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CATEGORICAL ABSTRACT ALGEBRAIC LOGIC: PSEUDO-REFERENTIAL MATRIX SYSTEM SEMANTICS

To the memory of A.I. Mal'cev,
50 years since his passing.

Abstract

This work adapts techniques and results first developed by Malinowski and by Marek in the context of referential semantics of sentential logics to the context of logics formalized as π-institutions. More precisely, the notion of a pseudo-referential matrix system is introduced and it is shown how this construct generalizes that of a referential matrix system. It is then shown that every π-institution has a pseudo-referential matrix system semantics. This contrasts with referential matrix system semantics which is only available for self-extensional π-institutions by a previous result of the author obtained as an extension of a classical result of Wójcicki. Finally, it is shown that it is possible to replace an arbitrary pseudo-referential matrix system semantics by a *discrete* pseudo-referential matrix system semantics.

Keywords: Referential Logics, Selfextensional Logics, Referential Semantics, Referential π-institutions, Selfextensional π-institutions, Pseudo-Referential Semantics, Discrete Referential Semantics.

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1. Introduction

Let $\mathcal{L} = \langle \Lambda, \rho \rangle$ be a logical signature/algebraic type, i.e., a set of logical connectives/operation symbols Λ with attached finite arities given by the function $\rho : \Lambda \to \omega$. Let also V be a countably infinite set of propositional variables and T a set of **reference/base points**. Wójcicki [10] defines a **referential algebra** $\boldsymbol{A}$ **over** T (or **based on** T) to be an $\mathcal{L}$-algebra with universe $A \subseteq \{0,1\}^T$, or, equivalently, $A \subseteq \mathcal{P}(T)$.

Let $\mathbf{Fm}_{\mathcal{L}}(V) = \langle \mathrm{Fm}_{\mathcal{L}}(V), \mathcal{L}\rangle$ be the free $\mathcal{L}$-algebra generated by the set V of variables. A homomorphism from $\mathbf{Fm}_{\mathcal{L}}(V)$ into a referential algebra $\boldsymbol{A}$ over T may be viewed as an interpretation of the formulas of $\mathrm{Fm}_{\mathcal{L}}(V)$ in A. We conceive of a formula $\alpha \in \mathrm{Fm}_{\mathcal{L}}(V)$ as **being true at point** $t \in T$ **under** h if and only if $t \in h(\alpha)$. This notion of truth gives rise to a consequence operation on $\mathrm{Fm}_{\mathcal{L}}(V)$. Namely, a referential algebra $\boldsymbol{A}$ determines the consequence operator $C^{\boldsymbol{A}}$ on $\mathrm{Fm}_{\mathcal{L}}(V)$ by setting, for all $X \cup \{\alpha\} \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$, $\alpha \in C^{\boldsymbol{A}}(X)$ iff, for all $h : \mathbf{Fm}_{\mathcal{L}}(V) \to \boldsymbol{A}$ and all $t \in T$,

$$h(\beta)(t) = 1, \text{ for all } \beta \in X, \quad \text{implies} \quad h(\alpha)(t) = 1,$$

or equivalently, iff, for all $h : \mathbf{Fm}_{\mathcal{L}}(V) \to \boldsymbol{A}$,

$$\bigcap_{\beta \in X} h(\beta) \subseteq h(\alpha).$$

Wójcicki calls a propositional logic $\mathcal{S} = \langle \mathcal{L}, C\rangle$, where $C = C^{\boldsymbol{A}}$, for a referential algebra $\boldsymbol{A}$, a **referential** (or **referentially truth-functional**) **propositional logic**.

Wójcicki shows in [10] that, given a class K of referential algebras, there exists a single referential algebra $\boldsymbol{A}$, such that $C^{\mathsf{K}} := \bigcap_{\boldsymbol{K} \in \mathsf{K}} C^{\boldsymbol{K}} = C^{\boldsymbol{A}}$. Thence follows that a propositional logic is referential if and only if it is defined by a class of referential algebras.

Given a propositional logic $\mathcal{S} = \langle \mathcal{L}, C\rangle$, the **Frege** or **interderivability relation** of $\mathcal{S}$, denoted $\Lambda(\mathcal{S})$, is the equivalence relation on $\mathrm{Fm}_{\mathcal{L}}(V)$, defined, for all $\alpha, \beta \in \mathrm{Fm}_{\mathcal{L}}(V)$, by

$$\langle \alpha, \beta\rangle \in \Lambda(\mathcal{S}) \quad \text{iff} \quad C(\alpha) = C(\beta).$$

The **Tarski congruence** $\widetilde{\Omega}(\mathcal{S})$ of $\mathcal{S}$ (see [5]) is the largest congruence relation on $\mathbf{Fm}_{\mathcal{L}}(V)$ that is compatible with all theories of $\mathcal{S}$. The Tarski congruence is a special case of the **Suszko congruence** $\widetilde{\Omega}^{\mathcal{S}}(T)$ associated with a given theory T of $\mathcal{S}$, which is defined as the largest congruence on $\mathbf{Fm}_{\mathcal{L}}(V)$ that is compatible with all theories of $\mathcal{S}$ that contain the given theory T (see [3]). In fact, by definition, $\widetilde{\Omega}(\mathcal{S}) = \widetilde{\Omega}^{\mathcal{S}}(C(\varnothing))$, i.e., the Tarski congruence of $\mathcal{S}$ is the Suszko congruence associated with the set of theorems of the logic $\mathcal{S}$. Font and Jansana [5], extending Czelakowski's [2] (see also [1]) well-known characterization of the *Leibniz congruence* $\Omega(T)$ associated with a theory T of a sentential logic, have shown that, for all $\alpha, \beta \in \mathrm{Fm}_{\mathcal{L}}(V)$,

$$\langle \alpha, \beta \rangle \in \widetilde{\Omega}(\mathcal{S}) \quad \text{iff} \quad \begin{array}{l} \text{for all } \varphi(p, \vec{q}) \in \mathrm{Fm}_{\mathcal{L}}(V), \\ C(\varphi(\alpha, \vec{q})) = C(\varphi(\beta, \vec{q})). \end{array}$$

Whereas $\widetilde{\Omega}(\mathcal{S}) \subseteq \Lambda(\mathcal{S})$, for every propositional logic $\mathcal{S}$, the reverse inclusion does not hold in general. A propositional logic is called **selfextensional** in [10] if $\Lambda(\mathcal{S}) \subseteq \widetilde{\Omega}(\mathcal{S})$. In fact, Wójcicki shows in what has become a fundamental theorem in the theory of referential semantics, Theorem 2 of [10], that a propositional logic is referential if and only if it is self-extensional. This result shows that, unless a propositional logic $\mathcal{S}$ is self-extensional, $\mathcal{S}$ cannot possess a referential algebraic semantics.

Let $\mathcal{L}$ be a logical signature. An $\mathcal{L}$**-g-matrix** $\mathbb{A} = \langle \boldsymbol{A}, \mathcal{C} \rangle$ consists of an $\mathcal{L}$-algebra $\boldsymbol{A}$ together with a collection $\mathcal{C} \subseteq \mathcal{P}(A)$. A g-matrix $\mathbb{A}$ generates a consequence operator $C^{\mathbb{A}}$ on $\mathrm{Fm}_{\mathcal{L}}(V)$ as follows: For all $X \cup \{\alpha\} \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$,

$$\alpha \in C^{\mathbb{A}}(X) \quad \text{iff} \quad \begin{array}{l} \text{for all } h : \mathbf{Fm}_{\mathcal{L}}(V) \to \boldsymbol{A} \text{ and all } F \in \mathcal{C}, \\ h(X) \subseteq F \text{ implies } h(\alpha) \in F. \end{array}$$

A g-matrix $\mathbb{A}$ is said to constitute a **g-matrix semantics** for a propositional logic $\mathcal{S} = \langle \mathcal{L}, C \rangle$ in case $C^{\mathbb{A}} = C$.

Consider now a referential algebra $\boldsymbol{A}$ over a set T of reference points. Let, for all $t \in T$,

$$D_t = \{a \in A : t \in a\}.$$

Define the collection $\mathcal{D} = \{D_t : t \in T\}$. We call $\langle \boldsymbol{A}, \mathcal{D} \rangle$ the **referential g-matrix** associated with the referential algebra $\boldsymbol{A}$.

