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LOGICS WITH IMPOSSIBILITY AS THE NEGATION AND REGULAR EXTENSIONS OF THE DEONTIC LOGIC D2*

Dedicated to Professor dr hab. Grzegorz Malinowski
with our thanks for having had the honour of being able to co-operate with him

Abstract

In [1] J.-Y. Béziau formulated a logic called **Z**. Béziau's idea was generalized independently in [6] and [7]. A family of logics to which **Z** belongs is denoted in [7] by $\mathcal{K}$. In particular, it has been shown in [6] and [7] that there is a correspondence between normal modal logics and logics from the class $\mathcal{K}$. Similar, but only partial results has been obtained also for regular logics (see [8] and [9]).

In (Došen, [2]) a logic **N** has been investigated in the language with negation, implication, conjunction and disjunction by axioms of positive intuitionistic logic, the right-to-left part of the second de Morgan law, and the rules of modus ponens and contraposition. From the semantical point of view the negation used by Došen is the modal operator of impossibility. It is known this operator is a characteristic of the modal interpretation of intuitionistic negation (see [3, p. 300]). In the present paper we consider an extension of **N** denoted by $\mathbf{N}^+$. We will prove that every extension of $\mathbf{N}^+$ that is closed under the same rules as $\mathbf{N}^+$, corresponds to a regular logic being an extension of the regular deontic logic **D2**[1]

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[1]Notice that '**D2**' has nothing to do with notation for Jaśkowski's logic D_2. **D2** was introduced by Lemmon ([4]).

(see [4] and [13]). The proved correspondence allows to obtain from soundness-completeness result for any given regular logic containing **D2**, similar adequacy theorem for the respective extension of the logic $\mathbf{N}^+$.

Keywords: non-classical negation, modalized negation, impossibility, correspondence, regular modal logics, the smallest regular deontic logic D2

Introduction

The main feature of the logic **Z** relies on understanding of negation as "it is not necessary" ([1]). While defining this logic, Béziau used modal logic **S**5. It appears that logics with "it is not necessary" or equivalently "it is possible that not" as the negation can be used to express any normal modal logic ([6, 7]). However this could not be repeated in a unified way in the case of regular logics ([8, 9]). Thus, in [10], next to "it is possible that not" the impossibility operator was also used to obtain more general result on the mentioned expressibility, but this time of some regular logics. For discussion of various negations in the context of the natural language one can consult [12]. Having in mind that the neighborhood semantics can be used in particular for characterisation of regular logics it is worth to mention that in [12, ch. 5] a framework by means of neighborhood semantics meant for analysis of various negative modalities is given.

In ([2]) a logic **N** has been investigated in the language with negation, implication, conjunction and disjunction by axioms of positive intuitionistic logic, the right-to-left part of the second de Morgan law, and the rules of modus ponens and contraposition. From the semantical point of view the negation used in the formulation of **N** is a modal operator of impossibility.

In the present paper we strengthen observations given in [10] using only impossibility connective. The smallest logic $\mathbf{N}^+$ that we are using here, is an extension of the mentioned logic **N**. The new translations presented in the current paper allow directly for obtaining an extension of $\mathbf{N}^+$ from any regular extension of the deontic logic **D2**.

1. Logics corresponding to regular extensions of D2

In the object language we can consider two negations: $\dot{\sim}$ (it is necessary that) and $\sim$ (it is possible that not).

Definition 1. Let $\mathrm{For}^{\dot{\sim}}_{\sim}$ be the set of all propositional formulas in the language with connectives $\{\dot{\sim}, \sim, \wedge, \vee, \rightarrow\}$ and the set of propositional variables Var.

Let us recall a class of logics considered in [10]:

Definition 2. Let $\mathcal{R}^{\dot{\sim}}_{\sim}$ be the class of all logics that are non-trivial subsets of $\mathrm{For}^{\dot{\sim}}_{\sim}$, containing the full positive classical logic in the language $\{\wedge, \vee, \rightarrow\}$, including the following formulas:

$$\dot{\sim}\, p \wedge \dot{\sim}\, q \rightarrow \dot{\sim}(p \vee q), \tag{$\mathrm{dM2}^{\dot{\sim}}_{\leftarrow}$}$$

$$\sim p \rightarrow (\dot{\sim}(p \rightarrow \dot{\sim}(q \rightarrow q)) \rightarrow \dot{\sim}(q \rightarrow q)) \tag{$\mathrm{df}^{\sim}_{\rightarrow}$}$$

$$(\dot{\sim}(p \rightarrow \dot{\sim}(q \rightarrow q)) \rightarrow \dot{\sim}(q \rightarrow q)) \rightarrow \sim p \tag{$\mathrm{df}^{\sim}_{\leftarrow}$}$$

$$\dot{\sim}\, p \rightarrow \sim p \tag{$\mathrm{D}^{\dot{\sim}}_{\sim}$}$$

$$((p \rightarrow \dot{\sim}(q \rightarrow q)) \rightarrow \dot{\sim}(q \rightarrow q)) \rightarrow p \tag{dneg}$$

and closed under modus ponens, ($\mathsf{CONTR}^{\dot{\sim}}$):

$$\frac{\vdash A \rightarrow B}{\vdash \dot{\sim}\, B \rightarrow \dot{\sim}\, A} \tag{$\mathsf{CONTR}^{\dot{\sim}}$}$$

and any substitution.

Remark 1. If we would put:

$$\bot_a := \dot{\sim}(a \rightarrow a) \tag{1.1}$$

$$\neg_a A := A \rightarrow \bot_a \tag{1.2}$$

we obtain respectively the following forms of formulas ($\mathrm{df}^{\sim}_{\rightarrow}$), ($\mathrm{df}^{\sim}_{\leftarrow}$), and (dneg):

$$\sim p \rightarrow \neg_q \dot{\sim} \neg_q p$$

$$\neg_q \dot{\sim} \neg_q p \rightarrow \sim p$$

$$\neg_q \neg_q p \rightarrow p$$

However, one should keep in mind that the above abbreviations are absent in the considered object language and should be only treated as shortcuts and a certain facilitation in reading formulas.

Let $\mathrm{For}^{\mathrm{M}}$ denote the set of all modal formulas in the language with $\{\neg, \wedge, \vee, \rightarrow, \Box\}$. A regular logic is a set $\boldsymbol{L} \subseteq \mathrm{For}^{\mathrm{M}}$, such that $\boldsymbol{L}$ contains all classical tautologies, $(\mathrm{K}) \in \boldsymbol{L}$

$$\Box(p \to q) \to (\Box p \to \Box q) \tag{K}$$

and $\boldsymbol{L}$ is closed under modus ponens, substitution and the monotonicity rule (MON):

$$\frac{\vdash A \to B}{\vdash \Box A \to \Box B} \tag{MON}$$

D2 (see [4]) is the smallest regular logic containing the axiom (D):

$$\Box p \to \neg\Box\neg p \tag{D}$$

The intended meaning of $\dot{\sim}$ and $\sim$ is $\Box\neg$ and $\Diamond\neg$, respectively. In [10] a correspondence between elements of $\mathcal{R}^{\dot{\sim}}_{\sim}$ and regular extensions **D2** was investigated. In the present paper we consider a simplified version of the class $\mathcal{R}^{\dot{\sim}}_{\sim}$, with logics in the language with $\dot{\sim}$ as the only negation. Let $\mathrm{For}^{\dot{\sim}}$ denote the obtained, reduced language.

Definition 3 (Counterparts of extensions of **D2**). Let $\mathcal{R}^{\dot{\sim}}$ be the class of all logics being subsets of $\mathrm{For}^{\dot{\sim}}$, containing the full positive classical logic $\mathbf{CL}^+$ in the language $\{\wedge, \vee, \to\}$, including the formulas ($\mathrm{dM2}^{\dot{\sim}}_{\leftarrow}$) and (dn):

$$\big((p \to \bot_p) \to \bot_p\big) \to p \tag{dn}$$

and closed under modus ponens, ($\mathsf{CONTR}^{\dot{\sim}}$) and any substitution.

Let us denote by $\mathbf{N}^+$ the smallest logic in $\mathcal{R}^{\dot{\sim}}$

The fact that $A \to B \in \mathbf{S}$ and $B \to A \in \mathbf{S}$ is denoted as: $A \leftrightarrow B \in \mathbf{S}$.

Fact 1. *The following formulas belong to every logic in* $\mathcal{R}^{\dot{\sim}}$

$$p \to ((p \to \bot_q) \to r) \tag{$\mathrm{DS}^{\dot{\sim}}$}$$

$$\big((p \to \bot_q) \to \bot_r\big) \leftrightarrow p \tag{$\mathrm{dn}_{\leftrightarrow}$}$$

$$\dot{\sim}(p \to \bot_q) \to (\dot{\sim} p \to \bot_q) \tag{$\mathrm{D}^{\dot{\sim}}$}$$

Proof: The case of ($\mathrm{DS}^{\dot{\sim}}$). By positive logic we have

$$p \to ((p \to \bot_q) \to \bot_q) \tag{1.3}$$

$$\bot_r \to \big((r \to \bot_r) \to \bot_r\big) \tag{1.4}$$

but due to ($\mathsf{CONTR}^{\dot{\sim}}$) applied to $(r \to r) \to (q \to q)$, we have $\dot{\sim}(q \to q) \to \dot{\sim}(r \to r)$, i.e. $\bot_q \to \bot_r$, so by (dn), (1.4) and again using $\mathbf{CL}^+$

$$\bot_q \to r \tag{1.5}$$

Thus, the required thesis follows from (1.3) and (1.5) by $\mathbf{CL}^+$.

The case of $(\mathrm{dn}_{\leftrightarrow})$. Right to left implication is just a special case of $(\mathrm{DS}^{\dot\sim})$.

