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COMPLEX FUZZY SETS WITH APPLICATION IN BCK/BCI-ALGEBRAS

Abstract

As a generation of fuzzy set, the notion of complex fuzzy set which is an innovative concept is introduced by Ramot, Milo, Friedman and Kandel. The purpose of this article is to apply complex fuzzy set to BCK/BCI-algebras. The notions of a complex subalgebra and a complex left (right) reduced ideal in a BCK/BCI-algebra are introduced, and related properties are investigated. Characterizations of a complex subalgebra are provided, and the homomorphic image (preimage) of a complex subalgebra and a complex left (right) reduced ideal.

Keywords: complex t -norm, min-complex t -norm, complex subalgebra, complex left (right) reduced ideal, complex characteristic function.

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1. Introduction

The extension of crisp sets to fuzzy sets, in terms of membership functions, is mathematically comparable to the extension of the set of integers, \mathbb{Z} , to the set of real numbers, \mathbb{R} . That is, expanding the range of the membership function, $\mu_A(x)$, from $\{0, 1\}$ to $[0, 1]$ is mathematically analogous to the extension of \mathbb{Z} to \mathbb{R} . Another extension of fuzzy set theory, Romat et al. [1] introduced the innovative complex fuzzy set. The complex fuzzy set, A , is characterized by a membership function, μ_A , whose range is not limited to $[0, 1]$ but extended to the unit circle in the complex plane. Hence,

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$\mu_A(x)$ is a complex valued function that assigns a grade of membership of the form $r_A \cdot e^{j\omega_A(x)}$, where $j = \sqrt{-1}$, to any element x in the universe of discourse. The value of $\mu_A(x)$ is defined by the two variables, $r_A(x)$ and $\omega_A(x)$, both real-valued, with $r_A(x) \in [0, 1]$. Tamir and Kandel [2] proposed an axiomatic framework for first order predicate complex fuzzy logic and use this framework for axiomatic definition of complex fuzzy classes. Al-Qudah and Hassan [3] introduced the concept of complex multi-fuzzy sets as a generalization of the concept of multi-fuzzy sets by adding the phase term to the definition of multi-fuzzy sets, and provided the structure of distance measure on complex multi-fuzzy sets by extending the structure of distance measure of complex fuzzy sets.

The aim of this paper is to apply the notion of complex fuzzy sets to *BCK/BCI*-algebras, and to generalize the fuzzy set theory in *BCK/BCI*-algebras. We introduce the notion of a complex subalgebra and a complex reduced left (right) ideal in a *BCK/BCI*-algebra, and investigate related properties. We provide characterizations of a complex subalgebra. We discuss the homomorphic image (preimage) of a complex subalgebra and a complex left (right) reduced ideal.

2. Preliminaries

By a *BCI-algebra*, we mean an algebra $(X, *, 0)$ of type $(2, 0)$ satisfying the axioms:

- (I) $(\forall x, y, z \in X) (((x * y) * (x * z)) * (z * y) = 0)$,
- (II) $(\forall x, y \in X) ((x * (x * y)) * y = 0)$,
- (III) $(\forall x \in X) (x * x = 0)$,
- (IV) $(\forall x, y \in X) (x * y = y * x = 0 \Rightarrow x = y)$.

We can define a partial ordering \leq by $x \leq y$ if and only if $x * y = 0$. If a *BCI*-algebra X satisfies $0 * x = 0$ for all $x \in X$, then we say that X is a *BCK*-algebra. A nonempty subset L of a *BCK/BCI*-algebra X is called a subalgebra of X if $x * y \in L$ for all $x, y \in L$. We refer the reader to the books [4, 5] for further information regarding *BCK/BCI*-algebras.

In 2011, Azam et al. [6] introduced the notion of complex valued metric space which is a generalization of the classical metric space, by defining the partial order “ \preceq ” on the set of complex numbers.

