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PC-LATTICES: A CLASS OF BOUNDED BCK-ALGEBRAS

Abstract

In this paper, we define the notion of PC-lattice, as a generalization of finite positive implicative BCK-algebras with condition (S) and bounded commutative BCK-algebras. We investigate some results for PC-lattices being a new class of BCK-lattices. Specially, we prove that any Boolean lattice is a PC-lattice and we show that if X is a PC-lattice with condition S , then X is an involutory BCK-algebra if and only if X is a commutative BCK-algebra. Finally, we prove that any PC-lattice with condition (S) is a distributive BCK-algebra.

Keywords: PC-lattice, BCK-lattice, Involutory BCK-algebras, Bounded commutative BCK-algebras

2010 Mathematics Subject Classification: 06F35, 03G25

1. Introduction

A BCK-algebra is an important class of logical algebras introduced by Y. Imai and K. Iséki in 1966 [8]. This notion originated from two different sources: one of them is based on the set theory, the other is from the classical and non-classical propositional calculi. Non-classical logic has become a considerable formal tool for computer science and artificial intelligence to deal with fuzzy information and uncertain information. In the first half of the nineteenth century, George Boole's attempts to formalize propositional logic led to the concept of Boolean algebras. Investigating the axiomatics of Boolean algebras at the end of the nineteenth century, Charles S. Peirce and Ernst Schröder found it useful to introduce the concept of a lattice. Dedekind also introduced modularity, a weakened form

of distributivity. It was Garret Birkhoff's work in the mid-thirties that started the general development of lattice theory. In a brilliant series of papers he demonstrated the importance of lattice theory and showed that it provides a unifying framework for hitherto unrelated developments in many mathematical disciplines. In BCK-algebras some important lattices such as bounded commutative BCK-algebras, involutory BCK-algebras and bounded implicative BCK-algebras were defined and some of their important properties were proved [7, 12, 14]. In order to extend the theory of bounded BCK-algebras, we introduce the concept of PC-lattices and characterize their properties. We prove that the class of these lattices includes some currently known subclasses of BCK-lattices such as bounded commutative BCK-algebras (bounded commutative BCK-lattices), finite positive implicative BCK-algebras with condition (S) and bounded implicative BCK-algebras (Boolean lattices). We study the relation between involutory BCK-algebras and PC-lattices, and show that in PC-lattices with condition (S), commutative BCK-algebras and involutory BCK-algebras coincide. Finally, we prove that any PC-lattice with condition (S) is distributive and detect the relations between these BCK-lattices by a figure.

2. Preliminaries

In this section, we give some definitions and theorems that will be used in the next sections.

DEFINITION 2.1. [11] Let X be a set with a binary operation $*$ and a constant 0 . Then $(X, *, 0)$ is called a BCK-algebra if it satisfies the following axioms:

$$(BCK-1) \quad ((x * y) * (x * z)) * (z * y) = 0,$$

$$(BCK-2) \quad (x * (x * y)) * y = 0,$$

$$(BCK-3) \quad x * x = 0,$$

$$(BCK-4) \quad x * y = 0 \text{ and } y * x = 0 \text{ imply } x = y,$$

$$(BCK-5) \quad 0 * x = 0.$$

for all $x, y, z \in X$. A partial order \leq on X can be defined by $x \leq y$ if and only if $x * y = 0$.

PROPOSITION 2.2. [11] *In any BCK-algebra X , the following hold:*

$$(i) \quad (x * y) * z = (x * z) * y,$$

$$(ii) \quad x \leq y \text{ implies } x * z \leq y * z \text{ and } z * y \leq z * x,$$

- (iii) $x * y \leq x$,
 (iv) $x * y \leq z \Leftrightarrow x * z \leq y$.
 for all $x, y, z \in X$.

