\alpha \in \mathsf{IC}$ and $\mathfrak{N} = \langle W, f, u\rangle$ be an α-model received from $\mathfrak{M}$. Then, for any $A \in \mathsf{For}^+$, for any $(w_1, \ldots, w_n) \in \overrightarrow{W}$, $v((f(\alpha), w_1, \ldots, w_n), A) = 1$ iff $u((w_1, \ldots, w_n), A) = 1$.

Proof: *Base case.* By the definition of an α-model.

Inductive hypothesis. Let $m \in \mathbb{N}$. Suppose that for any $A \in \mathsf{For}^+$ such that $c^+(A) \leq m$, for any $(w_1, \ldots, w_n) \in \overrightarrow{W}$, $v((f(\alpha), w_1, \ldots, w_n), A) = 1$ iff $w((w_1, \ldots, w_n), A) = 1$.

Inductive step. Let $A \in \mathsf{For}^+$ and $c^+(A) = m + 1$.

Let $A = \neg B$. Then: $v((f(\alpha), w_1, \ldots, w_n), \neg B) = 1$, by the condition ($\mathrm{v}_1^+$), iff $v((f(\alpha), w_1, \ldots, w_n), B) = 0$, by the inductive hypothesis, iff $u((w_1, \ldots, w_n), B) = 0$, by the condition ($\mathrm{v}_1^+$), iff $u((w_1, \ldots, w_n), \neg B) = 1$.

Let $A = B * C$, where $* \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$. We consider only case for $* = \wedge$. For other cases we reason in the similar way. We have: $v((f(\alpha), w_1, \ldots, w_n), B \wedge C) = 1$, by the condition ($\mathrm{v}_2^+$), iff $v((f(\alpha), w_1, \ldots,$

$w_n), B) = v((f(\alpha), w_1, \ldots, w_n), C) = 1$, by the inductive hypothesis, iff $u((w_1, \ldots, w_n), B) = u((w_1, \ldots, w_n), C) = 1$, by the condition ($\mathrm{v}_1^+$), iff $u((w_1, \ldots, w_n), B \wedge C) = 1$.

Let $A = \mathcal{R}_\beta B$. Then: $v((f(\alpha), w_1, \ldots, w_n), \mathcal{R}_\beta B) = 1$, by the condition ($\mathrm{v}_6^+$), iff $v((f(\alpha), w_1, \ldots, w_n, f(\beta)), B) = 1$, by the inductive hypothesis, iff $u((w_1, \ldots, w_n, f(\beta)), B) = 1$, by the condition ($\mathrm{v}_6^+$), iff $u((w_1, \ldots, w_n), \mathcal{R}_\beta B) = 1$. □

By fact 3.1 we receive the following corollary:

Fact 3.2. Let $\mathfrak{M} = \langle W, f, v\rangle$ be a MR^+- model, $\alpha \in \mathsf{IC}$ and $\mathfrak{N} = \langle W, f, w\rangle$ be an α-model. Then, for any $A \in \mathsf{For}^+$, $\mathfrak{M} \vDash \mathcal{R}_\alpha A$ iff $\mathfrak{N} \vDash A$.

Proof: *Base case.* By the definition of an α-model.

Inductive hypothesis. Let $n \in \mathbb{N}$. Suppose that for any $A \in \mathsf{For}^+$ such that $c^+(A) \leq n$, $\mathfrak{M} \vDash \mathcal{R}_\alpha A$ iff $\mathfrak{N} \vDash A$.

Inductive step. Let $A \in \mathsf{For}^+$ and $c^+(A) = m + 1$.

Let $A = \neg B$. Then: $\mathfrak{M} \vDash \mathcal{R}_\alpha \neg B$, by the truth-condition (m_1), iff $v((f(\alpha), \neg B) = 1$, by the condition ($\mathrm{v}_1^+$), iff $v((f(\alpha), B) = 0$, by the truth-condition (m_1), iff $\mathfrak{M} \nvDash \mathcal{R}_\alpha B$, by the inductive hypothesis iff $\mathfrak{N} \nvDash B$, by the truth-condition (m_2), iff $\mathfrak{N} \nvDash \neg B$.

Let $A = B * C$, where $* \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$. We consider only case for $* = \wedge$. For other cases we reason in the similar way. We have: $\mathfrak{M} \vDash \mathcal{R}_\alpha B \wedge C$, by the truth-condition (m_1), iff $v((f(\alpha), B \wedge C) = 1$, by the condition ($\mathrm{v}_2^+$), iff $v((f(\alpha), B) = v((f(\alpha), C) = 1$, by the truth-condition (m_1), iff $\mathfrak{M} \nvDash \mathcal{R}_\alpha B$ and $\mathfrak{M} \nvDash \mathcal{R}_\alpha C$, by the inductive hypothesis iff $\mathfrak{N} \nvDash B$ and $\mathfrak{N} \nvDash C$, by the truth-condition (m_3), iff $\mathfrak{N} \nvDash B \wedge C$.

Let $A = \mathcal{R}_\beta B$. Then: $\mathfrak{M} \vDash \mathcal{R}_\alpha \mathcal{R}_\beta B$, by the truth-condition (m_1), iff $v(f(\alpha), \mathcal{R}_\beta B) = 1$, by the condition ($\mathrm{v}_6^+$), iff $v((f(\alpha), f(\beta)), B) = 1$, by fact 3.1, iff $u(f(\beta), B) = 1$, by the truth-condition (m_1), iff $\mathfrak{N} \nvDash \mathcal{R}_\beta B$. □

By fact 3.2, if there is a MR^+- model $\mathfrak{M}$ such that $\mathfrak{M} \nvDash \mathcal{R}_\alpha A$, for some $\alpha \in \mathsf{IC}$, then there is MR^+- model $\mathfrak{N}$ such that $\mathfrak{N} \nvDash A$. Therefore (‡) holds.

