Bulletin of the **S**ection of **L**ogic
Volume 48/2 (2019), pp. 117–135

http://dx.doi.org/10.18778/0138-0680.48.2.03

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SEMI-HEYTING ALGEBRAS AND IDENTITIES OF ASSOCIATIVE TYPE

Abstract

An algebra $\mathbf{A} = \langle A, \vee, \wedge, \to, 0, 1\rangle$ is a semi-Heyting algebra if $\langle A, \vee, \wedge, 0, 1\rangle$ is a bounded lattice, and it satisfies the identities : $x \wedge (x \to y) \approx x \wedge y$, $x \wedge (y \to z) \approx x \wedge [(x \wedge y) \to (x \wedge z)]$, and $x \to x \approx 1$. $\mathcal{SH}$ denotes the variety of semi-Heyting algebras. Semi-Heyting algebras were introduced by the second author as an abstraction from Heyting algebras. They share several important properties with Heyting algebras. An identity of *associative type of length* 3 is a groupoid identity, both sides of which contain the same three (distinct) variables that occur in any order and that are grouped in one of the two (obvious) ways. A subvariety of $\mathcal{SH}$ is *of associative type of length* 3 if it is defined by a single identity of associative type of length 3.

In this paper we describe all the distinct subvarieties of the variety $\mathcal{SH}$ of asociative type of length 3. Our main result shows that there are 3 such subvarities of $\mathcal{SH}$.

Keywords: semi-Heyting algebra, Heyting algebra, identity of associative type, subvariety of associative type.

2010 Mathematics Subject Classification: Primary 03G25, 06D20, 06D15; Secondary 08B26, 08B15.

1. Introduction

Semi-Heyting algebras were introduced by the second author in 1983–84, as a result of his research that went into [33] (which was still a preprint at the time). Some of the early results were announced in [35].

A closer look at the proofs of results proved in [33] led him to the following rather interesting observation:

The arguments in [33], for the most part, used only the following well known properties of Heyting algebras:

(1) Their lattice-reducts are pseudocomplemented,
(2) Their lattice-reducts are distributive, and
(3) Congruences on them are determined by filters.

This observation led him to the following conjecture.

Conjecture A: There exists a variety **V** of algebras such that

- it has the same language as that of Heyting algebras,
- it contains Heyting algebras, and
- it possesses the following well known properties of Heyting algebras:
  (1) The lattice reducts of the algebras in **V** are pseudocomplemented
  (2) The lattice-reducts are distributive,
  (3) Congruences on the algebras in **V** are determined by filters,

Around the same time (1983-85), he had also completed the research for [36] (which was still in the preprint form). Led by the striking similarities in the results and in the proofs of (the preprints of) the papers [33] and [36], he formulated the following conjecture which appeared in print much later in 1987:

Conjecture 1: There exists a variety **V** of algebras of type $\langle \vee, \wedge, \to, ', 0, 1\rangle$ which would provide a unifying framework to state and prove common generalizations of strikingly similar results, proved in the above-mentioned two papers.

The search for such a variety led him naturally to consider the following conjecture and strengthened his belief in the validity of Conjecture A.

Conjecture 2: There exists a common generalization of (dually) pseudocomplemented lattices and De Morgan algebras.

Conjecture 2 was easy to settle with the variety of semi-De Morgan algebras, since they were already known to the author in 1979. The results on these algebras, however, appeared in print in the paper [37].

Conjecture A was settled in 1983–84 with the discovery of semi-Heyting algebras. However, the first results on semi-Heyting algebras appeared in print only in 2008 in the Proceedings of 9th A. Monteiro Conference in Bahia Blanca, Argentina (see [38]), held in 2007. (It was predicted in [37] that semi-De Morgan algebras might be useful in resolving Conjecture 1. Indeed, it turned out to be the case. Conjecture 1 was settled later in [39], with the help of both semi-Heyting algebras and (a subvariety of) semi-De Morgan algebras.)

DEFINITION 1.1. *An algebra* $\langle A, \vee, \wedge, \to, 0, 1\rangle$ *is a semi-Heyting algebra if the following conditions hold:*

(SH1) $\langle A, \vee, \wedge, 0, 1\rangle$ *is a lattice with* 0 *and* 1*,*

(SH2) $x \wedge (x \to y) \approx x \wedge y$,

(SH3) $x \wedge (y \to z) \approx x \wedge [(x \wedge y) \to (x \wedge z)]$,

(SH4) $x \to x \approx 1$.

