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A NOTE ON DISTRIBUTIVE TRIPLES

Abstract

Even if a lattice L is not distributive, it is still possible that for particular elements $x, y, z \in L$ it holds $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$. If this is the case, we say that the triple (x, y, z) is distributive. In this note we provide some sufficient conditions for the distributivity of a given triple.

Keywords: Distributive triple, dually distributive triple, covering diamond.

Standard lattice-theoretic notions can be found in [3]. Let us recall basic definitions and facts. If L is a lattice and $a, b \in L$, then the set $[a, b] = \{c \in L : a \leqslant c \leqslant b\}$ is called an *interval* (in L). Clearly, any interval is a sublattice of L. If $X \subseteq L$, then $[X]$ stands for the sublattice generated by X, i.e., the smallest sublattice of L, which contains the subset X. For any subset $X \subseteq L$ and for any interval $[a, b]$ we define

$$[\![a, b]\!]_X := [a, b] \cap [X].$$

In particular, if $X = \{x, y, z\}$, then $[\![x \wedge y \wedge z, x \vee y \vee z]\!]_X = [X]$.

A lattice L is said to be *modular* if $x \leqslant z$ implies $(x \vee y) \wedge z = x \vee (y \wedge z)$, for all $x, y, z \in L$. Moreover, L is called *distributive* if $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$, for all $x, y, z \in L$. The Dedekind–Birkhoff Theorem (cf. [3], p. 59) states that a lattice L is modular if and only if L does not contain a sublattice isomorphic to N_5 (so-called *pentagon*), and moreover, and L is distributive if and only if L does not contain a sublattice isomorphic to N_5 nor M_3 (so-called *diamond*).

Let L be an arbitrary lattice and $x, y, z \in L$. We say that (x, y, z) is a *distributive triple*, $(x, y, z)D$ in symbols, if $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$. Similarly, (x, y, z) is called a *dually distributive triple*, $(x, y, z)D^*$ in symbols, if $(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$ (cf. [7], p. 76[1]). Clearly, L is distributive if and only if $(x, y, z)D$, for all x, y, z. G. Birkhoff proved the following.

THEOREM 1 ([1], Theorem II.12). *Let L be a modular lattice and $X = \{x, y, z\} \subseteq L$. Then:*

(i) $[\![x \wedge y \wedge z, x \vee y \vee z]\!]_X$ is distributive if and only if $(x, y, z)D$,

(ii) $[\![x \wedge y \wedge z, x \vee y \vee z]\!]_X$ is distributive if and only if $(x, y, z)D^*$.

The Dedekind–Birkhoff Theorem shows that the hypothesis of modularity is necessary as well as sufficient in Theorem 1 (cf. the lattice (a) in Figure 1).

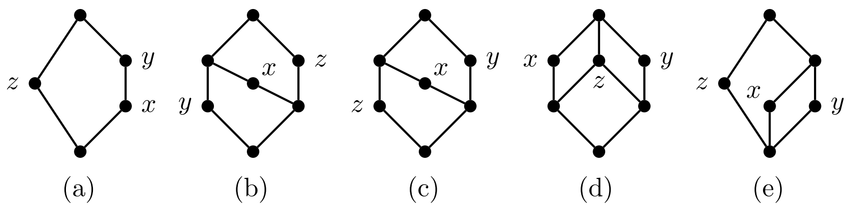

Fig. 1. Non-modular lattices satisfying $(x, y, z)D$ or $(x, y, z)D^*$.

Our result is the following.

THEOREM 2. *Let L be an arbitrary lattice and $X = \{x, y, z\} \subseteq L$. Then:*

(i) if $[\![x \wedge z, x \vee y \vee z]\!]_X$ and $[\![y \wedge z, x \vee y \vee z]\!]_X$ are distributive, then $(x, y, z)D$,

(ii) if $[\![x \wedge y \wedge z, x \vee y]\!]_X$ is distributive, then $(x, y, z)D$.

PROOF: To prove (i), assume that $[\![x \wedge z, x \vee y \vee z]\!]_X$ and $[\![y \wedge z, x \vee y \vee z]\!]_X$ are distributive sublattices of L. Then

[1] Note that Birkhoff in [1], p. 37, provides a different definition: a three-element subset $\{x, y, z\}$ of a lattice L is a distributive triple if $[\{x, y, z\}]$ is a distributive sublattice of L.

$$\begin{aligned}
z \wedge (x \vee y) &= z \wedge \Big(x \vee \big(y \vee (x \wedge z)\big)\Big) \\
&= (z \wedge x) \vee \Big(z \wedge \big(y \vee (x \wedge z)\big)\Big) \qquad \text{(by the 1st assumption)} \\
&= z \wedge \big(y \vee (x \wedge z)\big) \\
&= z \wedge \Big(y \vee \big((x \wedge z) \vee (y \wedge z)\big)\Big) \\
&= (z \wedge y) \vee \Big(z \wedge \big((x \wedge z) \vee (y \wedge z)\big)\Big) \\
&\qquad \text{(by the 2nd assumption)} \\
&= (z \wedge y) \vee \big((x \wedge z) \vee (y \wedge z)\big) \\
&= (z \wedge y) \vee (x \wedge z),
\end{aligned}$$

which completes the proof of (i).