Let \mathbb{C} be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order “ \preceq ” on \mathbb{C} as follows:

$z_1 \preceq z_2$ if and only if $\text{Re}(z_1) \leq \text{Re}(z_2)$ and $\text{Im}(z_1) \leq \text{Im}(z_2)$,

that is, $z_1 \preceq z_2$ if one of the following holds

- (C1) $\text{Re}(z_1) = \text{Re}(z_2)$ and $\text{Im}(z_1) = \text{Im}(z_2)$,
- (C2) $\text{Re}(z_1) < \text{Re}(z_2)$ and $\text{Im}(z_1) = \text{Im}(z_2)$,
- (C3) $\text{Re}(z_1) = \text{Re}(z_2)$ and $\text{Im}(z_1) < \text{Im}(z_2)$,
- (C4) $\text{Re}(z_1) < \text{Re}(z_2)$ and $\text{Im}(z_1) < \text{Im}(z_2)$.

Ramot et al. [1] introduced the notion of complex fuzzy sets.

A *complex fuzzy set* A , defined on a universe of discourse X , is characterized by a membership function $\mu_A(x)$ that assigns any element $x \in X$ a complex valued grade of membership in X , that is, the complex fuzzy set A may be represented as the set of ordered pairs

$$A = \{(x, \mu_A(x)) \mid x \in X\}, \tag{2.1}$$

where $\mu_A(x) = r_A(x) \cdot e^{j\omega_A(x)}$, $j = \sqrt{-1}$, $r_A(x)$ and $\omega_A(x)$ are both real-valued, and $r_A(x) \in [0, 1]$. Evidently, each complex grade of membership is defined by an amplitude term $r_A(x)$ and a phase term $\omega_A(x)$.

3. Complex subalgebras

Let A and B be complex fuzzy sets on X with complex valued membership functions μ_A and μ_B , respectively. We define

$$\mu_A(y) \odot \mu_B(z) = [r_A(y) \diamond r_B(z)] \cdot e^{j[\omega_A(y) \bar{\diamond} \omega_B(z)]}, \tag{3.1}$$

for all $y, z \in X$ where \diamond is a t -norm and $\bar{\diamond}$ is a function

$$\bar{\diamond} : [0, \pi] \times [0, \pi] \rightarrow [0, \pi]$$

satisfying the following conditions.

1. $(a \bar{\diamond} b) \bar{\diamond} c = a \bar{\diamond} (b \bar{\diamond} c)$,
2. $a \bar{\diamond} b = b \bar{\diamond} a$,
3. $b \leq c \Rightarrow a \bar{\diamond} b \leq a \bar{\diamond} c$,
4. $a \bar{\diamond} \pi = a$,

where a, b and c are elements of $[0, \pi]$. We say that the function $\bar{\diamond}$ is an *extended t -norm*, and the operation \odot is the *complex t -norm*.

In what follows, let X be a BCK/BCI-algebra and consider a complex fuzzy set A on X with complex valued membership function

$$(\forall x \in X) \left(\mu_A(x) = r_A(x) \cdot e^{j\omega_A(x)} \right)$$

where $j = \sqrt{-1}$, $r_A(x) \in [0, 1]$ and $\omega_A(x) \in [0, \pi]$. It will be denoted by

$$A = \{(x, \mu_A(x)) \mid x \in X\}.$$

For any $\delta = r \cdot e^{j\omega}$ with $r \in [0, 1]$ and $\omega \in [0, \pi]$, the δ -level set of A is denoted by $[A]_\delta$ and is defined to be the set

$$[A]_\delta := \{x \in X \mid \mu_A(x) \succeq \delta\}.$$

If, in the complex t -norm \odot , both the t -norm and extended t -norm are considered as “min”, it is denoted by \odot_{\min} and is called the *min-complex t -norm*.

DEFINITION 3.1. A complex fuzzy set $A = \{(x, \mu_A(x)) \mid x \in X\}$ on X is called a *complex subalgebra* of X if

$$\begin{cases} \operatorname{Re}(\mu_A(x * y)) \geq \operatorname{Re}(\mu_A(x) \odot \mu_A(y)), \\ \operatorname{Im}(\mu_A(x * y)) \geq \operatorname{Im}(\mu_A(x) \odot \mu_A(y)) \end{cases} \tag{3.2}$$

or, equivalently, $\mu_A(x * y) \succeq \mu_A(x) \odot \mu_A(y)$ for all $x, y \in X$.

