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TWO INFINITE SEQUENCES OF PRE-MAXIMAL EXTENSIONS OF THE RELEVANT LOGIC **E**

Abstract

The only maximal extension of the logic of relevant entailment **E** is the classical logic **CL**. A logic $L \subseteq [\mathbf{E}, \mathbf{CL}]$ called *pre-maximal* if and only if L is a coatom in the interval $[\mathbf{E}, \mathbf{CL}]$. We present two denumerable infinite sequences of pre-maximal extensions of the logic **E**. Note that for the relevant logic **R** there exist exactly three pre-maximal logics, i.e. coatoms in the interval $[\mathbf{R}, \mathbf{CL}]$.

Keywords: relevant logic, non-classical logics, lattice, universal algebra.

1. Preliminaries

Let FOR be the set of all the propositional formulae built up from the propositional variables $p, q, r, p_1 \ldots$ using the connectives $\neg, \wedge, \vee$ and $\rightarrow$. The first information about the logic of relevant entailment **E** can be found in [8]. The logic **E** is defined as a subset of the set FOR. **E** consists of formulae provable using the following list of axiom schemes:

$E1$ $\phi \rightarrow \phi$,
$E2$ $(\phi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow \phi \rightarrow \chi))$,
$E3$ $((\phi \rightarrow \phi) \rightarrow \psi) \rightarrow \psi$,
$E4$ $(\phi \rightarrow (\phi \rightarrow \psi)) \rightarrow (\phi \rightarrow \psi)$,
$E5$ $\phi \wedge \psi \rightarrow \phi$,
$E6$ $\phi \wedge \psi \rightarrow \psi$,
$E7$ $(\phi \rightarrow \psi) \wedge (\phi \rightarrow \chi) \rightarrow (\phi \rightarrow \psi \wedge \chi)$,

$E8$ $\phi \to \phi \vee \psi$,

$E9$ $\psi \to \phi \vee \psi$,

$E10$ $(\phi \to \psi) \wedge (\chi \to \psi) \to (\phi \vee \chi \to \psi)$,

$E11$ $(\phi \wedge (\psi \vee \chi)) \to ((\phi \wedge \psi) \vee \chi)$,

$E12$ $(\phi \to \neg\psi) \to (\psi \to \neg\phi)$,

$E13$ $\neg\neg\phi \to \phi$.

by application of the rule of modus ponens ($MP : \phi \to \psi, \phi \ / \ \psi$) and the rule of adjunction ($AD : \phi, \psi \ / \ \phi \wedge \psi$).

The definitions of proof and the metalogical are standard one.

There exists an equivalent version of the logic **E** with the same set of axioms, based on the substitution rule.

If we extend the logic **E** by adding the axiom

$$\phi \to ((\phi \to \psi) \to \psi),$$

then we obtain the well known relevant logic **R.**

The logic **R** and the structure of extensions of the logic **R** is rather well understood, (see A. R. Anderson, N. D. Belnap [2], W. Dziobiak [6], J. M. Font, G. Rodriguez [5], R. K. Meyer [10], L. L. Maksimowa [7],[8], K. Świrydowicz [11], [12]).

However, the logic **E** has not been fully described. One of the basic properties that have been proved is the lack of algebraizability (W.J. Blok and D.L. Pigozzi [4]). Moreover, the logic **E** is not structurally complete (see J.M. Dunn, R.M. Meyer [10]). There also exists method of proving theorems of **E** introduced by F.Fitch [13].

In addition, it has been shown that there exists exactly three pre-maximal extension of the logic **R**, i.e. extensions for which the only extension is the classical logic (see K. Świrydowicz). In the following manuscript we show that there exists infinitely many pre-maximal extensions of the logic **E.**

1.1. Syntactical matters

LEMMA 1. *The formulae listed below are theses of* ***E***:

(t1) $(p \to q) \to ((r \to s) \to ((s \to p) \to (r \to q)))$,
(t2) $(p \to q) \wedge (r \to s) \to ((p \wedge r) \to (q \wedge s))$,
(t3) $(p \to q) \wedge (r \to s) \to ((p \vee r) \to (q \vee s))$,
(t4) $(p \to q) \to (\neg q \to \neg p)$,
(t5) $(p \wedge (p \to q)) \to q$,
(t6) $(p \to \neg\neg p)$,
(t7) $((p \wedge q) \vee (p \wedge r)) \leftrightarrow (p \wedge (q \vee r))$, *where* $\leftrightarrow$ *denotes two implications*