DEFINITION 2.3. [1, 11, 13, 12] Let X be a BCK-algebra. Then, for all $x, y, z \in X$:

- (i) X is called a *positive implicative* BCK-algebra if $(x*y)*z = (x*z)*(y*z)$,
 (ii) X is called an *implicative* BCK-algebra if $x * (y * x) = x$,
 (iii) X is called a *commutative* BCK-algebra if $x * (x * y) = y * (y * x)$,
 (iv) X is called *bounded* BCK-algebra, if there exists the greatest element 1 of X , and for any $x \in X$, $1 * x$ is denoted by Nx ,
 (v) X is called *involutory* BCK-algebra, if for all $x \in X$, $NNx = x$.

PROPOSITION 2.4. [11, 12] Let X be a bounded BCK-algebra. Then, for all $x, y \in X$:

- (i) $N1 = 0$ and $N0 = 1$,
 (ii) $NNx \leq x$ that $NNx = N(Nx)$,
 (iii) $Nx * Ny \leq y * x$,
 (iv) $y \leq x$ implies $Nx \leq Ny$,
 (v) $Nx * y = Ny * x$,
 (vi) $NNNx = Nx$.

THEOREM 2.5. [7] Let X be a bounded BCK-algebra. Then for any $x, y \in X$, the following hold:

- (i) X is involutory,
 (ii) $x * y = Ny * Nx$,
 (iii) $x * Ny = y * Nx$,
 (iv) $x \leq Ny$ implies $y \leq Nx$.

THEOREM 2.6. [11] Every implicative BCK-algebra is a commutative and positive implicative BCK-algebra.

DEFINITION 2.7. [9, 6] Let X be a BCK-algebra. Then:

- (i) X is said to have condition (S) (S), if for any $x, y \in X$, the set $A(x, y) = \{t \in X : t * x \leq y\}$ has the greatest element which is denoted by $x \circ y$,
 (ii) $(X, *, \leq)$ is called a BCK-lattice, if (X, \leq) is a lattice, where \leq is the partial BCK-order on X , which has been introduced in Definition 2.1.

PROPOSITION 2.8. [9] Let X be a BCK-algebras with condition (S). Then, for all $x, y, z \in X$:

- (i) $y \leq x \circ (y * x)$,
- (ii) $(x \circ z) * (y \circ z) \leq x * y$,
- (iii) $(x * y) * z = x * (y \circ z)$,
- (iv) if $x \leq y$, then $x \circ z \leq y \circ z$,
- (v) $z * x \leq y \iff z \leq x \circ y$.

THEOREM 2.9. [10] Let X be a BCK-algebra with condition (S). Then, for all $x, y, z \in X$, the following are equivalent:

- (i) X is a positive implicative,
- (ii) $x \leq y$ implies $x \circ y = y$,
- (iii) $x \circ x = x$,
- (iv) $(x \circ y) * z = (x * z) \circ (y * z)$,
- (v) $x \circ y = x \circ (y * x)$.

THEOREM 2.10. [6, 7, 14] Let X be a BCK-algebra.

- (i) If X is a finite positive implicative BCK-algebra with condition (S), then (X, \leq) is a distributive lattice,
- (ii) If X is a BCK-algebra with condition (S), then X is positive implicative if and only if (X, \leq) is an upper semilattice with $x \vee y = x \circ y$, for any $x, y \in X$,
- (iii) If X is a bounded commutative BCK-algebra, then BCK-lattice (X, \leq) is a distributive lattice, where $x \wedge y = y * (y * x)$ and $x \vee y = N(Nx \wedge Ny)$.

THEOREM 2.11. [7] Let X be an involutory BCK-algebra. Then the following are equivalent:

- (i) (X, \leq) is a lower semilattice,
- (ii) (X, \leq) is an upper semilattice,
- (iii) (X, \leq) is a lattice.

THEOREM 2.12. [12] Let X be a bounded BCK-algebra. Then:

- (i) every commutative BCK-algebra is an involutory BCK-algebra.
- (ii) any implicative BCK-algebra is a Boolean lattice (a complemented distributive lattice).

THEOREM 2.13. [13, 14] *Let X be a BCK-algebra. Then, for all $x, y, z \in X$, the following are equivalent:*

- (i) X is commutative,
- (ii) $x * y = x * (y * (y * x))$,
- (iii) $x * (x * y) = y * (y * (x * (x * y)))$,
- (iv) $x \leq y$ implies $x = y * (y * x)$.