Note that the condition (3.2) is equivalent to the following condition:

$$\begin{cases} r_A(x * y) \cdot \cos \omega_A(x * y) \geq (r_A(x) \diamond r_A(y)) \cdot \cos(\omega_A(x) \bar{\diamond} \omega_A(y)), \\ r_A(x * y) \cdot \sin \omega_A(x * y) \geq (r_A(x) \diamond r_A(y)) \cdot \sin(\omega_A(x) \bar{\diamond} \omega_A(y)) \end{cases} \tag{3.3}$$

EXAMPLE 3.2. Let $X = \{0, a, b, c\}$ be a BCK-algebra with the operation $*$ which is described by Table 1 (see [5]).

Let A be a complex fuzzy set on X with the complex valued membership function μ_A defined by

$$\mu_A(x) = \begin{cases} 0.7e^{j\frac{3\pi}{8}} & \text{if } x = 0, \\ 0.5e^{j\frac{3\pi}{8}} & \text{if } x = a, \\ 0.3e^{j\frac{3\pi}{8}} & \text{if } x = b, \\ 0.1e^{j\frac{3\pi}{8}} & \text{if } x = c. \end{cases}$$

Table 1. Cayley table of the operation $*$

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	b	0	b
c	c	c	c	0

It is routine to verify that $A = \{(x, \mu_A(x)) \mid x \in X\}$ is a complex subalgebra of X .

EXAMPLE 3.3. Let $X = \{0, 1, a, b, c\}$ be a BCI-algebra in which the operation $*$ is described by Table 2 (see [5]).

Table 2. Cayley table of the operation $*$

$*$	0	1	a	b	c
0	0	0	a	b	c
1	1	0	a	b	c
a	a	a	0	c	b
b	b	b	c	0	a
c	c	c	b	a	0

Let A be a complex fuzzy set on X with the complex valued membership function μ_A defined by

$$\mu_A(x) = \begin{cases} 0.7e^{j\frac{\pi}{8}} & \text{if } x = 0, \\ 0.7e^{j\frac{3\pi}{16}} & \text{if } x = 1, \\ 0.7e^{j\frac{\pi}{4}} & \text{if } x = b, \\ 0.7e^{j\frac{5\pi}{16}} & \text{if } x \in \{a, c\}. \end{cases}$$

If we use the lexicographical order, denoted by \prec_l , on \mathbb{C} , that is, for any two complex numbers $z_1 = a_1 + jb_1$ and $z_2 = a_2 + jb_2$,

$$z_1 \prec_l z_2 \text{ provided either } a_1 < a_2 \text{ or } a_1 = a_2 \text{ and } b_1 < b_2,$$

then $A = \{(x, \mu_A(x)) \mid x \in X\}$ is a complex subalgebra of X .

PROPOSITION 3.4. *Let $A = \{(x, \mu_A(x)) \mid x \in X\}$ be a complex subalgebra of a BCK-algebra X . If we use the min-complex t -norm, then $\mu_A(0) \succeq \mu_A(x)$ for all $x \in X$.*

PROOF: Let $x \in X$. Using the conditions (III) and (3.3), we have

$$\begin{aligned} \operatorname{Re}(\mu_A(0)) &= r_A(0) \cdot \cos \omega_A(0) = r_A(x * x) \cdot \cos \omega_A(x * x) \\ &\geq (r_A(x) \diamond r_A(x)) \cdot \cos(\omega_A(x) \bar{\diamond} \omega_A(x)) \\ &= r_A(x) \cdot \cos \omega_A(x) = \operatorname{Re}(\mu_A(x)) \end{aligned}$$

and

$$\begin{aligned} \operatorname{Im}(\mu_A(0)) &= r_A(0) \cdot \sin \omega_A(0) = r_A(x * x) \cdot \sin \omega_A(x * x) \\ &\geq (r_A(x) \diamond r_A(x)) \cdot \sin(\omega_A(x) \bar{\diamond} \omega_A(x)) \\ &= r_A(x) \cdot \sin \omega_A(x) = \operatorname{Im}(\mu_A(x)). \end{aligned}$$

