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A NOTE ON CIUCIURA'S $\mathbf{mbC}^1$

Abstract

This note offers a non-deterministic semantics for $\mathbf{mbC}^1$, introduced by Janusz Ciuciura, and establishes soundness and (strong) completeness results with respect to the Hilbert-style proof system. Moreover, based on the new semantics, we briefly discuss an unexplored variant of $\mathbf{mbC}^1$ which has a contra-classical flavor.

Keywords: paraconsistent logic, non-deterministic semantics contra-classical logic

1. Introduction

In [10], Janusz Ciuciura introduces a system $\mathbf{mbC}^1$ of paraconsistent logic, formulated in the language of classical logic. The aim of this note is to present a non-deterministic semantics for $\mathbf{mbC}^1$, different from the semantics presented in [10], and prove its soundness and (strong) completeness. And in view of this new semantics, we will briefly discuss, in the last section, an unexplored variant of $\mathbf{mbC}^1$ which has a contra-classical flavor.

2. Proof system for $\mathbf{mbC}^1$

Let the languages $\mathcal{L}$ and $\mathcal{L}_\circ$ consist of a finite set $\{\sim, \wedge, \vee, \rightarrow\}$ and $\{\sim, \circ, \wedge, \vee, \rightarrow\}$ of propositional connectives respectively and a countable set Prop of propositional variables which we denote by p, q, etc. Furthermore, we denote by Form and $\mathsf{Form}_\circ$ the sets of formulas defined as usual in $\mathcal{L}$ and $\mathcal{L}_\circ$ respectively. We denote a formula of the languages by A, B, C, etc. and a set of formulas of the languages by Γ, Δ, Σ, etc.

First, we introduce **CLuN** which is the common core of many systems of paraconsistent logic, including $\mathbf{mbC}^1$.

Definition 1. The system **CLuN** consists of the following axioms and a rule of inference.

$$A \to (B \to A) \tag{A1}$$

$$(A \to (B \to C)) \to ((A \to B) \to (A \to C)) \tag{A2}$$

$$((A \to B) \to A) \to A \tag{A3}$$

$$A \to (A \vee B) \tag{A4}$$

$$B \to (A \vee B) \tag{A5}$$

$$(A \to C) \to ((B \to C) \to ((A \vee B) \to C)) \tag{A6}$$

$$(A \wedge B) \to A \tag{A7}$$

$$(A \wedge B) \to B \tag{A8}$$

$$(C \to A) \to ((C \to B) \to (C \to (A \wedge B))) \tag{A9}$$

$$A \vee {\sim} A \tag{A10}$$

$$\frac{A \quad A \to B}{B} \tag{MP}$$

Moreover, we write $\Gamma \vdash_{\mathbf{CLuN}} A$ if there is a sequence of formulas $B_1, \ldots, B_n, A$, $n \geq 0$, such that every formula in the sequence $B_1, \ldots, B_n, A$ either (i) belongs to Γ; (ii) is an axiom of **CLuN**; (iii) is obtained by (MP) from formulas preceding it in sequence.

Second, we introduce $\mathbf{mbC}^1$ and, for the sake of comparison, **mbC**, one of the basic systems within the family of Logics of Formal Inconsistency (cf. [8, 7]).

Definition 2. The system $\mathbf{mbC}^1$ is formulated in $\mathcal{L}$ and obtained by adding the following formula to **CLuN**.

$$A \to ({\sim} A \to ({\sim}{\sim} A \to B)) \tag{$*$}$$

Moreover, the system **mbC** is formulated in $\mathcal{L}_\circ$ and obtained by adding the following formula to **CLuN**.

$$\circ A \to (A \to ({\sim} A \to B))$$

We then define $\vdash_{\mathbf{mbC}^1}$ and $\vdash_{\mathbf{mbC}}$ in a similar manner.

Here are two remarks on the relation between $\mathbf{mbC}^1$ and **mbC**.

REMARK 3. Ciuciura notes that $\mathbf{mbC}^1$ is "an axiomatization of **mbC** formulated directly in the language of classical propositional logic" ([10, p. 173]). This is, however, not true due to the following result:

$$\nvdash_{\mathbf{mbC}} p \to (\sim p \to (\sim\sim p \to q)).$$

This may be observed by the following truth table for **LFI1**, an extension of **mbC**, introduced in [9].

