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ON INTERPOLATION IN $NEXT(\mathbf{KB}.\mathbf{Alt}(\mathbf{2}))$

Abstract

We prove that there is infinitely many tabular modal logics extending $\mathbf{KB}.\mathbf{Alt}(\mathbf{2})$ which have interpolation.

Keywords: symmetric Kripke frames, interpolation, amalgamation.

1. Introduction

There is a huge literature on interpolation in modal logics, see [1], pp. 443–470. The best known is the result by Maksimova [6] who showed that there is finitely many normal extensions of **S4** enjoying interpolation. Despite this, little is known about interpolation in intransitive modal logics (that are not extensions of **S4**). We try to fill the gap. We prove that there exists infinitely many logics without interpolation both in $NEXT(\mathbf{KTB}.4_2)$ and $NEXT(\mathbf{KTB}.\mathbf{Alt}(\mathbf{3}))$ see [4], [5] appropriately.

In this paper we study some normal extensions of the modal logic $\mathbf{KB} := \mathbf{K} \oplus B$, where $B := p \to \Box\Diamond p$. The logic **KB** is complete with respect to the class of symmetric Kripke frames.

The paper is a continuations of [4] and [5] concerning the interpolation in $NEXT(\mathbf{KTB})$. In [4] an infinite family of logics without interpolation is described. The logics studied there are weakly transitive and belong to $NEXT(\mathbf{KTB}.4_2)$. In [5] some, mostly negative, results concerning interpolation for intransitive logics are included. The logics considered there are characterized by reflexive, symmetric and linear Kripke frames. We considered extensions of $\mathbf{KTB}.\mathbf{Alt}(\mathbf{3}) := \mathbf{K} \oplus T \oplus B \oplus alt_3$, with

$$T := \Box p \to p$$
$$alt_3 := \Box p \vee \Box(p \to q) \vee \Box((p \wedge q) \to r) \vee \Box((p \wedge q \wedge r) \to s).$$

The axiom (alt_3) is a special case of a more general (alt_n), $n \geq 1$:

$$alt_n := \Box p_1 \vee \Box(p_1 \to p_2) \vee ... \vee \Box((p_1 \wedge ... \wedge p_n) \to p_{n+1}).$$

It is proven in [5] that the logic **KTB**.**Alt**(**3**) does not have interpolation and it has only two normal extensions having this property. These are the following: $L(\circ) = Triv$ and $L(\circ - - \circ)$ – logics determined by one-point or two-point cluster, appropriately. One may notice that the situation is analogous to that of **S5**, see [1], p. 463.

In this paper we contrast the negative results from [5] for $NEXT$(**KTB**.**Alt**(**3**)) with positive ones for $NEXT$(**KB**.**Alt**(**2**)). The common feature of the Kripke frames for both the families is their linearity. However, in the former case the appropriate Kripke frames are chains of reflexive points, whereas in the latter one, they are chains of (possibly) irreflexive points.

2. Preliminaries

Let us recall the basic definitions. The symbol $Var(\alpha)$ means the set of all propositional variables in the formula α.

Definition 1. *A logic L has the Craig interpolation property (CIP) if for every implication $\alpha \to \beta$ in L, there exists a formula γ (interpolant for $\alpha \to \beta$ in L) such that*

$$\alpha \to \gamma \in L \quad \textit{and} \quad \gamma \to \beta \in L$$

and $Var(\gamma) \subseteq Var(\alpha) \cap Var(\beta)$.

Definition 2. *A logic L has interpolation for deducibility (IPD) if for any α and β the condition $\alpha \vdash_L \beta$ implies that there exists a formula γ such that*

$$\alpha \vdash_L \gamma \quad \textit{and} \quad \gamma \vdash_L \beta$$

and $Var(\gamma) \subseteq Var(\alpha) \cap Var(\beta)$.