Therefore $\mu_A(0) \succeq \mu_A(x)$ for all $x \in X$. □

PROPOSITION 3.5. *Let $A = \{(x, \mu_A(x)) \mid x \in X\}$ be a complex subalgebra of a BCK-algebra X in which μ_A is increasing. If we use the min-complex t -norm, then μ_A is constant.*

PROOF: Straightforward. □

THEOREM 3.6. *Let $A = \{(x, \mu_A(x)) \mid x \in X\}$ be a complex subalgebra of X in which*

$$\mu_A(x) = r_A(x) \cdot e^{j\omega_A(x)}$$

with $j = \sqrt{-1}$, $r_A(x) \in [0, 1]$ and $\omega_A(x) \in [\frac{\pi}{2}, \pi]$. If we use the min-complex t -norm, then the δ -level set $[A]_\delta$ of A is a subalgebra of X for all $\delta := r \cdot e^{j\omega}$ with $r \in [0, 1]$, $\omega \in [\frac{\pi}{2}, \pi]$ and $[A]_\delta \neq \emptyset$.

PROOF: Assume that $A = \{(x, \mu_A(x)) \mid x \in X\}$ is a complex subalgebra of X . Let $x, y \in [A]_\delta$. Then $\mu_A(x) \succeq \delta$ and $\mu_A(y) \succeq \delta$. Thus

$$\begin{aligned} \operatorname{Re}(\mu_A(x)) &= r_A(x) \cdot \cos \omega_A(x) \geq r \cdot \cos \omega, \\ \operatorname{Im}(\mu_A(x)) &= r_A(x) \cdot \sin \omega_A(x) \geq r \cdot \sin \omega, \\ \operatorname{Re}(\mu_A(y)) &= r_A(y) \cdot \cos \omega_A(y) \geq r \cdot \cos \omega, \\ \operatorname{Im}(\mu_A(y)) &= r_A(y) \cdot \sin \omega_A(y) \geq r \cdot \sin \omega. \end{aligned}$$

Now, we consider the following four cases.

- (1) $r_A(x) \geq r_A(y)$ and $\omega_A(x) \geq \omega_A(y)$,
- (2) $r_A(x) \geq r_A(y)$ and $\omega_A(x) < \omega_A(y)$,
- (3) $r_A(x) < r_A(y)$ and $\omega_A(x) \geq \omega_A(y)$,
- (4) $r_A(x) < r_A(y)$ and $\omega_A(x) < \omega_A(y)$.

The case (1) implies that

$$(r_A(x) \diamond r_A(y)) \cdot \cos(\omega_A(x) \bar{\diamond} \omega_A(y)) = r_A(y) \cdot \cos \omega_A(y) \geq r \cdot \cos \omega$$

and

$$(r_A(x) \diamond r_A(y)) \cdot \sin(\omega_A(x) \bar{\diamond} \omega_A(y)) = r_A(y) \cdot \sin \omega_A(y) \geq r \cdot \sin \omega.$$

For the case (2), we have

$$\begin{aligned} (r_A(x) \diamond r_A(y)) \cdot \cos(\omega_A(x) \bar{\diamond} \omega_A(y)) &= r_A(y) \cdot \cos \omega_A(x) \\ &\geq r_A(y) \cdot \cos \omega_A(y) \geq r \cdot \cos \omega \end{aligned}$$

and

$$\begin{aligned} (r_A(x) \diamond r_A(y)) \cdot \sin(\omega_A(x) \bar{\diamond} \omega_A(y)) &= r_A(y) \cdot \sin \omega_A(x) \\ &\geq r_A(y) \cdot \sin \omega_A(y) \geq r \cdot \sin \omega \end{aligned}$$