3.2. Tableau system for logic MR^+

In the case of the elimination rules for standard connectives inside the scope of $\mathcal{R}$ operator (cf. Figure 1), the tableau rules for MR^+ are of the same form as rules for MR. The only difference is that the formulas in the numerator and denominator vary over For^+ instead of just For. The rest of tableau

For any $A, B \in \mathsf{For}^+$:

$$\mathrm{R}_{\mathcal{R}\neg}: \frac{\mathcal{R}_{\alpha_1}\ldots\mathcal{R}_{\alpha_n}\neg A}{\neg\mathcal{R}_{\alpha_1}\ldots\mathcal{R}_{\alpha_n}A} \qquad \mathrm{R}_{\mathcal{R}\wedge}: \frac{\mathcal{R}_{\alpha_1}\ldots\mathcal{R}_{\alpha_n}(A\wedge B)}{\mathcal{R}_{\alpha_1}\ldots\mathcal{R}_{\alpha_n}A, \mathcal{R}_{\alpha_1}\ldots\mathcal{R}_{\alpha n}B}$$

$$\mathrm{R}_{\mathcal{R}\vee}: \frac{\mathcal{R}_{\alpha_1}\ldots\mathcal{R}_{\alpha_n}(A\vee B)}{\mathcal{R}_{\alpha_1}\ldots\mathcal{R}_{\alpha n}A|\mathcal{R}_{\alpha_1}\ldots\mathcal{R}_{\alpha_n}B}$$

$$\mathrm{R}_{\mathcal{R}\rightarrow}: \frac{\mathcal{R}_{\alpha_1}\ldots\mathcal{R}_{\alpha_n}(A\rightarrow B)}{\mathcal{R}_{\alpha_1}\ldots\mathcal{R}_{\alpha_n}\neg A|\mathcal{R}_{\alpha_1}\ldots\mathcal{R}_{\alpha_n}B}$$

$$\mathrm{R}_{\mathcal{R}\leftrightarrow}: \frac{\mathcal{R}_{\alpha_1}\ldots\mathcal{R}_{\alpha_n}(A\leftrightarrow B)}{\mathcal{R}_{\alpha_1}\ldots\mathcal{R}_{\alpha_n}A, \mathcal{R}_{\alpha_1}\ldots\mathcal{R}_{\alpha_n}B|\mathcal{R}_{\alpha_1}\ldots\mathcal{R}_{\alpha_n}\neg A, \mathcal{R}_{\alpha_1}\ldots\mathcal{R}_{\alpha_n}\neg B}$$

$$\mathrm{R}_{\neg\mathcal{R}\neg}: \frac{\neg\mathcal{R}_{\alpha_1}\ldots\mathcal{R}_{\alpha_n}\neg A}{\neg\neg\mathcal{R}_{\alpha_1}\ldots\mathcal{R}_{\alpha_n}A} \qquad \mathrm{R}_{\neg\mathcal{R}\wedge}: \frac{\neg\mathcal{R}_{\alpha_1}\ldots\mathcal{R}_{\alpha_n}(A\wedge B)}{\mathcal{R}_{\alpha_1}\ldots\mathcal{R}_{\alpha_n}\neg A|\mathcal{R}_{\alpha_1}\ldots\mathcal{R}_{\alpha_n}\neg B}$$

$$\mathrm{R}_{\neg\mathcal{R}\vee}: \frac{\neg\mathcal{R}_{\alpha_1}\ldots\mathcal{R}_{\alpha_n}(A\vee B)}{\mathcal{R}_{\alpha_1}\ldots\mathcal{R}_{\alpha_n}\neg A, \mathcal{R}_{\alpha_1}\ldots\mathcal{R}_{\alpha_n}\neg B}$$

$$\mathrm{R}_{\neg\mathcal{R}\rightarrow}: \frac{\neg\mathcal{R}_{\alpha_1}\ldots\mathcal{R}_{\alpha_n}(A\rightarrow B)}{\mathcal{R}_{\alpha_1}\ldots\mathcal{R}_{\alpha_n}A, \mathcal{R}_{\alpha_1}\ldots\mathcal{R}_{\alpha_n}\neg B}$$

$$\mathrm{R}_{\neg\mathcal{R}\leftrightarrow}: \frac{\neg\mathcal{R}_{\alpha_1}\ldots\mathcal{R}_{\alpha_n}(A\leftrightarrow B)}{\mathcal{R}_{\alpha_1}\ldots\mathcal{R}_{\alpha_n}A, \mathcal{R}_{\alpha_1}\ldots\mathcal{R}_{\alpha_n}\neg B|\mathcal{R}_{\alpha_1}\ldots\mathcal{R}_{\alpha_n}\neg A, \mathcal{R}_{\alpha_1}\ldots\mathcal{R}_{\alpha_n}B}$$

Figure 3. Elimination rules for standard connectives inside the scope of $\mathcal{R}_{\alpha_1}\ldots\mathcal{R}_{\alpha_n}$, for some $n \in \mathbb{N}$.

rules, i.e. elimination rules for standard connectives inside the scope of operator $\mathcal{R}$, are presented in Figure 3. We will use, however, the same notion for rules. The set of tableau rules for logic MR^+ is denoted as $\mathbf{R}^+$. The notions of thesis and tableau consequence in MR^+ are defined as for MR but with respect to $\mathbf{R}^+$.

3.3. Soundness and completeness of tableau system for MR^+

In order to prove the soundness of our system we use the counterpart of lemma 2.1 for rules from $\mathbf{R}^+$. Notice that the first nine tableau rules in Figure 1 work just fine for formulas of CPL, for any $A, B \in \mathsf{F}$.

LEMMA 3.3. *Let $X \subseteq \mathsf{For}^+$ and $\mathfrak{M} = \langle W, f, v\rangle$ be a MR^+- model suitable for X. If any rule from $\mathbf{R}^+$ has been applied to X, then $\mathfrak{M}$ is suitable for the union of X and at least one output obtained by application of that rule.*

PROOF: Similarly as for lemma 2.1. $\square$

In order to prove completeness we use the same argument as before but with respect to the modified version of the generated model. Let $X \in \mathsf{For}^+$ be a $\mathbf{R}^+$-closed set and $\mathsf{IC}_X = \{\alpha \in \mathsf{IC} : \mathcal{R}_\alpha A \in s(X)\}$. A MR^+*-model generated by* X (for short: MR^+-X*-model*) is a MR^+-model $\langle W, f, v\rangle$ such that:

- W, f are as in the previous case,
- for any $A \in X \cap \mathsf{VAR}$, for any $(\alpha_1, \ldots, \alpha_n) \in \overrightarrow{W}$ we put:

$$v(\langle(\alpha_1, \ldots, \alpha_n), A\rangle) = \begin{cases} 1, & \text{if } \mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_n} A \in X \\ 0, & \text{if } \mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_n} A \notin X \end{cases}$$

$$v(A) = \begin{cases} 1, & \text{if } A \in X \\ 0, & \text{if } A \notin X \end{cases}$$

  we extend v on $(W \times \mathsf{For}^+) \cup \mathsf{F}$ by means of standard conditions for CPL formulas and conditions (v_1^+)–(v_6^+).

First we prove the following fact:

FACT 3.4. Let $X \subseteq \mathsf{For}^+$ be a t-consistent $\mathbf{R}^+$-closed set and $\mathfrak{M} = \langle W, f, v\rangle$ be a MR^+ X-model. Then, for any $A \in \mathsf{For}^+$, for any $\alpha_1, \ldots, \alpha_n \in \mathsf{IC}$:

- if $\mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_n} A \in X$ then $v(\langle(\alpha_1, \ldots, \alpha_n), A\rangle) = 1$,
- if $\neg\mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_n} A \in X$ then $v(\langle(\alpha_1, \ldots, \alpha_n), A\rangle) = 0$.