It is well known that (CIP) together with (MP) and deduction theorem implies (IPD). It is known that **K**, **T**, **K4** and **S4** have (CIP), see Gabbay [3]. Also the logics from $NEXT$(**S4**) are well recognized as regards interpolation (see [6], also [1], pp. 462–463). In particular, **S5** has (CIP). The last fact can be proven by applying a very general method of construction of inseparable tableaux (see i.e. [1], pp. 446–449). The same method

can be applied in the case of **KTB** and **KB**. Therefore, without getting into details, we conclude that both the logics **KTB** and **KB** have (CIP). The method of construction of inseparable tableaux is not applicable for the logics **KTB.Alt(n)** and **KB.Alt(n)**, $n \geq 2$. In [5] it is proven that **KTB.Alt(3)** does not have (CIP).

In the next section we shall look for tabular logics from $NEXT(\mathbf{KB.Alt(2)})$ having (IPD) or (CIP). Our approach is purely semantic. We shall consider logics determined by classes of Kripke frames $\mathcal{K}$. Formally, the logic determined by a class $\mathcal{K}$ is defined as follows:

$$L(\mathcal{K}) := \{\alpha \in Form : \ \mathfrak{F} \models \alpha \ \text{ for each } \ \mathfrak{F} \in \mathcal{K}\}.$$

Note that the class $\mathcal{K}$ may consist of one frame only. Hereafter symmetric Kripke frames will be called KB-frames. Symmetric and linear Kripke frames are said to be $KB.Alt(2)$-frames.

The properties (CIP) and (IPD) have an appropriate algebraic characterization, (see [6], [2]). The symbol $V(L)$ denotes the variety of modal algebras characterizing the logic L. From [6] the following equivalence is known for any logic $L \in NEXT(\mathbf{K})$: L possesses (CIP) iff $V(L)$ has the superamalgamation property. In [2] it is shown that L possesses (IPD) iff $V(L)$ has the amalgamation property.

By theory of duality between Kripke frames and modal algebras, amalgamation and superamalgamation properties are transformed into appropriate properties for classes $\mathcal{K}$ of Kripke frames. We need to recall the notion of p-morphism, first.

DEFINITION 3. *Let* $\mathfrak{F}_1 := \langle W_1, R_1\rangle$ *and* $\mathfrak{F}_2 := \langle W_2, R_2\rangle$ *be Kripke frames. A map* $f : W_1 \to W_2$ *is a p-morphism from* $\mathfrak{F}_1$ *to* $\mathfrak{F}_2$*, if it satisfies the following conditions:*

(p1) f *maps* W_1 *onto* W_2,

(p2) for all $x, y \in W_1$, xR_1y *implies* $f(x)R_2f(y)$,

(p3) for each $x \in W_1$ *and for each* $a \in W_2$, *if* $f(x)R_2a$ *then there exists* $y \in W_1$ *such that* xR_1y *and* $f(y) = a$.

It is said also that the frame $\mathfrak{F}_1$ *is reducible to* $\mathfrak{F}_2$ *or that the frame* $\mathfrak{F}_2$ *is a p-morphic reduct of* $\mathfrak{F}_1$.

The next lemma is a folklore in logic:

LEMMA 1. *Let $\mathfrak{F}_1$, $\mathfrak{F}_2$ be Kripke frames. If there exists a p-morphism from $\mathfrak{F}_1$ to $\mathfrak{F}_2$ then $L(\mathfrak{F}_1) \subseteq L(\mathfrak{F}_2)$.*

For some special Kripke frames Lemma 1 may be strengthened to an equivalence. The Jónsson's lemma, the congruence extension property of modal algebras, finiteness and simplicity of the dual algebra for $\mathfrak{F}_1$ are exploited. The proof is analogous to the one from [7]. We get as follows:

LEMMA 2. *Let $\mathfrak{F}_1$, $\mathfrak{F}_2$ be finite and connected KB-frames. Then $L(\mathfrak{F}_1) \subseteq L(\mathfrak{F}_2)$ iff there exists a p-morphism from $\mathfrak{F}_1$ onto $\mathfrak{F}_2$.*