A semi-Heyting algebra is a Heyting algebra if it satisfies the identity

(H) $(x \wedge y) \to x \approx 1$.

We will denote the variety of semi-Heyting algebras by $\mathcal{SH}$ and that of Heyting algebras by $\mathcal{H}$. It is clear that $\mathcal{H} \subset \mathcal{SH}$.

It turns out (see [38]) that semi-Heyting algebras share with Heyting algebras some rather strong properties, besides the three mentioned earlier. For example, semi-Heyting algebras share the following properties with Heyting algebras:
(1) every interval in a semi-Heyting algebra is also pseudocomplemented,
(2) the variety $\mathcal{SH}$ is arithmetical, and
(3) The variety $\mathcal{SH}$ has EDPC (equationally definable principal congruences).
Moreover, there is a rich supply of algebras in $\mathcal{SH}$. It is known that there are in $\mathcal{SH}$, up to isomorphism, two 2-element algebras, ten 3-element algebras, only one of which, of course, is a Heyting algebra and 160 algebras on a 4-element chain (see [38] and [4]).

It is well-known that Heyting algebras form an equivalent algebraic semantics for intuitionistic logic; and there is a vast literature on the lattice of subvarieties of $\mathcal{H}$ (equivalently, on the lattice of intermediate logics), both from algebraic and logical points of view. Recently, the first author, in

[11], has introduced "semi-intuitionistic logic", whose equivalent algebraic semantics is the the variety of semi-Heyting algebras and which has the intuitionistic logic and classical logic as extensions, thus implying that the lattice of intermediate logics (extensions of the intuitionistic logic) is an interval in the lattice of extensions of the semi-intuitionistic logic (see [13] for a more stream-lined version). These observations led us naturally to the following problem.

PROBLEM: Investigate the structure of the lattice of subvarieties of the variety of semi-Heyting algebras, algebraically and logically.

It should perhaps be mentioned here that already Problem 14.2 of [38] had called for an investigation into the structure of the lattice of subvarieties of the variety of semi-Heyting algebras (algebraically).

There exists already some literature related to this problem. The papers that deal with this problem algebraically include [38], [2], [3], [4], [5], [15] and [17]. The paper [4] investigates the properties of semi-Heyting chains and the structure of the variety $\mathcal{CSH}$ generated by all semi-Heyting chains. In [2], it is proved, among other things, that the variety of Boolean semi-Heyting algebras (algebras with an underlying structure of Boolean lattice) constitutes a reflective subcategory of $\mathcal{SH}$, extending the corresponding result for Heyting algebras (see [6, Corollary IX.5.4], and that the free algebras in a subvariety $\mathcal{V}$ of $\mathcal{SH}$ are directly indecomposable if and only if $\mathcal{V}$ satisfies the Stone identity, extending a known result for Heyting algebras. Article [3] presents two other subvarieties of semi-Heyting algebras that are term-equivalent to the variety of Goedel algebras (linear Heyting algebras), and that they are the only other subvarieties in L with this property. The variety of semi-Nelson algebras is introduced in [17] so that the well-known and well-exploited relationship between Heyting and Nelson algebras extends to semi-Heyting and semi-Nelson algebras. It is also proved that the variety of semi-Nelson algebras is arithmetical, has equationally definable principal congruences, has the congruence extension property, along with a description of the semisimple subvarieties. In [15] an equivalence is exhibited between the category of semi-Heyting algebras and the category of centered semi-Nelson algebras, extending Cignoli's result that the categories of Heyting algebras and centered Nelson algebras are equivalent.

The papers that deal with the above problem logically include [10], [11], [13] and [16]. In [10] the authors introduce a Gentzen style sequent calculus $\mathcal{LSJ}$ for the semi-intuitionistic logic. The advantage of this presentation of the logic is that they prove a cut-elimination theorem for $\mathcal{LSJ}$ that allows them to check the decidability of the logic. As a direct consequence, they also obtain the decidability of the equational theory of semi-Heyting algebras. In [16], a propositional calculus called "semi-intuitionistic logic with strong negation" is introduced and proved to be complete with respect to the variety of semi-Nelson algebras. It has intuitionstic logic with strong negation as an axiomatic extension.

The present paper is an addition to the above-mentioned papers. In the quest for finding new varieties of semi-Heyting algebras, we systematically investigate, in this paper, the identities of associative type.