Left to right implication follows again by application of $(\mathsf{CONTR}^{\dot\sim})$ to $(q \to q) \to (p \to p)$ and $(p \to p) \to (r \to r)$, and next (dn) and $\mathbf{CL}^{+}$.

The case of $(\mathrm{D}^{\dot\sim})$. By $\mathbf{CL}^{+}$ and substitution we have $(p \to \bot_q) \vee p$, so also $(q \to q) \to (p \to \bot_q) \vee p$. Using $(\mathsf{CONTR}^{\dot\sim})$ we have: $\dot\sim((p \to \bot_q) \vee p) \to \bot_q$. Thus by $(\mathrm{dM2}^{\dot\sim}_{\leftarrow})$ and $\mathbf{CL}^{+}$ we get $(\mathrm{D}^{\dot\sim})$. □

Remark 2. Using both abbreviations given in Remark 1 we can respectively write (dn), $(\mathrm{DS}^{\dot\sim})$, $(\mathrm{dn}_{\leftrightarrow})$ and $(\mathrm{D}^{\dot\sim})$ as follows:

$$\neg_p \neg_p p \to p$$
$$p \to (\neg_q p \to r)$$
$$\neg_r \neg_q p \leftrightarrow p$$
$$\dot\sim \neg_q p \to \neg_q \dot\sim p$$

2. Modalising and un-modalising translations

We will use the following translation as "modalisation":

DEFINITION 4. Let $-^{\mathtt{m}} : \mathrm{For}^{\dot\sim} \longrightarrow \mathrm{For}^{\mathrm{M}}$ be a function satisfying for any $a \in \mathrm{Var}, A, B \in \mathrm{For}^{\mathrm{M}}$ the following conditions:

1. $(a)^{\mathtt{m}} = a$,
2. $(\dot\sim A)^{\mathtt{m}} = \begin{cases} \Box((B)^{\mathtt{m}}) & \text{if } A = \neg_a B \text{ for some } B \in \mathrm{For}^{\dot\sim} \text{ and } a \in \mathrm{Var}, \\ \Box\neg((A)^{\mathtt{m}}) & \text{otherwise,} \end{cases}$
3. $(A \mathbin{\S} B)^{\mathtt{m}} = (A^{\mathtt{m}} \mathbin{\S} B^{\mathtt{m}})$, for $\S \in \{\wedge, \vee\}$,
4. $(A \to B)^{\mathtt{m}} = \begin{cases} \neg(A)^{\mathtt{m}} & \text{if } B = \bot_a \text{ for some } a \in \mathrm{Var}, \\ A^{\mathtt{m}} \to B^{\mathtt{m}} & \text{otherwise.} \end{cases}$

We will need translations that will be surjective. It is a modified version of translations used in [7, 10]:

DEFINITION 5. Let $-^{\mathtt{u}\dot\sim} : \mathrm{For}^{\mathrm{M}} \longrightarrow \mathrm{For}^{\dot\sim}$ be a function satisfying for any $a \in \mathrm{Var}$ and $A, B \in \mathrm{For}$ the following conditions:

1. $(a)^{\mathtt{u}\dot\sim} = a$,
2. $(\neg A)^{\mathtt{u}\dot\sim} = \neg_p((A)^{\mathtt{u}\dot\sim})$,
3. $(A \S B)^{\mathtt{u}\dot\sim} = ((A)^{\mathtt{u}\dot\sim}) \S ((B)^{\mathtt{u}\dot\sim})$, for $\S \in \{\wedge, \vee, \to\}$,

4. $(\Box A)^{\mathtt{u}\dot\sim} = \begin{cases} \dot\sim((B)^{\mathtt{u}\dot\sim}) & \text{if } A = \neg B, \text{ for some } B \in \mathrm{For}^{\mathrm{M}}, \\ \dot\sim \neg_p((A)^{\mathtt{u}\dot\sim}) & \text{otherwise.} \end{cases}$

Fact 2. *For any* $A \in \mathrm{For}^{\mathrm{M}}$*,* $a \in \mathrm{Var}$ *and any regular logic* $\mathbf{S}$ *containing* $\mathbf{D2}$ *we have that* $(A \to \Box\neg(a \to a)) \leftrightarrow \neg A \in \mathbf{S}$*.*

Lemma 1. *For any* $A \in \mathrm{For}^{\dot\sim}$*,* $D \in \mathrm{For}^{\mathrm{M}}$ *we have:*

$$((A)^{\mathtt{m}})^{\mathtt{u}\dot\sim} \leftrightarrow A \in \mathbf{N}^{+} \tag{2.1}$$

$$((D)^{\mathtt{u}\dot\sim})^{\mathtt{m}} \leftrightarrow D \in \mathbf{D2} \tag{2.2}$$

Proof: The case of a variable is obvious for both conditions.

The case of (2.1). For the case of negation consider a formula of the form $\dot\sim A$. Assume that $A = \neg_a B$ for some $B \in \mathrm{For}^{\dot\sim}$ and $a \in \mathrm{Var}$. We have: $((\dot\sim A)^{\mathtt{m}})^{\mathtt{u}\dot\sim} = (\Box((B)^{\mathtt{m}}))^{\mathtt{u}\dot\sim}$. We consider two cases: either $(B)^{\mathtt{m}}$ is of the form of negation, or not.

The first case means that $B = C \to \bot_b$, for some $C \in \mathrm{For}^{\dot\sim}$ and $b \in \mathrm{Var}$. Then $(\Box((B)^{\mathtt{m}}))^{\mathtt{u}\dot\sim} = (\Box\neg((C)^{\mathtt{m}}))^{\mathtt{u}\dot\sim} = \dot\sim((C)^{\mathtt{m}})^{\mathtt{u}\dot\sim}$. But by inductive hypothesis and the rule $(\mathsf{CONTR}^{\dot\sim})$ we have: $\dot\sim((C)^{\mathtt{m}})^{\mathtt{u}\dot\sim} \leftrightarrow \dot\sim C$. But by $(\mathrm{dn}_{\leftrightarrow})$ given in Fact 1 and $(\mathsf{CONTR}^{\dot\sim})$, $\dot\sim C \leftrightarrow \dot\sim((C \to \bot_b) \to \bot_a)$ and the right-hand side formula of the last equivalence is just the formula $\dot\sim A$.

In the second case one can see that: $(\Box((B)^{\mathtt{m}}))^{\mathtt{u}\dot\sim} = \dot\sim(((B)^{\mathtt{m}})^{\mathtt{u}\dot\sim} \to \bot_p)$. By the inductive hypothesis, positive logic and $(\mathsf{CONTR}^{\dot\sim})$ we have: $\dot\sim(((B)^{\mathtt{m}})^{\mathtt{u}\dot\sim} \to \bot_p) \leftrightarrow \dot\sim(B \to \bot_a)$. But the formula on the right-hand side of the last equivalence is the formula $\dot\sim A$.

Assume that A is not of the form of $\neg_a B$. By definitions and inductive hypothesis: $((\dot\sim A)^{\mathtt{m}})^{\mathtt{u}\dot\sim} = (\Box\neg((A)^{\mathtt{m}}))^{\mathtt{u}\dot\sim} = \dot\sim((A)^{\mathtt{m}})^{\mathtt{u}\dot\sim} \leftrightarrow \dot\sim A$.

For the case of implication of the form $(A \to \bot_a)$, we have: $((A \to \bot_a)^{\mathtt{m}})^{\mathtt{u}\dot\sim} = (\neg((A)^{\mathtt{m}}))^{\mathtt{u}\dot\sim} = (((A)^{\mathtt{m}})^{\mathtt{u}\dot\sim} \to \bot_p) \leftrightarrow (A \to \bot_p)$. Finally, $(A \to \bot_p) \leftrightarrow (A \to \bot_a) \in \mathbf{N}^{+}$.

For the other case of a formula of the form $A \to B$, assume that B is not of the form of $\bot_a$, where a would be a variable. We have $((A \to B)^{\mathtt{m}})^{\mathtt{u}\dot\sim} = ((A)^{\mathtt{m}} \to (B)^{\mathtt{m}})^{\mathtt{u}\dot\sim} = ((A)^{\mathtt{m}})^{\mathtt{u}\dot\sim} \to ((B)^{\mathtt{m}})^{\mathtt{u}\dot\sim} \leftrightarrow (A \to B)$. The cases of $\wedge$ and $\vee$ are being proved analogously to this case.

For (2.2), consider the case of negation. We see that $((\neg A)^{\mathtt{u}\dot\sim})^{\mathtt{m}} = ((A)^{\mathtt{u}\dot\sim} \to \bot_p)^{\mathtt{m}} = \neg((A)^{\mathtt{u}\dot\sim})^{\mathtt{m}} \leftrightarrow \neg A$, where the last equivalence holds by inductive hypothesis and extensionality.

Now, let us consider implication of the form $A \to B$. We have two cases: the first that $(B)^{\mathtt{u}\dot{\sim}}$ equals $\bot_a$ for some a, and the second that it does not. The first case means that B equals $\Box\neg(a \to a)$ for some variable a. We have: $((A \to \Box\neg(a \to a))^{\mathtt{u}\dot{\sim}})^{\mathtt{m}} = ((A)^{\mathtt{u}\dot{\sim}} \to (\Box\neg(a \to a))^{\mathtt{u}\dot{\sim}})^{\mathtt{m}} = ((A)^{\mathtt{u}\dot{\sim}} \to \bot_a)^{\mathtt{m}} = \neg(((A)^{\mathtt{u}\dot{\sim}})^{\mathtt{m}})$.

