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A USEFUL FOUR-VALUED EXTENSION OF THE TEMPORAL LOGIC $\mathsf{K_t T4}$

Abstract

The temporal logic $\mathsf{K_t T4}$ is the modal logic obtained from the minimal temporal logic $\mathsf{K_t}$ by requiring the accessibility relation to be reflexive (which corresponds to the axiom T) and transitive (which corresponds to the axiom $\mathsf{4}$). This article aims, firstly, at providing both a model-theoretic and a proof-theoretic characterisation of a four-valued extension of the temporal logic $\mathsf{K_t T4}$ and, secondly, at identifying some of the most useful properties of this extension in the context of partial and paraconsistent logics.

Keywords: temporal logic, many-valued logic, bi-intuitionistic logic, paraconsistent logic, sequent calculus; duality, cut-redundancy.

1. Introduction

Partiality and paraconsistency are metatheoretical properties which are sometimes attributed to non-classical logics. A logic is described as partial if it does not obey the law of excluded middle and it is described as paraconsistent if it does not obey the law of non-contradiction. These laws are inherited from a long philosophical tradition and give rise to various interpretations expressible in the language of formal logic (see [18]). In this discussion, they have the following meaning: the law of excluded middle states that every formula of the form (A *or not* A) is a theorem and the law of non-contradiction states that every formula of the form (A *and not* A) is a counter-theorem. Consequently, a logic is called partial if some sequent

(whose antecedent is empty) of the form $\vdash (A\,or\,not\,A)$ is not logically correct and a logic is called paraconsistent if some sequent (whose succedent is empty) of the form $(A\,and\,not\,A) \vdash$ is not logically correct.

Even if this characterisation makes the notions of partiality and paraconsistency precise, it still has a certain opacity. Beyond this definition in terms of classes of logically correct sequents, are there model-theoretic or proof-theoretic features explaining why a logic does not admit every formula of the form $(A\,or\,not\,A)$ as a theorem or every formula of the form $(A\,and\,not\,A)$ as a counter-theorem? In other words, are there underlying model-theoretic or proof-theoretic features that would be specific to partial or paraconsistent logics?

This issue sounds even more relevant when we notice that the generic name 'partial logic', just like 'paraconsistent logic', covers a range of logics that are very different in nature. For example, both intuitionistic logic and Kleene's strong three-valued logic are partial in the sense mentioned above. Yet we know from an argument of Gödel that intuitionistic logic cannot be understood as a finitely-valued logic (see [10]). Indeed, the model-theoretic interpretations of intuitionistic logic make use of a notion of model that is usually either topological or relational and that cannot be expressed by means of a function from the set of formulae to a finite set of truth-values.

Based on this observation, the present discussion is to identify some of the logico-philosophical meanings of partiality and paraconsistency, as well as their impact on the notion of logical consequence. In other words, our intention is to clarify some of the ways in which partial and paraconsistent logics violate the law of excluded middle and the law of non-contradiction respectively.

In this connection, three many-valued logics and three constructive logics are addressed. Among the many-valued logics, we consider Kleene's strong three-valued logic, Priest's logic of paradox, and Dunn-Belnap's four-valued logic. Among the constructive logics, we investigate intuitionistic logic, dual-intuitionistic logic, and bi-intuitionistic logic.

To propose a unified understanding of these partial and paraconsistent notions of logical consequence as well as a study of the relationship between them, two steps mark out this article.

First, we define a four-valued extension of the temporal logic $\mathsf{K_t T4}$. This logic provides a general framework from which the aforementioned logics can be investigated. According to the view that model-theoretic and proof-theoretic approaches are complementary and necessary for the

complete characterisation of a logic, a relational semantics and a labelled sequent calculus are set out.

Second, we show some useful properties of this four-valued extension of $\mathsf{K_t T4}$. We start by pointing out that bi-intuitionistic logic and Dunn-Belnap's four-valued logic can be faithfully embedded into that logic. Then, different forms of the original cut rule are shown to be admissible in the sequent calculus. Finally, we argue that this extension satisfies several duality properties that provide a fresh insight into the relationship between partiality and paraconsistency.

2.   A four-valued extension of $\mathsf{K_t T4}$

The temporal logic $\mathsf{K_t T4}$ is the modal logic obtained from the minimal temporal logic $\mathsf{K_t}$ (see [17]) by requiring the accessibility relation to be reflexive (which corresponds to the axiom T) and transitive (which corresponds to the axiom $\mathsf{4}$). This section aims at providing a relational semantics and a labelled sequent calculus for an extension of $\mathsf{K_t T4}$ based on Dunn-Belnap's four-valued logic (see [1]). This many-valued modal logic is here referred to as $\mathsf{K_t^4 T4}$.

2.1.   Relational semantics

A *language* $\mathcal{L}$ is composed of a countable set of propositional symbols p_n for every $n \in \mathbb{N}$ plus the propositional logical symbols $\neg$, $\wedge$, $\vee$ and the modal logical symbols $\square_\mathsf{F}$, $\lozenge_\mathsf{F}$, $\square_\mathsf{P}$, and $\lozenge_\mathsf{P}$ (where F stands for 'future' and P stands for 'past'). The formulae of $\mathcal{L}$ are recursively defined as follows:

$$A ::= p \mid \neg A \mid (A \wedge A) \mid (A \vee A) \mid \square_\mathsf{F} A \mid \lozenge_\mathsf{F} A \mid \square_\mathsf{P} A \mid \lozenge_\mathsf{P} A$$

A *frame* $\mathcal{F}$ is a structure $\langle W, R \rangle$ in which W is a non-empty set (of possible worlds) and R is an ordered pair $\langle R_\mathsf{F}, R_\mathsf{P} \rangle$ such that R_F is a reflexive and transitive binary relation on W and R_P is the inverse relation of R_F. Note that it follows immediately from this definition that R_P is also reflexive and transitive.