1.1. Identities of Associative Type

A look at the associative law would reveal at least the following characteristics:

(1) Length of the left side term = length of the right side term = 3,

(2) The number of distinct variables on the left = the number of distinct variables on right = the number of occurrences of variables on either side,

(3) The order of the variables on the left side is the same as the order of the variables on the right side,

(4) The bracketings used in the left side term and in the right side term are different from each other.

One way to generalize the associative law is to relax (3) and second half of (1), while keeping (2), (4) and the first half of (1). So, we are led to the following definition.

Definition 1.2. *An identity of associative type of length n is an identity of the form $p \approx q$ of length n such that*

(a) *each of p and q contains the same n (an integer ≥ 3) distinct variables,*

(b) *p and q are terms obtained by distinct bracketings of a permutation of the n variables.*

The above definition is taken from [14]. We do not know whether the notion of "identities of associative type of length n" in such a generality as given above has occurred in the literature earlier. However, we do know that specific instances of the identities of associative type have already appeared in the literature. We mention a few examples below, using $\cdot$ for the binary operation instead of $\to$. (The interested reader may refer to [14] for more such examples.)

- The identity $x{\cdot}(y{\cdot}z) \approx (z{\cdot}x){\cdot}y$ was considered in [42] by Suschkewitsch (see also [40, Theorem 11.5]).
- Abbott [1] uses the identity $x \cdot (y \cdot z) \approx y \cdot (x \cdot z)$ as one of the defining identities in his definition of implication algebras.
- The identities $x \cdot (y \cdot z) \approx z \cdot (y \cdot x)$, $x \cdot (y \cdot z) \approx y \cdot (x \cdot z)$, and $x \cdot (y \cdot z) \approx (z \cdot x) \cdot y$ were investigated for quasigroups by Hossuzú in [23].
- The identity $x \cdot (z \cdot y) \approx (x \cdot y) \cdot z$ is investigated by Pushkashu in [30].
- The identities $x \cdot (z \cdot y) \approx (x \cdot y) \cdot z$ and $x \cdot (y \cdot z) \approx z \cdot (y \cdot x)$ have appeared in [26] of Kazim and Naseeruddin.

The following problem was first mentioned in [14].

PROBLEM: Let $\mathcal{V}$ be a given variety of algebras (whose language includes a binary operation symbol, say, '$\to$'). Investigate the mutual relationships among the subvarieties of $\mathcal{V}$, each of which is defined by a single identity of associative type of length n.

We will now consider the above problem for the variety $\mathcal{SH}$. We begin a systematic analysis of the relationships among the identities of associative type of length 3 relative to the variety $\mathcal{SH}$. For reader's convenience we repeat the special case of Definition 1.2, when $n = 3$.

Definition 1.3. *An identity $p \approx q$, in the groupoid language $\langle \to \rangle$, is called an identity of associative type of length* 3 *if p and q have exactly* 3 *(distinct) variables, say x,y,z, and these variables are grouped according to one of the following two ways of grouping:*

(a) $o \to (o \to o)$ (b) $(o \to o) \to o$.

A subvariety $\mathcal{V}$ of $\mathcal{SH}$ is called a subvariety of associative type of length 3 if it is defined by a single identity of associative type of lenth 3.

In the rest of the paper, we refer to an "identity of associative type of length 3" and a variety of associative type of lenth 3 as simply an identity of associative type and a variety of associative type, respectively.

We wish to determine distinct subvarieties of associative type and their mutual relationships, as well as their relationships with other known subvarieties of $\mathcal{SH}$.

Our main theorem says that there are 3 such subvarieties of $\mathcal{SH}$ that are distinct from each other and describes explicitly, by a Hasse diagram, the poset formed by them.

2. Preliminaries

We refer to [9] for concepts and results in universal algebra and to [6] for distributive lattices.

In this section, we recall some known subvarieties of $\mathcal{SH}$ and also recall some results that will be useful in later sections.

LEMMA 2.1. [38] *Let* $\mathbf{A} \in \mathcal{SH}$ *and* $a, b \in A$.

(a) *If* $a \to b = 1$ *then* $a \leq b$.

(b) *If* $a \leq b$ *then* $a \leq a \to b$.

(c) $1 \to a = a$.

THEOREM 2.2. [5, Theorem 1.8] *Let* $\mathbf{A} \in \mathcal{SH}$. *The following conditions are equivalent:*

(1) $\mathbf{A} \models x \to y \approx y \to x$,

(2) $\mathbf{A} \models x \to 1 \approx x$,

(3) $\mathbf{A} \models y \wedge (x \to y) \approx x \wedge y$.