A	$\sim A$	$\circ A$
t	**f**	**t**
b	**b**	**f**
f	**t**	**t**

$A \wedge B$	**t**	**b**	**f**
t	**t**	**b**	**f**
b	**b**	**b**	**f**
f	**f**	**f**	**f**

$A \vee B$	**t**	**b**	**f**
t	**t**	**t**	**t**
b	**t**	**b**	**b**
f	**t**	**b**	**f**

$A \to B$	**t**	**b**	**f**
t	**t**	**b**	**f**
b	**t**	**b**	**f**
f	**t**	**t**	**t**

Note here that both **t** and **b** are designated values. Then, the axioms of **mbC** are all validated, and designated values are preserved by the above truth table. However, the concerned formula takes the non-designated value **f** when we assign **b** and **f** to p and q respectively.

REMARK 4. Note that if one takes the consistency to be *defined* as $\circ A =_{\text{def.}} A \to \sim\sim A$, then **mbC** *with this definition of consistency* becomes equivalent to $\mathbf{mbC}^1$ since $A \wedge \circ A$ is equivalent to $A \wedge \sim\sim A$ under the above definition of $\circ$. For a system of paraconsistent logic having this kind of definition of consistency, see [22, 21].

3. Non-deterministic semantics for $\mathbf{mbC}^1$

In [10], Ciuciura already offers a semantics for $\mathbf{mbC}^1$ along the line of what is sometimes called the bivaluational semantics which has been one of the most popular semantics for a wide range of LFIs. However, there is another semantics known in the literature of LFIs, namely the non-deterministic semantics, established systematically by Arnon Avron and Iddo Lev in [4] (see [5] for a survey on non-deterministic semantics). The semantics has a nice feature being an intuitive generalization of many-valued semantics. In this section, we present non-deterministic semantics for $\mathbf{mbC}^1$.

DEFINITION 5. A $\mathbf{mbC}^1$-*non-deterministic matrix* (*Nmatrix* for short) for $\mathcal{L}$ is a tuple $M = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, where:

(a) $\mathcal{V} = \{\mathbf{t}, \mathbf{b}, \mathbf{f}\}$,

(b) $\mathcal{D} = \{\mathbf{t}, \mathbf{b}\}$,

(c) For every n-ary connective $*$ of $\mathcal{L}$, $\mathcal{O}$ includes a corresponding n-ary function $\tilde{*}$ from $\mathcal{V}^n$ to $2^{\mathcal{V}} \setminus \{\emptyset\}$ as follows (we omit the brackets for sets):

A	$\tilde{\sim}A$
t	**f**
b	**t**
f	**t**, **b**

$A\tilde{\wedge}B$	**t**	**b**	**f**
t	**t**, **b**	**t**, **b**	**f**
b	**t**, **b**	**t**, **b**	**f**
f	**f**	**f**	**f**

$A\tilde{\vee}B$	**t**	**b**	**f**
t	**t**, **b**	**t**, **b**	**t**, **b**
b	**t**, **b**	**t**, **b**	**t**, **b**
f	**t**, **b**	**t**, **b**	**f**

$A\tilde{\to}B$	**t**	**b**	**f**
t	**t**, **b**	**t**, **b**	**f**
b	**t**, **b**	**t**, **b**	**f**
f	**t**, **b**	**t**, **b**	**t**, **b**

A *legal* $\mathbf{mbC}^1$*-valuation* in an $\mathbf{mbC}^1$-Nmatrix M is a function $v : \mathsf{Form} \to \mathcal{V}$ that satisfies the following condition for every n-ary connective $*$ of $\mathcal{L}$ and $A_1, \ldots, A_n \in \mathsf{Form}$:

$$v(*(A_1, \ldots, A_n)) \in \tilde{*}(v(A_1), \ldots, v(A_n)).$$

Finally, A is a *legal* $\mathbf{mbC}^1$ *consequence* of Γ ($\Gamma \models_{\mathbf{mbC}^1} A$) iff for every legal $\mathbf{mbC}^1$-valuation v, if $v(B) \in \mathcal{D}$ for every $B \in \Gamma$ then $v(A) \in \mathcal{D}$.