A *model* $\mathcal{M}$ for a language $\mathcal{L}$ is a structure $\langle W, R, V \rangle$ such that $\langle W, R \rangle$ is a frame and V is an ordered pair $\langle V^+, V^- \rangle$ such that V^+ and V^- are mappings from natural numbers to subsets of W. Thereby $V^+(n)$ denotes the set of possible worlds that verify the proposition p_n and $V^-(n)$ denotes the set of possible worlds that falsify the proposition p_n, for every $n \in \mathbb{N}$.

The *truth* and the *falsehood* of a formula of a language $\mathcal{L}$ are defined at a world in a model. Given a world α in a model $\mathcal{M} = \langle W, R, V \rangle$, the truth (denoted by $\mathcal{M}, \alpha \vDash^+$) and the falsehood (denoted by $\mathcal{M}, \alpha \vDash^-$) of the formulae of $\mathcal{L}$ at α in $\mathcal{M}$ are defined inductively:

$\mathcal{M}, \alpha \vDash^+ p_n$ iff $\alpha \in V^+(n)$, for $n \in \mathbb{N}$
$\mathcal{M}, \alpha \vDash^- p_n$ iff $\alpha \in V^-(n)$, for $n \in \mathbb{N}$
$\mathcal{M}, \alpha \vDash^+ \neg A$ iff $\mathcal{M}, \alpha \vDash^- A$
$\mathcal{M}, \alpha \vDash^- \neg A$ iff $\mathcal{M}, \alpha \vDash^+ A$
$\mathcal{M}, \alpha \vDash^+ (A \wedge B)$ iff $\mathcal{M}, \alpha \vDash^+ A$ and $\mathcal{M}, \alpha \vDash^+ B$
$\mathcal{M}, \alpha \vDash^- (A \wedge B)$ iff $\mathcal{M}, \alpha \vDash^- A$ or $\mathcal{M}, \alpha \vDash^- B$
$\mathcal{M}, \alpha \vDash^+ (A \vee B)$ iff $\mathcal{M}, \alpha \vDash^+ A$ or $\mathcal{M}, \alpha \vDash^+ B$
$\mathcal{M}, \alpha \vDash^- (A \vee B)$ iff $\mathcal{M}, \alpha \vDash^- A$ and $\mathcal{M}, \alpha \vDash^- B$
$\mathcal{M}, \alpha \vDash^+ \Box_{\mathsf{F}} A$ iff for all w in W, $\langle \alpha, w \rangle \in R_{\mathsf{F}}$ implies $\mathcal{M}, w \vDash^+ A$
$\mathcal{M}, \alpha \vDash^- \Box_{\mathsf{F}} A$ iff for some w in W, $\langle \alpha, w \rangle \in R_{\mathsf{F}}$ and $\mathcal{M}, w \vDash^- A$
$\mathcal{M}, \alpha \vDash^+ \Diamond_{\mathsf{F}} A$ iff for some w in W, $\langle \alpha, w \rangle \in R_{\mathsf{F}}$ and $\mathcal{M}, w \vDash^+ A$
$\mathcal{M}, \alpha \vDash^- \Diamond_{\mathsf{F}} A$ iff for all w in W, $\langle \alpha, w \rangle \in R_{\mathsf{F}}$ implies $\mathcal{M}, w \vDash^- A$
$\mathcal{M}, \alpha \vDash^+ \Box_{\mathsf{P}} A$ iff for all w in W, $\langle \alpha, w \rangle \in R_{\mathsf{P}}$ implies $\mathcal{M}, w \vDash^+ A$
$\mathcal{M}, \alpha \vDash^- \Box_{\mathsf{P}} A$ iff for some w in W, $\langle \alpha, w \rangle \in R_{\mathsf{P}}$ and $\mathcal{M}, w \vDash^- A$
$\mathcal{M}, \alpha \vDash^+ \Diamond_{\mathsf{P}} A$ iff for some w in W, $\langle \alpha, w \rangle \in R_{\mathsf{P}}$ and $\mathcal{M}, w \vDash^+ A$
$\mathcal{M}, \alpha \vDash^- \Diamond_{\mathsf{P}} A$ iff for all w in W, $\langle \alpha, w \rangle \in R_{\mathsf{P}}$ implies $\mathcal{M}, w \vDash^- A$

Several semantic approaches can be specified according to the class of models considered. Two of these approaches seem particularly relevant for our purposes, namely the gappy semantics and the glutty semantics. These two types of semantics can be distinguished by defining some conditions on the models.

Let $\mathcal{M}$ be a model such that $\mathcal{M} = \langle W, R, V \rangle$. Then, $\mathcal{M}$ is *consistent* if $V^+(n) \cap V^-(n) = \emptyset$, for every $n \in \mathbb{N}$ and $\mathcal{M}$ is *complete* if $V^+(n) \cup V^-(n) = W$, for every $n \in \mathbb{N}$. In this sense, the model $\mathcal{M}$ is called *classical* if it is both consistent and complete.

Depending on whether a semantics restricts the class of models to that of consistent or complete models, this semantics will be called *gappy* or *glutty*, respectively. The reason why we call these semantics gappy or glutty lies in the fact that they do not obey the 'metalinguistic' law of excluded middle (stating that any sentence of the object-language has at least one of the values true and false) or the 'metalinguistic' law of non-contradiction (stating that any sentence of the object-language has at most one of the

values true and false), respectively (see [8]). By induction on the complexity of formulae, we obtain:

PROPOSITION 1 (meta-law of excluded middle). *Let $\mathcal{M}$ be a complete model for a language $\mathcal{L}$. Then, for all formulae A of $\mathcal{L}$ and for all worlds w in $\mathcal{M}$, $\mathcal{M}, w \vDash^+ A$ or $\mathcal{M}, w \vDash^- A$.*

PROPOSITION 2 (meta-law of non-contradiction). *Let $\mathcal{M}$ be a consistent model for a language $\mathcal{L}$. Then, for all formulae A of $\mathcal{L}$ and for all worlds w in $\mathcal{M}$, $\mathcal{M}, w \nvDash^+ A$ or $\mathcal{M}, w \nvDash^- A$.*

2.2. Labelled sequent calculi

The labelled sequent calculi described hereafter are based on an internalisation of the relational semantics of $\mathsf{K_t T4}$ into a four-sided sequent calculus closely related to those developed by J.-Y. Girard (see [9]), R. Muskens (see [13]), and A. Bochman (see [2]). Similar approaches in the context of two-sided sequent calculi have been discussed, among others, by N. Bonnette and R. Goré (see [3]) as well as S. Negri (see [14]).