PROOF: Use the Fitch-style proofs. □

LEMMA 2. *Let* $\phi(p_1, \ldots, p_n)$ *be a formula constructed using variables* $p_1, \ldots, p_n$. *Then*

$$\vdash_E \phi(p_1, \ldots, p_n) \iff \vdash_E (p_1 \to p_1) \wedge \ldots \wedge (p_n \to p_n) \to \phi(p_1, \ldots, p_n)$$

Next we can prove the following lemma

LEMMA 3. $$\vdash_E \phi \iff \vdash_E (\phi_1 \to \phi_1) \wedge \ldots \wedge (\phi_n \to \phi_n) \to \phi$$
for some subformulae $\phi_1, \ldots, \phi_n$ *of the formula* ϕ. *In particular,*

$$\vdash_E (\phi \to \psi) \iff \vdash_E (\phi \to \phi) \to (\phi \to \psi)$$

1.2. Algebraic matters

DEFINITION 4. An Algebra $\mathbf{A} = \langle A, \wedge, \vee, \to, \neg \rangle$ is called an **E**-*algebra*, if $\langle A, \wedge, \vee \rangle$ is a distributive lattice and the following conditions are satisfied for all $x, y, z \in \mathbf{A}$:

(e1) $(x \to y) \leq ((y \to z) \to (x \to z))$,
(e2) $((x \to x) \to y) \leq y$,
(e3) $(x \to (x \to y)) \leq (x \to y)$,
(e4) $(x \to y) \wedge (v \to s) \leq ((x \wedge v) \to (y \wedge s))$,
(e5) $(x \to y) \wedge (v \to s) \leq ((x \vee v) \to (y \vee s))$,
(e6) $(x \to \neg y) \leq (y \to \neg x)$,
(e7) $x = \neg\neg x$.

In the expressions above, $\leq$ denotes partial order of the lattice $\langle A, \wedge, \vee \rangle$). The lattice $\langle A, \wedge, \vee \rangle$ of the algebra **A** is called *lattice of this* ***E***-*algebra.*

Definition 5. A pair $\langle \mathbf{A}, \nabla_{\mathbf{A}} \rangle$ is called an **E**-*matrix*, if **A** is an **E**-algebra, and $\nabla_A \subseteq A$ satisfies the condition

$$x \in \nabla_A \iff (x_1 \to x_1) \wedge \ldots \wedge (x_n \to x_n) \leq x,$$

for some $(x_1 \to x_1), \ldots, (x_n \to x_n)$. The set ∇_A is called *a set of the designated elements* of the algebra **A**.

Lemma 6. *The set ∇_A is a filter on* **A**.

Definition 7. Let **A** be an **E**-algebra. The *logic* $L(\mathbf{A})$ generated by the matrix $\langle \mathbf{A}, \nabla_{\mathbf{A}} \rangle$ is the set of the formulae which satisfy the following condition:

$$\phi \in L(\mathbf{A}) \iff \forall_{h:FOR \to A} \quad (h(\phi) \in \nabla_{\mathbf{A}}),$$

where $h : FOR \to \mathbf{A}$ is homomorphism.

Definition 8. If $h(\phi) \in \nabla_{\mathbf{A}}$ for any homomorphism $h : FOR \to \mathbf{A}$, then ϕ is called an $\langle \mathbf{A}, \nabla_{\mathbf{A}} \rangle$-*tautology* or simply **A**-*tautology.*

Theorem 9. (**Completeness of E**)

$$\vdash_E \phi \iff h(\phi) \in \nabla_{\mathbf{A}}$$

*for any **E**-algebra* **A***, and for any homomorphism* $h : FOR \to A$*, where* $\nabla_{\mathbf{A}}$ *is the set of designated elements of **A**.*

Proof: (⇒) Induction on the length of a proof of ϕ in **E**.
(⇐) Construction of the Lindenbaum algebra of **E** ($Lind_{\mathbf{E}}$). □

Recall that the Lindenbaum algebra for the logic **E** $Lind_{\mathbf{E}}$ is constructed of the set FOR by the equivalence relation defined by:

$$\psi \sim \phi \iff \vdash_{\mathbf{E}} \psi \to \phi \ \wedge \ \vdash_{\mathbf{E}} \phi \to \psi.$$

The partial order $\leq$ is defined by $\phi/_\sim \leq \psi/_\sim \iff \vdash_{\mathbf{E}} (\phi \to \psi)$.