By the inductive hypothesis, using extensionality for regular logics and Fact 2 we have $\neg((A)^{\mathtt{u}\dot{\sim}})^{\mathtt{m}} \leftrightarrow \neg A$ and $\neg A \leftrightarrow (A \to \Box\neg(a \to a)) \in \mathbf{D2}$.

If $(B)^{\mathtt{u}\dot{\sim}}$ is not of the form $\bot_a$ that is B does not equal $\Box\neg(a \to a)$, the proof goes as follows: $((A \to B)^{\mathtt{u}\dot{\sim}})^{\mathtt{m}} = ((A)^{\mathtt{u}\dot{\sim}} \to (B)^{\mathtt{u}\dot{\sim}})^{\mathtt{m}} = ((A)^{\mathtt{u}\dot{\sim}})^{\mathtt{m}} \to ((B)^{\mathtt{u}\dot{\sim}})^{\mathtt{m}}$. And by inductive hypothesis and positive logic $\ulcorner(((A)^{\mathtt{u}\dot{\sim}})^{\mathtt{m}} \to ((B)^{\mathtt{u}\dot{\sim}})^{\mathtt{m}}) \leftrightarrow (A \to B)\urcorner \in \mathbf{S}$. The cases of $\wedge$ and $\vee$ are also being proved straightforward. $\Box$

LEMMA 2. *For any* $a_1, \ldots, a_n \in \mathrm{Var}$, A, $C_1, \ldots, C_1 \in \mathrm{For}^{\mathrm{M}}$ *and* B, C $D_1, \ldots, D_1 \in \mathrm{For}^{\dot{\sim}}$

1. $(A(a_1/C_1, \ldots, a_n/C_n))^{\mathtt{u}\dot{\sim}} \leftrightarrow (A)^{\mathtt{u}\dot{\sim}}(a_1/(C_1)^{\mathtt{u}\dot{\sim}}, \ldots, a_n/(C_n)^{\mathtt{u}\dot{\sim}}) \in \mathbf{N}^+$
2. *If* $B \leftrightarrow C \in \mathbf{N}^+$, *then* $B(a_1/D_1, \ldots, a_n/D_n) \leftrightarrow C(a_1/D_1, \ldots, a_n/D_n) \in \mathbf{N}^+$.

PROOF: 1. The proof goes by induction on the complexity of a formula. For $a \in \mathrm{Var}$ assume that $a = a_i$ for some $1 \leqslant i \leqslant n$. We have $(a(a_1/(C_1), \ldots, a_n/(C_n)))^{\mathtt{u}\dot{\sim}} = (C_i)^{\mathtt{u}\dot{\sim}} = ((a_i)^{\mathtt{u}\dot{\sim}})(a_i/C_i)^{\mathtt{u}\dot{\sim}} = (a)^{\mathtt{u}\dot{\sim}}(a_1/(C_1)^{\mathtt{u}\dot{\sim}}, \ldots, a_n/(C_n)^{\mathtt{u}\dot{\sim}})$.

Assume that the thesis holds for formulas B and C. The case of negation:

$$
\begin{gathered}
((\neg B)(a_1/C_1, \ldots, a_n/C_n))^{\mathtt{u}\dot{\sim}} = (\neg(B(a_1/C_1, \ldots, a_n/C_n)))^{\mathtt{u}\dot{\sim}} = \\
((B(a_1/C_1, \ldots, a_n/C_n))^{\mathtt{u}\dot{\sim}} \to \bot_p) \overset{\text{ind. hyp. and } \mathbf{CL}^+}{\leftrightarrow} \\
(((B)^{\mathtt{u}\dot{\sim}}(a_1/(C_1)^{\mathtt{u}\dot{\sim}}, \ldots, a_n/(C_n)^{\mathtt{u}\dot{\sim}})) \to \bot_p) \leftrightarrow \\
\big(((B)^{\mathtt{u}\dot{\sim}}(a_1/(C_1)^{\mathtt{u}\dot{\sim}}, \ldots, a_n/(C_n)^{\mathtt{u}\dot{\sim}})) \to \\
\bot_p(a_1/(C_1)^{\mathtt{u}\dot{\sim}}, \ldots, a_n/(C_n)^{\mathtt{u}\dot{\sim}})\big) \leftrightarrow \\
((B)^{\mathtt{u}\dot{\sim}} \to \bot_p)(a_1/(C_1)^{\mathtt{u}\dot{\sim}}, \ldots, a_n/(C_n)^{\mathtt{u}\dot{\sim}}) = \\
(\neg B)^{\mathtt{u}\dot{\sim}}(a_1/(C_1)^{\mathtt{u}\dot{\sim}}, \ldots, a_n/(C_n)^{\mathtt{u}\dot{\sim}})
\end{gathered}
$$

The case of implication (cases of $\wedge$ and $\vee$ are being proved similarly):

$$((B \to C)(a_1/(C_1), \ldots, a_n/(C_n)))^{\mathtt{u}\dot{\sim}} =$$
$$((B(a_1/(C_1), \ldots, a_n/(C_n))) \to (C(a_1/(C_1), \ldots, a_n/(C_n))))^{\mathtt{u}\dot{\sim}} \leftrightarrow$$
$$(B(a_1/(C_1), \ldots, a_n/(C_n)))^{\mathtt{u}\dot{\sim}} \to (C(a_1/(C_1), \ldots, a_n/(C_n)))^{\mathtt{u}\dot{\sim}} \overset{\text{ind. hyp. and } \mathbf{CL}^+}{\leftrightarrow}$$
$$((B)^{\mathtt{u}\dot{\sim}}(a_1/(C_1)^{\mathtt{u}\dot{\sim}}, \ldots, a_n/(C_n)^{\mathtt{u}\dot{\sim}})) \to$$
$$\to ((C)^{\mathtt{u}\dot{\sim}}(a_1/(C_1)^{\mathtt{u}\dot{\sim}}, \ldots, a_n/(C_n)^{\mathtt{u}\dot{\sim}}))$$
$$((B)^{\mathtt{u}\dot{\sim}} \to (C)^{\mathtt{u}\dot{\sim}})(a_1/(C_1)^{\mathtt{u}\dot{\sim}}, \ldots, a_n/(C_n)^{\mathtt{u}\dot{\sim}}) =$$
$$(B \to C)^{\mathtt{u}\dot{\sim}}(a_1/(C_1)^{\mathtt{u}\dot{\sim}}, \ldots, a_n/(C_n)^{\mathtt{u}\dot{\sim}})$$

For the case of necessity assume first that B is of the form of negation $\neg D$:

$$((\Box B)(a_1/(C_1), \ldots, a_n/(C_n)))^{\mathtt{u}\dot{\sim}} = ((\Box\neg D)(a_1/(C_1), \ldots, a_n/(C_n)))^{\mathtt{u}\dot{\sim}} =$$
$$(\Box\neg(D(a_1/(C_1), \ldots, a_n/(C_n))))^{\mathtt{u}\dot{\sim}} =$$
$$\dot{\sim}(D(a_1/(C_1), \ldots, a_n/(C_n)))^{\mathtt{u}\dot{\sim}} \overset{\text{ind. hyp. and } (\mathsf{CONTR}^{\dot{\sim}})}{\leftrightarrow}$$
$$\dot{\sim}((D)^{\mathtt{u}\dot{\sim}}(a_1/(C_1)^{\mathtt{u}\dot{\sim}}, \ldots, a_n/(C_n)^{\mathtt{u}\dot{\sim}})) =$$
$$(\dot{\sim}((D)^{\mathtt{u}\dot{\sim}}))(a_1/(C_1)^{\mathtt{u}\dot{\sim}}, \ldots, a_n/(C_n)^{\mathtt{u}\dot{\sim}}) \overset{\text{def. of } \mathtt{u}\dot{\sim}}{=}$$
$$(\Box\neg D)^{\mathtt{u}\dot{\sim}}(a_1/(C_1)^{\mathtt{u}\dot{\sim}}, \ldots, a_n/(C_n)^{\mathtt{u}\dot{\sim}}) =$$
$$(\Box B)^{\mathtt{u}\dot{\sim}}(a_1/(C_1)^{\mathtt{u}\dot{\sim}}, \ldots, a_n/(C_n)^{\mathtt{u}\dot{\sim}})$$

And finally let B be not of the form of negation, so also $(\Box(B(a_1/(C_1), \ldots, a_n/(C_n))))$ is not of the form of negation. Thus, we have:

$$((\Box B)(a_1/(C_1), \ldots, a_n/(C_n)))^{\mathtt{u}\dot{\sim}} = (\Box(B(a_1/(C_1), \ldots, a_n/(C_n))))^{\mathtt{u}\dot{\sim}} =$$
$$\dot{\sim}((B(a_1/(C_1), \ldots, a_n/(C_n)))^{\mathtt{u}\dot{\sim}} \to \bot_p) \overset{\text{ind. hyp. and } (\mathsf{CONTR}^{\dot{\sim}})}{\leftrightarrow}$$
$$\dot{\sim}(((B)^{\mathtt{u}\dot{\sim}}(a_1/(C_1)^{\mathtt{u}\dot{\sim}}, \ldots, a_n/(C_n)^{\mathtt{u}\dot{\sim}})) \to \bot_p) \leftrightarrow$$
$$\dot{\sim}\big(((B)^{\mathtt{u}\dot{\sim}}(a_1/(C_1)^{\mathtt{u}\dot{\sim}}, \ldots, a_n/(C_n)^{\mathtt{u}\dot{\sim}})) \to$$
$$\to \bot_p(a_1/(C_1)^{\mathtt{u}\dot{\sim}}, \ldots, a_n/(C_n)^{\mathtt{u}\dot{\sim}})\big) \leftrightarrow$$
$$\big(\dot{\sim}((B)^{\mathtt{u}\dot{\sim}} \to \bot_p)\big)(a_1/(C_1)^{\mathtt{u}\dot{\sim}}, \ldots, a_n/(C_n)^{\mathtt{u}\dot{\sim}}) =$$
$$(\Box B)^{\mathtt{u}\dot{\sim}}(a_1/(C_1)^{\mathtt{u}\dot{\sim}}, \ldots, a_n/(C_n)^{\mathtt{u}\dot{\sim}})$$