3. PC-lattices

In this section, we define the notion of PC-lattice in BCK-algebras, which is a new class of bounded BCK-algebras.

DEFINITION 3.1. Let $(X, *, \leq)$ be a BCK-lattice. Then $(X, *, \leq)$ is called a PC-lattice, if

$$(z * x) * (y * x) = z * (x \vee y) \text{ for all } x, y, z \in X.$$

Note. In what follows, we show that the class of PC-lattices includes the finite **P**ositive implicative BCK-algebras with condition (S) and bounded **C**ommutative BCK-algebras that are two important classes of BCK-lattices. Hence this class of BCK-lattices is called **PC**-lattice.

EXAMPLE 3.2. (i) Let $X = \{0, a, b, 1\}$ be a chain, where $0 \leq a \leq b \leq 1$, and the operation $*$ on X is defined as follows:

$*$	0	a	b	1
0	0	0	0	0
a	a	0	0	0
b	b	b	0	0
1	1	1	1	0

It is easy to check that $(X, *, \leq)$ is a PC-lattice.

(ii) Let X be an interval $[0, 1]$ of real numbers and binary operation $*$ on X is defined as a follows:

$$x * y = \begin{cases} 0 & \text{if } x \leq y \\ x & \text{if } x > y \end{cases}$$

Then it is easy to check that $(X, *, \leq)$ is a PC-lattice, where $x \wedge y = \min\{x, y\}$ and $x \vee y = \max\{x, y\}$.

The following example shows that a BCK-lattice is not a PC-lattice, in general.

EXAMPLE 3.3. Let $X = \{0, a, b, 1\}$ be a chain, that is $0 \leq a \leq b \leq 1$, and the operation $*$ on X is defined as follows:

$*$	0	a	b	1
0	0	0	0	0
a	a	0	0	0
b	b	b	0	0
1	1	b	a	0

Then it is easy to check that $(X, *, \leq)$ is a BCK-lattice but it is not a PC-lattice, since

$$(1 * a) * (b * a) = b * b = 0 \neq a = 1 * b = 1 * (b \vee a).$$

THEOREM 3.4. *Let X be a bounded BCK-algebra. Then:*

(i) *every commutative BCK-algebra is a PC-lattice.*

(ii) *every finite positive implicative BCK-algebra with condition (S) is a PC-lattice*

PROOF: (i) Let X be a commutative BCK-algebra. By Theorem 2.10 (iii), X is a BCK-lattice and by Theorem 2.12 (i), X is an involutory BCK-algebra. Hence for any $x, y, z \in X$, we have:

$$\begin{aligned} (z * x) * (y * x) &= (Nx * Nz) * (Nx * Ny), \text{ by Theorem 2.5 (ii)} \\ &= (Nx * (Nx * Ny)) * Nz, \text{ by Proposition 2.2(i)} \\ &= (Nx \wedge Ny) * Nz, \text{ by Theorem 2.10 (iii)} \\ &= NN(Nx \wedge Ny) * Nz, \text{ by Definition 2.3(v)} \\ &= z * N(Nx \wedge Ny), \text{ by Theorem 2.5 (ii)} \\ &= z * (x \vee y). \text{ by Theorem 2.10 (iii)} \end{aligned}$$

Therefore, X is a PC-lattice.

(ii) Let X be a finite positive implicative BCK-algebra with condition (S). Then by Theorem 2.10 (i), X is a BCK-lattice and so for any $x, y, z \in X$,

$$\begin{aligned} (z * x) * (y * x) &= z * (x \circ (y * x)), \text{ by Proposition 2.8 (iii)} \\ &= z * (x \circ y), \text{ by Theorem 2.9(v)} \\ &= z * (x \vee y). \text{ by Theorem 2.10 (ii)} \end{aligned}$$

Hence X is a PC-lattice. □

The following example shows that the converse of Theorem 3.4 does not hold, in general.