PROOF: *Base case.* By the definition of the MR^+ X-model and since X is t-consistent.

Inductive hypothesis. Let $m \in \mathbb{N}$. Suppose that for any $A \in \mathsf{For}^+$ such that $c^+(A) \leq m$, for any $\alpha_1, \ldots, \alpha_n \in \mathsf{IC}$:

- if $\mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_n} A \in X$ then $v(\langle(\alpha_1, \ldots, \alpha_n), A\rangle) = 1$,
- if $\neg\mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_n} A \in X$ then $v(\langle(\alpha_1, \ldots, \alpha_n), A\rangle) = 0$.

Inductive step. Let $A \in \mathsf{For}^+$ and $c^+(A) = m + 1$.

Let $A = \neg B$. Suppose $\mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_n} \neg B \in X$. Hence, by the application of the rule $\mathrm{R}_{\neg\mathcal{R}} \in \mathbf{R}^+$, $\neg\mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_n} B \in X$. By the inductive hypothesis $v(\langle(f(\alpha_1), \ldots, f(\alpha_n)), B\rangle) = 0$. By the condition ($\mathrm{v}_1^+$) $v(\langle f(\alpha_1), \ldots, f(\alpha_n)), \neg B\rangle) = 1$. Suppose $\neg\mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_n} \neg B \in X$. Hence, by the application of the rule $\mathrm{R}_{\neg\mathcal{R}\neg} \in \mathbf{R}^+$, $\neg\neg\mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_n} B \in X$. By the application of the rule $\mathrm{R}_{\neg\neg} \in \mathbf{R}^+$, $\mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_n} B \in X$. By the inductive hypothesis $v(\langle(f(\alpha_1), \ldots, f(\alpha_n)), B\rangle) = 1$. Thus, by the condition (v_1^+), $v(\langle(f(\alpha_1), \ldots, f(\alpha_n)), \neg B\rangle) = 0$.

Let $A = B * C$, where $* \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$. We consider only case for $* = \wedge$. For other cases we reason in the similar way. Suppose $\mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_n} (B \wedge C) \in X$. Hence, by the application of the rule $\mathrm{R}_{\mathcal{R}\wedge} \in \mathbf{R}^+$, $\mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_n} B \in X$ and $\mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_n} C \in X$. Thus, by the inductive hypothesis, $v(\langle(f(\alpha_1), \ldots, f(\alpha_n)), B\rangle) = 1$ and $v(\langle(f(\alpha_1), \ldots, f(\alpha_n)), C\rangle) = 1$. Thus, by the condition (v_2^+), $v(\langle(f(\alpha_1), \ldots, f(\alpha_n)), B \wedge C\rangle) = 1$. Suppose $\neg\mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_n} (B \wedge C) \in X$. Hence, by the application of the rule $\mathrm{R}_{\neg\mathcal{R}\wedge} \in \mathbf{R}^+$, $\mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_n} \neg B \in X$ or $\mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_n} \neg C \in X$. By the application of the rule $\mathrm{R}_{\mathcal{R}\neg} \in \mathbf{R}^+$, $\neg\mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_n} B \in X$ or $\neg\mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_n} C \in X$. Thus, by the inductive hypothesis, either $v(\langle(f(\alpha_1), \ldots, f(\alpha_n)), B\rangle) = 0$ or $v(\langle(f(\alpha_1), \ldots, f(\alpha_n)), C\rangle) = 0$. Thus, by the condition (v_2^+), $v(\langle(f(\alpha_1), \ldots, f(\alpha_n)) B \wedge C\rangle) = 0$.

Let $A = \mathcal{R}_\beta B$. Suppose $\mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_n} A \in X$ (resp. $\neg\mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_n} A \in X$), so $\mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_n} \mathcal{R}_\beta B \in X$ (resp. $\neg\mathcal{R}_{\alpha_1} \ldots \mathcal{R}_{\alpha_n} \mathcal{R}_\beta B \in X$). Let us assume that that $\mathcal{R}_\beta$ is the longest iteration of $\mathcal{R}$ which appears right after $\mathcal{R}_{\alpha_n}$. If it is not then we can consider the longest since formula is a finite string of symbols. Hence B is a propositional variable or is of the form $\neg C$ or $C * D$, where $* \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$. Thus we can reason as in the previous cases. □

The following lemma enables one to prove completeness theorem:

LEMMA 3.5. *Let $X \subseteq \mathsf{For}^+$ be a t-consistent $\mathbf{R}^+$-closed set, $\mathfrak{M}$ be an MR^+ X-model. Then, for any $A \in \mathsf{For}^+$*

- *if $A \in X$ then $\mathfrak{M} \models A$,*
- *if $\neg A \in X$ then $\mathfrak{M} \not\models A$.*

Proof: We reason similarly as in the case of lemma 2.3. In the base case we consider a propositional variable and receive the required result by the definition of the MR^+ X-model. In the inductive step we additionally consider formulas of the form $\mathcal{R}_\alpha B$ and $\neg\mathcal{R}_\alpha B$. In such cases we receive the required result by fact 3.4. $\square$

As in the case for MR, by lemmas 3.3, 3.5 we get the following theorem:

Theorem 3.6. *Let* $X \cup \{A\} \subseteq \mathsf{For}^+$. *Then,* $X \rhd_{\mathsf{MR}^+} A$ *iff* $X \vDash_{\mathsf{MR}^+} A$.

4. Modal paradigm and logical omniscience

The standard approach to epistemic logic is based on modal logic, where the necessitation operator $\Box$ is rewritten as K. Formal interpretation of $\Box$ and K is the same if we consider modal logic at least as strong as logic T. Operator $\Box$ supposed to express a kind of necessity, very often called metaphysical or alethic one, while operator K supposed to enable one to express a propositional attitudes, that an agent knows this or that (see for instance [13]). The distinguishing feature of the standard epistemic logic is that it contains the schema (T): $\mathrm{K}A \rightarrow A$. By (T) the classical property of knowledge is expressed, i.e. what is known is true. Other interesting properties that are often considered on the ground of modal epistemic logic are so-called positive and negative introspection. The former means that if an agent knows that A, then he knows that he knows that A. On the formal ground it is expressed by the schema (4): $\mathrm{K}A \rightarrow \mathrm{KK}A$. The latter means that if an agent does not know that A, then he knows that he does not know that A. Such a property is expressed by the schema (5): $\neg\mathrm{K}A \rightarrow \mathrm{K}\neg\mathrm{K}A$.