A *sequent* Λ is a finite set of labelled formulae and structural elements. A *labelled formula* is a triple $\langle A, \lambda, x \rangle$ such that A is a formula, $\lambda \in \{\pi, \gamma, \delta, \sigma\}$, and x is a natural number. A *structural element* is an ordered pair $\langle x, y \rangle$ where x and y are natural numbers. If Λ_1 and Λ_2 are sequents and l is a labelled formula or a structural element, the sequents $\Lambda_1 \cup \Lambda_2$ and $\{l\}$ are respectively denoted by Λ_1, Λ_2 and l. The *antecedent* (respectively, *succedent*) of a sequent Λ is the set of labelled formulae $\langle A, \lambda, x \rangle$ in Λ such that $\lambda \in \{\pi, \gamma\}$ (respectively, $\lambda \in \{\delta, \sigma\}$).

A sequent Λ is *valid* if there is no counter-model to Λ. A model $\mathcal{M} = \langle W, R, V \rangle$ is a *counter-model* to Λ if there is a function $f : \mathbb{N} \to W$ such that:

- for every labelled formula $\langle A, \lambda, x \rangle$ in Λ:
  - $\ast$ $\mathcal{M}, f(x) \nvDash^- A$ if $\lambda = \pi$
  - $\ast$ $\mathcal{M}, f(x) \vDash^+ A$ if $\lambda = \gamma$
  - $\ast$ $\mathcal{M}, f(x) \nvDash^+ A$ if $\lambda = \delta$
  - $\ast$ $\mathcal{M}, f(x) \vDash^- A$ if $\lambda = \sigma$
- for every structural element $\langle x, y \rangle$ in Λ, $\langle f(x), f(y) \rangle \in R_{\mathsf{F}}$.

This definition of validity can be preserved for the gappy and the glutty semantics. Depending on whether the notion of valid sequent is restricted

to consistent models or to complete models, a sequent is called *gap-valid* or *glut-valid*, respectively. If only the class of classical models is taken into account, then a sequent is called *classic-valid*.

To define labelled sequent calculi which are sound and complete with respect to these model-theoretic notions, some rules of inference are to be set out. It is worth noting that these calculi are free of weakening and contraction structural rules. We could have also defined sequents as multisets and shown that these rules are admissible. Instead, we preferred to start with sequents as sets and avoid this exercise.

$$\frac{\Lambda, \langle A, \sigma, x\rangle}{\Lambda, \langle \neg A, \gamma, x\rangle} \, \gamma\neg \qquad\qquad \frac{\Lambda, \langle A, \pi, x\rangle}{\Lambda, \langle \neg A, \delta, x\rangle} \, \delta\neg$$

$$\frac{\Lambda, \langle A, \delta, x\rangle}{\Lambda, \langle \neg A, \pi, x\rangle} \, \pi\neg \qquad\qquad \frac{\Lambda, \langle A, \gamma, x\rangle}{\Lambda, \langle \neg A, \sigma, x\rangle} \, \sigma\neg$$

$$\frac{\Lambda, \langle A, \gamma, x\rangle, \langle B, \gamma, x\rangle}{\Lambda, \langle (A \wedge B), \gamma, x\rangle} \, \gamma\wedge \qquad\qquad \frac{\Lambda, \langle A, \delta, x\rangle \qquad \Lambda, \langle B, \delta, x\rangle}{\Lambda, \langle (A \wedge B), \delta, x\rangle} \, \delta\wedge$$

$$\frac{\Lambda, \langle A, \pi, x\rangle, \langle B, \pi, x\rangle}{\Lambda, \langle (A \wedge B), \pi, x\rangle} \, \pi\wedge \qquad\qquad \frac{\Lambda, \langle A, \sigma, x\rangle \qquad \Lambda, \langle B, \sigma, x\rangle}{\Lambda, \langle (A \wedge B), \sigma, x\rangle} \, \sigma\wedge$$

$$\frac{\Lambda, \langle A, \gamma, x\rangle \qquad \Lambda, \langle B, \gamma, x\rangle}{\Lambda, \langle (A \vee B), \gamma, x\rangle} \, \gamma\vee \qquad\qquad \frac{\Lambda, \langle A, \delta, x\rangle, \langle B, \delta, x\rangle}{\Lambda, \langle (A \vee B), \delta, x\rangle} \, \delta\vee$$

$$\frac{\Lambda, \langle A, \pi, x\rangle \qquad \Lambda, \langle B, \pi, x\rangle}{\Lambda, \langle (A \vee B), \pi, x\rangle} \, \pi\vee \qquad\qquad \frac{\Lambda, \langle A, \sigma, x\rangle, \langle B, \sigma, x\rangle}{\Lambda, \langle (A \vee B), \sigma, x\rangle} \, \sigma\vee$$

$$\frac{\Lambda, \langle \Box_{\mathsf{F}} A, \gamma, x\rangle, \langle x, y\rangle, \langle A, \gamma, y\rangle}{\Lambda, \langle \Box_{\mathsf{F}} A, \gamma, x\rangle, \langle x, y\rangle} \, \gamma\,\Box_{\mathsf{F}} \qquad\qquad \frac{\Lambda, \langle A, \delta, n\rangle, \langle x, n\rangle}{\Lambda, \langle \Box_{\mathsf{F}} A, \delta, x\rangle} \, \delta\,\Box_{\mathsf{F}}$$