EXAMPLE 3.5. Let $X = \{0, a, b, 1\}$ be a chain, where $0 \leq a \leq b \leq 1$, and the operation $*$ on X is defined as follows:

$*$	0	a	b	1
0	0	0	0	0
a	a	0	0	0
b	b	a	0	0
1	1	1	1	0

It is easy to check that X is a PC-lattice, but it is not a positive implicative BCK-algebra, since $(b * a) * (a * a) = a * 0 = a \neq 0 = a * a = (b * a) * a$. Also X is not a commutative BCK-algebra, since $1 * (1 * a) = 1 * 1 = 0 \neq a = a * 0 = a * (a * 1)$.

COROLLARY 3.6. Any bounded implicative BCK-algebra (Boolean lattice) is a PC-lattice.

PROOF: By Theorems 2.6, 2.12(i) and 3.4(i), the proof is clear. □

PROPOSITION 3.7. Let X be a PC-lattice. Then for all $x, y, z \in X$:

$$(x \vee y) * z = (x * z) \vee (y * z)$$

PROOF: Suppose that X is a PC-lattice, and $x, y, z \in X$. Since $x \leq x \vee y$ and $y \leq x \vee y$, by Proposition 2.2 (ii), we have $(x * z) \leq (x \vee y) * z$ and $(y * z) \leq (x \vee y) * z$. Therefore, $(x * z) \vee (y * z) \leq (x \vee y) * z$. On the other hand, since X is a PC-lattice, by Definition 3.1 used repeatedly, we have:

$$\begin{aligned} [(x \vee y) * z] * [(x * z) \vee (y * z)] &= [((x \vee y) * z) * (x * z)] * [(y * z) * (x * z)] \\ &= ((x \vee y) * (x \vee z)) * ((y * z) * (x * z)) \\ &= ((x \vee y) * (x \vee z)) * (y * (x \vee z)) \\ &= (x \vee y) * ((x \vee z) \vee y) \\ &= 0 \end{aligned}$$

Hence, $(x \vee y) * z \leq (x * z) \vee (y * z)$. Therefore $(x \vee y) * z = (x * z) \vee (y * z)$. □

THEOREM 3.8. *Let X be a bounded BCK-algebras.*

(i) *If X is a finite PC-lattice, then X is a BCK-algebra with condition (S).*

(ii) *If X is a BCK-algebra with condition (S), then $x \vee y = x \circ (y * x)$, for $x, y \in X$.*

PROOF: (i) Suppose that $(X, *, \leq)$ is a finite PC-lattice and $x, y \in X$. Then the set $A(x, y) = \{t : t * x \leq y\}$ is a finite subset of X . Let $A(x, y) = \{t_i | i \in I\}$, where $I = \{1, 2, \dots, n\}$. Since X is a lattice, there exists $z \in X$ such that $\bigvee \{t_i : t_i * x \leq y, i \in I\} = z$. Hence, by Proposition 3.7(ii),

$$z * x = \left(\bigvee_{i \in I} t_i \right) * x = \bigvee_{i \in I} (t_i * x) \leq \bigvee y = y$$

Therefore, $z \in A(x, y)$. It is clear that for every $w \in A(x, y)$, $w \leq z$, and so z is the greatest element of the set $A(x, y)$, which is defined by $x \circ y$. Hence every finite PC-lattice is a BCK-algebra with condition (S).

(ii) Suppose that X is a PC-lattice with condition (S). Then for $x, y \in X$, the set $A(x, y * x)$ has the greatest element. Now, since

$$((x \vee y) * x) * (y * x) = (x \vee y) * (x \vee y) = 0$$

we have $(x \vee y) * x \leq y * x$. Hence $x \vee y \in A(x, y * x)$. If $t \in A(x, y * x)$, then $t * x \leq y * x$, and so $(t * x) * (y * x) = 0$. Since X is a PC-lattice, $t * (x \vee y) = (t * x) * (y * x) = 0$ and so $t \leq x \vee y$.

Hence, $x \vee y$ is the greatest element of $A(x, y * x)$, that is $x \circ (y * x) = x \vee y$. \square

The following example shows that the converse of Theorem 3.8, is not correct in general.