The varieties of associative semi-Heyting algebras and commutive semi-Heyting algebras, denoted, respectively, by $\mathcal{A}$ and $\mathcal{C}$, are defined (see [38]), relative to $\mathcal{SH}$, by

(A) $x \to (y \to z) \approx (x \to y) \to z$,

(C) $x \to y \approx y \to x$.

In [5] it is proved that the identity $x \to (y \to z) \approx (x \to y) \to z$ characterizes the variety $\mathcal{V}(\mathbf{L}_2)$, where $\mathcal{V}(\mathbf{L}_2)$ is the variety generated by $\mathbf{L}_2$ (see bellow). The variety $\mathcal{C}$ has the interesting property that $0 \to 1 = 0$, quite opposite to the behavior of Heyting algebras. The variety $\mathcal{C}$ is, we think, also of interest from the philosophical point of view.

Theorem 2.3. [5, Theorem 1.12] $\mathcal{A} = \mathcal{V}(\mathbf{L}_2)$.

Lemma 2.4. [5, Lemma 1.10] *If* $\mathbf{A} \in \mathcal{A}$, *then* $\mathbf{A}$ *satisfies* $x \to 1 \approx x$.

The following examples of semi-Heyting algebras will be useful in the rest of the paper.

- Algebras defined on a 2-element chain $\{0, 1\}$ with $0 < 1$:

$\mathbf{L}_1$

→:	0	1
0	1	1
1	0	1

$\mathbf{L}_2$

→:	0	1
0	1	0
1	0	1

- Algebras defined on a 3-element chain $\{0, a, 1\}$, with $0 < a < 1$:

$\mathbf{L}_3$

→:	0	1	a
0	1	a	1
1	0	1	a
a	0	1	1

$\mathbf{L}_4$

→:	0	1	a
0	1	0	0
1	0	1	a
a	0	a	1

$\mathbf{L}_5$

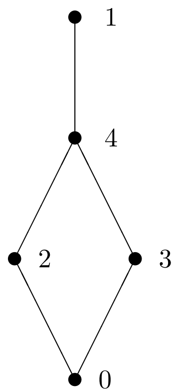
→:	0	1	a
0	1	1	1
1	0	1	a
a	0	a	1

- Algebras defined on a 4-element chain $\{0, a, b, 1\}$, with $0 < a < b < 1$:

$\mathbf{L}_6$

→:	0	1	b	a
0	1	1	1	b
1	0	1	b	a
b	0	1	1	a
a	0	1	1	1

- A 5-element algebra with the following lattice reduct and the $\to$ operation:



$\mathbf{L}_7$

$\to$:	0	1	2	3	4
0	1	0	3	2	0
1	0	1	2	3	4
2	3	2	1	0	2
3	2	3	0	1	3
4	0	1	2	3	1

3. Identities of Associative Type

We now turn our attention to identities of associative type of length 3. Recall that such an identity will contain three distinct variables that occur in any order and that are grouped in one of the two (obvious) ways. The following identities play a crucial role in the sequel.

Let Σ denote the set consisting of the following 14 identities of associative type in the binary language $\langle\to\rangle$):

($A1$) $x \to (y \to z) \approx (x \to y) \to z$, (Associative law,)

($A2$) $x \to (y \to z) \approx x \to (z \to y)$,

($A3$) $x \to (y \to z) \approx (x \to z) \to y$,

($A4$) $x \to (y \to z) \approx y \to (x \to z)$,

($A5$) $x \to (y \to z) \approx (y \to x) \to z$,

($A6$) $x \to (y \to z) \approx y \to (z \to x)$,

($A7$) $x \to (y \to z) \approx (y \to z) \to x$,

($A8$) $x \to (y \to z) \approx (z \to x) \to y$,

($A9$) $x \to (y \to z) \approx z \to (y \to x)$,

($A10$) $x \to (y \to z) \approx (z \to y) \to x$,

($A11$) $(x \to y) \to z \approx (x \to z) \to y$,

($A12$) $(x \to y) \to z \approx (y \to x) \to z$,

($A13$) $(x \to y) \to z \approx (y \to z) \to x$,

($A14$) $(x \to y) \to z \approx (z \to y) \to x$.

We will denote by $\mathcal{A}_i$ the subvariety of $\mathcal{SH}$ defined by the identity (Ai), for $1 \leq i \leq 14$. Such varieties will be referred to as subvarieties of $\mathcal{SH}$ of associative type. Sometimes we will use (A) for (A_1) and $\mathcal{A}$ for $\mathcal{A}_1$.