It can be shown that the consequence operator $C^{\langle \boldsymbol{A}, \mathcal{D} \rangle}$ generated by the g-matrix system $\langle \boldsymbol{A}, \mathcal{D} \rangle$ is identical to $C^{\boldsymbol{A}}$. Thus, it follows that, unless $\mathcal{S}$ is self-extensional it does not possess a referential g-matrix semantics.

To address this shortcoming of referential g-matrices in providing a semantics for arbitrary propositional logics, Malinowski introduced in [8] pseudo-referential g-matrices, as a generalization of referential g-matrices, and showed that every propositional logic possesses a pseudo-referential g-matrix semantics.

Let, once more, T be a set of reference points and consider, also, a collection $T^* \subseteq \mathcal{P}(T)$ of subsets of T. According to [8] a **pseudo-referential**

g-matrix $\mathbb{A} = \langle \boldsymbol{A}, \mathcal{D} \rangle$ **relative to** (T, T^*) is a g-matrix, such that $\boldsymbol{A}$ is a referential algebra based on T and

$$\begin{aligned} \mathcal{D} &= \{\{a \in A : (\exists t \in t^*)(t \in a)\} : t^* \in T^*\} \\ &= \{\{a \in A : a \cap t^* \neq \varnothing\} : t^* \in T^*\}. \end{aligned}$$

Note that this concept generalizes referentiality, since a referential g-matrix associated with a referential algebra $\boldsymbol{A}$ based on T is obtained as a special case of a pseudo-referential g-matrix relative to (T, T^*), with $T^* = \{\{t\} : t \in T\}$.

In the Theorem of [8] it is shown that every propositional logic $\mathcal{S}$ has a strongly adequate pseudo-referential g-matrix $\mathbb{A}$, which may be termed the **canonical pseudo-referential g-matrix associated with** $\mathcal{S}$.

Malinowski's work was followed by Marek [9]. Marek defines a **discrete pseudo-referential g-matrix** as a pseudo-referential g-matrix relative to a pair (T, T^*), such that $T^* \subseteq \{\{t\} : t \in T\}$. She then shows that every g-matrix is isomorphic to, and, hence, generates the same sentential logic as, a *discrete* pseudo-referential g-matrix. Thus, since, as is well-known, every propositional logic has a strongly adequate g-matrix semantics, it follows that it also has a strongly adequate discrete pseudo-referential g-matrix semantics (see Corollary of [9]).

The author, taking after the work of Wójcicki, showed in previous work [11, 12] that a logic formalized as a π-institution (see Section 2) is referential, i.e., has a referential g-matrix system semantics, if and only if it is self-extensional. Thus, it turns out that, similarly to the case of propositional logics, for these logics, unless the condition of self-extensionality is fulfilled, no referential g-matrix system semantics is available. The present work, inspired by the previously mentioned work of Malinowski [8] and Marek [9], addresses this constraint on the availability of a referential g-matrix system semantics by introducing a pseudo-referential g-matrix system semantics (see Section 4). It is shown in Theorem 5 that every π-institution possesses a pseudo-referential g-matrix system semantics. Finally, improving on this result, we show in Section 7, in a parallel to the Theorem of Marek [9], that, for every g-matrix system, there exists a discrete pseudo-referential g-matrix system that generates the same closure system (see Theorem 6). It then follows that every logic formalized as a π-institution has a discrete pseudo-referential g-matrix semantics.

2. π-Institutions and Closure Systems

We describe π-institutions [4] (see, also [6] for the closely related notion of an institution) on which our logical systems will be based.

Let $\mathbf{Sign}^\flat$ be a category, called the **category of signatures**. Let $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ be a set-valued functor, called the **sentence functor**. Let $N^\flat$ be a category of natural transformations on $\mathrm{SEN}^\flat$ (see Section 2 of [12]). We call the triple $\boldsymbol{A}^\flat = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ the **base algebraic system**.

A collection $T^\flat = \{T^\flat_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$, such that $T^\flat_\Sigma \subseteq \mathrm{SEN}^\flat(\Sigma)$, for all $\Sigma \in |\mathbf{Sign}^\flat|$, is called a **sentence family** of $\boldsymbol{A}^\flat$.

A **π-institution based on $\boldsymbol{A}^\flat$** is a pair $\mathcal{I} = \langle \boldsymbol{A}^\flat, C \rangle$, where

$$C = \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$$

is a **closure (operator) system**, i.e., a $|\mathbf{Sign}^\flat|$-indexed collection of closure operators $C_\Sigma : \mathcal{P}(\mathrm{SEN}^\flat(\Sigma)) \to \mathcal{P}(\mathrm{SEN}^\flat(\Sigma))$ that satisfy the **structurality condition**:

For all $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma_1, \Sigma_2)$ and $\Phi \subseteq \mathrm{SEN}^\flat(\Sigma_1)$,

$$\mathrm{SEN}^\flat(f)(C_{\Sigma_1}(\Phi)) \subseteq C_{\Sigma_2}(\mathrm{SEN}^\flat(f)(\Phi)).$$

For $\Sigma \in |\mathbf{Sign}^\flat|$, a set $T^\flat_\Sigma \subseteq \mathrm{SEN}^\flat(\Sigma)$ is called a **Σ-theory** of $\mathcal{I}$ if it is closed under consequence, i.e., if $C_\Sigma(T^\flat_\Sigma) = T^\flat_\Sigma$. The collection of all Σ-theories of $\mathcal{I}$ is denoted by $\mathrm{Th}_\Sigma(\mathcal{I})$. A collection $T^\flat = \{T^\flat_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$, such that $T^\flat_\Sigma \in \mathrm{Th}_\Sigma(\mathcal{I})$, for all $\Sigma \in |\mathbf{Sign}^\flat|$, is called a **theory family** of $\mathcal{I}$. The collection of all theory families of $\mathcal{I}$ is denoted by $\mathrm{ThFam}(\mathcal{I})$. It is well-known that they form a complete lattice under signature-wise inclusion $\leq$, whose meet coincides with signature-wise intersection.

Note that closure systems on $\boldsymbol{A}^\flat$ are ordered as follows:

$$C^1 \leq C^2 \quad \text{iff} \quad \text{for all} \quad \Sigma \in |\mathbf{Sign}^\flat|, \Phi \subseteq \mathrm{SEN}^\flat(\Sigma), \; C^1_\Sigma(\Phi) \subseteq C^2_\Sigma(\Phi).$$

Under this ordering the collection of all closure systems on $\boldsymbol{A}^\flat$ also forms a complete lattice whose meet is given by signature-wide intersection.

Given a base algebraic system $\boldsymbol{A}^\flat = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$, an **$N^\flat$-algebraic system** $\boldsymbol{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ is an algebraic system, such that there exists a surjective functor $N^\flat \to N$ preserving all projection natural transformations and, as a consequence, also all the arities of the natural transformations involved. We denote by $\sigma : \mathrm{SEN}^k \to \mathrm{SEN}$ the natural transformation that is the image of $\sigma^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ in $N^\flat$, an, in general use similar typing conventions to keep track of mappings of natural transformations in $N^\flat$ to those on $N^\flat$-algebraic systems.

An **interpreted $N^\flat$-algebraic system** is a pair $\mathcal{A} = \langle \boldsymbol{A}, \langle F, \alpha \rangle \rangle$, where

- $\boldsymbol{A}$ is an $N^\flat$-algebraic system and
- $\langle F, \alpha \rangle : \boldsymbol{A}^\flat \to \boldsymbol{A}$ is an algebraic system morphism.

We will use the term *algebraic system* to refer to both an $N^\flat$-algebraic system and an interpreted $N^\flat$-algebraic system relying on the context to clear the ambiguity.

Let $\boldsymbol{A}^\flat$ be an algebraic system and $\mathcal{I} = \langle \boldsymbol{A}^\flat, C \rangle$ a π-institution based on $\boldsymbol{A}^\flat$. We define, next, the notion of a matrix system and of a g-matrix system for $\boldsymbol{A}^\flat$ and of a matrix system model and g-matrix system model for $\mathcal{I}$.