Then we get the amalgamation property for class of frames (APK)

DEFINITION 4. *For any $\mathfrak{F}_0$, $\mathfrak{F}_1$ and $\mathfrak{F}_2$ in class $\mathcal{K}$ and for any p-morphism $f_1 : \mathfrak{F}_1 \to \mathfrak{F}_0$ and $f_2 : \mathfrak{F}_2 \to \mathfrak{F}_0$ there exist $\mathfrak{F}$ in $\mathcal{K}$ and p-morphisms $g_1 : \mathfrak{F} \to \mathfrak{F}_1$ and $g_2 : \mathfrak{F} \to \mathfrak{F}_2$ such that $f_1 \circ g_1 = f_2 \circ g_2$. See Figure 1.*

Superamalgamation property for frames except (APK) requires the additional condition (SAPK):

$$\forall_{x\in\mathfrak{F}_1}\forall_{y\in\mathfrak{F}_2}[f_1(x) = f_2(y) \;\Rightarrow\; \exists_{z\in\mathfrak{F}} g_1(z) = x \wedge g_2(z) = y].$$

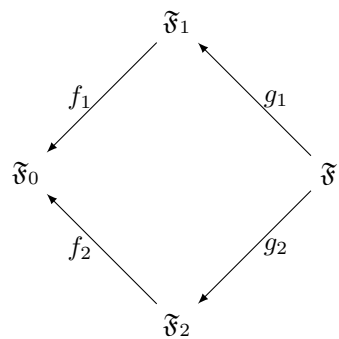

Fig. 1.

Dealing with logics determined by classes of finite frames we have the following equivalence.

THEOREM 1. *Let $L \in NEXT(\mathbf{KB})$, L be determined by a class $\mathcal{K}$ of finite Kripke frames and $\mathcal{K}$ be closed under p-morphisms. For the logic L the following is equivalent:*

- *L possesses (CIP),*
- *$\mathcal{K}$ has the superamalgamation property (SAPK).*

THEOREM 2. *Let $L \in NEXT(\mathbf{KB})$, L be determined by a class $\mathcal{K}$ of finite Kripke frames and $\mathcal{K}$ be closed under p-morphisms. For the logic L the following is equivalent:*

- *L possesses (IPD),*
- *$\mathcal{K}$ has the amalgamation property (APK).*

3. Interpolation for tabular logics from $NEXT(\mathbf{KB}.\mathbf{Alt}(\mathbf{2}))$

Let us recall that there are only two extensions of $\mathbf{KTB}.\mathbf{Alt}(\mathbf{3})$ with the interpolation property. In contrast, we shall prove that there is infinitely many tabular logics in $NEXT(\mathbf{KB}.\mathbf{Alt}(\mathbf{2}))$ with interpolation. Let us remind that the logic $\mathbf{KB}.\mathbf{Alt}(\mathbf{2})$ is determined by the class of symmetric Kripke frames (we allow irreflexivity) where each point sees at most two points (including itself). Hence if x is irreflexive, then it may see two distinct points; if it is reflexive then it can see only one distinct point. The Kripke frames for $\mathbf{KB}.\mathbf{Alt}(\mathbf{2})$ may be chains (possibly infinite) with irreflexive inside points. See Figure 2 where eight-point $KB.Alt(2)$-frames are presented.

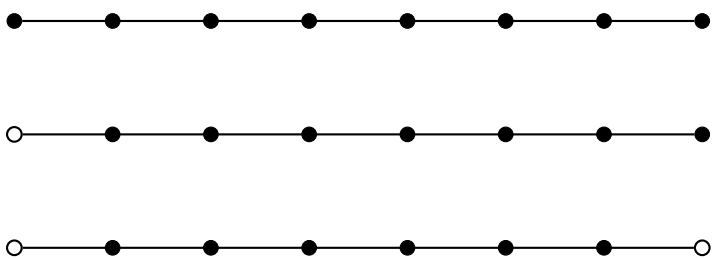

Fig. 2.