$Lind_{\mathbf{E}}$ is an **E**-algebra; in particular:

$(*)$ $\quad \phi/_\sim \leq \psi/_\sim \iff (\phi/_\sim \to \psi/_\sim) \in \nabla_{Lind_{\mathbf{E}}}$, ie.

$(**)$ $\quad x \leq y \iff (x \to y) \in \nabla_{Lind_{\mathbf{E}}}$

We point out that the equivalences $(*)$ and $(**)$ do not need to hold in each **E**-algebra.

Finally, we have

COROLLARY 10. Let $\vdash_{\mathbf{E}} (\phi \to \psi)$. Then for each **E**-algebra **A** and for each $h : FOR \to \mathbf{A}$ the following inequality holds

$$h(\phi) \leq h(\psi).$$

Thus, each **E**-theorem in the form $\phi \to \psi$ generates an inequality in each **E**-algebra.

For a given algebra **A** the filter $\nabla_{\mathbf{A}}$ is uniquely defined. Hence, now we show how to differentiate between **E**-algebras and **E**-matrices.

LEMMA 11. *Let* **A** *be an* ***E****-algebra and* $\nabla_A = \{x \in A : \exists t_k(t_k \leq x)\}$, *where* $t_k = \bigwedge_{1<i<k}(x_i \to x_i)$ *for some elements* $a_i \in \boldsymbol{A}$ *and let* $\nabla_{\mathbf{A}} \subseteq \nabla$. *Then the relation* $\theta(\nabla)$ *defined by the equivalence*

$$(x \equiv y)\theta(\nabla) \iff ((x \to y), (y \to x) \in \nabla)$$

is a congruence relation on $\boldsymbol{A}$.

LEMMA 12. *Let* θ *be a congruence relation on the* ***E****-algebra* $\boldsymbol{A}$. *Then the set* $\nabla(\theta) = \{x : \exists y(y \in \nabla_{\mathbf{A}}) \wedge (x \equiv_\theta y)\}$ *is a filter and* $\nabla_{\mathbf{A}} \subseteq \nabla(\theta)$.

PROOF: Easy. (cf. Definition 4) □

Let $\mathcal{F}(\nabla_{\mathbf{A}}) = \{\nabla : \nabla \text{ is a filter and } \nabla_{\mathbf{A}} \subseteq \nabla\}$. If **A** is an **R**-algebra, then the lattices $Con(\mathbf{A})$ and $\mathcal{F}(\nabla_{\mathbf{A}})$ are isomorphic. However, if **A** is an **E**-algebra, then $Con(\mathbf{A})$ and $\mathcal{F}(\nabla_{\mathbf{A}})$ do not have to be isomorphic (see W.J. Blok and D. Pigozzi) [4].

DEFINITION 13. An algebra **A** is called a *simple algebra,* if $Con(\mathbf{A})$ contains exactly two elements.

By Corollary 10 and the definition of **E**-algebra (refdef:1) we get the following useful lemma.

LEMMA 14. *The following inequalities hold in each* ***E****-algebra:*

(1) $x \wedge (x \to y) \leq y$,
(2) $(\neg x \to x) \leq x$,
(3) $(x \to ((y_1 \to y_2) \to z)) \leq ((y_1 \to y_2) \to (x \to z))$.

LEMMA 15. *Moreover, we have additional useful implications and inequalities:*

(i) $x \in \nabla \Longrightarrow x \to y \leq y$,
(ii) *Let* $y \to y = a$. *Then* $(x \to y) \leq a \to (x \to y)$.

LEMMA 16. *Let* $\boldsymbol{A}$ *satisfy the inequality* $((x \to x) \to (y \to z)) \leq (y \to ((x \to x) \to z))$. *Then the following conditions are equivalent*

($'$) $x \leq y \iff (x \to y) \in \nabla_{\mathbf{A}}$.
($''$) $(x \to x) \leq (y \to z) \iff y \leq ((x \to x) \to z)$.

COROLLARY 17. If $\mathbf{A}$ satisfies the inequality $(x \to x) \to (y \to z) \leq (y \to ((x \to x) \to z))$, then $\mathbf{A}$ satisfies $(x_1 \to x_2) \to ((y_1 \to y_2) \to z) \leq ((y_1 \to y_2) \to ((x_1 \to x_2) \to z))$.