$$\frac{\Lambda, \langle \Box_{\mathsf{F}} A, \pi, x\rangle, \langle x, y\rangle, \langle A, \pi, y\rangle}{\Lambda, \langle \Box_{\mathsf{F}} A, \pi, x\rangle, \langle x, y\rangle} \, \pi\,\Box_{\mathsf{F}} \qquad\qquad \frac{\Lambda, \langle A, \sigma, n\rangle, \langle x, n\rangle}{\Lambda, \langle \Box_{\mathsf{F}} A, \sigma, x\rangle} \, \sigma\,\Box_{\mathsf{F}}$$

$$\frac{\Lambda, \langle A, \gamma, n\rangle, \langle x, n\rangle}{\Lambda, \langle \Diamond_{\mathsf{F}} A, \gamma, x\rangle} \, \gamma\,\Diamond_{\mathsf{F}} \qquad\qquad \frac{\Lambda, \langle \Diamond_{\mathsf{F}} A, \delta, x\rangle, \langle x, y\rangle, \langle A, \delta, y\rangle}{\Lambda, \langle \Diamond_{\mathsf{F}} A, \delta, x\rangle, \langle x, y\rangle} \, \delta\,\Diamond_{\mathsf{F}}$$

$$\frac{\Lambda, \langle A, \pi, n\rangle, \langle x, n\rangle}{\Lambda, \langle \Diamond_{\mathsf{F}} A, \pi, x\rangle} \, \pi\,\Diamond_{\mathsf{F}} \qquad\qquad \frac{\Lambda, \langle \Diamond_{\mathsf{F}} A, \sigma, x\rangle, \langle x, y\rangle, \langle A, \sigma, y\rangle}{\Lambda, \langle \Diamond_{\mathsf{F}} A, \sigma, x\rangle, \langle x, y\rangle} \, \sigma\,\Diamond_{\mathsf{F}}$$

$$\frac{\Lambda, \langle \Box_\mathsf{P} A, \gamma, x \rangle, \langle y, x \rangle, \langle A, \gamma, y \rangle}{\Lambda, \langle \Box_\mathsf{P} A, \gamma, x \rangle, \langle y, x \rangle} \; \gamma \, \Box_\mathsf{P} \qquad \frac{\Lambda, \langle A, \delta, n \rangle, \langle n, x \rangle}{\Lambda, \langle \Box_\mathsf{P} A, \delta, x \rangle} \; \delta \, \Box_\mathsf{P}$$

$$\frac{\Lambda, \langle \Box_\mathsf{P} A, \pi, x \rangle, \langle y, x \rangle, \langle A, \pi, y \rangle}{\Lambda, \langle \Box_\mathsf{P} A, \pi, x \rangle, \langle y, x \rangle} \; \pi \, \Box_\mathsf{P} \qquad \frac{\Lambda, \langle A, \sigma, n \rangle, \langle n, x \rangle}{\Lambda, \langle \Box_\mathsf{P} A, \sigma, x \rangle} \; \sigma \, \Box_\mathsf{P}$$

$$\frac{\Lambda, \langle A, \gamma, n \rangle, \langle n, x \rangle}{\Lambda, \langle \Diamond_\mathsf{P} A, \gamma, x \rangle} \; \gamma \, \Diamond_\mathsf{P} \qquad \frac{\Lambda, \langle \Diamond_\mathsf{P} A, \delta, x \rangle, \langle y, x \rangle, \langle A, \delta, y \rangle}{\Lambda, \langle \Diamond_\mathsf{P} A, \delta, x \rangle, \langle y, x \rangle} \; \delta \, \Diamond_\mathsf{P}$$

$$\frac{\Lambda, \langle A, \pi, n \rangle, \langle n, x \rangle}{\Lambda, \langle \Diamond_\mathsf{P} A, \pi, x \rangle} \; \pi \, \Diamond_\mathsf{P} \qquad \frac{\Lambda, \langle \Diamond_\mathsf{P} A, \sigma, x \rangle, \langle y, x \rangle, \langle A, \sigma, y \rangle}{\Lambda, \langle \Diamond_\mathsf{P} A, \sigma, x \rangle, \langle y, x \rangle} \; \sigma \, \Diamond_\mathsf{P}$$

$$\frac{\Lambda, \langle x, x \rangle}{\Lambda} \; T$$

$$\frac{\Lambda, \langle x, y \rangle, \langle y, z \rangle, \langle x, z \rangle}{\Lambda, \langle x, y \rangle, \langle y, z \rangle} \; 4$$

Remark. The natural number n must not appear in the conclusion of the rules: $\delta \, \Box_\mathsf{F}$, $\sigma \, \Box_\mathsf{F}$, $\gamma \, \Diamond_\mathsf{F}$, $\pi \, \Diamond_\mathsf{F}$, $\delta \, \Box_\mathsf{P}$, $\sigma \, \Box_\mathsf{P}$, $\gamma \, \Diamond_\mathsf{P}$, and $\pi \, \Diamond_\mathsf{P}$.

The notion of *derivation* as well as those of *initial sequent* and *end-sequent* are defined inductively in the usual way. Roughly speaking, a derivation is a finite rooted tree in which the nodes are sequents. The root of the tree (at the bottom) is called the endsequent and the leaves of the tree (at the top) are called initial sequents. The *length* of a derivation is the number of sequents in that derivation.

Starting with the single set of rules of inference set out above, four notions of derivability are distinguished so that they differ only in the definition of axiomatic sequent. A sequent is *derivable*, *gap-derivable*, *glut-derivable*, or *classic-derivable* if there exists a derivation in which it is the endsequent and all initial sequents are respectively axiomatic, gap-axiomatic, glut-axiomatic, or classic-axiomatic.

