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PSEUDO-BCH SEMILATTICES

Abstract

In this paper we study pseudo-BCH algebras which are semilattices or lattices with respect to the natural relation $\leqslant$; we call them pseudo-BCH join-semilattices, pseudo-BCH meet-semilattices and pseudo-BCH lattices, respectively. We prove that the class of all pseudo-BCH join-semilattices is a variety and show that it is weakly regular, arithmetical at 1, and congruence distributive. In addition, we obtain the systems of identities defininig pseudo-BCH meet-semilattices and pseudo-BCH lattices.

Keywords: (pseudo-)BCK/BCI/BCH algebra, pseudo-BCH join (meet)-semilattice, weakly regular, arithmetical at 1.

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1. Introduction

In 1966, Imai and Iséki ([8, 11]) introduced BCK and BCI algebras as algebras connected to certain kinds of logics. In 1983, Hu and Li ([7]) defined BCH algebras. It is known that BCK and BCI algebras are contained in the class of BCH algebras. In [9, 10], Iorgulescu introduced many interesting generalizations of BCI or of BCK algebras.

In 2001, Georgescu and Iorgulescu ([6]) defined pseudo-BCK algebras as an extension of BCK algebras. In 2008, Dudek and Jun ([2]) introduced pseudo-BCI algebras as a natural generalization of BCI algebras and of pseudo-BCK algebras. These algebras have also connections with other algebras of logic such as pseudo-MV algebras and pseudo-BL algebras defined by Georgescu and Iorgulescu in [4] and [5], respectively. Recently, Walendziak ([14]) introduced pseudo-BCH algebras as an extension of BCH algebras.

In [13], Kühr investigated pseudo-BCK algebras whose underlying posets are semilattices. In this paper we study pseudo-BCH join-semilattices, that is. pseudo-BCH algebras which are join-semilattices with respect to the natural relation $\leqslant$. We prove that the class of all pseudo-BCH join-semilattices is a variety and show that it is weakly regular, arithmetical at 1, and congruence distributive. In addition, we obtain the systems of identities defininig pseudo-BCH meet-semilattices and pseudo-BCH lattices.

2. Preliminaries

We recall that an algebra $(X; \to, 1)$ of type $(2,0)$ is called a *BCH algebra* if it satisfies the following axioms:

(BCH-1) $x \to x = 1$;
(BCH-2) $x \to (y \to z) = y \to (x \to z)$;
(BCH-3) $x \to y = y \to x = 1 \Longrightarrow x = y$.

A *BCI algebra* is a BCH algebra $(X; \to, 1)$ satisfying the identity

(BCI) $(y \to z) \to ((z \to x) \to (y \to x)) = 1$.

A *BCK algebra* is a BCI algebra $(X; \to, 1)$ such that $x \to 1 = 1$ for all $x \in X$.

A *pseudo-BCI algebra* ([2]) is a structure $(X; \leq, \to, \rightsquigarrow, 1)$, where $\leq$ is a binary relation on the set X, $\to$ and $\rightsquigarrow$ are binary operations on X and 1 is an element of X, verifying the axioms:

(pBCI-1) $y \to z \leq (z \to x) \rightsquigarrow (y \to x)$, $y \rightsquigarrow z \leq (z \rightsquigarrow x) \to (y \rightsquigarrow x)$;
(pBCI-2) $x \leq (x \rightsquigarrow y) \to y$, $x \leq (x \to y) \rightsquigarrow y$;
(pBCI-3) $x \leq x$;
(pBCI-4) $x \leq y, y \leq x \Longrightarrow x = y$;
(pBCI-5) $x \leq y \Longleftrightarrow x \to y = 1 \Longleftrightarrow x \rightsquigarrow y = 1$.

A pseudo-BCI-algebra $(X; \leq, \to, \rightsquigarrow, 1)$ is called a *pseudo-BCK algebra* if it satisfies the identities

(pBCK) $x \to 1 = x \rightsquigarrow 1 = 1$.