2. Two infinite sequences of algebras

2.1. Introductory remarks

In this section we present the construction of two infinite sequences of **E**-algebras whose lattices are chains. For convenience, we us horizontal notation for chains (i.e., chains are written in a number like-line fashion).

Since all the **E**-algebras considered below are based on finite chains, hence these algebras have a smallest element (denoted by 0) and a greatest element (denoted by 1). Moreover, we use a, to denote an atom in all **E**-algebras. In addition, $\nabla = [a) = \{x : a \leq x\}$.

LEMMA 18. *The following equalities hold in* $\boldsymbol{E}$*-algebras:*

$1 \to 1 = 1, 0 \to x = 1, 1 \to 0 = 0, 0 \to 1 = 1.$

If $\boldsymbol{E}$*-algebra* $\mathbf{A}$ *is a chain,* $\nabla_{\mathbf{A}} = [a)$ *and* a *is an atom, then* $x \to 0 = 0, x \neq 0, if x \in \nabla_{\mathbf{A}}$.

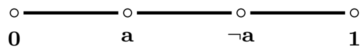
Since we examine only **E**-algebras based on chains and $\nabla_A = [a)$, where a is an atom, hence the equality $x \to 0$ holds for all x in **A.**

LEMMA 19. *The algebra* **2** *is a subalgebra of each nontrivial* $\boldsymbol{E}$*-algebra.*

2.2. Construction of $\mathbf{A_n}$-algebras

A_0-algebras

Let us consider the following lattice:



If the lattice of an **E**-algebra is a 4-elements chain the (operation $\neg$ is obvious here), then the operation $\to$ must be defined as the following table shows

$\to$	0	a	$\neg a$	1
0	1	1	1	1
a	0	a		1
$\neg a$	0	0	a	1
1	0	0	0	1

We observe $a \leq \neg a$, thus $(a \to \neg a) \in \nabla$, ie. $(a \to \neg a) \in [a)$, i.e. $a \leq a \to \neg a$. By the other hand, by the Clavius law $a \to \neg a \leq \neg a$. Summing it up, $a \leq (a \to \neg a) \leq \neg a$.

We conclude that the function $\to$ for $a \to \neg a$ can be defined in the following three ways:

1. $a \to \neg a = a$
2. $a \to \neg a = \neg a$
3. $a \to \neg a \neq a, a \to \neg a \neq \neg a$, i.e. $a \to \neg a$ is *a* new element different from $a, \neg a$.

If we assume that $a \to \neg a = a$ or $a \to \neg a = \neg a$, then we get two distinct **E**-algebras and the function $\to$ can be defined as in the following $\to$-tables:

$\to$	0	a	$\neg a$	1
0	1	1	1	1
a	0	a	a	1
$\neg a$	0	0	a	1
1	0	0	0	1

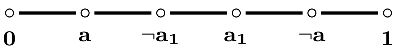
$\to$	0	a	$\neg a$	1
0	1	1	1	1
a	0	a	$\neg a$	1
$\neg a$	0	0	a	1
1	0	0	0	1

We encourage the reader to prove that the function $\to$ satisfies the inequalities which define **E**-algebras.

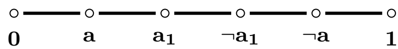
A_1-algebras

Let $a \to \neg a \neq a$ and $a \to \neg a \neq \neg a$; let $(a \to \neg a) := a_1$. Assume that $a_1 \leq \neg a_1$ or $\neg a_1 \leq a_1$, so we have two 6-elements chains. Thus there exists two possibilities:

1.



2.



We observe that if we assume that $a_1 \to a_1 = a_1$, then the algebrahas its own subalgebra that is different from **2**. Therefore, we assume that $a_1 \to a_1 = a$. Moreover, we determine the values for some of the elements in $\to$-table independently of the ordering of a_1 and $\neg a_1$.

1. We observe that $a \leq a \to a_1 \leq a_1$. By the syllogism, $a \to \neg a \leq (\neg a \to \neg a) \to (a \to \neg a)$, thus $a_1 \leq a \to a_1$. Hence $a \to a_1 = a_1$

2. Similarly, $a \leq a \to \neg a_1 \leq \neg a_1$. By the syllogism, $a \to \neg a_1 \leq (\neg a_1 \to \neg a) \to (a \to \neg a)$, thus $a \to \neg a_1 \leq a_1 \to a_1$ so $a \to \neg a_1 \leq a$. Therefore $a \to \neg a_1 = a$

3. Assume that $\neg a_1 \leq a_1$ (the first chain). It is clear that $a \leq \neg a_1 \to a_1 \leq a_1$. By (ii) in Lemma 15 $\neg a_1 \to a_1 \leq a \to (\neg a_1 \to a_1)$. If we take elements between a and a_1, then we obtain that $\neg a_1 \to a_1 = a$ or $\neg a_1 \to a_1 = a_1$, because other cases lead to a contradiction.