Let Λ be a sequent. Then:

- Λ is *axiomatic* if there exists an atomic formula p and a natural number x such that either $\langle p, \gamma, x \rangle$ and $\langle p, \delta, x \rangle$ belong to Λ or $\langle p, \pi, x \rangle$ and $\langle p, \sigma, x \rangle$ belong to Λ.
- Λ is *gap-axiomatic* if it is axiomatic or there exists an atomic formula p and a natural number x such that $\langle p, \gamma, x \rangle$ and $\langle p, \sigma, x \rangle$ belong to Λ.

- Λ is *glut-axiomatic* if it is axiomatic or there exists an atomic formula p and a natural number x such that $\langle p, \pi, x \rangle$ and $\langle p, \delta, x \rangle$ belong to Λ.
- Λ is *classic-axiomatic* if it is gap-axiomatic or glut-axiomatic.

The general sequent calculus is sound and complete with respect to the relational semantics. Moreover, these properties also hold for the gappy, glutty, and classical notions of validity and derivability.

THEOREM 1 (soundness and completeness). *Let Λ be a sequent.*

1. Λ *is valid if and only if Λ is derivable.*
2. Λ *is gap-valid if and only if Λ is gap-derivable.*
3. Λ *is glut-valid if and only if Λ is glut-derivable.*
4. Λ *is classic-valid if and only if Λ is classic-derivable.*

PROOF: The techniques for proving the soundness and completeness of such systems are well known. Also we refer the reader to [15] for a detailed proof of Theorem 1. Although the systems covered in the article do not involve past modalities, the proofs provided can be extended without difficulty to these two additional cases. □

3.   Some useful properties

This section is devoted to showing some properties of the four-valued extension of $\mathsf{K_t T4}$ described above. The most important and interesting results are related to duality, cut-redundancy, and embeddings of partial and paraconsistent logics.

3.1.   Embedding

Several well-known partial and paraconsistent logics can be faithfully embedded into $\mathsf{K_t^4 T4}$. In this connection, three many-valued logics and three constructive logics are addressed. Among the many-valued logics, we consider Kleene's strong three-valued logic ($\mathsf{K_3}$), Priest's logic of paradox (LP), and Dunn-Belnap's four-valued logic ($\mathsf{L_4}$). As far as the constructive logics are concerned, intuitionistic logic (H), dual-intuitionistic logic (B), and bi-intuitionistic logic (HB) are investigated (see [16]).

In order to state these embeddings precisely, some definitions are needed. A *Gentzen sequent* for a language $\mathcal{L}$ is an ordered pair $\langle \Gamma, \Delta \rangle$, where Γ and Δ are finite sequences of formulae of $\mathcal{L}$. The Gentzen sequent $\langle \Gamma, \Delta \rangle$ is

denoted $\Gamma \vdash \Delta$ and is said to be L-valid if it is logically correct with respect to the logic L. Moreover, if Σ is a sequence of formulae $A_1, \ldots, A_n$, then $\langle \Sigma, \lambda, x \rangle$ denotes the sequent $\{\langle A_i, \lambda, x \rangle | 1 \leq i \leq n\}$.

3.1.1 Some well-known many-valued logics

The language of the many-valued logics with which we are concerned is the language of $\mathsf{K_t^4 T4}$ without modal symbols. This language, here referred to as $\mathcal{L}(\mathsf{CL})$, is actually the language of classical propositional logic (CL). It is to be noted that such a language usually includes an additional logical symbol interpreted as material implication. In this context, this symbol is denoted $\rightarrow$ and a formula of the form $(A \rightarrow B)$ is regarded as an abbreviation of $(\neg A \lor B)$.

PROPOSITION 3. *Let* $\Gamma \vdash \Delta$ *be a Gentzen sequent for* $\mathcal{L}(\mathsf{CL})$.

1. $\Gamma \vdash \Delta$ *is* $\mathsf{L_4}$*-valid if and only if* $\langle \Gamma, \gamma, x \rangle, \langle \Delta, \delta, x \rangle$ *is derivable.*
2. $\Gamma \vdash \Delta$ *is* $\mathsf{K_3}$*-valid if and only if* $\langle \Gamma, \gamma, x \rangle, \langle \Delta, \delta, x \rangle$ *is gap-derivable.*
3. $\Gamma \vdash \Delta$ *is* LP*-valid if and only if* $\langle \Gamma, \gamma, x \rangle, \langle \Delta, \delta, x \rangle$ *is glut-derivable.*
4. $\Gamma \vdash \Delta$ *is* CL*-valid if and only if* $\langle \Gamma, \gamma, x \rangle, \langle \Delta, \delta, x \rangle$ *is classic-derivable.*

PROOF: This results from Theorem 1 and the fact that $\Gamma \vdash \Delta$ is respectively $\mathsf{L_4}$-valid, $\mathsf{K_3}$-valid, LP-valid, and CL-valid if and only if $\langle \Gamma, \gamma, x \rangle$, $\langle \Delta, \delta, x \rangle$ is valid, gap-valid, glut-valid, and classic-valid. For each of the many-valued logics discussed, we need to specify the notions of model, truth, falsehood, and validity. It then remains to establish by induction on the complexity of formulae that there is a counter-model, a consistent counter-model, a complete counter-model, and a classical counter-model to $\langle \Gamma, \gamma, x \rangle, \langle \Delta, \delta, x \rangle$ if and only if there is a counter-model to $\Gamma \vdash \Delta$ for $\mathsf{L_4}$, $\mathsf{K_3}$, LP, and CL, respectively. □

3.1.2 Some well-known constructive logics

Intuitionistic, dual-intuitionistic, and bi-intuitionistic propositional logics each have a different language. The language of intuitionistic logic, denoted by $\mathcal{L}(\mathsf{H})$, has $\sim$, $\cap$, $\cup$ and $\supset$ as logical symbols. The language of dual-intuitionistic logic, denoted by $\mathcal{L}(\mathsf{B})$, has $-$, $\cap$, $\cup$ and $\leftharpoonup$ as logical symbols. Finally, bi-intuitionistic logic involves all of these logical symbols and its language is denoted by $\mathcal{L}(\mathsf{HB})$. As for the syntax of these three languages, the formulae are defined inductively in the usual way.