Remark 6. Note that a three-valued non-deterministic semantics for **CLuN** is introduced in [2] by Avron. The only difference is that the table for negation is replaced by the following table:

A	$\tilde{\sim}A$
t	**f**
b	**t**, **b**
f	**t**, **b**

That is, there is one more non-determinacy when the negated sentence receives the value **b**. Furthermore, there is also a two-valued non-deterministic semantics for **CLuN** devised in [3]. Note also that non-deterministic semantics for **mbC** and its extensions are considered in [1] (the system **mbC** is referred to as **B** in [1]). However, the above matrix is not considered in the literature, at least to the best of author's knowledge.

Remark 7. The system $\mathbf{mbC}^1$ may be seen as a generalization of Sette's $\mathbf{P}^1$ developed in [19]. Indeed, the addition of the following formulas will eliminate the nonclassical value in the non-deterministic bits in the above matrix and give us the system $\mathbf{P}^1$.

- $\sim\sim A \to A$
- $(A * B) \to \sim\sim(A * B)$ where $* \in \{\wedge, \vee, \to\}$

For a recent discussion on discussive semantics for $\mathbf{P}^1$, see [16].

4. Soundness and completeness

We now turn to prove the soundness and completeness. The proof will be rather simple if the reader is already familiar with non-deterministic semantics, but for the purpose of making this note self-contained as much as possible, I will spell them out in some details.

The soundness is easy as usual.

PROPOSITION 1 (Soundness). *If* $\Gamma \vdash_{\mathbf{mbC}^1} A$ *then* $\Gamma \models_{\mathbf{mbC}^1} A$*.*

PROOF: Straightforward. □

For the completeness result, we first list some formulas that are provable in $\mathbf{mbC}^1$.

PROPOSITION 2. *The following formulas are provable in* $\mathbf{mbC}^1$*:*

$$A \vee (A \to B) \tag{4.1}$$

$$A \to (B \to (A \wedge B)) \tag{4.2}$$

PROOF: We safely leave the details to the readers. □

Second, we introduce the following standard notions.

DEFINITION 8. Let Σ be a set of formulas. Then,

- Σ is a *theory* iff it is closed under $\vdash$, i.e., if $\Sigma \vdash A$ then $A \in \Sigma$ for any formula A;
- Σ is *prime* iff $A \vee B \in \Sigma$ implies that $A \in \Sigma$ or $B \in \Sigma$ for any A and B;
- Σ is *non-trivial* iff for some formula A, $A \notin \Sigma$.

REMARK 9. Strictly speaking, we do need to specify the consequence relation in defining theories. However, in the following, we will omit that since contexts will disambiguate.

The following lemma is the well-known lemma of Lindenbaum.

LEMMA 1. *For any* $\Sigma \cup \{A\} \subseteq \mathsf{Form}$, *if* $\Sigma \nvdash A$ *then, there is a prime theory* $\Pi \supseteq \Sigma$ *such that* $\Pi \nvdash A$.

Moreover, we need the following lemma.

LEMMA 2. *Let* Σ *be a non-trivial prime theory, and define a function* v_0 *from* Form *to* $\mathcal{V}$ *as follows.*

$$v_0(B) := \begin{cases} \mathbf{t} & \text{if } \Sigma \vdash_{\mathbf{mbC^1}} B \text{ and } \Sigma \nvdash_{\mathbf{mbC^1}} {\sim}B \\ \mathbf{b} & \text{if } \Sigma \vdash_{\mathbf{mbC^1}} B \text{ and } \Sigma \vdash_{\mathbf{mbC^1}} {\sim}B \\ \mathbf{f} & \text{if } \Sigma \nvdash_{\mathbf{mbC^1}} B \end{cases}$$

Then, v_0 *is a legal* $\mathbf{mbC}^1$*-valuation.*

PROOF: By induction on the number n of connectives.
(Base): for atomic formulas, it immediately follows that v_0 is a function.
(Induction step): We split the cases based on the connectives.
Case 1. If $B = {\sim}C$, then we have the following three cases.