The following proposition, whose proof is routine, is crucial for the rest of the paper.

Proposition 3.1. [14] *Let $\mathcal{G}$ be the variety of all groupoids of type $\{\to\}$ and Let $\mathcal{V}$ denote the subvariety of $\mathcal{G}$ defined by a single identity of associative type. Then $\mathcal{V} = \mathcal{A}_i$, for some $i \in \{1, 2, \cdots, 14\}$.*

Our goal, in this paper, is to determine the distinct subvarieties of $\mathcal{SH}$ associative type and to describe the poset of subvarieties of $\mathcal{SH}$. It suffices to concentrate on the varieties defined by identities (A1)-(A14), in view of the above proposition.

3.1. Properties of subvarieties of $\mathcal{SH}$ of Associative type

In this section we present properties of several subvarieties of $\mathcal{SH}$ which will play a crucial role in our analysis of the identities of associative type relative to $\mathcal{SH}$.

The proof of the following lemma is straightforward.

Lemma 3.2. *If $\mathbf{A} \in \mathcal{SH}$ satisfies the identities $x \to y \approx y \to x$ and $x \to (y \to z) \approx (x \to y) \to z$ then $\mathbf{A} \in \mathcal{A}_j$ for all $j \in \{1, 2, \ldots, 14\}$.*

Lemma 3.3. *If $\mathbf{A} \in \mathcal{A}_j$ with $j \in \{1, 5, 8, 10, 12, 13, 14\}$, then $\mathbf{A} \models x \to 1 \approx x$.*

Proof: Let $a \in A$.

- $j = 1$: This case follows from Lemma 2.4.
- $j = 5$:

$$\begin{aligned} a \to 1 &= a \to (a \to a) && \text{by (SH4)} \\ &= (a \to a) \to a && \text{by (A5)} \\ &= 1 \to a && \text{by (SH4)} \\ &= a && \text{by Lemma 2.1 (c).} \end{aligned}$$

- $j = 8$:

$$\begin{aligned} a &= (1 \to 1) \to a && \text{by Lemma 2.1 (c)} \\ &= 1 \to (a \to 1) && \text{by (A8)} \\ &= a \to 1 && \text{by Lemma 2.1 (c).} \end{aligned}$$

- $j = 10$:

$$\begin{array}{rcll} a \to 1 & = & a \to (1 \to 1) & \text{by Lemma 2.1 (c)} \\ & = & (1 \to 1) \to a & \text{by (A10)} \\ & = & 1 \to a & \text{by Lemma 2.1 (c)} \\ & = & a & \text{by Lemma 2.1 (c).} \end{array}$$

- $j = 12$: By Lemma 2.1 (b), $a \leq a \to 1$. Also note that

$$\begin{array}{rcll} (a \to 1) \to a & = & (1 \to a) \to a & \text{by (A12)} \\ & = & a \to a & \text{by Lemma 2.1 (c)} \\ & = & 1 & \text{by (SH4).} \end{array}$$

  Hence, using Lemma 2.1 (a), $a \to 1 \leq a$.
- $j = 13$:

$$\begin{array}{rcll} a & = & 1 \to a & \text{by Lemma 2.1 (c)} \\ & = & (1 \to 1) \to a & \text{by Lemma 2.1 (c)} \\ & = & (1 \to a) \to 1 & \text{by (A13)} \\ & = & a \to 1 & \text{by Lemma 2.1 (c).} \end{array}$$

- $j = 14$: By Lemma 2.1 (b), $a \leq a \to 1$. Also note that

$$\begin{array}{rcll} a \to 1 & = & (a \to 1) \wedge 1 & \\ & = & (a \to 1) \wedge ((a \to 1) \to 1) & \text{by (SH2)} \\ & = & (a \to 1) \wedge ((1 \to 1) \to a) & \text{by (A14)} \\ & = & (a \to 1) \wedge a & \text{by Lemma 2.1 (c).} \end{array}$$

  Therefore, $a \to 1 \leq a$,

proving the lemma. □

Lemma 3.4. *If* $\mathbf{A} \in \mathcal{A}_j$ *for* $1 \leq j \leq 14$ *and* $j \neq 4$ *then* $\mathbf{A} \in \mathcal{C}$.

Proof: Let $a, b \in A$.