Definition 2.1. *([14]) A (dual) pseudo-BCH algebra is an algebra* $\mathfrak{X} = (X; \to, \rightsquigarrow, 1)$ *of type* $(2,2,0)$ *satisfying the axioms:*

(pBCH-1) $x \to x = x \rightsquigarrow x = 1$;
(pBCH-2) $x \to (y \rightsquigarrow z) = y \rightsquigarrow (x \to z)$;

(pBCH-3) $x \to y = y \rightsquigarrow x = 1 \Longrightarrow x = y$;
(pBCH-4) $x \to y = 1 \Longleftrightarrow x \rightsquigarrow y = 1$.

Remark 2.2. *Observe that if* $(X; \to, 1)$ *is a BCH algebra, then letting* $x \to y := x \rightsquigarrow y$*, produces a pseudo-BCH algebra* $(X; \to, \rightsquigarrow, 1)$*. Therefore, every BCH algebra is a pseudo-BCH algebra in a natural way. It is easy to see that if* $(X; \to, \rightsquigarrow, 1)$ *is a pseudo-BCH algebra, then* $(X; \rightsquigarrow, \to, 1)$ *is also a pseudo-BCH algebra. From Proposition 3.2 of [2] we conclude that if* $(X; \leq, \to, \rightsquigarrow, 1)$ *is a pseudo-BCI algebra, then* $(X; \to, \rightsquigarrow, 1)$ *is a pseudo-BCH algebra.*

In any pseudo-BCH algebra we can define a natural relation $\leqslant$ by putting

$$x \leqslant y \Longleftrightarrow x \to y = 1 \Longleftrightarrow x \rightsquigarrow y = 1.$$

It is easy to see that $\leqslant$ is reflexive and anti-symmetric but it is not transitive in general (see Example 2.3 below). We note that in pseudo-BCK/BCI algebras the relation $\leqslant$ is a partial order.

Example 2.3. *Let* $X = \{a, b, c, d, e, f, 1\}$*. We define the binary operations* $\to$ *and* $\rightsquigarrow$ *on* X *as follows*

$\to$	a	b	c	d	e	f	1
a	1	b	b	d	e	f	1
b	a	1	c	d	e	f	1
c	1	1	1	d	e	f	1
d	a	b	c	1	1	f	1
e	a	b	c	e	1	1	1
f	a	b	c	d	e	1	1
1	a	b	c	d	e	f	1

and

$\rightsquigarrow$	a	b	c	d	e	f	1
a	1	b	c	d	e	f	1
b	a	1	a	d	e	f	1
c	1	1	1	d	e	f	1
d	a	b	c	1	1	f	1
e	a	b	c	e	1	1	1
f	a	b	c	d	e	1	1
1	a	b	c	d	e	f	1

Then $\mathfrak{X} = (X; \to, \rightsquigarrow, 1)$ *is a pseudo-BCH algebra (see Example 2.6 of [15]). We have* $d \leqslant e$ *and* $e \leqslant f$ *but* $d \nleqslant f$*, and therefore* $\leqslant$ *is not transitive.*

Proposition 2.4. ([14]) Every pseudo-BCH algebra $\mathfrak{X}$ satisfies, for all $x, y \in X$, the following conditions:

(i) $1 \to x = 1 \rightsquigarrow x = x$,
(ii) $x \leqslant (x \rightsquigarrow y) \to y$, and $x \leqslant (x \to y) \rightsquigarrow y$.

Proposition 2.5. *([14]) Let $\mathfrak{X}$ be a pseudo-BCH algebra. Then $\mathfrak{X}$ is a pseudo-BCI algebra if and only if it verifies the following implication: for all $x, y, z \in X$,*

$$x \leqslant y \Longrightarrow (z \to x \leqslant z \to y,\ z \rightsquigarrow x \leqslant z \rightsquigarrow y). \tag{2.1}$$

3. Pseudo-BCH semilattices

Generalizing the notion of a pseudo-BCK semilattice (see [13]) we define pseudo-BCH join-semilattices.