since $\cos \omega_A(y) \leq \cos \omega_A(x)$ and $\sin \omega_A(y) \leq \sin \omega_A(x)$. The case (3) induces

$$\begin{aligned} (r_A(x) \diamond r_A(y)) \cdot \cos(\omega_A(x) \bar{\diamond} \omega_A(y)) &= r_A(x) \cdot \cos \omega_A(y) \\ &\geq r_A(x) \cdot \cos \omega_A(x) \geq r \cdot \cos \omega \end{aligned}$$

and

$$\begin{aligned} (r_A(x) \diamond r_A(y)) \cdot \sin(\omega_A(x) \bar{\diamond} \omega_A(y)) &= r_A(x) \cdot \sin \omega_A(y) \\ &\geq r_A(x) \cdot \sin \omega_A(x) \geq r \cdot \sin \omega. \end{aligned}$$

From the case (4), we have

$$(r_A(x) \diamond r_A(y)) \cdot \cos(\omega_A(x) \bar{\diamond} \omega_A(y)) = r_A(x) \cdot \cos \omega_A(x) \geq r \cdot \cos \omega$$

and

$$(r_A(x) \diamond r_A(y)) \cdot \sin(\omega_A(x) \bar{\diamond} \omega_A(y)) = r_A(x) \cdot \sin \omega_A(x) \geq r \cdot \sin \omega.$$

It follows from (3.2) that

$$r_A(x * y) \cdot \cos \omega_A(x * y) \geq (r_A(x) \diamond r_A(y)) \cdot \cos(\omega_A(x) \bar{\diamond} \omega_A(y)) \geq r \cdot \cos \omega$$

and

$$r_A(x * y) \cdot \sin \omega_A(x * y) \geq (r_A(x) \diamond r_A(y)) \cdot \sin(\omega_A(x) \bar{\diamond} \omega_A(y)) \geq r \cdot \sin \omega.$$

Hence $\mu_A(x * y) \succeq \delta$, and so $x * y \in [A]_\delta$. Therefore $[A]_\delta$ is a subalgebra of X . □

COROLLARY 3.7. Let $A = \{(x, \mu_A(x)) \mid x \in X\}$ be a complex subalgebra of a BCK-algebra X in which

$$\mu_A(x) = r_A(x) \cdot e^{j\omega_A(x)}$$

with $j = \sqrt{-1}$, $r_A(x) \in [0, 1]$ and $\omega_A(x) \in [\frac{\pi}{2}, \pi]$. If we use the min-complex t -norm, then the set

$$[X] := \{x \in X \mid \mu_A(x) = \mu_A(0)\}$$

is a subalgebra of X .

PROOF: Since $\mu_A(0) \succeq \mu_A(x)$ for all $x \in X$ by Proposition 3.4, we have

$$[A]_{\mu_A(0)} = \{x \in X \mid \mu_A(x) \succeq \mu_A(0)\} = \{x \in X \mid \mu_A(x) = \mu_A(0)\} = [X].$$

It follows from Theorem 3.6 that $[X]$ is a subalgebra of X . \square

THEOREM 3.8. Let $A = \{(x, \mu_A(x)) \mid x \in X\}$ be a complex fuzzy set on X such that the δ -level set $[A]_\delta$ of A is a subalgebra of X for all $\delta := r \cdot e^{j\omega}$ with $r \in [0, 1]$, $\omega \in [0, \pi]$ and $[A]_\delta \neq \emptyset$. If we use the min-complex t -norm, then $A = \{(x, \mu_A(x)) \mid x \in X\}$ is a complex subalgebra of X .