A **matrix system** for $\boldsymbol{A}^\flat$ is a pair $\mathfrak{A} = \langle \mathcal{A}, T \rangle$, where $\mathcal{A} = \langle \boldsymbol{A}, \langle F, \alpha \rangle \rangle$ is an interpreted algebraic system and T is a sentence family of $\mathcal{A}$.

A matrix system $\mathfrak{A}$ defines a closure system $C^{\mathfrak{A}}$ (and hence a π-institution $\mathcal{I}^{\mathfrak{A}} = \langle \boldsymbol{A}^\flat, C^{\mathfrak{A}} \rangle$) on $\boldsymbol{A}^\flat$ as follows: For all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\varphi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$,

$$\varphi \in C^{\mathfrak{A}}_\Sigma(\Phi) \quad \text{iff} \quad \Phi \vDash^{\mathfrak{A}}_\Sigma \varphi,$$

where the relation on the right means that, for all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$,

$$\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi)) \subseteq T_{F(\Sigma')} \quad \text{implies} \quad \alpha_{\Sigma'}(\mathrm{SEN}^\flat(\varphi)) \in T_{F(\Sigma')}.$$

A **generalized matrix system** for $\boldsymbol{A}^\flat$ (or **g-matrix system**, for short) is a pair $\mathbb{A} = \langle \mathcal{A}, \mathcal{T} \rangle$, where $\mathcal{A} = \langle \boldsymbol{A}, \langle F, \alpha \rangle \rangle$ is an interpreted algebraic system and $\mathcal{T}$ is a collection of sentence families of $\mathcal{A}$.

A g-matrix system $\mathbb{A}$ defines a closure system $C^{\mathbb{A}}$ (and hence a π-institution $\mathcal{I}^{\mathbb{A}} = \langle \boldsymbol{A}^\flat, C^{\mathbb{A}} \rangle$) on $\boldsymbol{A}^\flat$ by setting $C^{\mathbb{A}} = \bigcap_{\mathfrak{A} \in \mathbb{A}} C^{\mathfrak{A}}$, where $\mathfrak{A} = \langle \boldsymbol{A}, T \rangle \in \mathbb{A}$ means that $T \in \mathcal{T}$. Thus, equivalently, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\varphi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$,

$$\varphi \in C^{\mathfrak{A}}_\Sigma(\Phi) \quad \text{iff} \quad (\forall \mathfrak{A} \in \mathbb{A})(\Phi \vDash^{\mathfrak{A}}_\Sigma \varphi).$$

A **matrix system model for $\mathcal{I} = \langle \boldsymbol{A}^\flat, C \rangle$** or an **$\mathcal{I}$-matrix system** is a matrix system $\mathfrak{A} = \langle \mathcal{A}, T \rangle$ for $\boldsymbol{A}^\flat$, such that $C \leq C^{\mathfrak{A}}$.

Similarly, a **g-matrix system model for $\mathcal{I}$** or an **$\mathcal{I}$-g-matrix system** is a g-matrix system $\mathbb{A}$, such that $C \leq C^{\mathbb{A}}$.

3. Referential π-Institutions

In this work we focus on a special kind of (interpreted) $N^\flat$-algebraic system $\mathcal{A} = \langle \boldsymbol{A}, \langle F, \alpha\rangle\rangle$, $\boldsymbol{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$. We require that, for all $\Sigma \in |\mathbf{Sign}|$, there is a set $\mathrm{PTS}(\Sigma)$, called the set of **Σ-reference** or **Σ-base points**, and that, for all $\Sigma \in |\mathbf{Sign}|$, $\mathrm{SEN}(\Sigma) \subseteq \mathcal{P}(\mathrm{PTS}(\Sigma))$, i.e., each Σ-sentence is a set of Σ-points.

In this context, an interpretation $\langle F, \alpha\rangle : \boldsymbol{A}^\flat \to \boldsymbol{A}$ will be viewed as a valuation of sentences of $\boldsymbol{A}^\flat$ in the following way: For all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\varphi \in \mathrm{SEN}^\flat(\Sigma)$, φ is **true at** $p \in \mathrm{PTS}(F(\Sigma))$ **under** $\langle F, \alpha\rangle$ iff $p \in \alpha_\Sigma(\varphi)$.

An algebraic system of this special form is called a **referential algebraic system** and said to **be based on** PTS.

Note that this definition is a generalized version of the one given in Section 3 of [12]. The generalization stems from the fact that, in the present context, we no longer insist that the sentence functor SEN be a simple subfunctor (having the same domain) of the inverse powerset of a contravariant functor $\mathbf{Sign} \to \mathbf{Set}^{\mathsf{op}}$.

Let $\mathcal{A} = \langle \boldsymbol{A}, \langle F, \alpha\rangle\rangle$ be an interpreted referential $N^\flat$-algebraic system. Then $\mathcal{A}$ determines a closure system $C^{\mathcal{A}}$ on $\boldsymbol{A}^\flat$ according to the following definition:

For all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\varphi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, $\varphi \in C^{\mathcal{A}}_\Sigma(\Phi)$ iff, for all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$,

$$\bigcap_{\phi\in\Phi} \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi)) \subseteq \alpha_{\Sigma'}(\mathrm{SEN}(f)^\flat(\varphi)).$$

Essentially the same proof as that of Proposition 1 of [12] yields the following

Proposition 1 (Proposition 1 of [12]). *Let $\boldsymbol{A}^\flat = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be a base algebraic system and $\mathcal{A} = \langle \boldsymbol{A}, \langle F, \alpha\rangle\rangle$ an interpreted referential $N^\flat$-algebraic system. Then $C^{\mathcal{A}}$ is a closure system on $\boldsymbol{A}^\flat$.*

Since $C^{\mathcal{A}}$ is a closure system on $\boldsymbol{A}^\flat$, the pair $\mathcal{I}^{\mathcal{A}} = \langle \boldsymbol{A}^\flat, C^{\mathcal{A}}\rangle$ is a π-institution. We call an institution having this form a **referential π-institution**. Such π-institutions correspond in the theory of categorical abstract algebraic logic to the referential propositional logics of Wójcicki [10].

Let $\boldsymbol{A}^\flat = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be a base algebraic system and $\mathcal{I} = \langle \boldsymbol{A}^\flat, C\rangle$ a π-institution based on $\boldsymbol{A}^\flat$. We define the **Frege equivalence system**

$\Lambda(\mathcal{I})$ of $\mathcal{I}$, also known as the **interderivability equivalence system**, by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\varphi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\langle \varphi, \psi \rangle \in \Lambda_\Sigma(\mathcal{I}) \quad \text{if and only if} \quad C_\Sigma(\varphi) = C_\Sigma(\psi).$$

The **Tarski congruence system** $\widetilde{\Omega}(\mathcal{I})$ of $\mathcal{I}$ ([5] for the universal algebraic notion and [13] for its categorical extension) is the largest congruence system on $\boldsymbol{A}^\flat$ that is compatible with every theory family $T \in \mathrm{ThFam}(\mathcal{I})$.

Clearly, it is always the case that $\widetilde{\Omega}(\mathcal{I}) \le \Lambda(\mathcal{I})$. We call the π-institution $\mathcal{I}$ **self-extensional** if $\Lambda(\mathcal{I}) \le \widetilde{\Omega}(\mathcal{I})$. In view of the preceding remark, $\mathcal{I}$ is self-extensional if and only if $\Lambda(\mathcal{I}) = \widetilde{\Omega}(\mathcal{I})$.

A generalization to π-institutions of Wójcicki's Theorem (see Theorem 2 of [10], but, also, Theorem 2.2 of [7] for a complete proof), provides a characterization of referential sentential logics. This is essentially Theorem 8 of [12], with the aforementioned generalization pertaining to the signature category not affecting the proof.

THEOREM 2 (Theorem 8 of [12]). *A π-institution $\mathcal{I} = \langle \boldsymbol{A}^\flat, C \rangle$ is referential if and only if it is self-extensional.*

We recall here a version of the construction of the *canonical referential algebraic system* associated with a given selfextensional π-institution that witnesses one implication of Theorem 2.

Let $\mathcal{I} = \langle \boldsymbol{A}^\flat, C \rangle$, with $\boldsymbol{A}^\flat = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$, be a self-extensional π-institution. For each $\Sigma \in |\mathbf{Sign}^\flat|$, we take as the set of Σ-points the set $\mathrm{Th}_\Sigma(\mathcal{I})$ of Σ-theories of $\mathcal{I}$.