2. Follows by the fact that $\mathbf{N}^+$ is closed under substitution and standardly, that every substitution by definition is an automorphism on the set of formulas with respect to $\to$. $\Box$

Lemma 3. *For any* $a_1, \ldots, a_n \in \mathrm{Var}$, A, $C_1, \ldots, C_1 \in \mathrm{For}^{\dot{\sim}}$ *and* B, C $D_1, \ldots, D_1 \in \mathrm{For}^{\mathrm{M}}$

1. $(A(a_1/C_1, \ldots, a_n/C_n))^{\mathtt{m}} \leftrightarrow (A)^{\mathtt{m}}(a_1/(C_1)^{\mathtt{m}}, \ldots, a_n/(C_n)^{\mathtt{m}}) \in \mathbf{D2}$
2. *If* $B \leftrightarrow C \in \mathbf{D2}$, *then* $B(a_1/D_1, \ldots, a_n/D_n) \leftrightarrow C(a_1/D_1, \ldots, a_n/D_n)$ *belongs to* $\mathbf{D2}$.

Proof: The point 2 is a standard fact. The proof of 1 goes by induction on the complexity of a formula A. For a variable assume that $a = a_i$; we have

$$(a(a_1/C_1, \ldots, a_n/C_n))^{\mathtt{m}} = (C_i)^{\mathtt{m}} = (a_i)^{\mathtt{m}}(a_i/(C_1)^{\mathtt{m}})$$
$$(a)^{\mathtt{m}}(a_1/(C_1)^{\mathtt{m}}, \ldots, a_n/(C_n)^{\mathtt{m}})$$

Assume that the inductive thesis holds for any formula of the complexity not exceeding complexities of given formulas B and C.

The case of negation. Assume that $B = \neg_a D$.

$$(\dot{\sim} B(a_1/C_1, \ldots, a_n/C_n))^{\mathtt{m}} = (\dot{\sim}(B(a_1/C_1, \ldots, a_n/C_n)))^{\mathtt{m}}$$

If $a \notin \{a_1, \ldots, a_n\}$, then $(\dot{\sim}(B(a_1/C_1, \ldots, a_n/C_n)))^{\mathtt{m}} =$

$$\big(\dot{\sim}(D(a_1/C_1, \ldots, a_n/C_n) \to \bot_a)\big)^{\mathtt{m}} =$$
$$\Box(D(a_1/C_1, \ldots, a_n/C_n))^{\mathtt{m}} \overset{\text{ind. hyp. and extensionality}}{\leftrightarrow}$$
$$\Box((D)^{\mathtt{m}}(a_1/(C_1)^{\mathtt{m}}, \ldots, a_n/(C_n)^{\mathtt{m}})) =$$
$$(\Box(D)^{\mathtt{m}})(a_1/(C_1)^{\mathtt{m}}, \ldots, a_n/(C_n)^{\mathtt{m}}) = (\dot{\sim} B)^{\mathtt{m}}(a_1/(C_1)^{\mathtt{m}}, \ldots, a_n/(C_n)^{\mathtt{m}})$$

If $a \in \{a_1, \ldots, a_n\}$ say $a = a_i$, but C_i is a variable we act similarly as above. If $a = a_i$ and C_i is not a variable, while applying function $(-)^{\mathtt{m}}$ for the formula $\dot{\sim} B$ we have to use the second variant.

$$(\dot{\sim}(B(a_1/C_1, \ldots, a_n/C_n)))^{\mathtt{m}} =$$
$$\Box\neg\big((D \to \bot_a)(a_1/C_1, \ldots, a_n/C_n)\big)^{\mathtt{m}} \overset{\text{ind. hyp. and extensionality}}{\leftrightarrow}$$
$$\Box\neg\big((D \to \bot_a)^{\mathtt{m}}(a_1/(C_1)^{\mathtt{m}}, \ldots, a_n/(C_n)^{\mathtt{m}})\big) \leftrightarrow$$
$$\Box\neg\neg((D)^{\mathtt{m}}(a_1/(C_1)^{\mathtt{m}}, \ldots, a_n/(C_n)^{\mathtt{m}})) \leftrightarrow$$
$$(\Box(D)^{\mathtt{m}})(a_1/(C_1)^{\mathtt{m}}, \ldots, a_n/(C_n)^{\mathtt{m}}) = (\dot{\sim} B)^{\mathtt{m}}(a_1/(C_1)^{\mathtt{m}}, \ldots, a_n/(C_n)^{\mathtt{m}})$$

Assume that B is not of the form of $\neg_a D$.

$$((\dot\sim B)(a_1/C_1,\ldots,a_n/C_n))^{\mathtt{m}} = (\dot\sim(B(a_1/C_1,\ldots,a_n/C_n)))^{\mathtt{m}} =$$
$$\Box\neg\big(B(a_1/C_1,\ldots,a_n/C_n)\big)^{\mathtt{m}} \overset{\text{ind. hyp.}}{\leftrightarrow}$$
$$\Box\neg((B)^{\mathtt{m}}(a_1/(C_1)^{\mathtt{m}},\ldots,a_n/(C_n)^{\mathtt{m}})) \leftrightarrow$$
$$(\Box\neg(B)^{\mathtt{m}})(a_1/(C_1)^{\mathtt{m}},\ldots,a_n/(C_n)^{\mathtt{m}}) \leftrightarrow (\dot\sim B)^{\mathtt{m}}(a_1/(C_1)^{\mathtt{m}},\ldots,a_n/(C_n)^{\mathtt{m}})$$

The case of implication — consider the formula $B \to C$. Assume that $C = \bot_a$

$$((B\to C)(a_1/C_1,\ldots,a_n/C_n))^{\mathtt{m}} =$$
$$((B(a_1/C_1,\ldots,a_n/C_n)) \to (C(a_1/C_1,\ldots,a_n/C_n)))^{\mathtt{m}}$$

If $a \notin \{a_1,\ldots,a_n\}$

$$((B(a_1/C_1,\ldots,a_n/C_n)) \to (C(a_1/C_1,\ldots,a_n/C_n)))^{\mathtt{m}} =$$
$$((B(a_1/C_1,\ldots,a_n/C_n)) \to \bot_a)^{\mathtt{m}} =$$
$$\neg(B(a_1/C_1,\ldots,a_n/C_n))^{\mathtt{m}} \overset{\text{ind. hyp.}}{\leftrightarrow} \neg((B)^{\mathtt{m}}(a_1/(C_1)^{\mathtt{m}},\ldots,a_n/(C_n)^{\mathtt{m}})) =$$
$$(\neg(B)^{\mathtt{m}})(a_1/(C_1)^{\mathtt{m}},\ldots,a_n/(C_n)^{\mathtt{m}}) = (B\to C)^{\mathtt{m}}(a_1/(C_1)^{\mathtt{m}},\ldots,a_n/(C_n)^{\mathtt{m}})$$

If $a \in \{a_1,\ldots,a_n\}$ and respective substituted formula, let say C_i, is not a variable, then:

$$((B(a_1/C_1,\ldots,a_n/C_n)) \to (C(a_1/C_1,\ldots,a_n/C_n)))^{\mathtt{m}} =$$
$$(B(a_1/C_1,\ldots,a_n/C_n))^{\mathtt{m}} \to (C(a_1/C_1,\ldots,a_n/C_n))^{\mathtt{m}} \overset{\text{ind. hyp.}}{\leftrightarrow}$$
$$((B)^{\mathtt{m}}(a_1/(C_1)^{\mathtt{m}},\ldots,a_n/(C_n)^{\mathtt{m}})) \to ((C)^{\mathtt{m}}(a_1/(C_1)^{\mathtt{m}},\ldots,a_n/(C_n)^{\mathtt{m}})) =$$
$$((B)^{\mathtt{m}}(a_1/(C_1)^{\mathtt{m}},\ldots,a_n/(C_n)^{\mathtt{m}})) \to \Box\neg((C_i)^{\mathtt{m}} \to (C_i)^{\mathtt{m}}) \overset{\text{Fact 2 and substitution}}{\leftrightarrow}$$
$$(\neg(B)^{\mathtt{m}})(a_1/(C_1)^{\mathtt{m}},\ldots,a_n/(C_n)^{\mathtt{m}}) = (B\to C)^{\mathtt{m}}(a_1/(C_1)^{\mathtt{m}},\ldots,a_n/(C_n)^{\mathtt{m}})$$

Assume that C is not of the form of $\bot_a$ (cases of $\wedge$ and $\vee$ are being proved similarly):

$$((B\to C)(a_1/(C_1),\ldots,a_n/(C_n)))^{\mathtt{m}} =$$
$$((B(a_1/(C_1),\ldots,a_n/(C_n))) \to (C(a_1/(C_1),\ldots,a_n/(C_n))))^{\mathtt{m}} \leftrightarrow$$
$$(B(a_1/(C_1),\ldots,a_n/(C_n)))^{\mathtt{m}} \to (C(a_1/(C_1),\ldots,a_n/(C_n)))^{\mathtt{m}} \overset{\text{ind. hyp.}}{\leftrightarrow}$$
$$((B)^{\mathtt{m}}(a_1/(C_1)^{\mathtt{m}},\ldots,a_n/(C_n)^{\mathtt{m}})) \to ((C)^{\mathtt{m}}(a_1/(C_1)^{\mathtt{m}},\ldots,a_n/(C_n)^{\mathtt{m}})) \leftrightarrow$$
$$((B)^{\mathtt{m}} \to (C)^{\mathtt{m}})(a_1/(C_1)^{\mathtt{m}},\ldots,a_n/(C_n)^{\mathtt{m}}) =$$
$$(B\to C)^{\mathtt{m}}(a_1/(C_1)^{\mathtt{m}},\ldots,a_n/(C_n)^{\mathtt{m}})$$

□

3. Semantical correspondence

Let us recall notions used in [8].