Definition 3.1. *We say that an algebra $(X; \vee, \to, \rightsquigarrow, 1)$ is a pseudo-BCH join-semilattice if $(X; \vee)$ is a join-semilattice, $(X; \to, \rightsquigarrow, 1)$ is a pseudo-BCH-algebra and $x \vee y = y \Longleftrightarrow x \to y = 1$ for all $x, y \in X$.*

Example 3.2. *Let $X = \{a, b, c, 1\}$. We define the binary operations $\to$ and $\rightsquigarrow$ on X as follows:*

$\to$	a	b	c	1
a	1	b	b	1
b	1	1	b	1
c	1	1	1	1
1	a	b	c	1

and

$\rightsquigarrow$	a	b	c	1
a	1	b	c	1
b	1	1	a	1
c	1	1	1	1
1	a	b	c	1

It is easy to check that $(X; \to, \rightsquigarrow, 1)$ is a pseudo-BCH algebra. Since X is a join-semilattice with respect to $\vee$ (under $\leqslant$), we conclude that $\mathfrak{X} = (X; \vee, \to, \rightsquigarrow, 1)$ is a pseudo-BCH join-semilattice; it is even a chain with $c < b < a < 1$.

Example 3.3. *Let $\mathfrak{X} = (\{a, b, c, d, e, f, 1\}; \to, \rightsquigarrow, 1)$ be the pseudo-BCH algebra from Example 2.3. Since the relation $\leqslant$ is not transitive, X is not a join-semilattice with respect to $\leqslant$. Therefore it is not a pseudo-BCH join-semilattice.*

Proposition 3.4. *Let $(X; \vee, \to, \rightsquigarrow, 1)$ be a pseudo-BCH join-semilattice. The following properties hold (for all $x, y, z \in X$):*

(a1) $x \vee y = y \vee x$,
(a2) $(x \vee y) \vee z = x \vee (y \vee z)$,
(a3) $x \to (y \rightsquigarrow z) = y \rightsquigarrow (x \to z)$,

(a4) $1 \to x = 1 \rightsquigarrow x = x$,
(a5) $x \to (x \vee y) = x \rightsquigarrow (x \vee y) = 1$,
(a6) $((x \rightsquigarrow y) \to y) \vee x = (x \rightsquigarrow y) \to y$,
(a7) $((x \to y) \rightsquigarrow y) \vee x = (x \to y) \rightsquigarrow y$.

Proof: (a1)–(a3) and (a5) are obvious. By Proposition 2.4 (i) we get (a4). Identities (a6) and (a7) follow from Proposition 2.4 (ii). □

Proposition 3.5. *Let* $(X; \vee, \to, \rightsquigarrow, 1)$ *be an algebra of type* $(2, 2, 2, 0)$ *satisfying (a1)–(a7). Define* $\leqslant$ *on* X *by*

$$x \leqslant y \Longleftrightarrow x \vee y = y.$$

Then, for all $x, y, z \in X$, *we have:*

(1) $x \leqslant y$ *and* $y \leqslant x$ *imply* $x = y$,
(2) $x \leqslant y$ *and* $y \leqslant z$ *imply* $x \leqslant z$,
(3) $x \leqslant y \Longleftrightarrow x \to y = 1$,
(4) $x \leqslant y \Longleftrightarrow x \rightsquigarrow y = 1$,
(5) $x \vee 1 = 1 \vee x = 1$ (*that is*, $x \leqslant 1$),
(6) $x \to 1 = x \rightsquigarrow 1 = 1$,
(7) $x \to x = x \rightsquigarrow x = 1$ (*that is*, $x \leqslant x$).

Proof: Statements (1) and (2) follow from (a1) and (a2), respectively.

To prove (3), let $x, y \in X$ and $x \vee y = y$. Applying (a5), we get $x \to y = 1$.

Conversely, suppose that $x \to y = 1$. Hence $(x \to y) \rightsquigarrow y = 1 \rightsquigarrow y = y$ by (a4). From (a7) we see that $x \vee y = y$, that is, $x \leqslant y$.

(4) The proof of (4) is similar to that of (3).