Let us define a translation function τ (see [12]) from the set of formulae of $\mathcal{L}(\mathsf{HB})$ to the set of formulae of $\mathcal{L}$ such that:

$$
\begin{array}{lcl}
\tau[p] & = & \Box_{\mathsf{F}}\, p \text{ or } \Diamond_{\mathsf{P}}\, p \\
\tau[\sim A] & = & \Box_{\mathsf{F}}\, \neg\tau[A] \\
\tau[-A] & = & \Diamond_{\mathsf{P}}\, \neg\tau[A] \\
\tau[(A \cap B)] & = & (\tau[A] \wedge \tau[B]) \\
\tau[(A \cup B)] & = & (\tau[A] \vee \tau[B]) \\
\tau[(A \supset B)] & = & \Box_{\mathsf{F}}(\neg\tau[A] \vee \tau[B]) \\
\tau[(A \leftharpoondown B)] & = & \Diamond_{\mathsf{P}}(\neg\tau[B] \wedge \tau[A])
\end{array}
$$

To simplify the notation, we adopt the convention that if Σ is a sequence of formulae $A_1, \ldots, A_n$, then $\tau[\Sigma]$ denotes the sequence $\tau[A_1], \ldots, \tau[A_n]$.

PROPOSITION 4. *Let $\Gamma \vdash \Delta$ be a Gentzen sequent for $\mathcal{L}(\mathsf{H})$. Then, $\Gamma \vdash \Delta$ is H-valid if and only if $\langle\tau[\Gamma], \gamma, x\rangle, \langle\tau[\Delta], \delta, x\rangle$ is classic-derivable.*

PROPOSITION 5. *Let $\Gamma \vdash \Delta$ be a Gentzen sequent for $\mathcal{L}(\mathsf{B})$. Then, $\Gamma \vdash \Delta$ is B-valid if and only if $\langle\tau[\Gamma], \gamma, x\rangle, \langle\tau[\Delta], \delta, x\rangle$ is classic-derivable.*

PROPOSITION 6. *Let $\Gamma \vdash \Delta$ be a Gentzen sequent for $\mathcal{L}(\mathsf{HB})$. Then, $\Gamma \vdash \Delta$ is HB-valid if and only if $\langle\tau[\Gamma], \gamma, x\rangle, \langle\tau[\Delta], \delta, x\rangle$ is classic-derivable.*

PROOF: As Propositions 4–5 are special cases of Proposition 6, we only sketch the proof of the latter. By Theorem 1, it suffices to show that $\Gamma \vdash \Delta$ is HB-valid if and only if $\langle\tau[\Gamma], \gamma, x\rangle, \langle\tau[\Delta], \delta, x\rangle$ is classic-valid.

A bi-intuitionistic model for a language $\mathcal{L}(\mathsf{HB})$ is defined as a classical model $\mathcal{M} = \langle W, R, V \rangle$ which satisfies the following persistence condition: for all α and β in W, if $\langle\alpha, \beta\rangle \in R_{\mathsf{F}}$, then $\alpha \in V^+(n)$ implies $\beta \in V^+(n)$ (for every $n \in \mathbb{N}$). The bi-intuitionistic truth (denoted by $\mathcal{M}, \alpha \vDash$) and falsehood (denoted by $\mathcal{M}, \alpha \nvDash$) of the formulae of $\mathcal{L}(\mathsf{HB})$ as well as the HB-validity are defined as usual (see [11]).

To prove that $\Gamma \vdash \Delta$ is not HB-valid if and only if $\langle\tau[\Gamma], \gamma, x\rangle, \langle\tau[\Delta], \delta, x\rangle$ is not classic-valid, we establish that, for all classical models $\mathcal{M}$ and for all formulae A of $\mathcal{L}(\mathsf{HB})$, $\mathcal{M}, \alpha \vDash^+ \tau[A]$ if and only if $\mathcal{M}$ satisfies the persistence condition and $\mathcal{M}, \alpha \vDash A$. This is done by induction on the complexity of formulae. $\square$

3.2. Cut-redundancy

Several formulations of the redundancy of cut are possible in the sequent calculi mentioned in Section 2.2. According to the label of the cut formula, four different forms of the original cut rule are distinguished (see [6]).

$$\frac{\Lambda, \langle A, \delta, x \rangle \qquad \Lambda, \langle A, \gamma, x \rangle}{\Lambda} \, cut_{\delta - \gamma}$$

$$\frac{\Lambda, \langle A, \sigma, x \rangle \qquad \Lambda, \langle A, \pi, x \rangle}{\Lambda} \, cut_{\sigma - \pi}$$

$$\frac{\Lambda, \langle A, \delta, x \rangle \qquad \Lambda, \langle A, \pi, x \rangle}{\Lambda} \, cut_{\delta - \pi}$$

$$\frac{\Lambda, \langle A, \sigma, x \rangle \qquad \Lambda, \langle A, \gamma, x \rangle}{\Lambda} \, cut_{\sigma - \gamma}$$

We prove that only the first two rules are admissible in the general sequent calculus while all of them are redundant in the classical sequent calculus. As for the gappy and the glutty sequent calculi, we prove that they only admit one form of cut in addition to the two admissible in the general sequent calculus.

THEOREM 2. *Let A be a formula of a language* $\mathcal{L}$.

1. *If* $\Lambda, \langle A, \delta, x \rangle$ *and* $\Lambda, \langle A, \gamma, x \rangle$ *are derivable, then* Λ *is derivable.*
2. *If* $\Lambda, \langle A, \sigma, x \rangle$ *and* $\Lambda, \langle A, \pi, x \rangle$ *are derivable, then* Λ *is derivable.*

PROOF: A semantic proof of the first assertion is here given. The second assertion is obtained by a symmetric treatment. We show that if Λ is not derivable and $\Lambda, \langle A, \delta, x \rangle$ is derivable, then $\Lambda, \langle A, \gamma, x \rangle$ is not derivable.