Definition 6. 1. A *relational frame* (in short a *frame*) is a triple $\langle W, R, N\rangle$ consisting of a nonempty set W, a binary relation R on W, and a subset N of W. Elements of W, N, and $W \backslash N$ are called *worlds*, *normal worlds*, and *non-normal worlds*[2], respectively, while R is an *accessibility relation*.

2. A *valuation* is any function $v : \mathrm{Var} \longrightarrow 2^W$.
3. A *model* is a quadruple $\langle W, R, N, v\rangle$, where $\langle W, R, N\rangle$ is a frame and v is a valuation. We say that $\langle W, R, N, v\rangle$ is *based* on the frame $\langle W, R, N\rangle$.

We are using a reduct of the language with two negations (considered in [10]), so also validity and truth are meant accordingly. To keep the paper self-contained we recall these definitions.

Definition 7. A formula A is *true* in a world $w \in W$ under a valuation v (notation: $w \Vdash_v A$) iff

1. if A is a propositional variable,
   $w \Vdash_v A \Longleftrightarrow w \in v(A)$.
2. if A has the form $\dot{\sim} B$, for some formula B, then
   for $w \in N$:
   $w \Vdash_v \dot{\sim} B \Longleftrightarrow$ for every world w' such that wRw', it is not the case that $w' \Vdash_v B$ ($w' \nVdash_v B$ for short);
   for $w \in W \backslash N$: $w \nVdash_v \dot{\sim} B$,
3. if A is of the form $B \wedge C$, for some formulas B and C, then
   $w \Vdash_v B \wedge C \Longleftrightarrow w \Vdash_v B$ and $w \Vdash_v C$,
4. if A is of the form $B \vee C$, for some formulas B and C, then
   $w \Vdash_v B \vee C \Longleftrightarrow w \Vdash_v B$ or $w \Vdash_v C$,
5. if A is of the form $B \rightarrow C$, for some formulas B and C, then
   $w \Vdash_v B \rightarrow C \Longleftrightarrow w \nVdash_v B$ or $w \Vdash_v C$.

Definition 8. 1. A formula A is *true* in a model $M = \langle W, R, N, v\rangle$ (notation $M \Vdash_{\mathcal{R}} A$) iff $w \Vdash_v A$, for each $w \in W$.

[2]Since there are frames for which $N = W$, the considered class can be naturally treated as a superclass of the class of frames in the sense of [7].

2. A formula A is *valid* in a frame $\langle W, R, N\rangle$ iff it is true in all models based on $\langle W, R, N\rangle$.

For the notion of a model given in the point 3 of Definition 6 one can apply notions of truth and validity used for standard modal regular logics. Truth of a modal formula A in a world w by a valuation v will be denoted as usually: $w \vDash_v A$. Let us only recall the case for $\Box$:

1. if A has the form $\Box B$, for some formula B, then
   for $w \in N$: $w \Vdash_v \Box B \iff \forall_{w' \in R(w)} w' \Vdash_v B$;
   for $w \in W \backslash N$: $w \nVdash_v \Box B$.

The other cases are classical. We refer to seriality, but it is easily seen that for the case of considered logics one can equivalently use $(\mathrm{Ser_N})$.

$$\forall_{w \in N} \exists_{u \in W} (wRu) \tag{$\mathrm{Ser_N}$}$$

Lemma 4. *For any model $\langle W, R, N, v\rangle$ with serial accessibility relation, any $w \in W$, $A \in \mathrm{For}^{\dot{\sim}}$ and $B \in \mathrm{For}^{\mathrm{M}}$*

$$w \Vdash_v A \textit{ iff } w \vDash_v (A)^{\mathtt{m}}$$
$$w \vDash_v B \textit{ iff } w \Vdash_v (B)^{\mathtt{u}\dot{\sim}}$$

Proof: In both cases the proof goes by induction on the complexity of a formula. Since the case of variables is obvious, for the first equivalence assume that the thesis holds for any world in W and any formula of the complexity smaller then the complexity of a given formula A. We will consider the case of $\dot{\sim}$ and $\rightarrow$. Other cases are straightforward.

Assume that $A = \dot{\sim} B$, for some $B \in \mathrm{For}^{\dot{\sim}}$. Let us consider a world $w_0 \in W \backslash N$. We have $w_0 \nVdash_v \dot{\sim} B$. On the other hand by definition of $(-)^{\mathtt{m}}$ we have that either $(\dot{\sim} B)^{\mathtt{m}} = \Box(C)^{\mathtt{m}}$ if $B = C \rightarrow \bot_a$ for some $C \in \mathrm{For}^{\dot{\sim}}$ and $a \in \mathrm{Var}$, or $(\dot{\sim} B)^{\mathtt{m}} = \Box\neg(B)^{\mathtt{m}}$ otherwise. In both cases we have $w_0 \nVdash_v (\dot{\sim} B)^{\mathtt{m}}$. So the equivalence holds.

Now assume, that $w_0 \in N$. Let $w_0 \Vdash_v \dot{\sim} B$. Thus, for any $w' \in R(w_0)$: $w' \nVdash_v B$. By the inductive hypothesis for any $w' \in R(w_0)$: $w' \nvDash_v (B)^{\mathtt{m}}$, thus $w_0 \vDash_v \Box\neg((B)^{\mathtt{m}})$. If $(\dot{\sim} B)^{\mathtt{m}} = \Box\neg((B)^{\mathtt{m}})$ the implication has been proved, otherwise $B = C \rightarrow \bot_a$ for some formula C and a variable a, in this case $(\dot{\sim} B)^{\mathtt{m}} = \Box((C)^{\mathtt{m}})$. We see that $(B)^{\mathtt{m}} = (C \rightarrow \bot_a)^{\mathtt{m}} = \neg((C)^{\mathtt{m}})$, thus $w_0 \vDash_v \Box\neg\neg((C)^{\mathtt{m}})$, so also $w_0 \vDash_v \Box((C)^{\mathtt{m}})$ i.e., $w_0 \vDash_v (\dot{\sim} B)^{\mathtt{m}}$.

For the reverse direction assume that $w_0 \vDash_v (\dot{\sim} B)^{\mathtt{m}}$. If $B = \neg_a C$ for some formula C and a variable a, then $(\dot{\sim} B)^{\mathtt{m}} = \Box((C)^{\mathtt{m}})$, i.e. for every world w' accessible by R from w_0: $w' \models_v (C)^{\mathtt{m}}$, so by inductive hypothesis $w' \Vdash_v C$ for every such world. Since R is serial we also have $w' \nVdash_v \bot_a$, thus $w' \nVdash_v C \to \bot_a$ for every $w' \in R(w_0)$, hence $w_0 \Vdash_v \dot{\sim}(C \to \bot_a)$, i.e., $w_0 \Vdash_v \dot{\sim} B$. If B is not of the form $\neg_a C$, where $C \in \mathrm{For}^{\dot{\sim}}$ and $a \in \mathrm{Var}$, then $(\dot{\sim} B)^{\mathtt{m}} = \Box\neg((B)^{\mathtt{m}})$. So for every world w' accessible by R from w_0: $w' \not\models_v (B)^{\mathtt{m}}$ and by inductive hypothesis $w' \nVdash_v B$, i.e. $w_0 \Vdash_v \dot{\sim} B$.

For the functor of implication consider only the case of a formula of the form $B \to \bot_a$ assuming that $w_0 \Vdash_v B \to \bot_a$ for a given $w_0 \in W$. Since R is serial (what is relevant if $w_0 \in N$) and due to definition of validity in non-normal worlds we have $w \nVdash_v \bot_a$ for every $w \in W$, so $w_0 \nVdash_v B$. By inductive hypothesis $w_0 \not\models_v (B)^{\mathtt{m}}$, so $w_0 \models_v \neg((B)^{\mathtt{m}})$ and by definition of $(-)^{\mathtt{m}}$, $w_0 \models_v (B \to \bot_a)^{\mathtt{m}}$.

For the reverse implication assume that $w_0 \models_v (B \to \bot_a)^{\mathtt{m}}$ i.e. $w_0 \models_v \neg((B)^{\mathtt{m}})$, so $w_0 \not\models_v ((B)^{\mathtt{m}})$. By inductive hypothesis $w_0 \nVdash_v B$, that is $w_0 \Vdash_v B \to \bot_a$.

The case if $B \to C$ where C is not of the form $\bot_a$ and the cases of $\wedge$ and $\vee$ can be proved straightforward in both directions.

Assume that the second equivalence holds for any world in W and for every formula of the complexity smaller then the complexity of a given formula A. We will consider the cases of $\neg$, $\Box$ and $\to$. Other cases are straightforward. For the case of negation assume that $A = \neg B$. Due to seriality we know that for any world w it holds that $w \nVdash_v \bot_p$ and $w \not\models_v \Box\neg(p \to p)$. The following holds: $w \models_v \neg B$ iff $w \not\models_v B$ iff ($w \not\models_v B$ or $w \models_v \Box\neg(p \to p)$) iff ($w \nVdash_v (B)^{\mathtt{u}\dot{\sim}}$ or $w \Vdash_v \bot_p$) iff $w \Vdash_v ((B)^{\mathtt{u}\dot{\sim}} \to \bot_p)$ iff $w \Vdash_v (\neg B)^{\mathtt{u}\dot{\sim}}$.