Remark. Assume now that $a_1 \leq \neg a_1$. Thus $a \leq a_1 \to \neg a_1$. By the syllogism, $a_1 \to \neg a_1 \leq (\neg a_1 \to \neg a) \to (a_1 \to \neg a)$ and we obtain $a_1 \to \neg a_1 \leq a_1 \to a$, i.e. $a_1 \to \neg a_1 \leq 0$, which is a contradiction. We conclude that algebra does not exist.

As a consequence of the reasoning presented above, only the first chain in which $\neg a_1 \leq a_1$ can be the basis of our 6-element algebras.

We use $\mathbf{A_1}$ to denote the algebras based on our 6-element chain. For simplicity of notation, we omit the first and the last lines and the first and the last column in this $\to$-tables (cf. Lemma 18).

We infer that $\to$-tables for $\mathbf{A_1}$-algebras:

$\to$	a	$\neg a_1$	a_1	$\neg a$
a	a	a	a_1	a_1
$\neg a_1$	0	a	a	a_1
a_1	0	0	a	a
$\neg a$	0	0	0	a

$\to$	a	$\neg a_1$	a_1	$\neg a$
a	a	a	a_1	a_1
$\neg a_1$	0	a	a_1	a_1
a_1	0	0	a	a
$\neg a$	0	0	0	a

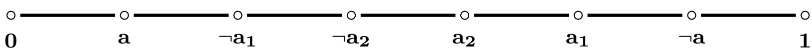
We use $\mathbf{A_{1,a}}$ to denote the algebra in which $\neg a_1 \to a_1 = a$; in $\mathbf{A_{1,a_1}}$, $\neg a_1 \to a_1 = a_1$.

The Reader can check that these $\mathbf{A_1}$-algebras are **E**-algebras.

The algebras $\mathbf{A_{1,a}}$ and $\mathbf{A_{1,a_1}}$ are called $\mathbf{A_1}$-*algebras.*

A_2-algebras

We have $a \leq \neg a_1 \to a_1 \leq a_1$. Assume that $\neg a_1 \to a_1 \neq a$ and $\neg a_1 \to a_1 \neq a_1$. Let us consider a new element $\neg a_1 \to a_1 := a_2$. Hence we consider an 8-elements chain in which $\neg a_2 \leq a_2$ (the case $a_2 \leq \neg a_2$ is impossible):



We observe that $a \leq \neg a_1 \to a_2 \leq a_2$. By (ii) in Lemma 15, $\neg a_1 \to a_2 \leq a \to (\neg a_1 \to a_2)$. Therefore, we have two possibilities: $\neg a_1 \to a_2 = a$ or $\neg a_1 \to a_2 = a_2$.

As a result, we define two $\mathbf{A_2}$-algebras based on our 8-elements lattice (cf. the picture above). In the first algebra, $\mathbf{A_{2,a}}$ we have $\neg a_1 \to a_2 = a$ and in the second algebra $\mathbf{A_{2,a_2}}$ we have $\neg a_1 \to a_2 = a_2$.

$\to$	a	$\neg a_1$	$\neg a_2$	a_2	a_1	$\neg a$
a	a	a	a	a_2	a_1	a_1
$\neg a_1$	0	a	a	a	a_2	a_1
$\neg a_2$	0	0	a	a	a	a_2
a_2	0	0	0	a	a	a
a_1	0	0	0	0	a	a
$\neg a$	0	0	0	0	0	a

$\to$	a	$\neg a_1$	$\neg a_2$	a_2	a_1	$\neg a$
a	a	a	a	a_2	a_1	a_1
$\neg a_1$	0	a	a	a_2	a_2	a_1
$\neg a_2$	0	0	a	a	a_2	a_2
a_2	0	0	0	a	a	a
a_1	0	0	0	0	a	a
$\neg a$	0	0	0	0	0	a

A_n-Algebras

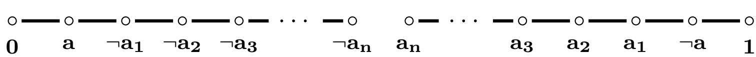
So far, we only considered chains have with even numbers of elements. In addition, each chain has a smallest element and a greatest element, and it has an element a and an element $\neg a$. The remaining elements are of the form a_k and $\neg a_k$. Thus all our chains have $2+2+2k$ elements. If our chain has $2+2+2k$, then the algebra generated by this chain will be denoted by $\mathbf{A_k}$, for example, algebras $\mathbf{A_3}$ have $2+2+2\cdot 3$ elements.