Define the functor $\mathrm{SEN} : \mathbf{Sign}^\flat \to \mathbf{Set}$ as follows:

For every $\Sigma \in |\mathbf{Sign}^\flat|$,

$$\mathrm{SEN}(\Sigma) = \{\mathrm{Th}_\Sigma(\varphi) : \varphi \in \mathrm{SEN}^\flat(\Sigma)\},$$

where $\mathrm{Th}_\Sigma(\varphi) = \{T \in \mathrm{Th}_\Sigma(\mathcal{I}) : \varphi \in T\}$, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\varphi \in \mathrm{SEN}^\flat(\Sigma)$.

Moreover, for all $\Sigma, \Sigma' \in |\mathbf{Sign}^\flat|$, and all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$, we define $\mathrm{SEN}(f) : \mathrm{SEN}(\Sigma) \to \mathrm{SEN}(\Sigma')$ by setting

$$\mathrm{SEN}(f)(\mathrm{Th}_\Sigma(\varphi)) = \mathrm{Th}_{\Sigma'}(\mathrm{SEN}^\flat(f)(\varphi)),$$

for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\varphi \in \mathrm{SEN}^\flat(\Sigma)$.

Define the category of natural transformations N on SEN as follows: For every $\sigma^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ in $N^\flat$, let $\sigma : \mathrm{SEN}^k \to \mathrm{SEN}$ be defined by letting, for all $\Sigma \in |\mathbf{Sign}^\flat|$, $\sigma_\Sigma : \mathrm{SEN}(\Sigma)^k \to \mathrm{SEN}(\Sigma)$ be given by

$$\sigma_\Sigma(\mathrm{Th}_\Sigma(\varphi_0),\ldots,\mathrm{Th}_\Sigma(\varphi_{k-1})) = \mathrm{Th}_\Sigma(\sigma^\flat_\Sigma(\varphi_0,\ldots,\varphi_{k-1})),$$

for all $\varphi_0,\ldots,\varphi_{k-1} \in \mathrm{SEN}^\flat(\Sigma)$. Using self-extensionality one may show that this is well-defined. Moreover, σ is a natural transformation and the collection of natural transformations, thus defined, forms a category of natural transformations on SEN. So the triple $\boldsymbol{A} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}, N\rangle$ constitutes an $N^\flat$-algebraic system.

Finally, the **canonical referential algebraic system** associated with $\mathcal{I}$ is defined by $\mathcal{A} = \langle \boldsymbol{A}, \langle I, \alpha\rangle\rangle$, where:

- $I : \mathbf{Sign}^\flat \to \mathbf{Sign}^\flat$ is the identity functor;
- $\alpha : \mathrm{SEN}^\flat \to \mathrm{SEN}$ is the natural transformation defined by letting, for all $\Sigma \in |\mathbf{Sign}^\flat|$, $\alpha_\Sigma : \mathrm{SEN}^\flat(\Sigma) \to \mathrm{SEN}(\Sigma)$ be given by

$$\alpha_\Sigma(\varphi) = \mathrm{Th}_\Sigma(\varphi),\ \text{for all } \varphi \in \mathrm{SEN}^\flat(\Sigma).$$

Note, now, that, for all $\Sigma, \Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$ and $\varphi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{ccc}
\mathrm{SEN}^\flat(\Sigma) & \xrightarrow{\alpha_\Sigma} & \mathrm{SEN}(\Sigma) \\
{\scriptstyle \mathrm{SEN}^\flat(f)}\downarrow & & \downarrow{\scriptstyle \mathrm{SEN}(f)} \\
\mathrm{SEN}^\flat(\Sigma') & \xrightarrow[\alpha_{\Sigma'}]{} & \mathrm{SEN}(\Sigma')
\end{array}$$

$$\begin{array}{rcl}
\mathrm{SEN}(f)(\alpha_\Sigma(\varphi)) & = & \mathrm{SEN}(f)(\mathrm{Th}_\Sigma(\varphi)) \\
& = & \mathrm{Th}_{\Sigma'}(\mathrm{SEN}^\flat(f)(\varphi)) \\
& = & \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\varphi)).
\end{array}$$

It can be shown that, if $\mathcal{I}$ is self-extensional, then $\mathcal{A}$ is well-defined and, moreover, $\mathcal{I} = \mathcal{I}^{\mathcal{A}}$. Thus, $\mathcal{I}$ is referential.

4. Pseudo-Referential Matrix Systems

Let $\boldsymbol{A}^\flat = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be a base algebraic system and $\mathcal{A} = \langle \boldsymbol{A}, \langle F, \alpha\rangle\rangle$ a referential $N^\flat$-algebraic system based on PTS. The algebraic system $\mathcal{A}$ will be said to be **supported** if it is endowed with a collection $\mathcal{S} = \{S^i : i \in I\}$ of base point families

$$S^i = \{S^i_\Sigma\}_{\Sigma \in |\mathbf{Sign}|},$$

where $S^i_\Sigma \subseteq \mathrm{PTS}(\Sigma)$, for all $i \in I$ and all $\Sigma \in |\mathbf{Sign}|$. We refer to $\mathcal{S}$ as the **support** of $\mathcal{A}$ in this case.

Given a supported algebraic system $\mathcal{A}$, with support $\mathcal{S}$, a **pseudo-referential g-matrix system relative to** $(\mathrm{PTS}, \mathcal{S})$ is a pair

$$\mathbb{A} = \langle \mathcal{A}, \mathcal{T} \rangle,$$

where $\mathcal{T} = \{T^i : i \in I\}$ is a collection of sentence families $T^i = \{T^i_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$, such that, for all $i \in I$ and all $\Sigma \in |\mathbf{Sign}|$,

$$T^i_\Sigma = \{X \in \mathrm{SEN}(\Sigma) : X \cap S^i_\Sigma \neq \varnothing\}.$$

We close this section with two properties of pseudo-referential g-matrix systems. The first states that, in a precise model-theoretic sense, pseudo-referential g-matrix systems encompass referential algebraic systems. The second characterizes the closure system $C^{\mathbb{A}}$ induced by a pseudo-referential g-matrix system on the base algebraic system $\boldsymbol{A}^\flat$.

Let $\mathcal{A} = \langle \boldsymbol{A}, \langle F, \alpha \rangle \rangle$ be a referential algebraic system, based on PTS. Consider the set $\mathcal{P}$ of all $|\mathbf{Sign}|$-indexed tuples P, such that, for some $\Sigma \in |\mathbf{Sign}|$,

$$P_{\Sigma'} \begin{cases} \in \{\{p\} : p \in \mathrm{PTS}(\Sigma)\}, & \text{if } \Sigma' = \Sigma \\ = \varnothing, & \text{if } \Sigma' \neq \Sigma \end{cases}$$

Consider the pseudo-referential g-matrix system $\mathbb{A}(\mathcal{A}) = \langle \mathcal{A}, \mathcal{T} \rangle$ relative to $(\mathrm{PTS}, \mathcal{P})$. This is called the **pseudo-referential g-matrix system associated with** $\mathcal{A}$. Then we have the following:

Lemma 3. *Let $\boldsymbol{A}^\flat = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be a base algebraic system, $\mathcal{A} = \langle \boldsymbol{A}, \langle F, \alpha \rangle \rangle$ a referential $N^\flat$-algebraic system and $\mathbb{A}(\mathcal{A}) = \langle \mathcal{A}, \mathcal{T} \rangle$ the pseudo-referential g-matrix system associated with $\mathcal{A}$. Then $C^{\mathcal{A}} = C^{\mathbb{A}(\mathcal{A})}$.*

Proof: This follows easily from the fact that, according to the definitions involved, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\varphi \in \mathrm{SEN}^\flat(\Sigma)$, we have

$$p \in \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\varphi)) \quad \text{iff} \quad \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\varphi)) \cap \{p\} \neq \varnothing,$$

for all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $p \in \mathrm{PTS}(\Sigma')$. □

Thus, by identifying $\mathcal{A}$ with $\mathbb{A}(\mathcal{A})$ we may view referential algebraic semantics in the sense of [12] as a special case of pseudo-referential g-matrix system semantics.