(5) Applying (a5) and (a4), we obtain $1 = 1 \to (1 \vee x) = 1 \vee x$. This clearly forces (5).

(6) By (5), $x \leqslant 1$. Using (3) and (4), we get (6).

(7) We have

$$\begin{aligned} 1 &= ((1 \rightsquigarrow x) \to x) \vee 1 && [\text{by (5)}] \\ &= (1 \rightsquigarrow x) \to x && [\text{by (a6)}] \\ &= x \to x. && [\text{by (a4)}] \end{aligned}$$

Similarly, $x \rightsquigarrow x = 1$. □

Combining Propositions 3.4 and 3.5 we get

Theorem 3.6. *An algebra* $(X; \vee, \to, \rightsquigarrow, 1)$ *of type* $(2, 2, 2, 0)$ *is a pseudo-BCH join-semilattice if and only if it satisfies the identities (a1)–(a7).*

From Proposition 3.5 (6) we have

Corollary 3.7. *Every pseudo-BCH join-semilattice verifies (pBCK).*

Let us denote by $\mathcal{J}$ the class of all pseudo-BCH join-semilattices.

Remark 3.8. *The class* $\mathcal{J}$ *is a variety. Therefore* $\mathcal{J}$ *is closed under the formation of homomorphic images, subalgebras, and direct products.*

The disjont union of BCK algebras was introduced by Iséki and Tanaka in [12] and next generalized to BCH algebras ([3]) and pseudo-BCH algebras ([15]). Below we extend this concept to the case of pseudo-BCH join-semilattices.

Let T be any set and, for each $t \in T$, let $\mathfrak{X}_t = (X_t; \vee_t, \to_t, \rightsquigarrow_t, 1)$ be a pseudo-BCH join-semilattice. Suppose that $X_s \cap X_t = \{1\}$ for $s, t \in T$, $s \neq t$. Set $X = \bigcup_{t \in T} X_t$ and define the binary operations $\vee, \to$ and $\rightsquigarrow$ on X via

$$x \vee y = \begin{cases} x \vee_t y & \text{if } x, y \in X_t,\ t \in T, \\ 0 & \text{if } x \in X_s,\ y \in X_t,\ s, t \in T,\ s \neq t. \end{cases}$$

$$x \to y = \begin{cases} x \to_t y & \text{if } x, y \in X_t,\ t \in T, \\ x & \text{if } x \in X_s,\ y \in X_t,\ s, t \in T,\ s \neq t. \end{cases}$$

and

$$x \rightsquigarrow y = \begin{cases} x \rightsquigarrow_t y & \text{if } x, y \in X_t,\ t \in T, \\ x & \text{if } x \in X_s,\ y \in X_t,\ s, t \in T,\ s \neq t. \end{cases}$$

It is easily seen that $\mathfrak{X} = (X; \vee, \to, \rightsquigarrow, 1)$ is a pseudo-BCH join-semilattice; it will be called the *disjont union* of $(\mathfrak{X}_t)_{t \in T}$.

Example 3.9. *Let* $\mathfrak{X}_1 = \mathfrak{X}$*, where* $\mathfrak{X} = (\{a, b, c, 1\}; \vee, \to, \rightsquigarrow, 1)$ *is the pseudo-BCH join-semilattice from Example 3.2. Consider the set* $X_2 = \{d, e, f, 1\}$ *with the operations* $\to_2$ *and* $\vee_2$ *defined by the following tables:*

$\to_2$	d	e	f	1
d	1	e	d	1
e	d	1	f	1
f	1	1	1	1
1	d	e	f	1

and

$\vee_2$	d	e	f	1
d	d	1	d	1
e	1	e	e	1
f	d	e	f	1
1	1	1	1	1

Let $\rightsquigarrow_2 := \rightarrow_2$. *Routine calculations show that* $\mathfrak{X}_2 = (X_2; \vee_2, \rightarrow_2, \rightsquigarrow_2, 1)$ *is a (pseudo)-BCH join-semilattice. Let* $X' = \{a, b, c, d, e, f, 1\}$. *We define the binary operations* $\rightarrow'$ *and* $\rightsquigarrow'$ *on* X' *as follows*