Assume that A is of the form $\Box\neg B$ then: $w \models_v \Box\neg B$ iff $\forall_{w' \in R(w)} w' \models_v \neg B$ iff $\forall_{w' \in R(w)} w' \not\models_v B$ iff $\forall_{w' \in R(w)} w' \nVdash_v (B)^{\mathtt{u}\dot{\sim}}$ iff $w \Vdash_v \dot{\sim}((B)^{\mathtt{u}\dot{\sim}})$ iff $w \Vdash_v (\Box\neg B)^{\mathtt{u}\dot{\sim}}$.

Assume that A is of the form $\Box B$ but B is not a negation.

$$\begin{aligned} w \models_v \Box B \text{ iff } \forall_{w' \in R(w)} w' \models_v B \text{ iff } \forall_{w' \in R(w)} (w' \models_v B \text{ and } w' \not\models_v \Box\neg(p \to p)) \\ \text{iff } \forall_{w' \in R(w)} (w' \Vdash_v (B)^{\mathtt{u}\dot{\sim}} \text{ and } w' \nVdash_v \bot_p) \\ \text{iff } \forall_{w' \in R(w)} w' \nVdash_v ((B)^{\mathtt{u}\dot{\sim}} \to \bot_p) \\ \text{iff } w \Vdash_v \dot{\sim}((B)^{\mathtt{u}\dot{\sim}} \to \bot_p) \text{ iff } w \Vdash_v (\Box B)^{\mathtt{u}\dot{\sim}} \end{aligned}$$

Consider the case $A = B \to C$.

$$w \vDash_v B \to C \text{ iff } (w \nvDash_v B \text{ or } w \vDash_v C) \text{ iff } (w \nVdash_v (B)^{\mathtt{u}\dot\sim} \text{ or } w \Vdash_v (C)^{\mathtt{u}\dot\sim}) \text{ iff}$$
$$w \Vdash_v (B)^{\mathtt{u}\dot\sim} \to (C)^{\mathtt{u}\dot\sim} \text{ iff } w \Vdash_v (B \to C)^{\mathtt{u}\dot\sim}$$

The cases of $\wedge$ and $\vee$ can be also proved straightforward. □

4. Surjectivity of the translations

Below, we will show that both considered translations are surjective.

Lemma 5. *For any $A \in \mathrm{For}^{\mathrm{M}}$, there is $A' \in \mathrm{For}^{\dot\sim}$ such that $(A')^{\mathtt{m}} = A$.*

Proof: The proof goes by induction A. For any $a \in \mathrm{Var}$, we have $(a)^{\mathtt{m}} = a$.

For the inductive step let us assume that for formulas B and C there are formulas B' and C' such that $(B')^{\mathtt{m}} = B$ and $(C')^{\mathtt{m}} = C$.

We have

1. $\neg B = \neg((B')^{\mathtt{m}}) = (B' \to \bot_p)^{\mathtt{m}}$
2. $(B \,\S\, C) = ((B')^{\mathtt{m}} \,\S (C')^{\mathtt{m}}) = ((B' \,\S\, C'))^{\mathtt{m}}$ for $\S \in \{\wedge, \vee\}$
3. Consider the case of $C = \Box\neg(a \to a)$:

$$\big(B' \to \dot\sim \big(((a \to a) \to \bot_a) \to \bot_a\big)\big)^{\mathtt{m}} =$$
$$(B')^{\mathtt{m}} \to \big(\dot\sim \big(((a \to a) \to \bot_a) \to \bot_a\big)\big)^{\mathtt{m}} =$$
$$B \to \Box(((a \to a) \to \bot_a)^{\mathtt{m}}) = (B \to \Box\neg(a \to a)) = (B \to C)$$

The case that C is not of the form $\Box\neg(a \to a)$:

$$(B \to C) = ((B')^{\mathtt{m}} \to (C')^{\mathtt{m}}) = (B' \to C')^{\mathtt{m}}$$

4. $\Box B = \Box((B')^{\mathtt{m}}) = (\dot\sim(B' \to \bot_a))^{\mathtt{m}}$ □

Lemma 6. *For any $A \in \mathrm{For}^{\dot\sim}$, there is $A' \in \mathrm{For}^{\mathrm{M}}$ such that $(A')^{\mathtt{u}\dot\sim} = A$.*

Proof: The proof goes by induction a formula A. For any $a \in \mathrm{Var}$, we have $(a)^{\mathtt{u}\dot\sim} = a$.

For the inductive step let us assume that for B and $C \in \mathrm{For}^{\dot\sim}$ there are B' and $C' \in \mathrm{For}^{\mathrm{M}}$ such that $(B')^{\mathtt{u}\dot\sim} = B$ and $(C')^{\mathtt{u}\dot\sim} = C$.

We have

1. $\dot\sim B = \dot\sim((B')^{\mathtt{u}\dot\sim}) = (\Box\neg B')^{\mathtt{u}\dot\sim}$
2. $(B \,\S\, C) = ((B')^{\mathtt{u}\dot\sim} \,\S (C')^{\mathtt{u}\dot\sim}) = ((B' \,\S\, C'))^{\mathtt{u}\dot\sim}$ for $\S \in \{\wedge, \vee, \to\}$ □

Using transitivity of implication and ($\mathsf{CONTR}^{\dot\sim}$) we obtain:

Lemma 7. *For any logic* $\boldsymbol{L} \in \mathcal{R}^{\dot{\sim}}$, $\boldsymbol{L}$ *is closed on the rule* ($\mathsf{CONTR}^{\dot{\sim}2}$)*:*

$$\frac{A \to B}{\dot{\sim}(A \to \bot_p) \to \dot{\sim}(B \to \bot_p)} \qquad (\mathsf{CONTR}^{\dot{\sim}2})$$

Lemma 8. *For any logic* $\mathbf{S} \in \mathcal{R}^{\dot{\sim}}$, *the image* $[\mathbf{S}]^{\mathtt{m}}$ *of* $\mathbf{S}$ *under* $(-)^{\mathtt{m}}$ *is a regular logic containing* $\mathbf{D2}$.

Proof: First observe that full positive classical logic $\mathbf{CL}^{+}$ is contained in $[\mathbf{S}]^{\mathtt{m}}$ since $[\mathbf{CL}^{+}]^{\mathtt{m}} = \mathbf{CL}^{+}$. The following proofs, show that the whole propositional classical logic can be obtained. Consider

$$(p \to q) \to ((p \to (q \to \bot_p)) \to (p \to \bot_p)) \qquad (\natural)$$

It is a substitution of $(p \to q) \to ((p \to (q \to r)) \to (p \to r)) \in \mathbf{CL}^{+}$, so $(\natural) \in \mathbf{S}$, but $(\natural)^{\mathtt{m}} = (p \to q) \to ((p \to \neg q) \to \neg p)$

Besides, $p \to (\neg p \to q) \in [\mathbf{S}]^{\mathtt{m}}$, by $(\mathrm{DS}^{\dot{\sim}})$ and $\neg\neg p \to p \in [\mathbf{S}]^{\mathtt{m}}$, by (dn): $\big(((p \to \bot_q) \to \bot_q) \to p\big)^{\mathtt{m}} = \neg(p \to \bot_q)^{\mathtt{m}} \to p = \neg\neg p \to p$.

Now we show that the set $[\mathbf{S}]^{\mathtt{m}}$ is closed under the monotonicity rule. Assume that $A \to B \in [\mathbf{S}]^{\mathtt{m}}$. This means that there are formulas A' and B' such that $(A' \to B')^{\mathtt{m}} = A \to B$, $(A')^{\mathtt{m}} = A$, $(B')^{\mathtt{m}} = B$, and $A' \to B' \in \mathbf{S}$. By ($\mathsf{CONTR}^{\dot{\sim}2}$) we have that also $\dot{\sim}(A' \to \bot_p) \to \dot{\sim}(B' \to \bot_p) \in \mathbf{S}$. By the definition of $(-)^{\mathtt{m}}$ we obtain that $\Box((A')^{\mathtt{m}}) \to \Box((B')^{\mathtt{m}}) \in [\mathbf{S}]^{\mathtt{m}}$, i.e. $\Box A \to \Box B \in [\mathbf{S}]^{\mathtt{m}}$.

For the axiom (K): by $\mathbf{CL}^{+}$ and substitution we see that $\big(((p \to q) \wedge p) \to \bot_p\big) \to \big(((p \to q) \to \bot_p) \vee (p \to \bot_p)\big)$ belongs to $\mathbf{S}$, as well as $\dot{\sim}\big(((p \to q) \to \bot_p) \vee (p \to \bot_p)\big) \to \dot{\sim}\big(((p \to q) \wedge p) \to \bot_p\big)$ by ($\mathsf{CONTR}^{\dot{\sim}}$).

Now, we know that $(p \to q) \wedge p \to q \in \mathbf{CL}^{+}$. By ($\mathsf{CONTR}^{\dot{\sim}2}$) we have $\dot{\sim}(((p \to q) \wedge p) \to \bot_p) \to \dot{\sim}(q \to \bot_p)$. Thus, using $\mathbf{CL}^{+}$, ($\mathsf{CONTR}^{\dot{\sim}}$), $(\mathrm{dM2}^{\dot{\sim}}_{\leftarrow})$, by transitivity of $\to$ and the law of exportation we get $\dot{\sim}((p \to q) \to \bot_p) \to (\dot{\sim}(p \to \bot_p) \to \dot{\sim}(q \to \bot_p))$ as a thesis of $\mathbf{S}$. But by Definition 4 the result of the application of $(-)^{\mathtt{m}}$ to the last formula gives the axiom (K).