Suppose that Λ is not derivable and $\Lambda, \langle A, \delta, x \rangle$ is derivable. Then, by Theorem 1, there are a model $\mathcal{M} = \langle W, R, V \rangle$ and a function $f : \mathbb{N} \to W$ such that, for every structural element $\langle x, y \rangle$ in Λ, $\langle f(x), f(y) \rangle \in R_\mathsf{F}$ and such that, for every labelled formula $\langle B, \lambda, x \rangle$ in Λ, $\mathcal{M}, f(x) \nvDash^- B$ if $\lambda = \pi$, $\mathcal{M}, f(x) \vDash^+ B$ if $\lambda = \gamma$, $\mathcal{M}, f(x) \nvDash^+ B$ if $\lambda = \delta$, and $\mathcal{M}, f(x) \vDash^- B$ if $\lambda = \sigma$. In addition, since $\Lambda, \langle A, \delta, x \rangle$ is valid, it follows that $\mathcal{M}, f(x) \vDash^+ A$. In other words, there is a counter-model to $\Lambda, \langle A, \gamma, x \rangle$. Therefore, by Theorem 1, $\Lambda, \langle A, \gamma, x \rangle$ is not derivable. □

THEOREM 3. *Let A be a formula of a language* $\mathcal{L}$.

1. *If* $\Lambda, \langle A,\delta,x \rangle$ *and* $\Lambda, \langle A,\pi,x \rangle$ *are gap-derivable, then* Λ *is gap-derivable.*
2. *If* $\Lambda, \langle A,\sigma,x \rangle$ *and* $\Lambda, \langle A,\gamma,x \rangle$ *are glut-derivable, then* Λ *is glut-derivable.*

PROOF: The proof proceeds in the same way as in Theorem 2. For the first assertion, it suffices to note that, by Proposition 2, every consistent counter-model to Λ such that $\mathcal{M}, f(x) \vDash^+ A$ is a counter-model to $\Lambda, \langle A, \pi, x \rangle$. Symmetrically, for the second assertion, we only need to mention that, by Proposition 1, every complete counter-model to Λ such that $\mathcal{M}, f(x) \nvDash^- A$ is a counter-model to $\Lambda, \langle A, \gamma, x \rangle$.   □

In view of Theorems 2 and 3, it is interesting to note that the $cut_{\delta-\gamma}$ and $cut_{\sigma-\pi}$ rules preserve derivability, gap-derivability, glut-derivability, and classic-derivability. By contrast, the $cut_{\delta-\pi}$ and $cut_{\sigma-\gamma}$ rules do not preserve derivability. In addition, the $cut_{\sigma-\gamma}$ rule does not preserve gap-derivability and the $cut_{\delta-\pi}$ rule does not preserve glut-derivability. In other words, the general sequent calculus admits only $cut_{\delta-\gamma}$ and $cut_{\sigma-\pi}$, while the gappy and the glutty sequent calculi admit, in addition to these rules, the $cut_{\delta-\pi}$ and $cut_{\sigma-\gamma}$ rules, respectively. As for the classical sequent calculus, it admits the four cut rules.

3.3. Duality

The four-valued modal logic $\mathsf{K}_t^4\mathsf{T}4$ satisfies many duality properties. Three of them are pointed out in this section: the first relies on a *temporal symmetry* between future and past modalities (see [5]); the second is based on an *alethic symmetry* between truth and non-falsehood on the one hand and falsehood and non-truth on the other hand (see [7]); the third relates to an *inferential symmetry* between antecedent and succedent of sequents (see [4]). These three duality properties are primitive and can be freely combined to form more complex notions of duality.

3.3.1 Temporal duality

Temporal duality consists in switching the future and past modalities while reversing the temporal order. Proposition 7 shows that the derivability, gap-derivability, glut-derivability, and classic-derivability of sequents are not sensitive to this temporal inversion. This is due to the fact that the order relation with respect to the past is the inverse of the order relation with respect to the future. In this sense, the future and past structures are the mirror of each other.

Let Λ be a sequent. Then, the *temporal dual* of Λ, denoted by $[\Lambda]^t$, is the set $\{\langle [A]^t, \lambda, x\rangle | \langle A, \lambda, x\rangle \in \Lambda\} \cup \{\langle x, y\rangle | \langle y, x\rangle \in \Lambda\}$ where $[A]^t$ is the formula which results from A by applying the following recursive function:

$$
\begin{aligned}
[p]^t &= p & [\Box_{\mathsf{F}} A]^t &= \Box_{\mathsf{P}}[A]^t \\
[\neg A]^t &= \neg [A]^t & [\Diamond_{\mathsf{F}} A]^t &= \Diamond_{\mathsf{P}}[A]^t \\
[(A \wedge B)]^t &= ([A]^t \wedge [B]^t) & [\Box_{\mathsf{P}} A]^t &= \Box_{\mathsf{F}}[A]^t \\
[(A \vee B)]^t &= ([A]^t \vee [B]^t) & [\Diamond_{\mathsf{P}} A]^t &= \Diamond_{\mathsf{F}}[A]^t
\end{aligned}
$$

PROPOSITION 7. *Let Λ be a sequent.*

1. Λ *is derivable if and only if* $[\Lambda]^t$ *is derivable.*
2. Λ *is gap-derivable if and only if* $[\Lambda]^t$ *is gap-derivable.*
3. Λ *is glut-derivable if and only if* $[\Lambda]^t$ *is glut-derivable.*
4. Λ *is classic-derivable if and only if* $[\Lambda]^t$ *is classic-derivable.*

PROOF: By induction on the length of derivations. $\Box$

3.3.2 Alethic duality

Alethic duality inverts truth and non-falsehood on the one hand and falsehood and non-truth on the other hand. This highlights the symmetry between the notions of logical consequence conceived as truth preservation from antecedent to succedent and as falsehood preservation from succedent to antecedent. Proposition 8 states that the properties of derivability and classic-derivability are indifferent to this choice while the properties of gap-derivability and glut-derivability are exchanged according to whether we are dealing with a sequent or its alethic dual.