One of the big questions with respect to propositional attitudes is the logical omniscience, i.e. a problem of the deductive closure of agent's knowledge and a problem of knowing by an agent all thesis of a given logic. On the formal ground the schema (K) $\mathrm{K}(A \rightarrow B) \rightarrow (\mathrm{K}A \rightarrow \mathrm{K}B)$ and the Necessitation Rule (RN): if A is a thesis than $\mathrm{K}A$ is a thesis, also known as the Gödel's Rule, enable one to prove the Monotonicity Rule (RM): if $A \rightarrow B$ is a thesis, then $\mathrm{K}A \rightarrow \mathrm{K}B$ is a thesis. The rule (RM) simply says that an agent's knowledge is deductively closed. And let us remember that metavariables A and B represent formulas of arbitrary complexity. While the bigger the complexity of formula is, the harder the reasoning to perform. The deductive closure, however, makes sure that no matter how hard the reasoning is, the agent is able to derive the consequence. It seems highly unintuitive with regard to empirical agents like human beings. From this perspective even sole (K) and (RN) might seem to be unintuitive. For (K) says that the agent's knowledge is closed under the Modus Ponens and no matter what formulas are taken into account. And (RN) says that agent knows each thesis of a given logic which is rather impossible.

5. Epistemic positional logics

Let us stick to the language of MR^+. Formulas of the form $\mathcal{R}_\alpha A$ might be read: agent α knows that A. By counterparts of modal schemata (K), (T), (4) and (5) in the positional language we mean the following schemata:

$$\mathcal{R}_\alpha(A \to B) \to (\mathcal{R}_\alpha A \to \mathcal{R}_\alpha B) \tag{$\mathcal{R}$K}$$

$$\mathcal{R}_\alpha A \to A \tag{$\mathcal{R}$T}$$

$$\mathcal{R}_\alpha A \to \mathcal{R}_\alpha \mathcal{R}_\alpha A \tag{$\mathcal{R}$4}$$

$$\neg\mathcal{R}_\alpha A \to \mathcal{R}_\alpha \neg\mathcal{R}_\alpha A. \tag{$\mathcal{R}$5}$$

The counterpart of (RN) is of the following form:

$$\text{if } A \text{ is valid, then } \mathcal{R}_\alpha A \text{ is valid.} \tag{$\mathcal{R}$RN}$$

The rule ($\mathcal{R}$RN) is not only positional but also a semantic counterpart of (RN).

Our main goal is to obtain epistemic logic based on positional logic such that:

- it contains the positional counterpart of (T),
- it does not contain the positional counterpart of (K),
- the positional semantic counterpart of (RN) is not satisfied,
- some of its extensions contain counterparts of (4) and (5).

5.1. Semantics

Notice that MR^+ contains ($\mathcal{R}$K). Suppose $\mathfrak{M} \models \mathcal{R}_\alpha(A \to B)$ and $\mathfrak{M} \models \mathcal{R}_\alpha A$. Then, by the truth-condition (m_1), $v(f(\alpha), A \to B) = 1$ and $v(f(\alpha), A) = 1$. Hence, by the condition (v_4^+), $v(f(\alpha), B) = 1$. Thus, by the truth-condition (m$_1$), $\mathfrak{M} \models \mathcal{R}_\alpha B$. Moreover, by (‡) for MR^+ ($\mathcal{R}$RN) is satisfied. Thus we have to change the notion of a MR^+-model.

A *non-standard* MR^+*-model* (for short: a *non-standard model*) is a triple $\langle W, f, v\rangle$ such that:

- W, f are as in the previous cases,
- $v(\overrightarrow{W} \times \mathsf{For}^+) \cup \mathsf{F} \longrightarrow \{0, 1\}$ is such that:
  - $v\restriction_{\overrightarrow{W}\times\mathsf{For}^+}$ is such that (v_6^+) is satisfied and for other cases is arbitrary,
  - $v\restriction_{\mathsf{F}}$ is a classical CPL valuation.

The truth-conditions are the same as in the case of MR^+-models.

By means of non-standard models we avoid the problem of logical omniscience. For instance, we still have that $p\vee\neg p$ is valid but since the valuation of a non-standard model is arbitrary on $\langle w, p \vee \neg p\rangle$ it does not have to be the case that for any $\alpha \in \mathsf{IC}$, $\mathcal{R}_\alpha p \vee \neg p$. By means of such models we also can falsify ($\mathcal{R}$K). Consider a non-standard model $\mathfrak{M} = \langle W, f, v\rangle$ such that $W = \{w\}$, $f(\mathsf{IC}) = \{w\}$ and v is such that, for any $(w_1, \ldots, w_n) \in \overrightarrow{W}$:

- $v(\langle(w_1 \ldots, w_n), q\rangle) = 0$, if $n = 1$,
- $v(\langle(w_1 \ldots, w_n), q\rangle) = 1$, if $n > 1$,
- $v(\langle(w_1 \ldots, w_n), A\rangle) = 1$, for any $A \in \mathsf{For}^+ \setminus \{q\}$,
- $v(\mathsf{VAR}) = \{1\}$ and is extended on F in the standard way.

Then $\mathfrak{M} \vDash \mathcal{R}_\alpha(p \to q)$ and $\mathfrak{M} \vDash \mathcal{R}_\alpha p$ but $\mathfrak{M} \nvDash \mathcal{R}_\alpha q$.

In order to validate formulas of the schema ($\mathcal{R}$T) we need to stipulate some additional restrictions on non-standard models. Let us consider the following condition, for any $\alpha \in \mathsf{IC}$, for any $A \in \mathsf{For}^+$:

$$\text{if } v(\langle f(\alpha), A\rangle) = 1 \text{ then } \mathfrak{M} \vDash A. \qquad (\star)$$

In order to validate formulas of the schema ($\mathcal{R}$4) and ($\mathcal{R}$5) we use the following conditions, for any $\alpha \in \mathsf{IC}$, for any $A \in \mathsf{For}^+$:

$$\text{if } v(\langle f(\alpha), A\rangle) = 1 \text{ then } v(\langle f(\alpha), \mathcal{R}_\alpha A\rangle) = 1 \qquad (\star\star)$$
$$\text{if } v(\langle f(\alpha), A\rangle) = 0 \text{ then } v(\langle f(\alpha), \neg\mathcal{R}_\alpha A\rangle) = 1. \qquad (\star\star\star)$$

We receive the following fact:

FACT 5.1. Let $\mathfrak{M} = \langle W, f, v\rangle$ be a non-standard model, $A \in \mathsf{For}^+$ and $\alpha \in \mathsf{IC}$. Then:

(1) $(\star)$ is satisfied iff $\mathfrak{M} \models \mathcal{R}_\alpha A \rightarrow A$,

(2) $(\star\star)$ is satisfied iff $\mathfrak{M} \models \mathcal{R}_\alpha A \rightarrow \mathcal{R}_\alpha\mathcal{R}_\alpha A$,

(3) $(\star\star\star)$ is satisfied iff $\mathfrak{M} \models \neg\mathcal{R}_\alpha A \rightarrow \mathcal{R}_\alpha\neg\mathcal{R}_\alpha A$.