PROOF: Suppose that the δ -level set $[A]_\delta$ of A is a subalgebra of X for all $\delta := r \cdot e^{j\omega}$ with $r \in [0, 1]$ and $\omega \in [0, \pi]$. Assume that there exist $a, b \in X$ such that

$$\mu_A(a * b) \prec \mu_A(a) \odot \mu_A(b).$$

We take $\delta := r \cdot e^{j\omega}$ with $r \in [0, 1]$ and $\omega \in [0, \pi]$ such that

$$\mu_A(a * b) \prec \delta \preceq \mu_A(a) \odot \mu_A(b).$$

Then $a * b \notin [A]_\delta$,

$$(r_A(a) \diamond r_A(b)) \cdot \cos(\omega_A(a) \bar{\diamond} \omega_A(b)) = \operatorname{Re}(\mu_A(a) \odot \mu_A(b)) \geq \operatorname{Re}(\delta) = r \cdot \cos \omega$$

and

$$(r_A(a) \diamond r_A(b)) \cdot \sin(\omega_A(a) \bar{\diamond} \omega_A(b)) = \operatorname{Im}(\mu_A(a) \odot \mu_A(b)) \geq \operatorname{Im}(\delta) = r \cdot \sin \omega.$$

It follows that

$$r_A(a) \cdot \cos \omega_A(a) \geq r \cdot \cos \omega, \quad r_A(a) \cdot \sin \omega_A(a) \geq r \cdot \sin \omega$$

and

$$r_A(b) \cdot \cos \omega_A(b) \geq r \cdot \cos \omega, \quad r_A(b) \cdot \sin \omega_A(b) \geq r \cdot \sin \omega.$$

This shows that $a, b \in [A]_\delta$ and this is a contradiction. Therefore $\mu_A(x * y) \succeq \mu_A(x) \odot \mu_A(y)$ for all $x, y \in X$, and $A = \{(x, \mu_A(x)) \mid x \in X\}$ is a complex subalgebra of X . \square

COROLLARY 3.9. For any subset L of X , let $A = \{(x, \mu_A(x)) \mid x \in X\}$ be a complex fuzzy set on a BCK-algebra X with the complex valued membership function μ_A defined by

$$\mu_A(x) = \begin{cases} r_1 \cdot e^{j\omega_1} & \text{if } x \in L, \\ r_2 \cdot e^{j\omega_2} & \text{otherwise} \end{cases}$$

where $r_1 \cdot e^{j\omega_1} \succeq r_2 \cdot e^{j\omega_2}$. If L is a subalgebra of X and we use the min-complex t -norm, then $A = \{(x, \mu_A(x)) \mid x \in X\}$ is a complex subalgebra of X .

THEOREM 3.10. Let $\delta_1, \delta_2, \dots, \delta_n, \dots$ be a strictly increasing sequence of complex numbers, where $\delta_k = r_k \cdot e^{j\omega_k}$ with $r_k \in [0, 1]$ and $\omega_k \in [\frac{\pi}{2}, \pi]$. For a strictly decreasing sequences $L_1(= X), L_2, \dots, L_n, \dots$ of subalgebras of X , there is a complex subalgebra $A = \{(x, \mu_A(x)) \mid x \in X\}$ of X in which

$$\mu_A(x) = r_A(x) \cdot e^{j\omega_A(x)}$$

with $j = \sqrt{-1}$, $r_A(x) \in [0, 1]$ and $\omega_A(x) \in [\frac{\pi}{2}, \pi]$ such that $[A]_{\delta_n} = L_n$ for $n \in \mathbb{N}$ if we use the min-complex t -norm.

PROOF: Define a complex fuzzy set A on X with the complex valued membership function μ_A defined by

$$\mu_A(x) = \begin{cases} \delta_n & \text{if } x \in L_n \setminus L_{n+1}, \\ \lim_{n \rightarrow \infty} \delta_n & \text{if } x \in \bigcap_{n=1}^{\infty} L_n. \end{cases}$$

It is easy to verify that $A = \{(x, \mu_A(x)) \mid x \in X\}$ is a complex subalgebra of X and $[A]_{\delta_n} = L_n$ for $n = 1, 2, \dots$. □

Let $f : X \rightarrow Y$ be a mapping of sets. If B is a complex fuzzy set on Y with the complex valued membership function μ_B , then the preimage of B under f , denoted by $f^{-1}(B)$, is also a complex fuzzy set on X with the complex valued membership function $\mu_{f^{-1}(B)}$ which is defined by $\mu_{f^{-1}(B)}(x) = \mu_B(f(x))$ for all $x \in X$.