Cases	$v(C)$	condition for C	$v(B)$	condition for B i.e. ${\sim}C$
(i)	$\mathbf{t}$	$\Sigma \vdash C$ and $\Sigma \nvdash {\sim}C$	$\mathbf{f}$	$\Sigma \nvdash {\sim}C$
(ii)	$\mathbf{b}$	$\Sigma \vdash C$ and $\Sigma \vdash {\sim}C$	$\mathbf{t}$	$\Sigma \vdash {\sim}C$ and $\Sigma \nvdash {\sim}{\sim}C$
(iii)	$\mathbf{f}$	$\Sigma \nvdash C$	$\mathbf{t}, \mathbf{b}$	$\Sigma \vdash {\sim}C$

By induction hypothesis, we have the conditions for C, and it is easy to see that the conditions for B i.e. ${\sim}C$ are provable. Indeed, (i) is obvious. For (ii), note that we have ($*$) and that Σ is non-trivial. Finally, for (iii), note that we have (A10) and that Σ is prime.
Case 2. If $B = C \vee D$, then we have the following three cases.

Cases	$v(C)$	condition for C	$v(D)$	condition for D	$v(B)$	condition for B i.e. $C \vee D$
(i)	$\mathbf{t}, \mathbf{b}$	$\Sigma \vdash C$	any	—	$\mathbf{t}, \mathbf{b}$	$\Sigma \vdash C \vee D$
(ii)	any	—	$\mathbf{t}, \mathbf{b}$	$\Sigma \vdash D$	$\mathbf{t}, \mathbf{b}$	$\Sigma \vdash C \vee D$
(iii)	$\mathbf{f}$	$\Sigma \nvdash C$	$\mathbf{f}$	$\Sigma \nvdash D$	$\mathbf{f}$	$\Sigma \nvdash C \vee D$

By induction hypothesis, we have the conditions for C and D, and we can see that the conditions for B i.e. $C \vee D$ are provable in view of (A4), (A5) and that Σ is a prime theory for (i), (ii) and (iii) respectively.
Case 3. If $B = C \wedge D$, then we have the following three cases.

Cases	$v(C)$	condition for C	$v(D)$	condition for D	$v(B)$	condition for B i.e. $C \wedge D$
(i)	$\mathbf{f}$	$\Sigma \nvdash C$	any	—	$\mathbf{f}$	$\Sigma \nvdash C \wedge D$
(ii)	any	—	$\mathbf{f}$	$\Sigma \nvdash D$	$\mathbf{f}$	$\Sigma \nvdash C \wedge D$
(iii)	$\mathbf{t}, \mathbf{b}$	$\Sigma \vdash C$	$\mathbf{t}, \mathbf{b}$	$\Sigma \vdash D$	$\mathbf{f}$	$\Sigma \vdash C \wedge D$

By induction hypothesis, we have the conditions for C and D, and we can see that the conditions for B i.e. $C \wedge D$ are provable in view of (A7), (A8) and (4.2) for (i), (ii) and (iii) respectively.
Case 4. If $B = C \to D$, then we have the following three cases.

Cases	$v(C)$	condition for C	$v(D)$	condition for D	$v(B)$	condition for B i.e. $C \to D$
(i)	$\mathbf{f}$	$\Sigma \nvdash C$	any	—	$\mathbf{t}, \mathbf{b}$	$\Sigma \vdash C \to D$
(ii)	any	—	$\mathbf{t}, \mathbf{b}$	$\Sigma \vdash D$	$\mathbf{t}, \mathbf{b}$	$\Sigma \vdash C \to D$
(iii)	$\mathbf{t}, \mathbf{b}$	$\Sigma \vdash C$	$\mathbf{f}$	$\Sigma \nvdash D$	$\mathbf{f}$	$\Sigma \nvdash C \to D$

By induction hypothesis, we have the conditions for C and D, and we can see that the conditions for B i.e. $C \to D$ are provable in view of (4.1) and that Σ is prime, (A1) and (MP) for (i), (ii) and (iii) respectively.

This completes the proof. □

We are now ready to prove the completeness result.

THEOREM 1 (Completeness). *If* $\Gamma \models_{\mathbf{mbC^1}} A$ *then* $\Gamma \vdash_{\mathbf{mbC^1}} A$.