In the class of $KB.Alt(2)$-frames we shall distinguish finite chain frames with one of the end point being reflexive. We call them almost irreflexive chains. Formally:

DEFINITION 5. *The almost irreflexive chain frame* $\mathfrak{Ch}_m$ *is defined as follows:* $\mathfrak{Ch}_m := \langle\{1, 2, ..., m\}, R\rangle$, $m \geq 1$ *and*

$$R := \{(i, j),\ i, j = 1, 2, ..., n,\ |i - j| = 1\} \cup \{(1, 1)\}.$$

On may see that $\mathfrak{Ch}_1 = \circ$.

Below, we shall describe the p-morphisms between almost irreflexive chain frames $\{\mathfrak{Ch}_m, m \geq 1\}$.

LEMMA 3. *There is a unique p-morphism from* $\mathfrak{Ch}_{3m-1}$ *onto* $\mathfrak{Ch}_m$ *for any* $m \geq 1$.

PROOF: Let $\mathfrak{Ch}_m = \langle\{1, 2, ..., m\}, R\rangle$ and $\mathfrak{Ch}_{3m-1} = \langle\{1, 2, ..., 3m - 1\}, S\rangle$. Point 1 is the only reflexive point in both $\mathfrak{Ch}_m$ and $\mathfrak{Ch}_{3m-1}$. We define:

$$\begin{aligned} f(1) &= 1\ , f(2) = 2,\ \ f(3) = 3,\ ...\ f(m) = m\ , \\ f(m+1) &= m - 1,\ \ f(m+2) = m - 2,\ ...\ f(2m-1) = 1 \\ f(2m) &= 1,\ \ f(2m+1) = 2,\ \ f(2m+2) = 3,\ ...\ f(3m-1) = m\ . \end{aligned}$$

One may easily see that f is indeed a p-morphism from $\mathfrak{Ch}_{3m-1}$ onto $\mathfrak{Ch}_m$. For $m = 3$ the p-morphism f is presented in Figure 3.

We shall explain why the p-morphism f is a unique one. First, from the condition (p2) of Definition 3 we know that the reflexive point 1 from $\mathfrak{Ch}_{3m-1}$ must be mapped onto 1 from $\mathfrak{Ch}_m$.

Second, we prove that any function f gluing two neighboring irreflexive points in $\mathfrak{Ch}_{3m-1}$ and mapping them onto some irreflexive point from $\mathfrak{Ch}_m$ is not a p-morphism. Suppose, on the contrary, that $iSi + 1$ for $1 < i \leq 3m - 4$, $m \geq 2$ and $f(i) = f(i + 1) = j$, $j \neq 1$. Then we get jRj and it is a contradiction. Let us notice that gluing two neighboring irreflexive points by a p-morphism causes reflexivity of the point they are mapped onto. Similarly, one may prove that any function gluing more than two neighboring points in $\mathfrak{Ch}_{3m-1}$ and mapping them onto any point from $\mathfrak{Ch}_m$ is not a p-morphism.

Then we see that the p-morphism from $\mathfrak{Ch}_{3m-1}$ onto $\mathfrak{Ch}_m$ has to start at 1 and then has to move along $\mathfrak{Ch}_{3m-1}$ with a short (1-step) stop at the final point m and a long (2-step) stop at 1 and further until m. The point m is the final point of the whole journey.

■

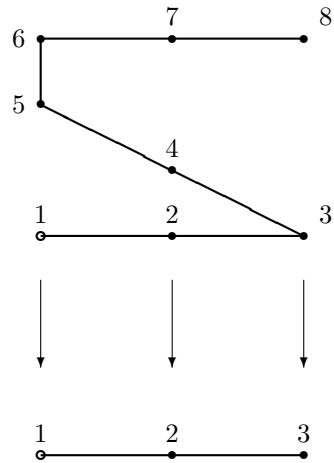

Fig. 3.