For (ii), we assume that $[\![x \wedge y \wedge z, x \vee y]\!]_X$ is distributive and calculate as follows:

$$\begin{aligned}
z \wedge (x \vee y) &= \big(z \wedge (x \vee y)\big) \wedge (x \vee y) \\
&= \Big(\big(z \wedge (x \vee y)\big) \wedge x\Big) \vee \Big(\big(z \wedge (x \vee y)\big) \wedge y\Big) \\
&\qquad \text{(by the assumption)} \\
&= (z \wedge x) \vee (z \wedge y).
\end{aligned}$$

■

By the duality principle we obtain

THEOREM 3. *Let L be an arbitrary lattice and $X = \{x, y, z\} \subseteq L$. Then:*

(*i*) *if $[\![x \wedge y \wedge z, x \vee z]\!]_X$ and $[\![x \wedge y \wedge z, y \vee z]\!]_X$ are distributive, then $(x, y, z)D^*$,*

(*ii*) *if $[\![x \wedge y, x \vee y \vee z]\!]_X$ is distributive, then $(x, y, z)D^*$.*

REMARK 1. *Lattices (b) and (c) in Figure 1 disprove the converses of Theorems 2 and 3, respectively.*

REMARK 2. *Theorem 2 allows the conclusion that $(x, y, z)D$ in lattices (d) and (e) in Figure 1. On the other hand, this fact cannot be justified on the basis of Theorem 1.*

In order to illustrate a possible use of Theorem 2 we will provide an easy inductive proof of the following

Theorem 4. *Let L be a lattice of finite length. If L is modular but non-distributive lattice, then L contains a covering diamond, i.e., a diamond $D = \{o, a, b, c, i\}$, such that $o \prec a, b, c \prec i$.*

In the literature of lattice theory the preceding theorem is known as "folklore" (cf. [4], p. 111, or [2], p. 270). This theorem easily follows from [5] (cf. Theorem 1.4 for the case $n = 2$), or from [3] (cf. Lemma 8, p. 247). Note that [6] generalizes the theorem to the class of weakly atomic lattices.

Proof of Theorem 4: Induction on $l(L)$—the length of L. If $l(L) = 1$ or $l(L) = 2$ the theorem is obvious. For the induction step, assume that for any modular, non-distributive lattice K if $l(K) < n$, then K contains a covering diamond. Moreover, fix a modular, non-distributive lattice L such that $l(L) = n \geqslant 3$. Then, by Dedekind–Birkhoff Theorem, L contains a diamond $D = \{o, a, b, c, i\}$. If $0 < o$ or $i < 1$, then $[o, i]$ satisfies premises of our induction hypothesis, thus it contains a covering diamond, so L does. If not, i.e., $D = \{0, a, b, c, 1\}$, since $l(L) \geqslant 3$ there exists some intermediate element $x \notin D$; we may assume without loss of generality that $b < x < 1$.

Let us observe that $a \wedge x > 0$, because if not, the set $\{0, a, x, b, 1\}$ would be a pentagon. For similar reasons, $c \wedge x > 0$. Now, consider intervals $[a \wedge x, 1]$ and $[c \wedge x, 1]$. If one of them is non-distributive, then by the induction hypothesis, it contains a covering diamond, so L does. On the other hand, if both intervals are distributive, then by Theorem 2, the triple (a, c, x) is distributive, thus we obtain

$$(a \wedge x) \vee (c \wedge x) = (a \vee c) \wedge x = 1 \wedge x = x.$$

Moreover, by modularity, we get $(a \wedge x) \vee b = x$ and $(c \wedge x) \vee b = x$, and obviously $(a \wedge x) \wedge (c \wedge x) = (a \wedge x) \wedge b = (c \wedge x) \wedge b = 0$, so the set $\{0, a \wedge x, b, c \wedge x, x\}$ forms a diamond. Therefore, by the induction hypothesis, the interval $[0, x]$ contains a covering diamond, and hence L does. ∎

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