We now obtain the following characterization of $C^{\mathbb{A}}$ for an arbitrary pseudo-referential g-matrix system $\mathbb{A}$.

Proposition 4. *Let* $\mathbf{A}^\flat = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be a base algebraic system,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ *an* $N^\flat$*-referential algebraic system based on* PTS*, and* $\mathbb{A} = \langle \mathcal{A}, \mathcal{T}\rangle$ *a pseudo-referential g-matrix system relative to a pair* $(\mathrm{PTS}, \mathcal{S})$*, with* $\mathcal{S} = \{S^i : i \in I\}$*. Then, for all* $\Sigma \in |\mathbf{Sign}^\flat|$*,* $\Phi \cup \{\varphi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$*,* $\varphi \in C^{\mathbb{A}}_\Sigma(\Phi)$ *iff, for all* $\Sigma' \in |\mathbf{Sign}^\flat|$*,* $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ *and all* $i \in I$*,*

$$\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi)) \cap S^i_{F(\Sigma')} \neq \varnothing, \text{ for all } \phi \in \Phi, \\ \text{implies } \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\varphi)) \cap S^i_{F(\Sigma')} \neq \varnothing.$$

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\Phi \cup \{\varphi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$. We have $\varphi \in C^{\mathbb{A}}_\Sigma(\Phi)$ iff, by definition, for all $\Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $i \in I$,

$$\alpha_{\Sigma'}(\mathrm{SEN}(f)(\Phi)) \subseteq T^i_{F(\Sigma')} \quad \text{implies} \quad \alpha_{\Sigma'}(\mathrm{SEN}(f)(\varphi)) \in T^i_{F(\Sigma')}.$$

By the definition of T^i, this is equivalent to having, for all $\Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $i \in I$,

$$\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi)) \subseteq \{X \in \mathrm{SEN}(F(\Sigma')) : X \cap S^i_{F(\Sigma')} \neq \varnothing\} \\ \text{implies } \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\varphi)) \in \{X \in \mathrm{SEN}(F(\Sigma')) : X \cap T^i_{F(\Sigma')} \neq \varnothing\}.$$

Equivalently, for all $\Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $i \in I$,

$$\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi)) \cap S^i_{F(\Sigma')} \neq \varnothing, \text{ for all } \phi \in \Phi, \\ \text{implies } \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\varphi)) \cap S^i_{F(\Sigma')} \neq \varnothing.$$

□

5. Universality of the Semantics

In this section we show that every π-institution has a pseudo-referential matrix semantics. This contrasts with Theorem 2, which implies that not every π-institution has a referential algebraic semantics.

Theorem 5. *Let* $\mathcal{I} = \langle \mathbf{A}^\flat, C\rangle$ *be a* π*-institution based on an algebraic system* $\mathbf{A}^\flat = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$*. Then, there exists a pseudo-referential g-matrix system* $\mathbb{A} = \langle \mathcal{A}, \mathcal{T}\rangle$ *relative to a pair* $(\mathrm{PTS}, \mathcal{S})$*, such that* $\mathcal{I} = \mathcal{I}^{\mathbb{A}}$*, i.e.,* $C = C^{\mathbb{A}}$*.*

Proof: Let $\mathbf{Sign} = \mathbf{Sign}^\flat$. For all $\Sigma \in |\mathbf{Sign}^\flat|$, let $\mathrm{PTS}(\Sigma) = \mathrm{SEN}^\flat(\Sigma)$. Now we define $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ based on PTS as follows:

- $\text{SEN}(\Sigma) = \{\{\varphi\} : \varphi \in \text{SEN}^\flat(\Sigma)\}$, for all $\Sigma \in |\mathbf{Sign}^\flat|$. And, given, $\Sigma, \Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$,

$$\text{SEN}(f)(\{\varphi\}) = \{\text{SEN}^\flat(f)(\varphi)\}, \text{ for all } \varphi \in \text{SEN}^\flat(\Sigma).$$

- For all $\sigma^\flat : (\text{SEN}^\flat)^k \to \text{SEN}^\flat$ in $N^\flat$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\varphi_0, \dots, \varphi_{k-1} \in \text{SEN}^\flat(\Sigma)^k$,

$$\sigma_\Sigma(\{\varphi_0\}, \dots, \{\varphi_{k-1}\}) = \{\sigma^\flat_\Sigma(\varphi_0, \dots, \varphi_{k-1})\}.$$

We let N consist of all natural transformations of this form.

It is not difficult to see that, with these definitions, the triple $\boldsymbol{A} = \langle \mathbf{Sign}, \text{SEN}, N\rangle$ becomes a referential $N^\flat$-algebraic system based on PTS.

Next, define $\langle I, \alpha\rangle : \boldsymbol{A}^\flat \to \boldsymbol{A}$ by setting

- $I : \mathbf{Sign}^\flat \to \mathbf{Sign}$ the identity functor;
- For all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\varphi \in \text{SEN}^\flat(\Sigma)$, $\alpha_\Sigma(\varphi) = \{\varphi\}$.

Now $\mathcal{A} = \langle \boldsymbol{A}, \langle I, \alpha\rangle\rangle$ is an interpreted referential $N^\flat$-algebraic system.

Let $\mathcal{S} = \text{ThFam}(\mathcal{I}) = \{S^i : i \in I\}$. This determines the pseudo-referential g-matrix system $\mathbb{A} = \langle \mathcal{A}, \mathcal{T}\rangle$ relative to $(\text{PTS}, \mathcal{S})$. We have that $\mathcal{T} = \{T^i : i \in I\}$, with $T^i = \{T^i_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$ given, for all $i \in I$ and all $\Sigma \in |\mathbf{Sign}|$, by

$$\begin{array}{rcl} T^i_\Sigma & = & \{\{\varphi\} \in \text{SEN}(\Sigma) : \{\varphi\} \cap S^i_\Sigma \neq \varnothing\} \\ & = & \{\{\varphi\} \in \text{SEN}(\Sigma) : \varphi \in S^i_\Sigma\}, \end{array}$$

We prove that $C = C^{\mathbb{A}}$, i.e., that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\varphi\} \subseteq \text{SEN}^\flat(\Sigma)$,

$$\varphi \in C_\Sigma(\Phi) \quad \text{iff} \quad \varphi \in C^{\mathbb{A}}_\Sigma(\Phi).$$

$\Rightarrow$: Suppose that $\varphi \in C_\Sigma(\Phi)$. Let $\Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and $i \in I$, such that $\alpha_{\Sigma'}(\text{SEN}^\flat(f)(\phi)) \subseteq T^i_{\Sigma'}$, for all $\phi \in \Phi$. By the definition of α, this holds iff $\{\text{SEN}^\flat(f)(\phi)\} \in T^i_{\Sigma'}$, for all $\phi \in \Phi$. By the expression given above for T^i, this holds iff $\text{SEN}^\flat(f)(\phi) \in S^i_{\Sigma'}$, for all $\phi \in \Phi$, i.e., iff $\text{SEN}^\flat(f)(\Phi) \subseteq S^i_{\Sigma'}$. Then, since by hypothesis $\varphi \in C_\Sigma(\Phi)$, we get $\text{SEN}^\flat(f)(\varphi) \in S^i_{\Sigma'}$. This shows that $\{\text{SEN}^\flat(f)(\varphi)\} \in T^i_{\Sigma'}$ or, equivalently, $\alpha_{\Sigma'}(\text{SEN}^\flat(f)(\varphi)) \in T^i_{\Sigma'}$. Therefore, $\varphi \in C^{\mathbb{A}}_\Sigma(\Phi)$.

$\Leftarrow$: Suppose that $\varphi \in C^{\mathbb{A}}_\Sigma(\Phi)$. Let $i \in I$, such that $\Phi \subseteq S^i_\Sigma$. This is equivalent to $\{\phi\} \in T^i_\Sigma$, for all $\phi \in \Phi$. Since, by hypothesis $\varphi \in C^{\mathbb{A}}_\Sigma(\Phi)$, we get that $\{\varphi\} \in T^i_\Sigma$. Equivalently, $\varphi \in S^i_\Sigma$. Since $i \in I$ was arbitrary, we get that $\varphi \in C_\Sigma(\Phi)$. □

We call the pseudo-referential g-matrix system $\mathbb{A}$, constructed in the proof of Theorem 5, such that $\mathcal{I}^{\mathbb{A}} = \mathcal{I}$, the **canonical pseudo-referential g-matrix system associated with** $\mathcal{I}$.