EXAMPLE 3.9. Let $X = \{0, a, b, 1\}$ be a chain, where $0 \leq a \leq b \leq 1$, and the operation $*$ on X is defined as follows:

$*$	0	a	b	1
0	0	0	0	0
a	a	0	0	0
b	b	a	0	0
1	1	a	a	0

\circ	0	a	b	1
0	0	a	b	1
a	a	1	1	1
b	b	1	1	1
1	1	1	1	1

Then X is a BCK-algebra with condition (S), but it is not a PC-lattice. Since

$$(1 * a) * (b * a) = a * a = 0 \neq a = 1 * b = 1 * (b \vee a)$$

Also we have $a \circ (b * a) = a \circ a = 1 \neq a \vee b = b$.

THEOREM 3.10. *Let X be a BCK-lattice. Then:*

- (i) $x * (y \wedge z) = (x * y) \vee (x * z)$, for $x, y, z \in X$,
(ii) if X is with condition (S), then for $x, y, z \in X$, $x \circ (y \wedge z) = (x \circ y) \wedge (x \circ z)$.

PROOF: (i) Suppose that X is a BCK-lattice and $x, y, z \in X$. Since $y \wedge z \leq y$ and $y \wedge z \leq z$, by Proposition 2.2 (ii), we have $x * z \leq x * (y \wedge z)$ and $x * y \leq x * (y \wedge z)$. Hence

$$(x * y) \vee (x * z) \leq x * (y \wedge z)$$

Now it remains to prove that $x * (y \wedge z) \leq (x * y) \vee (x * z)$. Since $(x * y) \leq (x * y) \vee (x * z)$, by Proposition 2.2(ii), and BCK-2, we have $x * ((x * y) \vee (x * z)) \leq x * (x * y) \leq y$, and similarly we have $x * ((x * y) \vee (x * z)) \leq x * (x * z) \leq z$. Hence $x * ((x * y) \vee (x * z)) \leq y \wedge z$. By Proposition 2.2 (iv), we conclude that

$$x * (y \wedge z) \leq (x * y) \vee (x * z)$$

Therefore

$$x * (y \wedge z) = (x * y) \vee (x * z)$$

(ii) Suppose that X is a BCK-lattice with condition (S) and $x, y, z \in X$. Since $y \wedge z \leq y$ and $y \wedge z \leq z$, by Proposition 2.8 (iv), we have $x \circ (y \wedge z) \leq (x \circ y)$ and $x \circ (y \wedge z) \leq (x \circ z)$, and so

$$x \circ (y \wedge z) \leq (x \circ y) \wedge (x \circ z)$$

Now it remains to prove that $(x \circ y) \wedge (x \circ z) \leq x \circ (y \wedge z)$. Since $(x \circ y) \wedge (x \circ z) \leq x \circ y$, by Proposition 2.8(v), we have $((x \circ y) \wedge (x \circ z)) * x \leq y$. In the same way we have $((x \circ y) \wedge (x \circ z)) * x \leq z$. Hence $((x \circ y) \wedge (x \circ z)) * x \leq y \wedge z$, and so

$$(x \circ y) \wedge (x \circ z) \leq x \circ (y \wedge z)$$

Therefore

$$(x \circ y) \wedge (x \circ z) = x \circ (y \wedge z)$$

□

THEOREM 3.11. *Let X be a PC-lattice with condition (S). Then the following are equivalent:*

- (i) X is an involutory BCK-algebra,
(ii) X is a commutative BCK-algebra.