PROOF: We prove the contrapositive. Suppose that $\Gamma \nvdash_{\mathbf{mbC^1}} A$. Then by Lemma 1, we have a non-trivial prime theory Σ_0 such that $\Gamma \subseteq \Sigma_0$ and $\Sigma_0 \nvdash_{\mathbf{mbC^1}} A$. In view of Lemma 2, we can define a legal valuation v_0. Since we have $v_0(\Gamma) \in \mathcal{D}$ and $v_0(A) \notin \mathcal{D}$, we obtain $\Gamma \not\models_{\mathbf{mbC^1}} A$, as desired. □

REMARK 10. Note that Ciuciura develops a hierarchy of systems $\mathbf{mbC}^n$ obtained by adding the following axiom scheme to **CLuN**:

$$A \to ({\sim}A \to ({\sim\sim}A \to (\cdots \to ({\sim}^{n+1}A \to B)\dots))$$

where ${\sim}^{n+1}A$ abbreviates the formula with $n+1$ iterated $\sim$ in front of A. The task of devising a non-deterministic semantics for $\mathbf{mbC}^n$ is left for interested readers.

5. Concluding remarks: a contra-classical variant of $\mathbf{mbC}^1$

Let us assume the three-valued non-deterministic semantics for **CLuN**, and in particular, focus on the table for negation. Then, there are two cases with non-deterministic values. Avron already observed in [2] the following.

- The refinement $\tilde{\sim}\mathbf{b} = \{\mathbf{b}\}$ corresponds to the addition of $A \to {\sim}{\sim}A$.
- The refinement $\tilde{\sim}\mathbf{f} = \{\mathbf{t}\}$ corresponds to the addition of ${\sim}{\sim}A \to A$.

Moreover, we observed in this note the following through the system $\mathbf{mbC}^1$.

- The refinement $\tilde{\sim}\mathbf{b} = \{\mathbf{t}\}$ corresponds to the addition of $A \to ({\sim}A \to ({\sim}{\sim}A \to B))$.

Then, from a purely combinatoric perspective, one may wonder what kind of formula is required in order to obtain a refinement of the three-valued non-deterministic matrix for **CLuN** with $\tilde{\sim}\mathbf{f} = \{\mathbf{b}\}$. Quick answer: $A \vee {\sim}{\sim}A$. What we need to check are the following two items.

- $A \vee {\sim}{\sim}A$ is validated in the refined matrix, and;
- (iii) of Case 1 in Lemma 2 holds for the modified case.

The first item is easy to check, and for the second item, we may confirm that if Σ is a non-trivial prime theory, then $\Sigma \nvdash C$ implies $\Sigma \vdash {\sim}{\sim}C$ thanks to the presence of $A \vee {\sim}{\sim}A$ and that Σ is prime.

Therefore, what we obtain by the unexplored refinement is a contra-classical logic obtained by adding the formula $A \vee {\sim}{\sim}A$ to **CLuN**. Note here that a logic is contra-classical "just in case not everything provable in the logic is provable in classical logic" ([12, p.438]). Moreover, the formula $A \vee {\sim}{\sim}A$ is not discussed here for the first time, but already discussed in the literature, for example, in [11, 13, 17, 18].

Finally, it is not the case that we obtain contra-classical refinements only through negation. For example, the following truth table can be seen as a refinement of the conditional of **CLuN**.

$A \to B$	**t**	**b**	**f**
t	**t**	**b**	**f**
b	**t**	**b**	**f**
f	**b**	**b**	**b**

If we combine this conditional with the negation of the *Logic of Paradox*, one of the refinements of the negation of **CLuN**, then the above conditional is *connexive* in the sense that theses of Aristotle (i.e. ${\sim}(A{\to}{\sim}A)$

and $\sim(\sim A \to A)$) and Boethius (i.e. $(A \to B) \to \sim(A \to \sim B)$ and $(A \to \sim B) \to \sim(A \to B)$) are validated. And, connexive logics are of course one of the families of contra-classical logics (see [20] for connexive logics in general, and [6, 14, 15] for systems of connexive logic with the above conditional).

A more systematic study of contra-classicality in the context of non-deterministic semantics, possibly starting with a weaker language, is yet to be seen, even for three- and four-valued logics. However, this goes well beyond the scope of this note, and I will need to leave it for another occasion.

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References

[1] A. Avron, *Non-deterministic Matrices and Modular Semantics of Rules*, [in:] J.-Y. Béziau (ed.), **Logica Universalis**, Birkhüser Verlag, 2005, pp. 149–167.

[2] A. Avron, *Non-deterministic Semantics for Families of Paraconsistent Logics*, [in:] J. Y. Béziau, W. A. Carnielli and D. Gabbay (eds.), **Handbook of Paraconsistency**, College Publications, 2007, pp. 285–320.