6. Selfextensional π-Institutions

In this section, we start with a selfextensional π-institution $\mathcal{I}$ and show how, starting from the canonical pseudo-referential g-matrix system associated with $\mathcal{I}$, a process of dividing out by the Frege equivalence system of $\mathcal{I}$ (which is a congruence system due to selfextensionality), leads to the canonical referential g-matrix system for $\mathcal{I}$ constructed in [12]. We present an outline, omitting some of the details that are easy to check.

Let $\mathcal{I} = \langle \boldsymbol{A}^\flat, C\rangle$ be a selfextensional π-institution based on the algebraic system $\boldsymbol{A}^\flat = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$. Consider the canonical pseudo-referential g-matrix system $\mathbb{A} = \langle \mathcal{A}, \mathcal{T}\rangle$ associated with $\mathcal{I}$, based on $(\mathrm{PTS}, \mathcal{S})$, with $\mathcal{A} = \langle \boldsymbol{A}, \langle F, \alpha\rangle\rangle$ and $\boldsymbol{A} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}, N\rangle$, as constructed in the proof of Theorem 5.

Recall that the Frege equivalence system $\Lambda(\mathcal{I}) = \{\Lambda_\Sigma(\mathcal{I})\}_{\Sigma \in |\mathbf{Sign}|}$ of $\mathcal{I}$ is defined, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\varphi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, by

$$\langle \varphi, \psi\rangle \in \Lambda_\Sigma(\mathcal{I}) \quad \text{iff} \quad C_\Sigma(\varphi) = C_\Sigma(\psi).$$

By selfextensionality, $\Lambda(\mathcal{I})$ is a congruence system on $\boldsymbol{A}^\flat$ and, in fact, coincides with the Tarski congruence system $\widetilde{\Omega}(\mathcal{I})$.

We define on the underlying algebraic system $\boldsymbol{A} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}, N\rangle$ of the canonical pseudo-referential g-matrix system $\mathbb{A}$ associated with $\mathcal{I}$ the relation family $\equiv^{\mathcal{I}} = \{\equiv^{\mathcal{I}}_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$, by setting, , for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\varphi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\{\varphi\} \equiv^{\mathcal{I}}_\Sigma \{\psi\} \quad \text{iff} \quad \langle \varphi, \psi\rangle \in \Lambda_\Sigma(\mathcal{I}).$$

Clearly, $\equiv^{\mathcal{I}}$ is an equivalence family on $\boldsymbol{A}$. Moreover, it is an equivalence system because of structurality. This establishes that the quotient functor $\mathrm{SEN}^{\equiv^{\mathcal{I}}} := \mathrm{SEN}/{\equiv^{\mathcal{I}}} : \mathbf{Sign}^\flat \to \mathbf{Set}$ is well-defined (see [13]).

Note that $\mathrm{SEN}^{\equiv^{\mathcal{I}}}$ may be considered as a point-based functor, based on $\mathrm{Th}(\mathcal{I}) = \{\mathrm{Th}_\Sigma(\mathcal{I})\}_{\Sigma\in|\mathbf{Sign}^\flat|}$ under the identification

$$\{\varphi\}/\equiv^{\mathcal{I}}_\Sigma \quad\longleftrightarrow\quad \mathrm{Th}_\Sigma(\varphi),$$

for all $\varphi \in \mathrm{SEN}^\flat(\Sigma), \Sigma \in |\mathbf{Sign}^\flat|$ (which is well-defined by the definition of $\equiv^{\mathcal{I}}$).

Next, observe that, by the self-extensionality of $\mathcal{I}$, the equivalence system $\equiv^{\mathcal{I}}$ is actually a congruence system on $\boldsymbol{A}$. In fact, for all $\sigma^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ in $N^\flat$, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\varphi_0,\psi_0,\ldots,\varphi_{k-1},\psi_{k-1} \in \mathrm{SEN}^\flat(\Sigma)$, such that $\{\varphi_i\} \equiv^{\mathcal{I}}_\Sigma \{\psi_i\}$, for all $i < k$, we get that $C_\Sigma(\varphi_i) = C_\Sigma(\psi_i)$, for all $i \in I$, whence by self-extensionality, $C_\Sigma(\sigma^\flat_\Sigma(\varphi_0,\ldots,\varphi_{k-1}) = C_\Sigma(\sigma^\flat_\Sigma(\psi_0,\ldots,\psi_{k-1}))$, giving that $\{\sigma^\flat_\Sigma(\varphi_0,\ldots,\varphi_{k-1})\} \equiv^{\mathcal{I}}_\Sigma \{\sigma^\flat_\Sigma(\psi_0,\ldots,\psi_{k-1})\}$. But, by the definition of $\sigma : \mathrm{SEN}^k \to \mathrm{SEN}$, the latter is equivalent to $\sigma_\Sigma(\{\varphi_0\},\ldots,\{\varphi_{k-1}\}) \equiv^{\mathcal{I}}_\Sigma \sigma_\Sigma(\{\psi_0\},\ldots,\{\psi_{k-1}\})$.

Now we conclude that the quotient $\boldsymbol{A}^{\equiv^{\mathcal{I}}} := \boldsymbol{A}/{\equiv^{\mathcal{I}}} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^{\equiv^{\mathcal{I}}}, N^{\equiv^{\mathcal{I}}}\rangle$ is a well-defined $N^\flat$-algebraic system.

Finally, recall that $\mathcal{T} = \{T^i : i \in I\}$, with $T^i = \{T^i_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$ given, for all $i \in I$ and all $\Sigma \in |\mathbf{Sign}|$, by

$$T^i_\Sigma = \{\{\varphi\} \in \mathrm{SEN}(\Sigma) : \varphi \in S^i_\Sigma\}.$$

We note that $\equiv^{\mathcal{I}}$ is compatible with T^i, for all i, and , therefore, it is a (g-matrix) congruence system of $\mathbb{A} = \langle \mathcal{A}, \mathcal{T}\rangle$. In fact, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\varphi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\{\varphi\} \equiv^{\mathcal{I}}_\Sigma \{\psi\}$ and $\{\varphi\} \in T^i_\Sigma$, we get that $C_\Sigma(\varphi) = C_\Sigma(\psi)$ and $\varphi \in S^i_\Sigma \in \mathrm{Th}_\Sigma(\mathcal{I})$. Hence, we obtain $\psi \in S^i_\Sigma$, which shows that $\{\psi\} \in T^i_\Sigma$.

It follows that the quotient g-matrix system $\mathbb{A}^{\equiv^{\mathcal{I}}} = \langle \mathcal{A}^{\equiv^{\mathcal{I}}}, \mathcal{T}^{\equiv^{\mathcal{I}}}\rangle$ is well-defined.

To establish the equivalence of the canonical referential g-matrix system associated with $\mathcal{I}$ with the quotient $\mathbb{A}^{\equiv^{\mathcal{I}}}$ of the canonical pseudo-referential g-matrix system $\mathbb{A}$ associated with $\mathcal{I}$ it suffices to note that the mapping

$$\mathrm{Th}_\Sigma(\varphi) \mapsto \{\varphi\}/{\equiv^{\mathcal{I}}_\Sigma},$$

for all $\Sigma \in |\mathbf{Sign}^\flat|$, $\varphi \in \mathrm{SEN}^\flat(\Sigma)$, determines an isomorphism between these two g-matrix systems.

7. Discrete Pseudo-Referential Matrix Systems

Let $\boldsymbol{A}^\flat = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathbb{A} = \langle \mathcal{A}, \mathcal{T} \rangle$ a pseudo-referential g-matrix system relative to some $(\mathrm{PTS}, \mathcal{S})$, with $\mathcal{S} = \{S^i : i \in I\}$, i.e., such that $\mathcal{T} = \{T^i : i \in I\}$, with

$$T^i_\Sigma = \{X \in \mathrm{SEN}(\Sigma) : X \cap S^i_\Sigma \neq \varnothing\},$$

for all $\Sigma \in |\mathbf{Sign}|$ and all $i \in I$.