One may notice that the frame $\mathfrak{Ch}_{3m-1}$ is the minimal almost irreflexive chain frame which can be non-trivially reducible onto $\mathfrak{Ch}_m$, $m \geq 1$.

We may generalize the above lemma by proving the existence of a suitable p-morphism from $\mathfrak{Ch}_{m+(2m-1)k}$ onto $\mathfrak{Ch}_m$, for $k \geq 1$. Then the frame with $m+(2m-1)k$ points is folded k-times. See Figure 4 for $m = 3$.

LEMMA 4. *There is a unique p-morphism from* $\mathfrak{Ch}_{m+(2m-1)k}$ *onto* $\mathfrak{Ch}_m$ *for any* $m, k \geq 1$.

PROOF: Obvious.

The above lemma may be strengthened to the following equivalence.

THEOREM 3. *Let* $n \geq 2$. *Then* $\mathfrak{Ch}_n$ *is reducible onto* $\mathfrak{F}$ *iff* $\mathfrak{F} = \mathfrak{Ch}_m$ *for some* $k \geq 0$ *and* $m \geq 1$ *such that* $n = m + (2m-1)k$.

PROOF: ($\Leftarrow$) See Lemma 4.

($\Rightarrow$)

Case 1. $\mathfrak{F} = \circ$. It means that $\mathfrak{F} = \mathfrak{Ch}_1$. It is easy to observe that any almost irreflexive frame $\mathfrak{Ch}_n$ is reducible onto $\mathfrak{Ch}_1$. Each natural number n can be written as $n := 1 + k$, $k \geq 0$.

Case 2. $\mathfrak{F} \neq \circ$. Let us observe that $\mathfrak{F}$ as a reduct of some $\mathfrak{Ch}_n$, $n \geq 2$ must be a $KB.Alt(2)$-frame. It is because any p-morphism can not move out

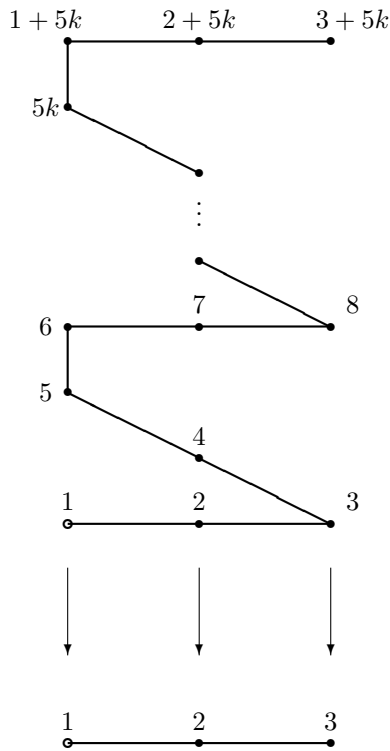

Fig. 4.

outside the class of $KB.Alt(2)$-frames. We shall prove that $\mathfrak{F}$ is an almost irreflexive chain frame.

Let $\mathfrak{Ch}_n = \langle\{x_1, x_2, ..., x_n\}, R\rangle$ with x_1 being the reflexive point and $n \geq 3$. From the condition (p2) of Definition 3 we see that x_1 must be mapped onto a reflexive point. Then $\mathfrak{F}$ has to include at least one reflexive point (and it must be the end point). Let $\mathfrak{F} = \langle\{y_1, y_2, ..., y_m\}, S\rangle$ and point y_1 is reflexive. Because $\mathfrak{F} \neq \circ$ then $m \geq 2$. We show that the point y_m must be irreflexive. Suppose, on the contrary, that y_m for $m \geq 2$ is a reflexive point, $f(x_1) = y_1$, and $f(x_n) = y_m$. Then $f(x_n)Sy_m$ and from (p3) of