$\rightarrow'$	a	b	c	d	e	f	1
a	1	b	b	d	e	f	1
b	1	1	b	d	e	f	1
c	1	1	1	d	e	f	1
d	a	b	c	1	e	d	1
e	a	b	c	d	1	f	1
f	a	b	c	1	1	1	1
1	a	b	c	d	e	f	1

and

$\rightsquigarrow'$	a	b	c	d	e	f	1
a	1	b	c	d	e	f	1
b	1	1	a	d	e	f	1
c	1	1	1	d	e	f	1
d	a	b	c	1	e	d	1
e	a	b	c	d	1	f	1
f	a	b	c	1	1	1	1
1	a	b	c	d	e	f	1

It is clear that $\mathfrak{X}' = (X'; \vee', \rightarrow', \rightsquigarrow', 1)$, *where the operation* $\vee'$ *is illustrated in Figure 1, is the disjont union of* $\mathfrak{X}_1$ *and* $\mathfrak{X}_2$.

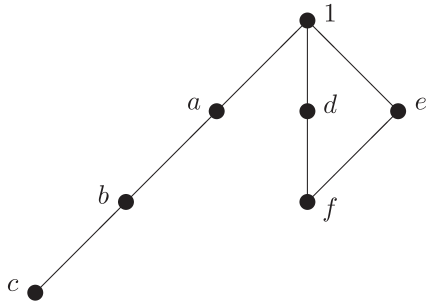

Figure 1

Proposition 3.10. *Let* $\mathfrak{X} = (X; \vee, \rightarrow, \rightsquigarrow, 1)$ *be a pseudo-BCH join-semilattice. Then the following statements are equivalent:*

(i) $\mathfrak{X}$ *is a pseudo-BCK join-semilattice.*

(ii) $\mathfrak{X}$ *satisfies (2.1) for all* $x, y, z \in X$.

Proof: Follows immediately from Proposition 2.5 and Corollary 3.7. □

Proposition 3.11. *Let* $\mathfrak{X} = (X; \vee, \rightarrow, \rightsquigarrow, 1)$ *be a pseudo-BCH join-semilattice satisfying the following implication: for all* $x, y, z \in X$,

$$x \leqslant y \Longrightarrow (y \rightarrow x) \rightsquigarrow x = (y \rightsquigarrow x) \rightarrow x = y. \tag{3.1}$$

Then $\mathfrak{X}$ *is a pseudo-BCK join-semilattice.*

Proof: Let $x, y, z \in X$ and $x \leqslant y$. By (pBCH-2), (pBCH-1) and (pBCK),

$$\begin{aligned}(z \to x) \to (z \to y) &= (z \to x) \to (z \to ((y \to x) \rightsquigarrow x)) \\ &= (y \to x) \rightsquigarrow ((z \to x) \to (z \to x)) \\ &= (y \to x) \rightsquigarrow 1 \\ &= 1.\end{aligned}$$

Then $z \to x \leqslant z \to y$. Similarly, $z \rightsquigarrow x \leqslant z \rightsquigarrow y$. From Proposition 3.10 we see that $\mathfrak{X}$ is a pseudo-BCK join-semilattice. □

Remark 3.12. *The converse of Proposition 3.11 is false. Indeed, let $\mathfrak{X}$ be the pseudo-BCH join-semilattice from Example 3.2. It is easy to check that $\mathfrak{X}$ satisfies implication (2.1), and therefore it is a pseudo-BCK join-semilattice. However, (3.1) does not hold in $\mathfrak{X}$, because we have $c < a$ and $(a \rightsquigarrow c) \to c = 1$.*

Definition 3.13. *An algebra $(X; \wedge, \to, \rightsquigarrow, 1)$ is called a pseudo-BCH meet-semilattice if $(X; \wedge)$ is a meet-semilattice, $(X; \to, \rightsquigarrow, 1)$ is a pseudo-BCH algebra, and $x \wedge y = x \Longleftrightarrow x \to y = 1$ for all $x, y \in X$.*

Denote by $\mathcal{M}$ the class of all pseudo-BCH meet-semilattices.