The pseudo-referential g-matrix system $\mathbb{A}$ will be called **discrete** if, for all $i \in I$, there exists $\Sigma_i \in |\mathbf{Sign}|$, such that, for all $\Sigma \in |\mathbf{Sign}|$,

$$S^i_\Sigma \begin{cases} \in \{\{p\} : p \in \mathrm{PTS}(\Sigma_i)\}, & \text{if } \Sigma = \Sigma_i, \\ = \varnothing, & \text{otherwise.} \end{cases}$$

In this section, taking after the work of Marek [9], we show that every π-institution $\mathcal{I} = \langle \boldsymbol{A}^\flat, C \rangle$ has a strongly adequate discrete pseudo-referential matrix system semantics. This is done by exhibiting, for every g-matrix system, an equivalent discrete pseudo-referential g-matrix system.

Theorem 6. *Let $\boldsymbol{A}^\flat = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system. For every $N^\flat$-g-matrix system $\mathbb{A}^\# = \langle \mathcal{A}^\#, \mathcal{T}^\# \rangle$, with $\mathcal{A}^\# = \langle \boldsymbol{A}^\#, \langle F^\#, \alpha^\# \rangle \rangle$, $\boldsymbol{A}^\# = \langle \mathbf{Sign}^\#, \mathrm{SEN}^\#, N^\# \rangle$, there exists a discrete pseudo referential g-matrix system $\mathbb{A} = \langle \mathcal{A}, \mathcal{T} \rangle$ relative to some $(\mathrm{PTS}, \mathcal{S})$, such that $\mathcal{I}^{\mathbb{A}} = \mathcal{I}^{\mathbb{A}^\#}$.*

Proof: Let $\boldsymbol{A}^\flat = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system. Consider an $N^\flat$-g-matrix system $\mathbb{A}^\# = \langle \mathcal{A}^\#, \mathcal{T}^\# \rangle$, with $\mathcal{A}^\# = \langle \boldsymbol{A}^\#, \langle F^\#, \alpha^\# \rangle \rangle$, $\boldsymbol{A}^\# = \langle \mathbf{Sign}^\#, \mathrm{SEN}^\#, N^\# \rangle$ and $\mathcal{T}^\# = \{T^{\# i} : i \in I\}$.

For all $\Sigma \in |\mathbf{Sign}^\#|$, consider a collection $\{x^i_\Sigma : i \in I\}$, where, for all $i \in I$, $x^i_\Sigma \notin \mathrm{SEN}^\#(\Sigma)$ and, for all $i, j \in I$, with $i \neq j$, $x^i_\Sigma \neq x^j_\Sigma$.

Now define

$$\mathrm{PTS}(\Sigma) = \mathrm{SEN}^\#(\Sigma) \cup \{x^i_\Sigma : i \in I\}, \text{ for all } \Sigma \in |\mathbf{Sign}^\#|.$$

Moreover, let $\mathcal{S} = \{S^{\Sigma,i} : \Sigma \in |\mathbf{Sign}^\#|, i \in I\}$, where, for all $\Sigma \in |\mathbf{Sign}^\#|$ and all $i \in I$, $S^{\Sigma,i} = \{S^{\Sigma,i}_{\Sigma'}\}_{\Sigma' \in |\mathbf{Sign}^\#|}$ is defined by setting

$$S^{\Sigma,i}_{\Sigma'} = \begin{cases} \{x^i_\Sigma\}, & \text{if } \Sigma' = \Sigma \\ \varnothing, & \text{if } \Sigma' \neq \Sigma \end{cases}, \text{ for all } \Sigma' \in |\mathbf{Sign}^\#|.$$

Next, define, for all $\Sigma \in |\mathbf{Sign}^\#|$ and all $\varphi \in \mathrm{SEN}^\#(\Sigma)$, $X_\varphi \subseteq \mathrm{PTS}(\Sigma)$, by

$$p \in X_\varphi \quad \Leftrightarrow \quad p = \varphi \text{ or } (\exists i \in I)(p = x^i_\Sigma \text{ and } \varphi \in T^{\#,i}_\Sigma).$$

Claim: For all $\Sigma \in |\mathbf{Sign}^\#|$ and all $\varphi, \psi \in \mathrm{SEN}^\#(\Sigma)$, $X_\varphi = X_\psi$ if and only if $\varphi = \psi$.

Proof of the Claim: The "if" direction is obvious. For the "only if", reasoning by contraposition, we note that if $\varphi \neq \psi$, then $\varphi \in X_\varphi$, whereas $\varphi \notin X_\psi$. Therefore $X_\varphi \neq X_\psi$. □

Now define, for all $\Sigma \in |\mathbf{Sign}^\#|$,

$$\mathrm{SEN}(\Sigma) = \{X_\varphi : \varphi \in \mathrm{SEN}^\#(\Sigma)\}$$

and, moreover, for all $\Sigma, \Sigma' \in |\mathbf{Sign}^\#|$ and all $f \in \mathbf{Sign}^\#(\Sigma, \Sigma')$, let $\mathrm{SEN}(f) : \mathrm{SEN}(\Sigma) \to \mathrm{SEN}(\Sigma')$ be given, for all $\varphi \in \mathrm{SEN}^\#(\Sigma)$, by

$$\mathrm{SEN}(f)(X_\varphi) = X_{\mathrm{SEN}^\#(f)(\varphi)}.$$

The fact that $\mathrm{SEN} : \mathbf{Sign}^\# \to \mathbf{Set}$, thus defined, is a functor follows from the fact that $\mathrm{SEN}^\#$ is a functor.

Next, for all $\sigma : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ in $N^\flat$, we define $\sigma : \mathrm{SEN}^k \to \mathrm{SEN}$ by letting, for all $\Sigma \in |\mathbf{Sign}^\#|$, $\sigma_\Sigma : \mathrm{SEN}(\Sigma)^k \to \mathrm{SEN}(\Sigma)$ be given by

$$\sigma_\Sigma(X_{\varphi_0}, \ldots, X_{\varphi_{k-1}}) = X_{\sigma^\#_\Sigma(\varphi_0, \ldots, \varphi_{k-1})},$$

for all $\varphi_0, \ldots, \varphi_{k-1} \in \mathrm{SEN}^\#(\Sigma)$.

This is well-defined by the preceding claim and, moreover, it is a bona fide natural transformation, since, for all $\Sigma, \Sigma' \in |\mathbf{Sign}^\#|$, $f \in \mathbf{Sign}^\#(\Sigma, \Sigma')$ and all $\varphi_0, \ldots, \varphi_{k-1} \in \mathrm{SEN}^\#(\Sigma)$, we have according to the preceding definitions,

$$\begin{array}{ccc}
\mathrm{SEN}^k(\Sigma) & \xrightarrow{\sigma_\Sigma} & \mathrm{SEN}(\Sigma) \\
{\scriptstyle \mathrm{SEN}^k(f)} \downarrow & & \downarrow {\scriptstyle \mathrm{SEN}(f)} \\
\mathrm{SEN}^k(\Sigma') & \xrightarrow[\sigma_{\Sigma'}]{} & \mathrm{SEN}(\Sigma')
\end{array}$$

$$\begin{aligned}
\mathrm{SEN}(f)(\sigma_\Sigma(X_{\varphi_0},\ldots,X_{\varphi_{k-1}}))
&= \mathrm{SEN}(f)(X_{\sigma^\#_\Sigma(\varphi_0,\ldots,\varphi_{k-1})})\\
&= X_{\mathrm{SEN}^\#(f)(\sigma^\#_\Sigma(\varphi_0,\ldots,\varphi_{k-1}))}\\
&= X_{\sigma^\#_{\Sigma'}(\mathrm{SEN}^\#(f)(\varphi_0),\ldots,\mathrm{SEN}^\#(f)(\varphi_{k-1}))}\\
&= \sigma_{\Sigma'}(X_{\mathrm{SEN}^\#(f)(\varphi_0)},\ldots,X_{\mathrm{SEN}^\#(f)(\varphi_{k-1})})\\
&= \sigma_{\Sigma'}(\mathrm{SEN}(f)(X_{\varphi_0},\ldots,X_{\varphi_{k-1}})).
\end{aligned}$$

Let N be the category consisting of all natural transformations σ, for $\sigma^\#$ in $N^\#$. Then the triple $\boldsymbol{A} = \langle \mathbf{Sign}^\#, \mathrm{SEN}, N\rangle$ is a referential $N^\flat$-algebraic system.