Let us generalize the procedure of defining operation $\to$ for A_n-chains.

Let us consider the $\mathbf{A_n}$-chain. In fact the algebra $\mathbf{A_n}$ is an 'extension' of the algebra $\mathbf{A_{n-1}}$, i.e. the values of the operation $\to$ on the elements of $\mathbf{A_n}$ and on the elements of $\mathbf{A_{n-1}}$ are exactly the same with the exception of the element $\neg a_1 \to a_{n-1}$ and its negation; $\neg a_1 \to a_{n-1}$ in $\mathbf{A_n}$ equals a or a_{n-1}, but equals a_n in $\mathbf{A_n}$. We obtain

1. $\neg a_1 \to a_{n-1} = a$ in $\mathbf{A_{n-1,a}}$
2. $\neg a_1 \to a_{n-1} = a_{n-1}$ in $\mathbf{A_{n-1,a_{n-1}}}$
3. $\neg a_1 \to a_{n-1} = a_n$ in $\mathbf{A_n}$.

In other words, in the algebra $\mathbf{A_n}$ the element $\neg a_1 \to a_{n-1} = a_n$ differs from a and a_{n-1}. Thus we must consider A_n-chain where $\neg a_n \leq a_n$ (the case $a_n \leq \neg a_n$ is impossible):



In fact there are two $\mathbf{A_n}$-algebras, i.e. the first, $\mathbf{A_{n,a}}$, in which $\neg a_1 \to a_n = a$:

$\to$	a	$\neg a_1$	$\neg a_2$	$\ldots$	$\neg a_{n-1}$	$\neg a_n$	a_n	a_{n-1}	$\ldots$	a_2	a_1	$\neg a$
a	a	a	a	$\ldots$	a	a	a_n	a_{n-1}	$\ldots$	a_2	a_1	a_1
$\neg a_1$	0	a	a	$\ldots$	a	a	a	a_n	$\ldots$	a_3	a_2	a_1
$\neg a_2$	0	0	a	$\ldots$	a	a	a	a	$\ldots$	a	a_3	a_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
$\neg a_{n-1}$	0	0	0	$\ldots$	a	a	a	a	$\ldots$	a	a_n	a_{n-1}
$\neg a_n$	0	0	0	$\ldots$	0	a	a	a	$\ldots$	a	a	a_n
a_n	0	0	0	$\ldots$	0	0	a	a	$\ldots$	a	a	a
a_{n-1}	0	0	0	$\ldots$	0	0	0	a	$\ldots$	a	a	a
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
a_2	0	0	0	$\ldots$	0	0	0	0	$\ldots$	a	a	a
a_1	0	0	0	$\ldots$	0	0	0	0	$\ldots$	0	a	a
$\neg a$	0	0	0	$\ldots$	0	0	0	0	$\ldots$	0	0	a

and the second, $\mathbf{A_{n,a_n}}$, in which $\neg a_1 \to a_n = a_n$:

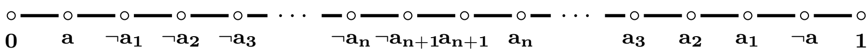
$\to$	a	$\neg a_1$	$\neg a_2$	$\ldots$	$\neg a_{n-1}$	$\neg a_n$	a_n	a_{n-1}	$\ldots$	a_2	a_1	$\neg a$
a	a	a	a	$\ldots$	a	a	a_n	a_{n-1}	$\ldots$	a_2	a_1	a_1
$\neg a_1$	0	a	a	$\ldots$	a	a	a_n	a_n	$\ldots$	a_3	a_2	a_1
$\neg a_2$	0	0	a	$\ldots$	a	a	a	a	$\ldots$	a	a_3	a_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
$\neg a_{n-1}$	0	0	0	$\ldots$	a	a	a	a	$\ldots$	a	a_n	a_{n-1}
$\neg a_n$	0	0	0	$\ldots$	0	a	a	a	$\ldots$	a	a_n	a_n
a_n	0	0	0	$\ldots$	0	0	a	a	$\ldots$	a	a	a
a_{n-1}	0	0	0	$\ldots$	0	0	0	a	$\ldots$	a	a	a
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
a_2	0	0	0	$\ldots$	0	0	0	0	$\ldots$	a	a	a
a_1	0	0	0	$\ldots$	0	0	0	0	$\ldots$	0	a	a
$\neg a$	0	0	0	$\ldots$	0	0	0	0	$\ldots$	0	0	a