- If $j \in \{1, 5, 8, 10, 12, 13, 14\}$ the result follows from Theorem 2.2 and Lemma 3.3.
- $j = 2$:

$$\begin{array}{rcll} a \to b & = & 1 \to (a \to b) & \text{by Lemma 2.1 (c)} \\ & = & 1 \to (b \to a) & \text{by (A2)} \\ & = & b \to a & \text{by Lemma 2.1 (c).} \end{array}$$

- $j = 3$:

$$\begin{array}{rcll} a \to b & = & 1 \to (a \to b) & \text{by Lemma 2.1 (c)} \\ & = & (1 \to b) \to a & \text{by (A3)} \\ & = & b \to a & \text{by Lemma 2.1 (c).} \end{array}$$

- $j = 6$:

$$\begin{array}{rcll} a \to b & = & a \to (1 \to b) & \text{by Lemma 2.1 (c)} \\ & = & 1 \to (b \to a) & \text{by (A6)} \\ & = & b \to a & \text{by Lemma 2.1 (c).} \end{array}$$

- $j = 7$:

$$\begin{array}{rcll} a \to b & = & a \to (1 \to b) & \text{by Lemma 2.1 (c)} \\ & = & (1 \to b) \to a & \text{by (A7)} \\ & = & b \to a & \text{by Lemma 2.1 (c).} \end{array}$$

- $j = 9$:

$$\begin{array}{rcll} a \to b & = & a \to (1 \to b) & \text{by Lemma 2.1 (c)} \\ & = & b \to (1 \to a) & \text{by (A9)} \\ & = & b \to a & \text{by Lemma 2.1 (c).} \end{array}$$

- $j = 11$:

$$\begin{array}{rcll} a \to b & = & (1 \to a) \to b & \text{by Lemma 2.1 (c)} \\ & = & (1 \to b) \to a & \text{by (A11)} \\ & = & b \to a & \text{by Lemma 2.1 (c),} \end{array}$$

proving the lemma. □

LEMMA 3.5. *If* $\mathbf{A} \in \mathcal{A}_j$ *with* $j \in \{3, 5, 6, 8, 9, 11, 13, 14\}$ *then* $\mathbf{A} \in \mathcal{A}$.

PROOF: If $\mathbf{A} \in \mathcal{A}_3$, then

$$\begin{array}{rcll} a \to (b \to c) & = & a \to (c \to b) & \text{by Lemma 3.4} \\ & = & (a \to b) \to c & \text{by (A3).} \end{array}$$

If $\mathbf{A} \in \mathcal{A}_5$, then

$$\begin{array}{rcll} a \to (b \to c) & = & (b \to a) \to c & \text{by (A5)} \\ & = & (a \to b) \to c & \text{by Lemma 3.4.} \end{array}$$

If $\mathbf{A} \in \mathcal{A}_6$, then

$$\begin{array}{rcll} a \to (b \to c) & = & a \to (c \to b) & \text{by Lemma 3.4} \\ & = & c \to (b \to a) & \text{by (A6)} \\ & = & c \to (a \to b) & \text{by Lemma 3.4} \\ & = & (a \to b) \to c & \text{by Lemma 3.4.} \end{array}$$

If $\mathbf{A} \in \mathcal{A}_8$, then

$$\begin{array}{rcll} a \to (b \to c) & = & a \to (c \to b) & \text{by Lemma 3.4} \\ & = & (b \to a) \to c & \text{by (A8)} \\ & = & (a \to b) \to c & \text{by Lemma 3.4.} \end{array}$$

If $\mathbf{A} \in \mathcal{A}_9$, then

$$\begin{array}{rcll} a \to (b \to c) & = & c \to (b \to a) & \text{by (A9)} \\ & = & c \to (a \to b) & \text{by Lemma 3.4} \\ & = & (a \to b) \to c & \text{by Lemma 3.4.} \end{array}$$

If $\mathbf{A} \in \mathcal{A}_{11}$, then

$$\begin{array}{rcll} (a \to b) \to c & = & (b \to a) \to c & \text{by Lemma 3.4} \\ & = & (b \to c) \to a & \text{by (A11)} \\ & = & a \to (b \to c) & \text{by Lemma 3.4.} \end{array}$$

If $\mathbf{A} \in \mathcal{A}_{13}$, then

$$\begin{array}{rcll} (a \to b) \to c & = & (b \to c) \to a & \text{by (A13)} \\ & = & a \to (b \to c) & \text{by Lemma 3.4.} \end{array}$$