Definition 3 we need another point, say x_k in $\mathfrak{Ch}_n$ such that $x_k R x_n$ to map it onto y_m. Hence $k = n-1$ and $f(x_{n-1}) = y_m$. Further we have two possibilities for the point x_{n-2}: $f(x_{n-2}) = y_{m-1}$ or $f(x_{n-2}) = y_m$. But in both these cases we get $f(x_n) S y_{m-1}$ and it means that there should exist an element $x_s R x_n$ such that $f(x_s) = y_{m-1}$. So, we must take $s = n-1$. This leads to a contradiction because we have already defined: $f(x_{n-1}) = y_m$. Analogously, one may prove that the irreflexive point x_n from $\mathfrak{Ch}_n$ can not be mapped onto the reflexive point y_1 from $\mathfrak{F}$. Hence, indeed $\mathfrak{F} = \mathfrak{Ch}_m$ for some $m < n$.

Further, we shall prove the existence of $k \geq 0$ and $m \geq 1$ such that $n = m + (2m-1)k$.

Defining the needed p-morphism we have to map x_1 onto y_1. Then the mapping goes further along the $\mathfrak{Ch}_m$, turn over at y_m. Then we need $m-1$ points $y_{m+1}, ..., y_{2m-1}$ to reach again the point x_1. Because it is reflexive then we map onto it two consecutive points x_{2m-1} and x_{2m}. The mapping can not stop at this point, we need again $m-1$ points to reach the point y_m in $\mathfrak{Ch}_m$. So, we see that in such a case $n = m+(m-1)+1+(m-1) = 3m-1$. But if $n > 3m-1$ then the whole journey repeats and we need another $(m-1)+1+(m-1)$ points. We would get $n = 3m-1+(m-1)+1+(m-1) = 5m-2$ and we see that after two 'full turns' $k = 2$ and $n = m+(2m-1)2$. If $n > m+(2m-1)2$ we have to continue the mapping and so on.

Depending on the number k of 'full turns' we need $m+(2m-1)k$ points in the larger frame. ∎

Let $\mathbb{P} := \{n \in \mathbb{N} : \exists_{p \geq 3}\ n = \frac{p+1}{2}\ \ \&\ \ p - \text{prime}\}$.

LEMMA 5. *There is infinitely many distinct logics in* $NEXT(\mathbf{KB.Alt(2)})$ *which are* $L(\mathfrak{Ch}_n)$, $n \in \mathbb{P}$.

PROOF: Let us take the sequence $p_1, p_2, ...$ of the consecutive prime numbers larger than 2 and consider the following almost irreducible frames $\mathfrak{Ch}_2$, $\mathfrak{Ch}_3$, $\mathfrak{Ch}_4$, $\mathfrak{Ch}_6$, $\mathfrak{Ch}_7$, $\mathfrak{Ch}_9$, $\mathfrak{Ch}_{10}$, $\mathfrak{Ch}_{12}$... , indexed with numbers from $\mathbb{P}$. We shall prove that each of them is reducible onto $\circ$, only. Suppose, on the contrary, that there is some $\mathfrak{Ch}_{\frac{p+1}{2}}$ reducible onto some $\mathfrak{Ch}_m$, $m \geq 2$ (see Theorem 3). We get that there must exist $k \geq 1$ such that it holds: $\frac{p+1}{2} = m + (2m-1)k$. After simple calculation we get:

$$p + 1 = 2m + (2m-1)2k\ ,$$
$$p = 2m - 1 + (2m-1)2k\ ,$$
$$p = (2m-1)(2k+1)\ .$$

But in the last line we get a contradiction since p is a prime number. From the above and from Lemma 2 we conclude that $L(\mathfrak{Ch}_n) \not\subset L(\mathfrak{Ch}_m)$ and $L(\mathfrak{Ch}_m) \not\subset L(\mathfrak{Ch}_n)$ for $n \neq m$, $n, m \in \mathbb{P}$. ∎

COROLLARY 1. *If $n \in \mathbb{P}$ then there is no other reduct of $\mathfrak{Ch}_n$ than $\mathfrak{Ch}_1 = \circ$.*

Let us go back to the interpolation property. We are ready to prove:

THEOREM 4. *All the logics $L(\mathfrak{Ch}_n)$ with $n \in \mathbb{P}$ have (IPD).*

PROOF: The logic $L(\mathfrak{Ch}_n)$ with $n \in \mathbb{P}$ is determined by the the class consisting of one almost irreflexive chain frame. After closing this class under p-morphisms we add the one point frame $\circ$. Suppose we take as $\mathfrak{F}_1$ and $\mathfrak{F}_2$ the frame $\mathfrak{Ch}_n$ (twice) and as $\mathfrak{F}_0$ the frame $\circ$. The p-morphisms $f_1 : \mathfrak{F}_1 \to \mathfrak{F}_0$ and $f_2 : \mathfrak{F}_2 \to \mathfrak{F}_0$ glue all the points into $\circ$. As $\mathfrak{F}$ we have to again take $\mathfrak{Ch}_n$. The p-morphisms $g_1 : \mathfrak{F} \to \mathfrak{F}_1$ and $g_2 : \mathfrak{F} \to \mathfrak{F}_2$ are isomorphisms. Since $\mathfrak{F}_0 = \circ$ then obviously $f_1 \circ g_1 = f_2 \circ g_2$.

One may notice that other choices from $\{\mathfrak{Ch}_n, \circ\}$, $n \in \mathbb{P}$ for $\mathfrak{F}_i$, $i := 0, 1, 2$ lead to the same equalities. We see that in all these cases the classes of frames have (APK), hence the logics determined by them have (IPD), (see Theorem 2). ∎

One may easily see that the logics $L(\mathfrak{Ch}_n)$ with $n \in \mathbb{P}$ do not have (CIP). From Lemma 5 and Theorem 4 we get:

COROLLARY 2. *There is infinitely many tabular logics with (IPD) in $NEXT(\mathbf{KB.Alt(2)})$.*

One may ask question about interpolation of the other logics than $L(\mathfrak{Ch}_n)$ with $n \in \mathbb{P}$. They are the following: $L(\mathfrak{Ch}_5)$, $L(\mathfrak{Ch}_8)$, $L(\mathfrak{Ch}_{11})$,... . The answer is a positive one.

LEMMA 6. *The logics $L(\mathfrak{Ch}_n)$ with $n \in \mathbb{N} \setminus \mathbb{P}$ have (IPD).*

PROOF: Let $n \in \mathbb{N} \setminus \mathbb{P}$ and let $\mathfrak{Ch}_n \in \mathcal{K}$. First, we closed $\mathcal{K}$ under p-morphisms. From Theorem 3 we know that if $\mathfrak{Ch}_n$ is reducible then $n = m + (2m - 1)k$ for some $k \geq 0$ and $m \geq 1$. The reduct of $\mathfrak{Ch}_n$ is then the frame $\mathfrak{Ch}_m$. Because $n \in \mathbb{N} \setminus \mathbb{P}$ then $m \geq 2$.

Case 1. Suppose that there is not another $m' \geq 2$ such that $n = m' + (2m' - 1)k'$ for some $k' \geq 0$ (and $m' \geq 2$). Then $\mathcal{K} = \{\mathfrak{Ch}_n, \mathfrak{Ch}_m, \circ\}$. For any choices of frames from $\mathcal{K}$ for $\mathfrak{F}_1$, $\mathfrak{F}_2$, $\mathfrak{F}_0$ and $\mathfrak{F}$ (APK) holds.

Such a situation takes place, i.e. for $\mathfrak{Ch}_5$ which is reducible onto $\mathfrak{Ch}_2$. In turn $\mathfrak{Ch}_2$ is reducible only onto $\circ$.

Case 2. Suppose that there are another pair of numbers $k' \geq 0$ and $m' \geq 1$ such that $n = m' + (2m' - 1)k'$ ($m' \neq m$, $m' \geq 2$) and $\mathfrak{Ch}_n$ is reducible also onto the frame $\mathfrak{Ch}_{m'}$.