THEOREM 3.11. Let $f : X \rightarrow Y$ be a homomorphism from X to a BCK/BCI-algebra Y . If B is a complex subalgebra of Y with the complex valued membership function μ_B , then the homomorphic preimage $f^{-1}(B)$ of B under f is a complex subalgebra of X with the complex valued membership function $\mu_{f^{-1}(B)}$.

PROOF: Assume that B is a complex subalgebra of Y with the complex valued membership function μ_B . For any $x, y \in X$, we have

$$\begin{aligned} \operatorname{Re}(\mu_{f^{-1}(B)}(x * y)) &= \operatorname{Re}(\mu_B(f(x * y))) = \operatorname{Re}(\mu_B(f(x) * f(y))) \\ &\geq \operatorname{Re}(\mu_B(f(x)) \odot \mu_B(f(y))) \\ &= \operatorname{Re}(\mu_{f^{-1}(B)}(x) \odot \mu_{f^{-1}(B)}(y)) \end{aligned}$$

and

$$\begin{aligned} \operatorname{Im}(\mu_{f^{-1}(B)}(x * y)) &= \operatorname{Im}(\mu_B(f(x * y))) = \operatorname{Im}(\mu_B(f(x) * f(y))) \\ &\geq \operatorname{Im}(\mu_B(f(x)) \odot \mu_B(f(y))) \\ &= \operatorname{Im}(\mu_{f^{-1}(B)}(x) \odot \mu_{f^{-1}(B)}(y)). \end{aligned}$$

Therefore the homomorphic preimage $f^{-1}(B)$ of B under f is a complex subalgebra of X with the complex valued membership function $\mu_{f^{-1}(B)}$. \square

THEOREM 3.12. *Let f be an endomorphism of X . If A is a complex subalgebra of X with the complex valued membership function μ_A , then the complex fuzzy set $A[f]$ on X with the complex valued membership function $\mu_{A[f]}$ defined by*

$$\mu_{A[f]}(x) = \mu_A(f(x))$$

for all $x \in X$ is a complex subalgebra of X .

PROOF: Let A be a complex subalgebra of X with the complex valued membership function μ_A . For any $x, y \in X$, we get

$$\begin{aligned} \operatorname{Re}(\mu_{A[f]}(x * y)) &= \operatorname{Re}(\mu_A(f(x * y))) = \operatorname{Re}(\mu_A(f(x) * f(y))) \\ &\geq \operatorname{Re}(\mu_A(f(x)) \odot \mu_A(f(y))) \\ &= \operatorname{Re}(\mu_{A[f]}(x) \odot \mu_{A[f]}(y)) \end{aligned}$$

and

$$\begin{aligned} \operatorname{Im}(\mu_{A[f]}(x * y)) &= \operatorname{Im}(\mu_A(f(x * y))) = \operatorname{Im}(\mu_A(f(x) * f(y))) \\ &\geq \operatorname{Im}(\mu_A(f(x)) \odot \mu_A(f(y))) \\ &= \operatorname{Im}(\mu_{A[f]}(x) \odot \mu_{A[f]}(y)). \end{aligned}$$

Therefore $\mu_{A[f]}$ is a complex subalgebra of X . \square

DEFINITION 3.13. A complex fuzzy set $A = \{(x, \mu_A(x)) \mid x \in X\}$ on X is called a *complex left reduced ideal* of X if

$$\operatorname{Re}(\mu_A(x * y)) \geq \operatorname{Re}(\mu_A(y)) \quad \text{and} \quad \operatorname{Im}(\mu_A(x * y)) \geq \operatorname{Im}(\mu_A(y)) \quad (3.4)$$

for all $x, y \in X$. If $A = \{(x, \mu_A(x)) \mid x \in X\}$ satisfies the condition

$$\operatorname{Re}(\mu_A(x * y)) \geq \operatorname{Re}(\mu_A(x)) \quad \text{and} \quad \operatorname{Im}(\mu_A(x * y)) \geq \operatorname{Im}(\mu_A(x))$$

for all $x, y \in X$, then we say $A = \{(x, \mu_A(x)) \mid x \in X\}$ is a *complex right reduced ideal* of X .