If $\mathbf{A} \in \mathcal{A}_{14}$,

$$\begin{array}{rcll} (a \to b) \to c & = & (c \to b) \to a & \text{by (A14)} \\ & = & a \to (c \to b) & \text{by Lemma 3.4} \\ & = & a \to (b \to c) & \text{by Lemma 3.4.} \end{array}$$

The lemma is now proved. □

THEOREM 3.6. $\mathcal{A} = \mathcal{A}_1 = \mathcal{A}_3 = \mathcal{A}_5 = \mathcal{A}_6 = \mathcal{A}_8 = \mathcal{A}_9 = \mathcal{A}_{11} = \mathcal{A}_{13} = \mathcal{A}_{14}$.

PROOF: The proof follows directly from Lemma 3.2, Lemma 3.4 and Lemma 3.5. □

The proof of the following lemma is straightforward.

LEMMA 3.7. *If* $\mathbf{A} \in \mathcal{C}$ *then* $\mathbf{A} \in \mathcal{A}_j$ *with* $j \in \{2, 7, 10, 12\}$.

THEOREM 3.8. $\mathcal{C} = \mathcal{A}_2 = \mathcal{A}_7 = \mathcal{A}_{10} = \mathcal{A}_{12}$.

PROOF: This result is easy to check by using Lemma 3.4 and Lemma 3.7. □

4. Main Theorem

We are now ready to present the main theorem of this paper.

THEOREM 4.1. *We have*

(a) *The following are the* 3 *subvarieties of* $\mathcal{SH}$ *of associative type that are distinct from each other:*

$$\mathcal{A}, \mathcal{C} \text{ and } \mathcal{A}_4.$$

(b) *They satisfy the following relationships:*

1. $\mathcal{T} \subset \mathcal{A} \subset \mathcal{C} \subset \mathcal{SH}$, *where* $\mathcal{T}$ *denotes the trivial variety,*
2. $\mathcal{A} \subset \mathcal{A}_4 \subset \mathcal{SH}$,
3. $\mathcal{C} \mid\mid \mathcal{A}_4$,
4. $\mathcal{H} \subset \mathcal{A}_4$.

PROOF: Observe that, in view of Theorem 3.6 and Theorem 3.8 we can conclude that each of the 14 subvarieties of associative type of $\mathcal{SH}$ is equal to one of the following varieties:

$$\mathcal{A}, \mathcal{C} \text{ and } \mathcal{A}_4.$$

We first wish to prove (b). By Lemma 3.2, $\mathcal{A} \subseteq \mathcal{A}_4$. The algebra $\mathbf{L}_1$ shows that the inclusion is proper using $x = 0, y = 0, z = 0$ in the identity (A). It is clear that $\mathcal{A}_4 \subseteq \mathcal{SH}$. The algebra $\mathbf{L}_3$ shows that the inclusion is proper using $x = 0, y = a, z = 1$ in the identity (A4) proving b2.

Let us check item (b3). The algebra $\mathbf{L}_4$ shows that $\mathcal{C} \not\subseteq \mathcal{A}_4$ using $x = 0, y = a, z = 0$ in the identity (A4). The algebra $\mathbf{L}_1$ shows that $\mathcal{A}_4 \not\subseteq \mathcal{C}$ using $x = 0, y = 1$ in the identity (C).

The condition $\mathcal{H} \subseteq \mathcal{A}_4$ is clear since if $\mathbf{A} \in \mathcal{H}$ and $a, b \in A$ then $a \to (b \to c) = (a \wedge b) \to c = (b \wedge a) \to c = b \to (a \to c)$, the latter being well-known. Let us consider the algebra $\mathbf{L}_2$. It shows that $\mathcal{A}_4 \not\subseteq \mathcal{H}$ using $x = 0, y = 1$ in the identity (H). Then the proof of item (b4) is done.

The inclusion $\mathcal{A} \subseteq \mathcal{C}$ follows from Lemma 2.4 and Theorem 2.2. The algebra $\mathbf{L}_4$ shows that the inclusion is proper since (A) fails in it at $x = 0$, $y = 0$, $z = a$. The proof of the theorem is now complete since (a) is an immediate consequence of (b). □

Further relationship between $\mathcal{C}$, $\mathcal{A}_4$ and $\mathcal{A}$ is given in the following theorem.