A_{n+1}-algebras

The construction of the algebras $\mathbf{A_{n+1}}$ is very similar. As in the case of $\mathbf{A_n}$-algebras, we observe that $\to$ can satisfy the conditions:

1. $\neg a_1 \to a_{n+1} = a$
2. $\neg a_1 \to a_{n+1} = a_{n+1}$
3. $\neg a_1 \to a_{n+1} \neq a$ and $\neg a_1 \to a_{n+1} \neq a_{n+1}$

Hence, we obtain the following A_{n+1}-chain (the case $a_{n+1} \leq \neg a_{n+1}$ is impossible):



Thus we have two $\mathbf{A_{n+1}}$-algebras, i.e. the first, $\mathbf{A_{n+1,a}}$, in which $\neg a_1 \to a_{n+1} = a$:

$\to$	a	$\neg a_1$	$\neg a_2$	$\ldots$	$\neg a_n$	$\neg a_{n+1}$	a_{n+1}	a_n	$\ldots$	a_2	a_1	$\neg a$
a	a	a	a	$\ldots$	a	a	a_{n+1}	a_{n-1}	$\ldots$	a_2	a_1	a_1
$\neg a_1$	0	a	a	$\ldots$	a	a	a	a_{n+1}	$\ldots$	a_3	a_2	a_1
$\neg a_2$	0	0	a	$\ldots$	a	a	a	a	$\ldots$	a	a_3	a_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
$\neg a_n$	0	0	0	$\ldots$	a	a	a	a	$\ldots$	a	a_{n+1}	a_{n-1}
$\neg a_{n+1}$	0	0	0	$\ldots$	0	a	a	a	$\ldots$	a	a	a_{n+1}
a_{n+1}	0	0	0	$\ldots$	0	0	a	a	$\ldots$	a	a	a
a_n	0	0	0	$\ldots$	0	0	0	a	$\ldots$	a	a	a
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
a_2	0	0	0	$\ldots$	0	0	0	0	$\ldots$	a	a	a
a_1	0	0	0	$\ldots$	0	0	0	0	$\ldots$	0	a	a
$\neg a$	0	0	0	$\ldots$	0	0	0	0	$\ldots$	0	0	a

and the second, $\mathbf{A_{n+1,a_{n+1}}}$, in which $\neg a_1 \to a_{n+1} = a_{n+1}$:

$\to$	a	$\neg a_1$	$\neg a_2$	$\dots$	$\neg a_n$	$\neg a_{n+1}$	a_{n+1}	a_n	$\dots$	a_2	a_1	$\neg a$
a	a	a	a	$\dots$	a	a	a_{n+1}	a_{n-1}	$\dots$	a_2	a_1	a_1
$\neg a_1$	0	a	a	$\dots$	a	a	a_{n+1}	a_{n+1}	$\dots$	a_3	a_2	a_1
$\neg a_2$	0	0	a	$\dots$	a	a	a	a	$\dots$	a	a_3	a_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
$\neg a_n$	0	0	0	$\dots$	a	a	a	a	$\dots$	a	a_{n+1}	a_{n-1}
$\neg a_{n+1}$	0	0	0	$\dots$	0	a	a	a	$\dots$	a	a_{n+1}	a_{n+1}
a_{n+1}	0	0	0	$\dots$	0	0	a	a	$\dots$	a	a	a
a_n	0	0	0	$\dots$	0	0	0	a	$\dots$	a	a	a
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
a_2	0	0	0	$\dots$	0	0	0	0	$\dots$	a	a	a
a_1	0	0	0	$\dots$	0	0	0	0	$\dots$	0	a	a
$\neg a$	0	0	0	$\dots$	0	0	0	0	$\dots$	0	0	a

3. Fundamental theorem

PROPOSITION. Each $\mathbf{A_n}$-algebra is an **E**-algebra.