Sub-case 2a. Suppose that the common reduct of $\mathfrak{Ch}_m$ and $\mathfrak{Ch}_{m'}$ is only $\mathfrak{Ch}_1 = \circ$. Then $\mathcal{K} = \{\mathfrak{Ch}_n, \mathfrak{Ch}_m, \mathfrak{Ch}_{m'}, \circ\}$. To show that (APK) holds we have to choose as the frames $\mathfrak{F}_1$ and $\mathfrak{F}_2$ some frames from $\mathcal{K}$. The less trivial choice is: $\mathfrak{F}_1 = \mathfrak{Ch}_m$ and $\mathfrak{F}_2 = \mathfrak{Ch}_{m'}$. Then obviously $\mathfrak{F}_0 = \circ$. Then, we have to select: $\mathfrak{F} = \mathfrak{Ch}_n$. Because $\mathfrak{F}_0 = \circ$ then (APK) holds. For other choices (APK) holds as well.

Such a situation takes place, i.e. for $\mathfrak{Ch}_8$ which is reducible onto $\mathfrak{Ch}_3$ and $\mathfrak{Ch}_2$. In turn $\mathfrak{Ch}_3$ and $\mathfrak{Ch}_2$ are both reducible only onto $\circ$.

Sub-case 2b. Suppose that the common reduct of $\mathfrak{Ch}_m$ and $\mathfrak{Ch}_{m'}$ is some $\mathfrak{Ch}_k$, $k \neq 1$. Again the less trivial choice is: $\mathfrak{F}_1 = \mathfrak{Ch}_m$, $\mathfrak{F}_2 = \mathfrak{Ch}_{m'}$ and $\mathfrak{F}_0 = \mathfrak{Ch}_k$. Then we select: $\mathfrak{F} = \mathfrak{Ch}_n$. We denote the existing reductions as follows:

$$f_1 : \mathfrak{Ch}_m \to \mathfrak{Ch}_k, \quad f_2 : \mathfrak{Ch}_{m'} \to \mathfrak{Ch}_k,$$
$$g_1 : \mathfrak{Ch}_n \to \mathfrak{Ch}_m, \quad g_2 : \mathfrak{Ch}_n \to \mathfrak{Ch}_{m'},$$

and all the functions: f_1, f_2, g_1, g_2 are unique. Then their superpositions $f_1 \circ g_1$ and $f_2 \circ g_2$ are unique and $f_1 \circ g_1 = f_2 \circ g_2$. For other choices of frames (APK) holds as well.

Such a situation takes place, i.e. for $\mathfrak{Ch}_{53}$ which is reducible onto $\mathfrak{Ch}_8$ and $\mathfrak{Ch}_{11}$. In turn $\mathfrak{Ch}_8$ and $\mathfrak{Ch}_{11}$ are both reducible only onto $\mathfrak{Ch}_2$.

We allow here that $k = m$ or $k = m'$. The example is the following: $n = 14$, $m = 5$, $m' = k = 2$. Then f_2 is isomorphism. ∎

COROLLARY 3. *All the logics* $L(\mathfrak{Ch}_n)$ *with* $n \in \mathbb{N}$ *have (IPD).*

We proved that there exist countably infinitely many logics with (IPD) in $NEXT(\mathbf{KB}.\mathbf{Alt}(\mathbf{2}))$. It is quite opposite than in $NEXT(\mathbf{KTB}.\mathbf{Alt}(\mathbf{3}))$ where there are only two logics with interpolation.

It seems that reflexivity of Kripke frames negatively affects the amalgamation property (and hence interpolation for the determined logics).

The following problems are left open:

PROBLEM 1. *Whether the logic* $\mathbf{KB}.\mathbf{Alt}(\mathbf{2})$ *has interpolation?*

PROBLEM 2. *Is there a tabular logic (different from* $L(\circ)$ *or* $L(\circ - -\circ)$*) with interpolation in* $NEXT(\mathbf{KTB}.\mathbf{Alt}(\mathbf{n}))$ *with* $n \geq 4$*?*

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