[3] A. Avron and B. Konikowska, *Multi-valued calculi for logics based on non-determinism*, **Logic Journal of IGPL**, Vol. 13, No. 4 (2005), pp. 365–387.

[4] A. Avron and I. Lev, *Non-Deterministic Multiple-valued Structures*, **Journal of Logic and Computation**, Vol. 15, No. 3 (2005), pp. 241–261.

[5] A. Avron and A. Zamansky, *Non-Deterministic Semantics for Logical Systems*, [in:] D. Gabbay and F. Guenthner (eds.), **Handbook of Philosophical Logic**, Vol. 16, Springer, 2011, pp. 227–304.

[6] J. Cantwell, *The Logic of Conditional Negation*, **Notre Dame Journal of Formal Logic**, Vol. 49 (2008), pp. 245–260.

[7] W. Carnielli, M. Coniglio and J. Marcos, *Logics of Formal Inconsistency*, [in:] D. Gabbay and F. Guenthner (eds.), **Handbook of Philosphical Logic**, Vol. 14, Dordrecht: Springer-Verlag, 2007, pp. 1–93.

[8] W. Carnielli and J. Marcos, *A Taxonomy of C-systems*, [in:] W. A. Carnielli and M. E. Coniglio and I. M. L. d'Ottaviano (eds.), **Paraconsistency: The Logical Way to the Inconsistent, Proceedings of the II World Congress on Paraconsistency**, Marcel Dekker, 2002, pp. 1–94.

[9] W. Carnielli, J. Marcos and S. de Amo, *Formal Inconsistency and Evolutionary Databases*, **Logic and Logical Philosophy**, Vol. 8 (2000), pp. 115–152.

[10] J. Ciuciura, *Paraconsistent heap. A Hierarchy of* mbC^n*-systems*, **Bulletin of the Section of Logic**, Vol. 43, No. 3/4 (2014), pp. 173–182.

[11] L. Humberstone, *Negation by iteration*, **Theoria**, Vol. 61, No. 1 (1995), pp. 1–24.

[12] L. Humberstone, *Contra-classical logics*, **Australasian Journal of Philosophy**, Vol. 78, No. 4 (2000), pp. 438–474.

[13] N. Kamide, *Paraconsistent Double Negations as Classical and Intuitionistic Negations*, **Studia Logica**, Vol. 105, No. 6 (2017), pp. 1167–1191.

[14] G. Olkhovikov, *On a new three-valued paraconsistent logic*, **IfCoLog Journal of Logics and their Applications**, Vol. 3, No. 3 (2016), pp. 317–334.

[15] H. Omori, *From paraconsistent logic to dialetheic logic*, [in:] Holger Andreas and Peter Verdée (eds.), **Logical Studies of Paraconsistent Reasoning in Science and Mathematics**, Springer, 2016, pp. 111–134.

[16] H. Omori, *Sette's Logics, Revisited*, [in:] A. Baltag, J. Seligman and T. Yamada (eds.), **Proceedings of LORI 2017**, 2017, pp. 451–465.

[17] H. Omori and H. Wansing, *On Contra-classical variants of Nelson logic N4 and its classical extension*, **The Review of Symbolic Logic**, Vol. 11, No. 4 (2018), pp. 805–820.

[18] F. Paoli, *Bilattice Logics and Demi-Negation*, [in:] Hitoshi Omori and Heinrich Wansing (eds.), **New Essays on Belnap-Dunn Logic**, Synthese Library, Springer, forthcoming.

[19] A. Sette, *On the propositional calculus* P^1, **Mathematica Japonicae**, Vol. 16 (1973), pp. 173–180.

[20] W. Heinrich, *Connexive Logic*, [in:] Edward N. Zalta, **The Stanford Encyclopedia of Philosophy**, 2014, Fall 2014, http://plato.stanford.edu/archives/fall2014/entries/logic-connexive/

[21] T. Waragai and H. Omori, *Some New Results on PCL1 and its Related Systems*, **Logic and Logical Philosophy**, Vol. 19, No. 1/2 (2010), pp. 129–158.

[22] T. Waragai and T. Shidori, *A system of paraconsistent logic that has the notion of "behaving classically" in terms of the law of double negation and its relation to S5*, [in:] J.-Y. Béziau, W. A. Carnielli and D. Gabbay (eds.), **Handbook of Paraconsistency**, 2007, College Publications, pp. 177–187.

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