Define $\langle F,\alpha\rangle : \boldsymbol{A}^\flat \to \boldsymbol{A}$ as follows:

- $F : \mathbf{Sign}^\flat \to \mathbf{Sign}^\#$ is equal to $F^\# : \mathbf{Sign}^\flat \to \mathbf{Sign}^\#$;
- $\alpha : \mathrm{SEN}^\flat \to \mathrm{SEN}\circ F$ is defined by letting, for all $\Sigma \in |\mathbf{Sign}^\flat|$, $\alpha_\Sigma : \mathrm{SEN}^\flat(\Sigma) \to \mathrm{SEN}(F(\Sigma))$ be given by

$$\alpha_\Sigma(\varphi) = X_{\alpha^\#_\Sigma(\varphi)}, \quad \text{for all } \varphi \in \mathrm{SEN}^\#(\Sigma).$$

Again this definition makes $\alpha : \mathrm{SEN}^\flat \to \mathrm{SEN}\circ F$ a bona fide natural transformation, since, for all $\Sigma,\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$ and all $\varphi \in \mathrm{SEN}^\flat(\Sigma)$, we have

$$\begin{array}{ccc}
\mathrm{SEN}^\flat(\Sigma) & \xrightarrow{\ \alpha_\Sigma\ } & \mathrm{SEN}(F(\Sigma))\\
{\scriptstyle \mathrm{SEN}^\flat(f)}\downarrow & & \downarrow{\scriptstyle \mathrm{SEN}(F(f))}\\
\mathrm{SEN}^\flat(\Sigma') & \xrightarrow[\ \alpha_{\Sigma'}\ ]{} & \mathrm{SEN}(F(\Sigma'))
\end{array}$$

$$\begin{aligned}
\mathrm{SEN}(F(f))(\alpha_\Sigma(\varphi)) &= \mathrm{SEN}(F(f))(X_{\alpha^\#_\Sigma(\varphi)})\\
&= X_{\mathrm{SEN}^\#(F^\#(f))(\alpha^\#_\Sigma(\varphi))}\\
&= X_{\alpha^\#_{\Sigma'}(\mathrm{SEN}^\flat(f)(\varphi))}\\
&= \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\varphi)).
\end{aligned}$$

Moreover, $\langle F,\alpha\rangle : \boldsymbol{A}^\flat \to \boldsymbol{A}$ is an algebraic system morphism. Indeed, for all $\sigma^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ in $N^\flat$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\varphi_0,\ldots,\varphi_{k-1} \in \mathrm{SEN}^\flat(\Sigma)$, we have

$$\begin{array}{ccc}
\mathrm{SEN}^\flat(\Sigma)^k & \xrightarrow{\sigma^\flat_\Sigma} & \mathrm{SEN}^\flat(\Sigma) \\
\Big\downarrow{\scriptstyle \alpha^k_\Sigma} & & \Big\downarrow{\scriptstyle \alpha_\Sigma} \\
\mathrm{SEN}(F(\Sigma))^k & \xrightarrow[\sigma_{F(\Sigma)}]{} & \mathrm{SEN}(F(\Sigma))
\end{array}$$

$$\begin{array}{rcl}
\alpha_\Sigma(\sigma^\flat_\Sigma(\varphi_0,\dots,\varphi_{k-1})) & = & X_{\alpha^\#_\Sigma(\sigma^\flat_\Sigma(\varphi_0,\dots,\varphi_{k-1}))} \\
 & = & X_{\sigma^\#_{F^\#(\Sigma)}(\alpha^\#_\Sigma(\varphi_0),\dots,\alpha^\#_\Sigma(\varphi_{k-1}))} \\
 & = & \sigma_{F(\Sigma)}(X_{\alpha^\#_\Sigma(\varphi_0)},\dots,X_{\alpha^\#_\Sigma(\varphi_{k-1})}) \\
 & = & \sigma_{F(\Sigma)}(\alpha_\Sigma(\varphi_0),\dots,\alpha_\Sigma(\varphi_{k-1})).
\end{array}$$

Thus, the pair $\mathcal{A} = \langle \boldsymbol{A}, \langle F, \alpha\rangle\rangle$ is an interpreted referential $N^\flat$-algebraic system.

Let $\mathbb{A} = \langle \mathcal{A}, \mathcal{T}\rangle$ be the discrete pseudo-referential $N^\flat$-g-matrix system relative to $(\mathrm{PTS}, \mathcal{S})$, where $\mathcal{S} = \{S^{\Sigma,i} : \Sigma \in |\mathbf{Sign}^\#|, i \in I\}$, as before, with $\mathcal{T}^\# = \{T^{\#i} : i \in I\}$ being the collection of filter families of the g-matrix system $\mathbb{A}^\#$.

Then, for all $i \in I$ and for all $\Sigma \in |\mathbf{Sign}^\#|$, we have $T^{\Sigma,i} = \{T^{\Sigma,i}_{\Sigma'}\}_{\Sigma' \in |\mathbf{Sign}^\#|}$, where, for all $\Sigma' \in |\mathbf{Sign}^\#|$,

$$\begin{array}{rcl}
T^{\Sigma,i}_{\Sigma'} & = & \{X \in \mathrm{SEN}(\Sigma') : X \cap S^{\Sigma,i}_{\Sigma'} \neq \varnothing\} \\
 & = & \left\{ \begin{array}{ll} \varnothing, & \text{if } \Sigma' \neq \Sigma, \\ \{X_\varphi : x^i_\Sigma \in X_\varphi, \varphi \in \mathrm{SEN}^\#(\Sigma)\}, & \text{if } \Sigma' = \Sigma \end{array} \right. \\
 & = & \left\{ \begin{array}{ll} \varnothing, & \text{if } \Sigma' \neq \Sigma \\ \{X_\varphi : \varphi \in T^{\#i}_\Sigma\}, & \text{if } \Sigma' = \Sigma \end{array} \right.
\end{array}$$

Now notice that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\varphi \in \mathrm{SEN}^\flat(\Sigma)$, we have that, for all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $i \in I$,

$$\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\varphi)) \in T^{F(\Sigma'),i}_{F(\Sigma')} \quad \text{iff} \quad \alpha^\#_{\Sigma'}(\mathrm{SEN}^\flat(f)(\varphi)) \in T^{\#i}_{F(\Sigma')}. \tag{7.1}$$

Equation (7.1) is true because, from the expression obtained from $T^{\Sigma,i}$ above, we obtain

$$\begin{array}{rll}
\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\varphi)) \in T^{F(\Sigma'),i}_{F(\Sigma')} & \text{iff} & X_{\alpha^\#_{\Sigma'}(\mathrm{SEN}^\flat(f)(\varphi))} \in \{X_\varphi : \varphi \in T^{\#i}_{F(\Sigma')}\} \\
 & \text{iff} & \alpha^\#_{\Sigma'}(\mathrm{SEN}^\flat(f)(\varphi)) \in T^{\#i}_{F(\Sigma')}.
\end{array}$$

Finally, we get the desired conclusion expressed in the following

Claim: $\mathcal{I}^{\mathbb{A}} = \mathcal{I}^{\mathbb{A}^{\#}}$.

Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\Phi \cup \{\varphi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$. Then we have $\varphi \in C^{\mathbb{A}}_\Sigma(\Phi)$ iff, for all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $i \in I$,

$$\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi)) \subseteq T^{F(\Sigma'),i}_{F(\Sigma')} \quad \text{implies} \quad \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\varphi)) \in T^{F(\Sigma'),i}_{F(\Sigma')},$$

iff, by Equivalence (7.1), for all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $i \in I$,

$$\alpha^{\#}_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi)) \subseteq T^{\#i}_{F(\Sigma')} \quad \text{implies} \quad \alpha^{\#}_{\Sigma'}(\mathrm{SEN}^\flat(f)(\varphi)) \in T^{\#i}_{F(\Sigma')},$$

iff $\varphi \in C^{\mathbb{A}^{\#}}_\Sigma(\Phi)$. Since $C^{\mathbb{A}} = C^{\mathbb{A}^{\#}}$, we conclude that $\mathcal{I}^{\mathbb{A}} = \mathcal{I}^{\mathbb{A}^{\#}}$, as required. □

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