For axiom (D) consider $(\mathrm{D}^{\dot{\sim}})$: $\dot{\sim}(p \to \bot_p) \to (\dot{\sim} p \to \bot_p)$. We see that $(\dot{\sim}(p \to \bot_p) \to (\dot{\sim} p \to \bot_p))^{\mathtt{m}} = \Box p \to \neg\Box\neg p$.

Now we prove that $[\mathbf{S}]^{\mathtt{m}}$ is closed under substitution. Assume that $A \in [\mathbf{S}]^{\mathtt{m}}$. Let us consider $s(A)$ a result of substitution of modal formulas $C_1, \ldots, C_n$ respectively for variables $a_1, \ldots, a_n$ in the formula A, i.e., $A(a_1/C_1, \ldots, a_n/C_n) = s(A)$. By the definition of image, there is $A' \in \mathbf{S}$

such that $(A')^{\mathtt{m}} = A$. For every $1 \leqslant i \leqslant n$ let us consider $(C_i)^{\mathtt{u}\dot{\sim}}$ and the formula $(A'(a_1/(C_1)^{\mathtt{u}\dot{\sim}}, \ldots, a_n/(C_n)^{\mathtt{u}\dot{\sim}}))^{\mathtt{m}}$.

By Lemma 5 there is $B' \in \mathrm{For}^{\dot{\sim}}$ such that $(B')^{\mathtt{m}} = s(A)$. Thus, by Lemmas 1 and 2 the following equivalences hold on the basis of $\mathbf{N}^{+}$: $B' \leftrightarrow ((B')^{\mathtt{m}})^{\mathtt{u}\dot{\sim}} = (A(a_1/C_1, \ldots, a_n/C_n))^{\mathtt{u}\dot{\sim}} \leftrightarrow ((A)^{\mathtt{u}\dot{\sim}}(a_1/(C_1)^{\mathtt{u}\dot{\sim}}, \ldots, a_n/(C_n)^{\mathtt{u}\dot{\sim}}) = ((A')^{\mathtt{m}})^{\mathtt{u}\dot{\sim}}(a_1/(C_1)^{\mathtt{u}\dot{\sim}}, \ldots, a_n/(C_n)^{\mathtt{u}\dot{\sim}}) \leftrightarrow A'(a_1/(C_1)^{\mathtt{u}\dot{\sim}}, \ldots, a_n/(C_n)^{\mathtt{u}\dot{\sim}})$. But $A'(a_1/(C_1)^{\mathtt{u}\dot{\sim}}, \ldots, a_n/(C_n)^{\mathtt{u}\dot{\sim}}) \in \mathbf{S}$, since $A' \in \mathbf{S}$ and $\mathbf{S}$ is closed under substitution, so also $B' \in \mathbf{S}$ and finally $s(A) \in [\mathbf{S}]^{\mathtt{m}}$.

Finally let us consider the case of modus ponens. Assume that $A, A \to B \in [\mathbf{S}]^{\mathtt{m}}$. By the definition of image, there are $A', C' \in \mathbf{S}$ such that $(A')^{\mathtt{m}} = A$ and $(C')^{\mathtt{m}} = A \to B$. By the definition of $(-)^{\mathtt{m}}$ we can see that $C' = D \to B'$ for $D, B' \in \mathrm{For}^{\dot{\sim}}$ such that $(D)^{\mathtt{m}} = A$ and $(B')^{\mathtt{m}} = B$. Although $(-)^{\mathtt{m}}$ is not injective, but we have $((A')^{\mathtt{m}})^{\mathtt{u}\dot{\sim}} = (A)^{\mathtt{u}\dot{\sim}}$ and $(A)^{\mathtt{u}\dot{\sim}} = ((D)^{\mathtt{m}})^{\mathtt{u}\dot{\sim}}$, thus by (2.1) given in Lemma 1 we have $A' \leftrightarrow D \in \mathbf{N}^{+} \subseteq \mathbf{S}$, thus since $C' = D \to B' \in \mathbf{S}$, also $A' \to B' \in \mathbf{S}$ and due to the fact that $\mathbf{S}$ is a logic, $B' \in \mathbf{S}$, hence $B \in [\mathbf{S}]^{\mathtt{m}}$. $\Box$

Lemma 9. *For any regular logic* $\mathbf{S}$ *containing* $\mathbf{D2}$*, the image* $[\mathbf{S}]^{\mathtt{u}\dot{\sim}}$ *of* $\mathbf{S}$ *under* $(-)^{\mathtt{u}\dot{\sim}}$ *belongs to* $\mathcal{R}^{\dot{\sim}}$.

Proof: First observe that full positive classical logic $\mathbf{CL}^{+}$ is contained in $[\mathbf{S}]^{\mathtt{u}\dot{\sim}}$ since $[\mathbf{CL}^{+}]^{\mathtt{u}\dot{\sim}} = \mathbf{CL}^{+}$.

Now we will show that $(\mathrm{dM2}^{\dot{\sim}}_{\leftarrow}) \in [\mathbf{S}]^{\mathtt{u}\dot{\sim}}$. First, let us recall that $\Box\neg p \wedge \Box\neg q \to \Box\neg(p \vee q)$ is a thesis of any regular logic. Besides $(\Box\neg p \wedge \Box\neg q \to \Box\neg(p \vee q))^{\mathtt{u}\dot{\sim}} = (\Box\neg p \wedge \Box\neg q)^{\mathtt{u}\dot{\sim}} \to (\Box\neg(p \vee q))^{\mathtt{u}\dot{\sim}} = (\Box\neg p)^{\mathtt{u}\dot{\sim}} \wedge (\Box\neg q)^{\mathtt{u}\dot{\sim}} \to (\dot{\sim}(p \vee q)) = \dot{\sim} p \wedge \dot{\sim} q \to \dot{\sim}(p \vee q)$.

(dn) $\in [\mathbf{S}]^{\mathtt{u}\dot{\sim}}$: first one can see that $\big(((p \to \Box\neg(p \to p)) \to \Box\neg(p \to p)\big) \to p\big)$ belongs to $\mathbf{D2}$. Moreover, we have: $\big(((p \to \Box\neg(p \to p)) \to \Box\neg(p \to p)\big) \to p\big)^{\mathtt{u}\dot{\sim}} = \big((p \to \bot_p) \to \bot_p\big) \to p$.

For $(\mathsf{CONTR}^{\dot{\sim}})$ assume that $A \to B \in [\mathbf{S}]^{\mathtt{u}\dot{\sim}}$. This means that there is $C \in \mathrm{For}^{\mathrm{M}}$ that belongs to $\mathbf{S}$, for which $(C)^{\mathtt{u}\dot{\sim}} = A \to B$. Analysing the definition of $(-)^{\mathtt{u}\dot{\sim}}$ we see, we have two cases: first $C = \neg D$, $(D)^{\mathtt{u}\dot{\sim}} = A$ and $\bot_p = B$, and second $C = (D \to E)$, $(D)^{\mathtt{u}\dot{\sim}} = A$, and $(E)^{\mathtt{u}\dot{\sim}} = B$.

For the first case, sine $\neg D \in \mathbf{S}$, so $\neg\Box\neg(p \to p) \to \neg D \in \mathbf{S}$ and also $\Box\neg\Box\neg(p \to p) \to \Box\neg D \in \mathbf{S}$. This means that $(\Box\neg\Box\neg(p \to p) \to \Box\neg D)^{\mathtt{u}\dot{\sim}} \in [\mathbf{S}]^{\mathtt{u}\dot{\sim}}$, but $(\Box\neg\Box\neg(p \to p) \to \Box\neg D)^{\mathtt{u}\dot{\sim}} = \dot{\sim}\bot_p \to \dot{\sim}((D)^{\mathtt{u}\dot{\sim}}) = \dot{\sim} B \to \dot{\sim} A \in [\mathbf{S}]^{\mathtt{u}\dot{\sim}}$.

For the second case, by contraposition and monotonicity we have: $\Box\neg E \to \Box\neg D \in \mathbf{S}$, but again $(\Box\neg E \to \Box\neg D)^{\mathtt{u}\dot{\sim}} = (\Box\neg E)^{\mathtt{u}\dot{\sim}} \to (\Box\neg D)^{\mathtt{u}\dot{\sim}} = \dot{\sim}((E)^{\mathtt{u}\dot{\sim}}) \to \dot{\sim}((D)^{\mathtt{u}\dot{\sim}}) = \dot{\sim} B \to \dot{\sim} A \in [\mathbf{S}]^{\mathtt{u}\dot{\sim}}$.

The case of modus ponens. Assume that $A, A \to B \in [\mathbf{S}]^{\mathtt{u}\dot{\sim}}$. It means that there are $A', C' \in \mathbf{S}$, such that $(A')^{\mathtt{u}\dot{\sim}} = A$ and $(C')^{\mathtt{u}\dot{\sim}} = A \to B$. Again we have two cases: the first $C' = \neg D$, $(D)^{\mathtt{u}\dot{\sim}} = A$ and $\bot_p = B$, the second $C' = (D \to E)$, $(D)^{\mathtt{u}\dot{\sim}} = A$ and $(E)^{\mathtt{u}\dot{\sim}} = B$.