We point out that we have two infinite sequences of algebras, i.e. a sequence $\mathbf{A_{n,a}}$ and the sequence $\mathbf{A_{n,a_n}}$. In addition, none of these algebras have a proper subalgebra with the exception of the two-element subalgebra.

Each of these algebras is generated by the element a. Moreover, none of $\mathbf{A_n}$-algebras have a non-trivial homomorphic image.

It entails the following theorem:

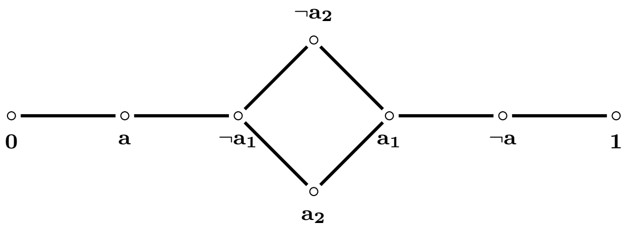
THEOREM 20. *There exists two infinite sequences of finite simple* ***E****-algebras such that the only proper subalgebra is* **2**.

COROLLARY 21. The interval [**E**, **2**] has infinitely many coatoms.

REMARK. Note that for the logic **RM** there exist one pre-maximal extension and for the logic **R** there exist three pre-maximal extensions.

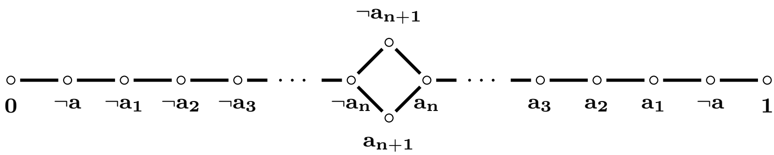
4. Another example of infinite sequences of E-algebras

Let us consider the following lattice



and an algebra based on this lattice. Of course, in this algebra the elements a_2 and $\neg a_2$ are not comparable. If we define the operation $\rightarrow$ as in $\mathbf{A_n}$-algebras, then we get an **E**-algebra.

In general, for the following lattice



if we define the operation $\rightarrow$ as for the $\mathbf{A_n}$-algebras, then we obtain an **E**-algebra.

References

[1] W. Ackermann, *Begründung Einer Strengen Implikation*, **The Journal of Symbolic Logic**, Vol. 21, 2 (1956), pp. 113–128.

[2] A. R. Anderson, N. D. Belnap, Jr., *Entailment. The Logic of relevance and necessity*, **Princeton University Press**, Vol. I (1975).

[3] N. D. Belnap, Jr., *Intesional Models for First Degree Formula*, **The Journal of Symbolic Logic**, Vol. 32, 1 (1967), pp. 1–22.

[4] W. J. Blok, D. Pigozzi, *Algbebraizable logics*, **Memoirs of the American Mathematical Society**, 1989.

[5] J. M. Font, G. B. Rodriguez, *Note on algebraic models for relevance logic*, **Zeitschrift für Matematische Logik und Grundlagen der Mathematic**, Vol. 36, 6 (1990), pp. 535–540.

[6] W. Dziobiak, *There are* $2^{\aleph_0}$ *Logics with the Relevance Principle Between* **R** *and* **RM**, **Studia Logica**, Vol. XLII (1983), pp. 49–61.

[7] L. Maksimowa, *Struktury s implikacjiej*, **Algebra and Logic**, Vol. 12, 4 (1973), pp. 445–467.

[8] L. Maksimowa, *O Modeljach iscislenija E*, **Algebra and Logic**, Vol. 6, 6 (1967), pp. 5–20.

[9] R. M. Martin, *Twenty-Third Annual Meeting of the Association for Symbolic Logic*, **The Journal of Symbolic Logic**, Vol. 23, 4 (1958), pp. 456–461.

[10] R. K. Meyer, **E** *and* **S4**, **Noter Dame Journal of Formal Logic**, Vol. XI, 2 (1970), pp. 181–199.

[11] K. Świrydowicz, *There exists exactly two maximal strictly relevant extensions of the relevant logic* **R**, **The Journal of Symbolic Logic**, Vol. 64, 3 (1999), pp. 1125–1154.

[12] K. Świrydowicz, *A Remark on the Maximal Extensions of the Relevant Logic* ***R***, **Reports on Mathematical Logic**, 29 (1995), pp. 19–33.

[13] M. Tokarz, *Essays in matrix semantics of relevant logics*, **The Institute of Philosophy and Sociology of the Polish Academy of Sciences**, Warsaw 1980.

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