Proposition 3.14. *An algebra $\mathfrak{X} = (X; \wedge, \to, \rightsquigarrow, 1)$ of type $(2, 2, 2, 0)$ is a pseudo-BCH meet-semilattice if and only if it satisfies the following identities:*

(b1) $x \wedge x = x$,
(b2) $x \wedge y = y \wedge x$,
(b3) $x \wedge (y \wedge z) = (x \wedge y) \wedge z$,
(b4) $x \to (y \rightsquigarrow z) = y \rightsquigarrow (x \to z)$,
(b5) $1 \to x = 1 \rightsquigarrow x = x$,
(b6) $(x \wedge y) \to y = 1 = (x \wedge y) \rightsquigarrow y$,
(b7) $x \wedge ((x \rightsquigarrow y) \to y) = x = x \wedge ((x \to y) \rightsquigarrow y)$.

Proof: Obviously, every pseudo-BCH meet-semilattice satisfies the axioms (b1)–(b7).

Conversely, let (b1)–(b7) hold in $\mathfrak{X}$. Clearly, $(X; \wedge)$ is a meet-semilattice. Define $\leqslant$ on X by

$$x \leqslant y \Longleftrightarrow x = x \wedge y.$$

Observe that

$$x \leqslant y \Longleftrightarrow x \to y = 1 \Longleftrightarrow x \rightsquigarrow y = 1 \tag{3.2}$$

for all $x, y \in X$. Let $x \leqslant y$, that is, $x \wedge y = x$. By (b6), $x \to y = 1$ and $x \rightsquigarrow y = 1$. Suppose now that $x \to y = 1$. Applying (b7) and (b5), we get

$$x = x \wedge ((x \to y) \rightsquigarrow y) = x \wedge (1 \rightsquigarrow y) = x \wedge y.$$

Hence $x \leqslant y$. Similarly, if $x \rightsquigarrow y = 1$, then $x \leqslant y$. Thus (3.2) holds. Therefore, we deduce that $(X; \to, \rightsquigarrow, 1)$ is a pseudo-BCH algebra, and finally that $(X; \wedge, \to, \rightsquigarrow, 1)$ is a pseudo-BCH meet-semilattice. □

Corollary 3.15. *The class $\mathcal{M}$ is a variety.*

Definition 3.16. *An algebra $(X; \vee, \wedge, \to, \rightsquigarrow, 1)$ is called a pseudo-BCH lattice if $(X; \vee, \wedge)$ is a lattice, $(X; \to, \rightsquigarrow, 1)$ is a pseudo-BCH algebra, and $x \to y = 1 \Longleftrightarrow x \vee y = y \Longleftrightarrow x \wedge y = x$ for all $x, y \in X$.*

Denote by $\mathcal{L}$ the class of all pseudo-BCH lattices.

Example 3.17. *Let $X = \{a, b, c, d, 1\}$. Define binary operations $\to$ and $\rightsquigarrow$ on X by the following tables:*

$\to$	a	b	c	d	1
a	1	b	b	b	1
b	a	1	a	d	1
c	1	1	1	b	1
d	1	1	1	1	1
1	a	b	c	d	1

$\rightsquigarrow$	a	b	c	d	1
a	1	b	b	d	1
b	a	1	a	a	1
c	1	1	1	a	1
d	1	1	1	1	1
1	a	b	c	d	1

By routine calculation, $\mathfrak{X} = (X; \to, \rightsquigarrow, 1)$ is a pseudo-BCH algebra. We shall represent the set X and the binary relation $\leqslant$ by the following Hasse diagram:

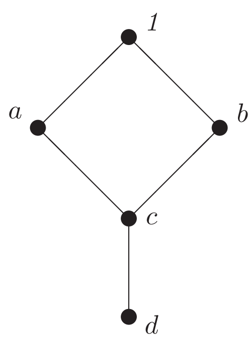

Figure 2

Therefore, $(X; \vee, \wedge, \rightarrow, \rightsquigarrow, 1)$ *is a pseudo-BCH lattice.*

Remark 3.18. *The class* $\mathcal{L}$ *is a variety that is axiomatized by the defining identities of lattices and by the identities (a3)–(a7) or by (b4)–(b7), respectively.*

Now we recall several universal algebraic notions (see e. g. [1]). We will denote by Con$\mathfrak{A}$ the congruence lattice of an algebra $\mathfrak{A}$. For $\theta \in$ Con$\mathfrak{A}$ and $x \in A$, let x/θ denote the equivalence class of x modulo θ. An algebra $\mathfrak{A}$ with a constant 1 is called:

- *weakly regular* (*at* 1) if $1/\theta = 1/\phi$ implies $\theta = \phi$, for all $\theta, \phi \in$ Con$\mathfrak{A}$;
- *permutable at* 1 if $1/(\theta \circ \phi) = 1/(\phi \circ \theta)$ for all $\theta, \phi \in$ Con$\mathfrak{A}$;
- *distributive at* 1 if $1/\theta \cap (\phi \vee \psi) = 1/(\theta \cap \phi) \vee (\theta \cap \psi)$ for all $\theta, \phi, \psi \in$ Con$\mathfrak{A}$;
- *arithmetical at* 1 if it is both permutable at 1 and distributive at 1.

Let $\mathcal{V}$ be a variety of algebras with a constant 1. We say that $\mathcal{V}$ is *weakly regular* (resp., *permutable at* 1, *distributive at* 1, and *arithmetical at* 1) if every algebra $\mathfrak{A} \in \mathcal{V}$ is weakly regular (resp., permutable at 1, distributive at 1, and arithmetical at 1). It is known that a variety $\mathcal{V}$ is weakly regular if and only if there exist binary terms $t_1, \ldots, t_n$ for some $n \in \mathbb{N}$ such that

$$t_1(x, y) = \cdots = t_n(x, y) = 1 \Longleftrightarrow x = y. \tag{3.3}$$

A variety is *arithmetical at* 1 if and only if there exists a binary term t satisfying $t(x, x) = t(1, x) = 1$ and $t(x, 1) = x$. A variety $\mathcal{V}$ is *congruence distributive* if Con$\mathfrak{A}$ is a distributive lattice for every $\mathfrak{A} \in \mathcal{V}$.

Theorem 3.19. *The variety* $\mathcal{J}$, $\mathcal{M}$ *and* $\mathcal{L}$ *are weakly regular. Moreover,* $\mathcal{J}$ *and* $\mathcal{L}$ *are arithmetical at 1 and congruence distributive.*

Proof: $\mathcal{J}$, $\mathcal{M}$ and $\mathcal{L}$ are weakly regular since the terms $t_1(x, y) = x \rightarrow y$ and $t_2(x, y) = y \rightsquigarrow x$ satisfy (3.3) for $n = 2$.

Let $\mathfrak{X}$ be a pseudo-BCH join-semilattice and $t(x, y) = y \rightarrow x$. Clearly, $t(x, x) = 1$ and $t(x, 1) = x$. By Corollary 3.7, $\mathfrak{X}$ satisfies (pBCK), and hence $t(1, x) = 1$. Then $\mathfrak{X}$ is arithmetical at 1, and consequently distributive at 1.

Let $\theta, \phi, \psi \in$ Con$\mathfrak{X}$. By distributivity at 1, $1/\theta \cap (\phi \vee \psi) = 1/(\theta \cap \phi) \vee (\theta \cap \psi)$. From weak regularity we obtain $\theta \cap (\phi \vee \psi) = (\theta \cap \phi) \vee (\theta \cap \psi)$. Therefore Con$\mathfrak{X}$ is a distributive lattice.

Thus pseudo-BCH join-semilattices (and hence pseudo-BCH lattices) are arithmetical at 1 and congruence distributive. □

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