Let Λ be a sequent. Then, the *alethic dual* of Λ, denoted by $[\Lambda]^a$, is the set $\{\langle A, \overline{\lambda}, x\rangle | \langle A, \lambda, x\rangle \in \Lambda\} \cup \{\langle x, y\rangle | \langle x, y\rangle \in \Lambda\}$ where $\overline{\lambda}$ is defined as follows:

$$
\overline{\lambda} = \begin{cases}
\pi & \text{if } \lambda = \gamma \\
\gamma & \text{if } \lambda = \pi \\
\delta & \text{if } \lambda = \sigma \\
\sigma & \text{if } \lambda = \delta
\end{cases}
$$

PROPOSITION 8. *Let Λ be a sequent.*

1. Λ *is derivable if and only if* $[\Lambda]^a$ *is derivable.*
2. Λ *is gap-derivable if and only if* $[\Lambda]^a$ *is glut-derivable.*

3. Λ *is glut-derivable if and only if* $[\Lambda]^a$ *is gap-derivable.*
4. Λ *is classic-derivable if and only if* $[\Lambda]^a$ *is classic-derivable.*

PROOF: By induction on the length of derivations.   □

3.3.3 Inferential duality

Inferential duality emphasizes the symmetry between antecedent and succe-dent of sequents as well as between some logical connectives. This sym-metry is particularly striking in light of the rules of inference governing the behavior of the logical connectives. Indeed, there is a one-to-one corre-spondence between the rules which introduce a connective in the π-side or the γ-side of a sequent and the rules which introduce a connective in the σ-side or the δ-side, respectively. For example, the following pairs of rules can be identified: $\gamma\neg$ and $\delta\neg$, $\sigma\wedge$ and $\pi\vee$, $\gamma\,\Box_\mathsf{F}$ and $\delta\,\Diamond_\mathsf{F}$. This property of duality is closely related to other properties such as the law of double negation, De Morgan laws, and the interdefinability of the necessity and possibility modal connectives.

Let Λ be a sequent. Then, the *inferential dual* of Λ, denoted by $[\Lambda]^i$, is the set $\{\langle [A]^i, \overline{\lambda}, x\rangle | \langle A, \lambda, x\rangle \in \Lambda\} \cup \{\langle x, y\rangle | \langle x, y\rangle \in \Lambda\}$ where $[A]^i$ is the formula obtained from A as follows:

$$
\begin{array}{llll@{\qquad}llll}
[p]^i & = & p & & [\Box_\mathsf{F}\,A]^i & = & \Diamond_\mathsf{F}[A]^i \\
[\neg A]^i & = & \neg [A]^i & & [\Diamond_\mathsf{F}\,A]^i & = & \Box_\mathsf{F}[A]^i \\
[(A \wedge B)]^i & = & ([A]^i \vee [B]^i) & & [\Box_\mathsf{P}\,A]^i & = & \Diamond_\mathsf{P}[A]^i \\
[(A \vee B)]^i & = & ([A]^i \wedge [B]^i) & & [\Diamond_\mathsf{P}\,A]^i & = & \Box_\mathsf{P}[A]^i
\end{array}
$$

and where $\overline{\lambda}$ is defined as follows:

$$
\overline{\lambda} = \left\{
\begin{array}{ll}
\pi & \text{if } \lambda = \sigma \\
\gamma & \text{if } \lambda = \delta \\
\delta & \text{if } \lambda = \gamma \\
\sigma & \text{if } \lambda = \pi
\end{array}
\right.
$$

PROPOSITION 9. *Let* Λ *be a sequent.*

1. Λ *is derivable if and only if* $[\Lambda]^i$ *is derivable.*
2. Λ *is gap-derivable if and only if* $[\Lambda]^i$ *is glut-derivable.*
3. Λ *is glut-derivable if and only if* $[\Lambda]^i$ *is gap-derivable.*
4. Λ *is classic-derivable if and only if* $[\Lambda]^i$ *is classic-derivable.*

PROOF: By induction on the length of derivations.   □

4. Concluding remarks

In the light of our study of the four-valued modal logic $\mathsf{K_t^4 T4}$, it appears that there are at least two different ways for a partial logic or a paraconsistent logic to infringe the law of excluded middle or the law of non-contradiction, respectively.

In this connection, the embeddings previously described are particularly illuminating. While the partiality and the paraconsistency of the constructive logics discussed involve an 'intensional' interpretation of negation and a classical notion of model, those of the many-valued logics considered relies on an 'extensional' interpretation of negation and a broader notion of model.

This distinction is also reflected in the fact that the four versions of the original cut rule are admissible in intuitionistic, dual-intuitionistic, and bi-intuitionistic logics. By contrast, only two of them are admissible in Dunn-Belnap's four-valued logic. As for Kleene's strong three-valued logic and Priest's logic of paradox, they each admit only one additional version.

Several logico-philosophical meanings of partiality and paraconsistency can therefore be distinguished. However the notions of duality that we have identified suggest that these logics share a common trait. In this regard, the composition of the temporal and the inferential duality function defined on the set of sequents gives rise to a general property of duality which can be applied to every partial or paraconsistent logic discussed here. Through this notion, a perfect symmetry is observed between Kleene's strong three-valued logic and Priest's logic of paradox on the one hand and intuitionistic logic and dual-intuitionistic logic on the other hand. In addition, Dunn-Belnap's four-valued logic, just like bi-intuitionistic logic, is its own counterpart.

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