PROOF: (i) \Rightarrow (ii) Let X be an involutory BCK-algebra and $y \leq x$. Since $Ny * Ny = 0$ and $Nx * Ny = (1 * x) * (1 * y) \leq y * x = 0$, it follows that $(Nx * Ny) \vee (Ny * Ny) = 0$. Hence:

$$\begin{aligned} (Nx \circ (x * y)) * Ny &= (Nx \circ (Ny * Nx)) * Ny, \text{ by Theorem 2.5(ii)} \\ &= (Nx \vee Ny) * Ny, \text{ by Theorem 3.8} \\ &= (Nx * Ny) \vee (Ny * Ny), \text{ by Proposition 3.7(ii)} \\ &= 0 \end{aligned}$$

Hence,

$$\begin{aligned} y * (x * (x * y)) &= y * (NNx * (x * y)) \\ &= y * ((1 * Nx) * (x * y)) \\ &= y * (1 * (Nx \circ (x * y))), \text{ by Proposition 2.8(iii)} \\ &= NNy * N(Nx \circ (x * y)) \\ &\leq (Nx \circ (x * y)) * Ny \\ &= 0 \end{aligned}$$

Therefore, $y \leq x * (x * y)$. On the other hand, by BCK-2, we have $x * (x * y) \leq y$. Consequently, $y = x * (x * y)$. Hence by Theorem 2.13, $(X, *, \leq)$ is a commutative BCK-algebra.

(ii) \Rightarrow (i). The proof holds by Theorem 2.12(i). \square

THEOREM 3.12. *Let X be a PC-lattice with condition (S). Then X is a distributive lattice.*

PROOF: Suppose that X is a PC-lattice with condition (S). It is clear that $x \vee (y \wedge z) \leq (x \vee y) \vee (x \vee z)$, for any $x, y, z \in X$. Now it remains to show that $x \vee (y \wedge z) \geq (x \vee y) \wedge (x \vee z)$. Since X satisfies the condition (S), we have;

$$\begin{aligned} x \vee (y \wedge z) = (y \wedge z) \vee x &= (y \wedge z) \circ (x * (y \wedge z)), \text{ by Theorem 3.8(ii)} \\ &= (x * (y \wedge z)) \circ (y \wedge z) \\ &= ((x * (y \wedge z)) \circ y) \wedge ((x * (y \wedge z)) \circ z), \\ &\qquad\qquad\qquad \text{by Proposition 3.7(i)} \\ &\geq ((x * y) \circ y) \wedge ((x * z) \circ z) \\ &= (x \vee y) \wedge (x \vee z), \text{ by Theorem 3.8(ii)} \end{aligned}$$

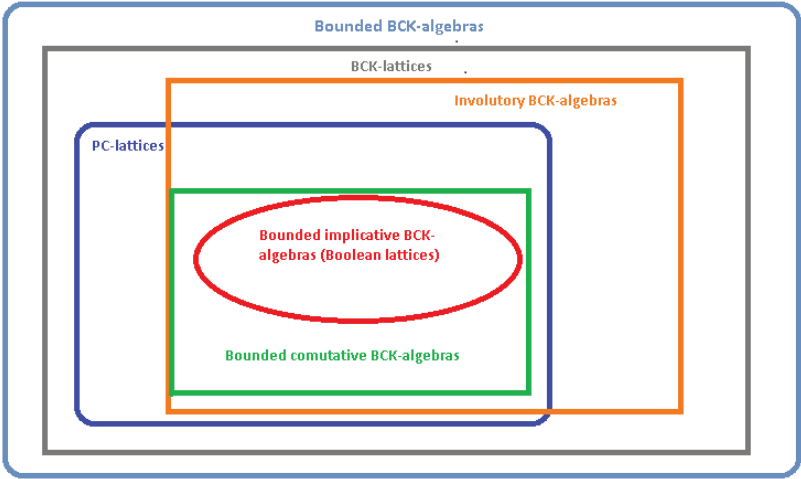
\square

Therefore, X is a distributive lattice.

4. Conclusion

It is well-known that the concept of the lattice theory has an important role in investigating the structure of a logical system. Also a Boolean lattice, that is a complemented distributive lattice, has many applications in the computer science. In order to extend the concept of BCK-algebras, we have proposed the concepts of PC-lattices. Then we have established the relationships between these lattices and other currently known lattices in bounded BCK-algebras and proved that PC-lattices are distributive.

In the following diagram we can see the relation between some kinds of BCK-lattices in bounded BCK-algebras.



Acknowledgments

The authors would like to thank the referees for their valuable suggestions and comments.

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