PROOF: Ad. (1). Suppose that $(\star)$ holds and $\mathfrak{M} \models \mathcal{R}_\alpha A$. Hence $v(f(\alpha), A)$ $= 1$. By $(\star)$ we get $\mathfrak{M} \models A$, hence $\mathfrak{M} \models A$. For the other direction suppose $\mathfrak{M} \models \mathcal{R}_\alpha A \rightarrow A$ and $v(\langle f(\alpha), A\rangle) = 1$. This means $\mathfrak{M} \models \mathcal{R}_\alpha A$, which gives us $\mathfrak{M} \models A$.

Ad. (2). Suppose that $(\star\star)$ holds and $\mathfrak{M} \models \mathcal{R}_\alpha A$. Thus $c(\langle f(\alpha), A\rangle) =$ 1. By $(\star\star)$ we get $v(\langle f(\alpha), \mathcal{R}_\alpha A\rangle) = 1$, hence $\mathfrak{M} \models \mathcal{R}_\alpha\mathcal{R}_\alpha A$. For the other direction suppose $\mathfrak{M} \models \mathcal{R}_\alpha A \rightarrow \mathcal{R}_\alpha\mathcal{R}_\alpha A$ and $v(\langle f(\alpha), A\rangle) = 1$. Hence $\mathfrak{M} \models \mathcal{R}_\alpha\mathcal{R}_\alpha A$. By (v_6^+) we obtain $v(\langle f(\alpha), \mathcal{R}_\alpha A\rangle) = 1$.

Ad. (3). Suppose that $(\star\star\star)$ holds and $\mathfrak{M} \models \neg\mathcal{R}_\alpha A$. Thus $v(\langle f(\alpha), A\rangle)$ $= 0$. By $(\star\star\star)$ $v(\langle f(\alpha), \neg\mathcal{R}_\alpha A\rangle) = 1$, so $\mathfrak{M} \models \mathcal{R}_\alpha\neg\mathcal{R}_\alpha A$. For the other direction suppose $\mathfrak{M} \models \neg\mathcal{R}_\alpha A \rightarrow \mathcal{R}_\alpha\neg\mathcal{R}_\alpha A$ and $v(\langle f(\alpha), A\rangle) = 0$. Hence $\mathfrak{M} \models \neg\mathcal{R}_\alpha A$, so $\mathfrak{M} \models \mathcal{R}_\alpha\neg\mathcal{R}_\alpha A$ which means $v(\langle f(\alpha), \neg\mathcal{R}_\alpha A\rangle) = 1$. □

Any non-standard model such that $(\star)$ is satisfied shall be called a *model of* ER (for short: an ER-*model*). Any ER-model such that $(\star\star)$ (resp. $(\star\star\star)$) is satisfied shall be called a *model of* ER4 (resp. a *model of* ER5) (for short: an ER4-*model*, resp. an ER5-*model*). A logic ER might be considered the minimal epistemic positional logic based in non-standard models. Logics ER4 and ER5 are the minimal epistemic positional logics based on non-standard models that contain $(\mathcal{R}4)$ and $(\mathcal{R}5)$ respectively.

5.2. Tableau systems of ER, ER4 and ER5

For logics ER, ER4 and ER5 the elimination rules for standard connectives outside the scope of operator $\mathcal{R}$ are the same as in the case of MR^+. For our logics we also have to include the rule $\mathrm{R}_{\mathcal{R}\neg}$ from Figure 3 and the rule $\mathrm{R}_{\mathcal{R}\mathrm{T}}$ from Figure 4. In the case of logic ER4 (resp. ER5) we additionally include $\mathrm{R}_{\mathcal{R}4}$ (resp. $\mathrm{R}_{\mathcal{R}5}$) from Figure 4. In the case of logic ER we assume all the specific rules. The sets of tableau rules for ER (resp. ER4, ER5) shall be denoted as $\mathbf{R}_{\mathsf{ER}}$ (resp. $\mathbf{R}_{\mathsf{ER4}}$, $\mathbf{R}_{\mathsf{ER5}}$).

For any $A, B \in \mathsf{For}^+$:

$$\mathrm{R}_{\mathcal{R}\mathrm{T}} : \frac{\mathcal{R}_\alpha A}{A} \qquad \mathrm{R}_{\mathcal{R}4} : \frac{\neg\mathcal{R}_\alpha \mathcal{R}_\alpha A}{\neg\mathcal{R}_\alpha A} \qquad \mathrm{R}_{\mathcal{R}5} : \frac{\neg\mathcal{R}_\alpha \neg\mathcal{R}_\alpha A}{\mathcal{R}_\alpha A}$$

Figure 4. Specific rules for $\mathcal{R}$ operator

Let us notice an interesting dependence. By means of rules $\mathrm{R}_{\mathcal{R}\mathrm{T}}$, $\mathrm{R}_{\neg\neg}$ and $\mathrm{R}_{\neg\mathcal{R}\neg}$ we can easily derive the rule $\mathrm{R}_{\mathcal{R}5}$.

1. $\neg\mathcal{R}_\alpha\neg\mathcal{R}_\alpha A$
2. $\neg\neg\mathcal{R}_\alpha\mathcal{R}_\alpha A$ — by the rule $\mathrm{R}_{\neg\mathcal{R}\neg}$ and 1
3. $\mathcal{R}_\alpha\mathcal{R}_\alpha A$ — by the rule $\mathrm{R}_{\neg\neg}$ and 2
4. $\mathcal{R}_\alpha A$ — by the rule $\mathrm{R}_{\mathcal{R}\mathrm{T}}$ and 3 □

Clearly the rule $\mathrm{R}_{\neg\mathcal{R}\neg}$ corresponds with the condition (v_1^+). We have that the logic determined by ER-models such that the condition (v_1^+) is satisfied contains ($\mathcal{R}5$). Suppose $\mathfrak{M} \models \neg\mathcal{R}_\alpha\neg\mathcal{R}_\alpha A$. Thus $v(f(\alpha), \neg\mathcal{R}_\alpha A) = 0$, by the condition ($\mathrm{v}_1^+$) $v(f(\alpha), \mathcal{R}_\alpha A) = 1$. By the condition ($\star$) $\mathfrak{M} \models \mathcal{R}_\alpha A$.

5.3. Soundness and completeness of tableau systems for ER, ER4 and ER5

A soundness theorem might be proved similarly as in the case of MR^+ and MR. Let us notice that by fact 5.1 by applications of new rules $\mathrm{R}_{\mathcal{R}\mathrm{T}}$, $\mathrm{R}_{\mathcal{R}4}$ and $\mathrm{R}_{\mathcal{R}5}$ from satisfiable formulas we receive some satisfiable formulas.

LEMMA 5.2. *Let $\Lambda \in \{\mathsf{ER}, \mathsf{ER4}, \mathsf{ER5}\}$, $X \subseteq \mathsf{For}^+$ and $\mathfrak{M} = \langle W, f, v\rangle$ be a Λ-model suitable for X. If any rule from $\mathbf{R}_\Lambda$ has been applied to X, then $\mathfrak{M}$ is suitable for the union of X and at least one output obtained by application of that rule.*

PROOF: Similarly as for lemma 2.1.