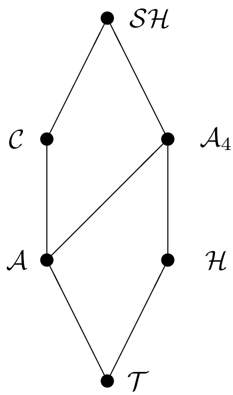
THEOREM 4.2. $\mathcal{C} \cap \mathcal{A}_4 = \mathcal{A}$.

PROOF: Let $\mathbf{A} \in \mathcal{A}_4$ and $a, b, c \in A$. Notice that

$$\begin{aligned} a \to (b \to c) &= a \to (c \to b) && \text{by (C)} \\ &= c \to (a \to b) && \text{by (A4)} \\ &= (a \to b) \to c && \text{by (C)} \end{aligned}$$

Hence $\mathcal{C} \cap \mathcal{A}_4 \subseteq \mathcal{A}$. In view of Theorem 4.1, $\mathcal{C} \cap \mathcal{A}_4 = \mathcal{A}$. □

The Hasse diagram of the poset (in fact, $\wedge$-semilattice) of subvarieties of $\mathcal{SH}$ of associative type, together with $\mathcal{SH}$, $\mathcal{T}$ and $\mathcal{H}$, is given below.



Next, we will study some relationships of this interesting new subvariety $\mathcal{A}_4$ with some of the other earlier known subvarieties of $\mathcal{SH}$.

In [38, Definition 8.1], Sankappanavar introduced the following subvarieties of $\mathcal{SH}$ by providing defining identities relative to $\mathcal{SH}$ for each of them (where * is the operation of pseudocomplementation):

Subvariety	Defining identity within $\mathcal{SH}$
$\mathcal{FTT}$ (**F**alse implies **T**rue is **T**rue)	$0 \to 1 \approx 1$
$\mathcal{FTD}$ (**F**alse implies **T**rue is **D**ense)	$(0 \to 1)^* \approx 0$
$\mathcal{QH}$ (Quasi-Heyting algebras)	$y \leq x \to y$
$\mathcal{SH}^S$ (Stone semi-Heyting algebras)	$x^* \vee x^{**} \approx 1$
$\mathcal{FTF}$ (**F**alse implies **T**rue is **F**alse)	$0 \to 1 \approx 0$

THEOREM 4.3. *The variety $\mathcal{A}_4$ is incomparable to each of the subvarieties*

$$\mathcal{FTT}, \mathcal{FTD}, \mathcal{QH}, \mathcal{SH}^S \text{ and } \mathcal{FTF}.$$

PROOF: The algebra $\mathbf{L}_2$ shows that $\mathcal{A}_4 \not\subseteq \mathcal{FTT}$, $\mathcal{A}_4 \not\subseteq \mathcal{FTD}$ and $\mathcal{A}_4 \not\subseteq \mathcal{QH}$.
The algebra $\mathbf{L}_3$ shows that $\mathcal{FTD} \not\subseteq \mathcal{A}_4$ and $\mathcal{SH}^S \not\subseteq \mathcal{A}_4$ using $x = 0$, $y = a$, $z = 1$.
The algebra $\mathbf{L}_5$ shows that $\mathcal{FTT} \not\subseteq \mathcal{A}_4$ using $x = 0$, $y = a$, $z = 0$.
The algebra $\mathbf{L}_6$ shows that $\mathcal{QH} \not\subseteq \mathcal{A}_4$ using $x = 0$, $y = b$, $z = a$.
The algebra $\mathbf{L}_7$ shows that $\mathcal{A}_4 \not\subseteq \mathcal{SH}^S$ with $x = 2$.
The algebra $\mathbf{L}_1$ shows that $\mathcal{A}_4 \not\subseteq \mathcal{FTF}$.
The algebra $\mathbf{L}_4$ shows that $\mathcal{FTF} \not\subseteq \mathcal{A}_4$ using $x = 0$, $y = a$, $z = 0$. □

Acknowledgment. The first author wants to thank the institutional support of CONICET (Consejo Nacional de Investigaciones Científicas y Técnicas). The authors are grateful to the anonymous referee for his/her careful reading of an earlier version of the paper.

Compliance with Ethical Standards:

Conflict of Interest. The first author declares that he has no conflict of interest. The second author declares that he has no conflict of interest.

Ethical approval. This article does not contain any studies with human participants or animals performed by any of the authors.

Funding. The work of Juan M. Cornejo was supported by CONICET (Consejo Nacional de Investigaciones Cientificas y Tecnicas) and Universidad Nacional del Sur.

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