THEOREM 3.14. *Let $A = \{(x, \mu_A(x)) \mid x \in X\}$ be a complex subalgebra of a BCK-algebra X . If we use the min-complex t -norm, then A is a complex left reduced ideal of X if and only if the complex valued membership function μ_A of A is constant, that is, $\mu_A(0) = \mu_A(x)$ for all $x \in X$.*

PROOF: The sufficiency is clear. Assume that A is a complex left reduced ideal of X . For any $x \in X$, we have

$$\operatorname{Re}(\mu_A(x)) = \operatorname{Re}(\mu_A(x * 0)) \geq \operatorname{Re}(\mu_A(0))$$

and

$$\operatorname{Im}(\mu_A(x)) = \operatorname{Im}(\mu_A(x * 0)) \geq \operatorname{Im}(\mu_A(0)).$$

Since $x * x = 0$ for all $x \in X$, the condition (3.4) implies that $\operatorname{Re}(\mu_A(x)) \leq \operatorname{Re}(\mu_A(x * x)) = \operatorname{Re}(\mu_A(0))$ and $\operatorname{Im}(\mu_A(x)) \leq \operatorname{Im}(\mu_A(x * x)) = \operatorname{Im}(\mu_A(0))$. Therefore $\mu_A(0) = \mu_A(x)$ for all $x \in X$, that is, the complex valued membership function μ_A of A is constant. \square

The proof of the following two theorems is the same as the proof of Theorems 3.11 and 3.12.

THEOREM 3.15. *Let $f : X \rightarrow Y$ be a homomorphism from X to a BCK/BCI-algebra Y . If B is a complex left (resp. right) reduced ideal of Y with the complex valued membership function μ_B , then the homomorphic preimage $f^{-1}(B)$ of B under f is a complex left (resp. right) reduced ideal of X with the complex valued membership function $\mu_{f^{-1}(B)}$.*

THEOREM 3.16. *Let f be an endomorphism of X . If A is a complex left (resp. right) reduced ideal of X with the complex valued membership function μ_A , then the complex fuzzy set $A[f]$ on X with the complex valued membership function $\mu_{A[f]}$ defined by*

$$\mu_{A[f]}(x) = \mu_A(f(x))$$

for all $x \in X$ is a complex left (resp. right) reduced ideal of X .

4. Conclusions

Generally, the extension of crisp sets to fuzzy sets, in terms of membership functions, is mathematically comparable to the extension of \mathbb{Z} (; the set of integers) to \mathbb{R} (; the set of real numbers). That is, expanding the range of the membership function from $\{0, 1\}$ to the unit interval $[0, 1]$ is mathematically analogous to the extension of \mathbb{Z} to \mathbb{R} . The development of the number set did not end with real numbers. Historically, the introduction of real numbers was followed by their extension to the set of complex numbers, \mathbb{C} . Hence, it may be suggested that a further development of fuzzy set theory should be based on this extension. In the context of set theory, the result of such an extension is the complex fuzzy set, i.e., a fuzzy set characterized by a complex-valued membership function. Based on such background, Ramot et al. introduced complex fuzzy set in their paper [1]. The complex fuzzy set is characterized by a membership function μ whose range is not limited to $[0, 1]$ but extended to the unit circle in the complex plane. In this paper, we have used complex fuzzy sets to obtain the generalization of fuzzy set theory in BCK/BCI-algebras. We have introduced the notion of a complex subalgebra and a complex reduced left (right) ideal in a BCK/BCI-algebra, and have investigated related properties. We have provided characterizations of a complex subalgebra, and have discussed the homomorphic image (preimage) of a complex subalgebra and a complex left (right) reduced ideal. We will use the ideas and results of this paper to study various types of sub-structure in algebras in the future.

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