Suppose that $\mathrm{R}_{\mathcal{R}\mathrm{T}}$ has been applied to X. Hence $\mathcal{R}_\alpha A \in X$. Since $\mathfrak{M}$ is suitable for X $\mathfrak{M} \models \mathcal{R}_\alpha A$. By 5.1 (1), we obtain $\mathfrak{M} \models A$.

Let $\Lambda = \mathsf{ER4}$. Suppose that $\mathrm{R}_{\mathcal{R}4}$ has been applied to X. Hence $\neg\mathcal{R}_\alpha\mathcal{R}_\alpha A \in X$. Since $\mathfrak{M}$ is suitable for X $\mathfrak{M} \models \neg\mathcal{R}_\alpha\mathcal{R}_\alpha A$. By 5.1 (2) we obtain $\mathfrak{M} \models \neg\mathcal{R}_\alpha A$.

Let $\Lambda = \mathsf{ER5}$. Suppose that $\mathrm{R}_{\mathcal{R}5}$ has been applied to X. Hence $\neg\mathcal{R}_\alpha\neg\mathcal{R}_\alpha A \in X$. Hence $\mathfrak{M} \models \neg\mathcal{R}_\alpha\neg\mathcal{R}_\alpha A$. Since $\mathfrak{M}$ is suitable for X $\mathfrak{M} \models \neg\mathcal{R}_\alpha\neg\mathcal{R}_\alpha A$. By 5.1 (3) and the truth-condition (m_2) we obtain $\mathfrak{M} \models \mathcal{R}_\alpha A$. □

In order to prove completeness we use the same argument as before but with respect to modified version of the generated model. Let us first present special extensions of sets closed under tableau rules. Let $\Lambda \in \{\mathsf{ER}, \mathsf{ER4}, \mathsf{ER5}\}$ and X be a $\mathbf{R}_\Lambda$-closed set. By X_Λ we shall denote a set such that:

- if $\Lambda = \mathsf{ER}$ then $X_\Lambda = X$,
- if $\Lambda = \mathsf{ER4}$ then X_Λ is the smallest set $Y \subseteq \mathsf{For}^+$ such that $X \subseteq Y$ and if $\mathcal{R}_\alpha A \in Y$ then $\mathcal{R}_\alpha\mathcal{R}_\alpha A \in Y$,
- if $\Lambda = \mathsf{ER5}$ then X_Λ is the smallest set $Y \subseteq \mathsf{For}^+$ such that $X \subseteq Y$ and if either $\neg\mathcal{R}_\alpha A \in X$ or $\mathcal{R}_\alpha A \notin Y$ then $\mathcal{R}_\alpha\neg\mathcal{R}_\alpha A \in Y$.

We have the following fact:

Fact 5.3. Let $\Lambda \in \{\mathsf{ER}, \mathsf{ER4}, \mathsf{ER5}\}$ and X be a $\mathbf{R}_\Lambda$-closed set. If X is t-consistent then X_Λ is t-consistent.

Proof: Let $\Lambda = \mathsf{ER4}$. Assume that X is t-consistent and X_{ER4} is t-inconsistent. Note that there are no formulas of the form $\neg A$ in $X_{\mathsf{ER4}} \setminus X$ – there are formulas preceded by an $\mathcal{R}$ operator only. For this reason X_{ER4} can be t-inconsistent only when $\neg\underbrace{\mathcal{R}_\alpha \ldots \mathcal{R}_\alpha}_{n} A \in X$ and $\underbrace{\mathcal{R}_\alpha \ldots \mathcal{R}_\alpha}_{n} A \in X_{\mathsf{ER4}} \setminus X$, for some $n \geq 1$. Hence by definition of X_{ER4}, $\underbrace{\mathcal{R}_\alpha \ldots \mathcal{R}_\alpha}_{k} A \in X$, for some $k < n$. But by application of rule $\mathrm{R}_{\mathcal{R}4}$ $n-k$ times we obtain $\neg\underbrace{\mathcal{R}_\alpha \ldots \mathcal{R}_\alpha}_{k} A \in X$, so X is t-inconsistent which gives us contradiction with the assumption.

Let $\Lambda = \mathsf{ER5}$. Reasoning in the same manner, we assume that X is t-consistent and X_{ER5} is t-inconsistent. Hence $\mathcal{R}_\alpha\neg\mathcal{R}_\alpha A \in X_{\mathsf{ER5}} \setminus X$ and $\neg\mathcal{R}_\alpha \neg\mathcal{R}_\alpha A \in X$. By the definition of X_{ER5}, either $\neg\mathcal{R}_\alpha A \in X$ or $\mathcal{R}_\alpha A \notin X_{\mathsf{ER5}}$.

In the second case we get $\mathcal{R}_\alpha A \notin X \subseteq X_{\mathsf{ER5}}$. By the application of the rule $\mathrm{R}_{\mathcal{R}5}$ we obtain $\mathcal{R}_\alpha A \in X$. In both cases we get a contradiction. $\square$

Let $\Lambda \in \{\mathsf{ER}, \mathsf{ER4}, \mathsf{ER5}\}$ and X be a $\mathbf{R}_\Lambda$-closed set. A Λ-*model generated by* X_Λ (for short: a X_Λ-*model*) is a Λ-model $\langle W, f, v\rangle$ such that:

- W, f are as in the previous case,
- for any for any $(\alpha_1, \dots, \alpha_n) \in \overrightarrow{W}$ and any $A \in \mathsf{For}^+$:

$$v(\langle(\alpha_1, \dots, \alpha_n), A\rangle) = \begin{cases} 1, & \text{if } \mathcal{R}_{\alpha_1} \dots \mathcal{R}_{\alpha_n} A \in X_\Lambda \\ 0, & \text{if } \mathcal{R}_{\alpha_1} \dots \mathcal{R}_{\alpha_n} A \notin X_\Lambda \end{cases}$$

- for any $A \in \mathsf{VAR}$ we stipulate:

$$v(A) = \begin{cases} 1, & \text{if } A \in X_\Lambda \\ 0, & \text{if } A \notin X_\Lambda \end{cases}$$

  we extend v^E on F in the standard way.

Because of the definition of valuation from X_Λ-model, implications of the fact 3.4 are obvious. The implications obviously hold if in the antecedents we change X on X_Λ.

Lemma 5.4. *Let* $\Lambda \in \{\mathsf{ER}, \mathsf{ER4}, \mathsf{ER5}\}$, X *be a* $\mathbf{R}_\Lambda$*-closed set and* $\mathfrak{M}$ *be a* X_Λ*-model. Then, for any* $A \in \mathsf{For}^+$*:*

- *if* $A \in X_\Lambda$ *then* $\mathfrak{M} \vDash A$,
- *if* $\neg A \in X_\Lambda$ *then* $\mathfrak{M} \nvDash A$.

Proof: *Base case.* Let $A \in \mathsf{For}^+$ and $c^+(A) = 1$. Thus $A \in \mathsf{VAR}$. Suppose $A \in X$. Then, by the definition of v, $v(A) = 1$.

Inductive hypothesis. Let $n \in \mathbb{N}$. Suppose that for any $A \in \mathsf{For}^+$ such that $c^+(A) \leq n$, if $A \in X$ then $\mathfrak{M} \vDash A$.

Inductive step. Let $A \in \mathsf{For}^+$ and $c^+(A) = n + 1$. We consider the following cases, the others are considered in a similar way.

Let $A = \neg\neg B$. Suppose $\neg\neg B \in X_\Lambda$. Thus, by the definition of X_Λ, $\neg\neg B \in X$. Hence, by the application of the rule $\mathrm{R}_{\neg\neg}$, $B \in X$. Thus, by the inductive hypothesis, $\mathfrak{M} \vDash B$.

Let $A = B * C$, where $* \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$. We consider only case for $* = \wedge$. For other cases we reason in the similar way. Suppose $B \wedge C \in X_\Lambda$.

Thus, by the definition of X_Λ, $B \wedge C \in X$. Since X is $\mathbf{R}_\Lambda$-closed set, by the application of the rule $\mathrm{R}_\wedge$, $B, C \in X$. Hence, by the inductive hypothesis, $\mathfrak{M} \vDash B$ and $\mathfrak{M} \vDash C$. Therefore, by the truth-condition (m$_3$), $\mathfrak{M} \vDash B \wedge C$.

Let $A = \neg(B * C)$, where $* \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$. We consider only case for $* = \wedge$. For other cases we reason in the similar way. Suppose $\neg(B \wedge C) \in X_\Lambda$. Thus, by the definition of X_Λ, $\neg(B \wedge C) \in X$. Since X is $\mathbf{R}_\Lambda$-closed set, by the application of the rule $\mathrm{R}_{\neg\wedge}$, either $\neg B \in X$ or $\neg C \in X$. Hence, by the inductive hypothesis and the truth-condition (m$_2$), either $\mathfrak{M} \nvDash B$ or $\mathfrak{M} \nvDash C$. Therefore, by the truth-condition (m$_3$), $\mathfrak{M} \vDash B \wedge C$.

Let $A = \mathcal{R}_{\alpha_1} \dots \mathcal{R}_{\alpha_n} B$. Suppose $\mathcal{R}_{\alpha_1} \dots \mathcal{R}_{\alpha_n} B \in X_\Lambda$. By the definition of a X_Λ-model, the truth-condition (m$_1$) and the condition (v_6^+) $\mathfrak{M} \vDash \mathcal{R}_{\alpha_1} \dots \mathcal{R}_{\alpha_n} B$.

Let $A = \neg\mathcal{R}_{\alpha_1} \dots \mathcal{R}_{\alpha_n} B$. Suppose $\neg\mathcal{R}_{\alpha_1} \dots \mathcal{R}_{\alpha_n} B \in X_\Lambda$. By the definition of a X_Λ-model the truth-condition (m$_1$) and the condition (v_6^+) $\mathfrak{M} \nvDash \mathcal{R}_{\alpha_1} \dots \mathcal{R}_{\alpha_n} B$. □

The following fact shows that generated models satisfy the proper conditions.

Fact 5.5. Let $\Lambda \in \{\mathsf{ER}, \mathsf{ER4}, \mathsf{ER5}\}$, $X \subseteq \mathsf{For}^+$ be a t-consistent $\mathbf{R}_\Lambda$-closed set and $\mathfrak{M} = \langle W, f, v \rangle$ be a X_Λ-model. Then:

(1) the condition ($\star$) is satisfied,

(2) if $\Lambda = \mathsf{ER4}$ then the condition ($\star\star$) is satisfied,

(2) if $\Lambda = \mathsf{ER5}$ then the condition ($\star\star\star$) is satisfied.

Proof: Ad (1). Suppose $v(\langle f(\alpha), A \rangle) = 1$. Hence, by the definition of a X_Λ-model, $\mathcal{R}_\alpha A \in X_\Lambda$. Assume $\mathcal{R}_\alpha A \in X$. Thus, by the rule $\mathrm{R}_{\mathcal{R}\mathrm{T}}$, $A \in X$. By lemma 5.4 (1) $\mathfrak{M} \vDash A$. Assume $\mathcal{R}_\alpha A \notin X$. We have two possible cases. Let $\mathcal{R}_\alpha A = \mathcal{R}_\alpha \mathcal{R}_\alpha \dots \mathcal{R}_\alpha B$. Thus, by the definition of a X_{ER4}-model, $\mathcal{R}_\alpha \dots \mathcal{R}_\alpha B \in X_{\mathsf{ER4}}$. By lemma 5.4 (1) $\mathfrak{M} \vDash \mathcal{R}_\alpha \dots \mathcal{R}_\alpha B$. Let $\mathcal{R}_\alpha A = \mathcal{R}_\alpha \neg\mathcal{R}_\alpha B$. Thus $A = \neg\mathcal{R}_\alpha B$ and either (a) $\neg\mathcal{R}_\alpha B \in X \subseteq X_{\mathsf{ER5}}$ or (b) $\mathcal{R}_\alpha B \notin X_{\mathsf{ER5}}$. If (a), then by lemma 5.4 (1) $\mathfrak{M} \vDash \neg\mathcal{R}_\alpha B$. If (b), then by the definition of X_{ER5}-model, $v(\langle f(\alpha), B \rangle) = 0$. Hence, by truth-conditions (m$_1$) and (m$_2$), $\mathfrak{M} \vDash \neg\mathcal{R}_\alpha B$.

Ad (2). Suppose $v(\langle f(\alpha), A \rangle) = 1$. Hence, by the definition of a X_{ER4}-model, $\mathcal{R}_\alpha A \in X_{\mathsf{ER4}}$. Thus $\mathcal{R}_\alpha \mathcal{R}_\alpha A \in X_{\mathsf{ER4}}$. By lemma 5.4 (1) $\mathfrak{M} \vDash \mathcal{R}_\alpha \mathcal{R}_\alpha A$. Hence, by the truth-condition (m$_1$), $v(\langle f(\alpha), \mathcal{R}_\alpha A \rangle) = 1$.

Ad (3). Suppose $v(\langle f(\alpha), A\rangle) = 0$. Thus, by truth-conditions (m_1) and (m_2), $\mathfrak{M} \nvDash \mathcal{R}_\alpha A$. By lemma 5.4 (1) $\mathcal{R}_\alpha A \notin X_{\mathsf{ER5}}$. Hence, by the definition of X_{ER5}, $\mathcal{R}_\alpha \neg \mathcal{R}_\alpha A \in X_{\mathsf{ER5}}$. By lemma 5.4 (1) $\mathfrak{M} \vDash \mathcal{R}_\alpha A \neg \mathcal{R}_\alpha A$. Hence, by the truth-condition (m_1), $v(\langle f(\alpha), \neg \mathcal{R}_\alpha A\rangle) = 1$. □

As before, by lemmas 5.2, 5.4, we get the following theorem:

THEOREM 5.6. *Let* $X \cup \{A\} \subseteq \mathsf{For}^+$. *Then:*

(1) $X \rhd_{\mathsf{ER}} A$ *iff* $X \vDash_{\mathsf{ER}} A$,

(2) $X \rhd_{\mathsf{ER4}} A$ *iff* $X \vDash_{\mathsf{ER4}} A$,

(3) $X \rhd_{\mathsf{ER5}} A$ *iff* $X \vDash_{\mathsf{ER5}} A$.

Acknowledgements. This work was supported by the National Science Centre, Poland, under Grants UMO-2015/19/N/HS1/02401 and UMO-2015/19/B/HS1/02478. The authors would like to thank Tomasz Jarmużek for all his ideas and valuable suggestions.

References

[1] T. Jarmużek, *Minimal Logical Systems With R-operator: Their Metalogical Properties and Ways of Extensions*, [in:] J. Bézieau, A. Costa-Leite (eds.), **Perspectives on Universal Logic**, Polimetrica Publisher (2007), pp. 319–333.

[2] T. Jarmużek, *Tableau Metatheorem for Modal Logics*, [in:] R. Ciuni, H. Wansing, C. Willkommen (eds.), **Trends in Logic Vol. 41. Recent Trends in Philosophical Logic**, Springer International Publishing (2014), pp. 103–126, DOI: https://doi.org/10.1007/978-3-319-06080-4_8.

[3] T. Jarmużek, **On the Sea Battle Tomorrow That May Not Happen**, Peter Lang, Berlin (2018), DOI: https://doi.org/10.3726/b14343.

[4] T. Jarmużek, M. Klonowski, *Some Intensional Logics Defined by Relating Semantics and Tableau Systems*, [in:] A. Giordani, J. Malinowski (eds.), **Logic in High Definition. Current Issues in Logical Semantics**, Springer International Publishing (2021), pp. 31–48, DOI: https://doi.org/10.1007/978-3-030-53487-5_3.

[5] T. Jarmużek, A. Parol, *On Some Language Extension of Logic MR: A Semantic and Tableau Approach*, **Roczniki Filozoficzne**, vol. 68 (2021), pp. 345–366, DOI: https://doi.org/10.18290/rf20684-16.

[6] T. Jarmużek, A. Pietruszczak, *Completeness of Minimal Positional Calculus*, **Logic and Logical Philosophy**, vol. 13 (2004), pp. 147–162, DOI: https://doi.org/10.12775/LLP.2004.009.

[7] T. Jarmużek, M. Tkaczyk, **Normalne logiki pozycyjne (Normal Positional Logics)**, Wydawnictwo KUL, Lublin (2015).

[8] T. Jarmużek, M. Tkaczyk, *Expressive Power of the Positional Operator R: a Case Study in Modal Logic and Modal Philosophy*, **Ruch Filozoficzny**, vol. 75 (2019), pp. 93–107, DOI: https://doi.org/10.12775/RF.2019.022.

[9] A. Karczewska, *Maximality of the Minimal R-Logic*, **Logic and Logical Philosophy**, vol. 27 (2017), pp. 193–203, DOI: https://doi.org/10.12775/LLP.2017.008.

[10] J. Łoś, *Podstawy analizy metodologicznej kanonów Milla (The Foundations of Methodological Analysis of Mill's Canons)*, **Annales Universitatis Mariae Curie-Skłodowska**, vol. 2 (1947), pp. 269–301.

[11] J. Łoś, *Logiki wielowartościowe a formalizacja funkcji intensjonalnych (Many-Valued Logics and Formalization of Intensional Functions)*, **Kwartalnik Filozoficzny**, vol. 17(1) (1948), pp. 59–68.

[12] J. Malinowski, K. Pietrowicz, J. Szalacha-Jarmużek, *Logic of Social Ontology and Łoś's Operator*, **Logic and Logical Philosophy**, vol. 29 (2020), pp. 239–258, DOI: https://doi.org/10.12775/LLP.2020.005.

[13] J. Meyer, *Epistemic Logic*, [in:] L. Goble (ed.), **The Blackwell Guide to Philosophical Logic**, Blackwell (2001), pp. 183–202, DOI: https://doi.org/10.1002/9781405164801.

[14] G. Priest, **An Introduction to Non-Classical Logic. From If to Is. 2nd ed**, Cambridge University Press, Cambridge (2008), DOI: https://doi.org/10.1017/CBO9780511801174.

[15] N. Rescher, *On the Logic of Chronological Propositions*, **Mind**, vol. 75(297) (1966), pp. 75–96, DOI: https://doi.org/10.2307/2251711.

[16] N. Rescher, *Topological Logic*, [in:] **Topics in Philosophical Logic**, Reidel Publishing Company, Dordrecht (1968), pp. 229–249, DOI: https://doi.org/10.1007/978-94-017-3546-9_13.

[17] N. Rescher, A. Urquhart, **Temporal Logic**, Springer Verlag, Vienna–New York (1971).

[18] M. Tkaczyk, **Logika czasu empirycznego: funktor realizacji czasowej w językach teorii fizykalnych** (**Logic of Empirical Time: Functor

of Temporal Realisation in the Languages of Phisical Theories), Wydawnictwo KUL, Lublin (2009).

[19] M. Tkaczyk, *Negation in Weak Positional Calculi*, **Logic and Logical Philosophy**, vol. 22 (2013), pp. 3–19, DOI: https://doi.org/10.12775/LLP.2013.001.

[20] M. Tkaczyk, *Distribution Laws in Weak Positional Logics*, **Roczniki Filozoficzne**, vol. 66 (2018), pp. 163–179, DOI: https://doi.org/10.18290/rf.2018.66.3-8.

Mateusz Klonowski
Nicolaus Copernicus University
Department of Logic
ul. Stanisława Moniuszki 16/20
87-100 Toruń, Poland
e-mail: mateusz.klonowski@umk.pl

Krzysztof Aleksander Krawczyk
Nicolaus Copernicus University
Department of Logic
ul. Stanisława Moniuszki 16/20
87-100 Toruń, Poland
e-mail: krawczyk@doktorant.umk.pl

Bożena Pięta
Nicolaus Copernicus University
Department of Logic
ul. Stanisława Moniuszki 16/20
87-100 Toruń, Poland
e-mail: b.pieta@doktorant.umk.pl