In the first case we have $(D)^{\mathtt{u}\dot{\sim}} = (A')^{\mathtt{u}\dot{\sim}}$, so $((D)^{\mathtt{u}\dot{\sim}})^{\mathtt{m}} = ((A')^{\mathtt{u}\dot{\sim}})^{\mathtt{m}}$, by Lemma 1, (2.2), we see that $D \leftrightarrow ((D)^{\mathtt{u}\dot{\sim}})^{\mathtt{m}} \in \mathbf{S}$, and $((A')^{\mathtt{u}\dot{\sim}})^{\mathtt{m}} \leftrightarrow A' \in \mathbf{S}$, so $(D \leftrightarrow A') \in \mathbf{S}$ and $D \in \mathbf{S}$, but by Duns Scotus law $\mathbf{S}$ equals $\mathrm{For}^{\mathrm{M}}$, in particular $\Box\neg(p \to p) \in \mathbf{S}$, so also $\bot_p \in [\mathbf{S}]^{\mathtt{u}\dot{\sim}}$.

For the second case we again have $(D)^{\mathtt{u}\dot{\sim}} = (A')^{\mathtt{u}\dot{\sim}}$, so $((D)^{\mathtt{u}\dot{\sim}})^{\mathtt{m}} = ((A')^{\mathtt{u}\dot{\sim}})^{\mathtt{m}} \in \mathbf{S}$ and by (2.2) from Lemma 1 and transitivity of $\to$ we conclude that $(D \leftrightarrow A') \in \mathbf{S}$. Hence by modus ponens $D \in \mathbf{S}$ and $E \in \mathbf{S}$, thus $(E)^{\mathtt{u}\dot{\sim}} = B \in [\mathbf{S}]^{\mathtt{u}\dot{\sim}}$.

Now we prove that $[\mathbf{S}]^{\mathtt{u}\dot{\sim}}$ is closed under substitution. Assume that $A \in [\mathbf{S}]^{\mathtt{u}\dot{\sim}}$. Let us consider $s(A)$ a result of substitution of formulas $C_1,\ldots, C_n \in \mathrm{For}^{\dot{\sim}}$ for variables $a_1,\ldots, a_n$ in the formula A, i.e., $A(a_1/C_1, \ldots, a_n/C_n) = s(A)$. By the definition of image, there is $A' \in \mathbf{S}$ such that $(A')^{\mathtt{u}\dot{\sim}} = A$. For any $1 \leqslant i \leqslant n$ let us consider formulas $(C_i)^{\mathtt{m}}$ and $(A'(a_1/(C_1)^{\mathtt{m}}, \ldots, a_n/(C_n)^{\mathtt{m}}))^{\mathtt{u}\dot{\sim}}$. Observe, that by Lemma 6 there is a formula $B' \in \mathrm{For}^{\mathrm{M}}$ such that $(B')^{\mathtt{u}\dot{\sim}} = s(A)$. Thus, by Lemmas 1 and 3 we have $B' \leftrightarrow ((B')^{\mathtt{u}\dot{\sim}})^{\mathtt{m}} = (A(a_1/C_1, \ldots, a_n/C_n))^{\mathtt{m}} \leftrightarrow (A)^{\mathtt{m}}(a_1/(C_1)^{\mathtt{m}},\ldots, a_n/(C_n)^{\mathtt{m}}) = ((A')^{\mathtt{u}\dot{\sim}})^{\mathtt{m}}(a_1/(C_1)^{\mathtt{m}}, \ldots, a_n/(C_n)^{\mathtt{m}}) \leftrightarrow A'(a_1/(C_1)^{\mathtt{m}},\ldots, a_n/(C_n)^{\mathtt{m}})$. The last formula belongs to $\mathbf{S}$, since $A' \in \mathbf{S}$ and $\mathbf{S}$ is closed under substitution, so also $B' \in \mathbf{S}$. Hence $(B')^{\mathtt{u}\dot{\sim}} \in [\mathbf{S}]^{\mathtt{u}\dot{\sim}}$, in other words $s(A) \in [\mathbf{S}]^{\mathtt{u}\dot{\sim}}$. □

Corollary 1. 1. $[\mathbf{N}^+]^{\mathtt{m}} = \mathbf{D2}$,

2. $[\mathbf{D2}]^{\mathtt{u}\dot{\sim}} = \mathbf{N}^+$.

Proof: For the first equation: by Lemma 8, $[\mathbf{N}^+]^{\mathtt{m}}$ is a regular logic and $\mathbf{D2} \subseteq [\mathbf{N}^+]^{\mathtt{m}}$. Assume that $A \in [\mathbf{N}^+]^{\mathtt{m}}$, i.e., there is $B \in \mathbf{N}^+$, such that $(B)^{\mathtt{m}} = A$. Since by Lemma 9, $\mathbf{N}^+ \subseteq [\mathbf{D2}]^{\mathtt{u}\dot{\sim}}$, there is $C \in \mathbf{D2}$, such that $(C)^{\mathtt{u}\dot{\sim}} = B$. So $((C)^{\mathtt{u}\dot{\sim}})^{\mathtt{m}} = (B)^{\mathtt{m}} = A$ and due to Lemma 1, $((C)^{\mathtt{u}\dot{\sim}})^{\mathtt{m}} \in \mathbf{D2}$, thus $A \in \mathbf{D2}$.

For the second equation, we see that by Lemma 9, $[\mathbf{D2}]^{\mathtt{u}\dot{\sim}} \in \mathcal{R}^{\dot{\sim}}$, so $\mathbf{N}^+ \subseteq [\mathbf{D2}]^{\mathtt{u}\dot{\sim}}$. Assume that $A \in [\mathbf{D2}]^{\mathtt{u}\dot{\sim}}$, i.e., there is $B \in \mathbf{D2}$, such that $(B)^{\mathtt{u}\dot{\sim}} = A$. Since by Lemma 8, $\mathbf{D2} \subseteq [\mathbf{N}^+]^{\mathtt{m}}$, there is $C \in \mathbf{N}^+$, such that

$(C)^{\mathtt{m}} = B$, so $((C)^{\mathtt{m}})^{\mathtt{u}\dot{\sim}} = (B)^{\mathtt{u}\dot{\sim}} = A$ and by Lemma 1, $((C)^{\mathtt{m}})^{\mathtt{u}\dot{\sim}} \in \mathbf{N}^+$, thus $A \in \mathbf{N}^+$. □

5. The main correspondence result

THEOREM 1. *For any regular logic* $\mathbf{S}$ *containing* $\mathbf{D2}$*, if* $\mathbf{S}$ *is complete (sound) with respect to a class of models* X*,* $[\mathbf{S}]^{\mathtt{u}\dot{\sim}}$ *is complete (sound) with respect to the class* X*.*

PROOF: Assume that $\mathbf{D2} \subseteq \mathbf{S}$ and that $\mathbf{S}$ is a regular logic complete (sound) with respect to a class of models X.

For completeness let $A \in \mathrm{For}^{\dot{\sim}}$. Assume that for any model $\langle W, R, N, v\rangle \in X$, and every $w \in W$, $w \Vdash_v A$. By Lemma 4 for any $\langle W, R, N, v\rangle \in X$ and $w \in W$, $w \vDash_v (A)^{\mathtt{m}}$. By assumed completeness $(A)^{\mathtt{m}} \in \mathbf{S}$. Hence, $((A)^{\mathtt{m}})^{\mathtt{u}\dot{\sim}} \in [\mathbf{S}]^{\mathtt{u}\dot{\sim}}$. By Lemma 9, $[\mathbf{S}]^{\mathtt{u}\dot{\sim}}$ is a logic and $[\mathbf{S}]^{\mathtt{u}\dot{\sim}} \in \mathcal{R}^{\dot{\sim}}$, so by Lemma 1, $(((A)^{\mathtt{m}})^{\mathtt{u}\dot{\sim}} \leftrightarrow A) \in \mathbf{N}^+ \subseteq [\mathbf{S}]^{\mathtt{u}\dot{\sim}}$. Thus $A \in [\mathbf{S}]^{\mathtt{u}\dot{\sim}}$.

Now we consider the case of soundness. Let $A \in [\mathbf{S}]^{\mathtt{u}\dot{\sim}}$. By the definition of image, there is $B \in \mathbf{S}$, such that $(B)^{\mathtt{u}\dot{\sim}} = A$. By the assumed soundness for $\mathbf{S}$, we have that for any $\langle W, R, N, v\rangle \in X$, and any $w \in W$, $w \vDash_v B$, by Lemma 4 for any $\langle W, R, N, v\rangle \in X$ and $w \in W$, $w \Vdash_v (B)^{\mathtt{u}\dot{\sim}} = A$. □

We also obtain:

THEOREM 2. *For any logic* $\mathbf{S} \in \mathcal{R}^{\dot{\sim}}$ *if* $\mathbf{S}$ *is complete (sound) with respect to a class of models* X*,* $[\mathbf{S}]^{\mathtt{m}}$ *is complete (sound) with respect to the class* X*.*

PROOF: Assume that $\mathbf{S} \in \mathcal{R}^{\dot{\sim}}$ and that $\mathbf{S}$ is complete (sound) with respect to a class of models X.

For the case of completeness let $B \in \mathrm{For}^{\mathrm{M}}$. Assume that for any model $\langle W, R, N, v\rangle \in X$, and every $w \in W$, $w \vDash_v B$. By Lemma 4 for any $\langle W, R, N, v\rangle \in X$ and $w \in W$, $w \Vdash_v (B)^{\mathtt{u}\dot{\sim}}$. By assumed completeness, $(B)^{\mathtt{u}\dot{\sim}} \in \mathbf{S}$. Hence, $((B)^{\mathtt{u}\dot{\sim}})^{\mathtt{m}} \in [\mathbf{S}]^{\mathtt{m}}$. By Lemma 8, $[\mathbf{S}]^{\mathtt{m}}$ is a regular logic and $\mathbf{D2} \subseteq [\mathbf{S}]^{\mathtt{m}}$, so by Lemma 1 $(((B)^{\mathtt{u}\dot{\sim}})^{\mathtt{m}} \leftrightarrow B) \in \mathbf{D2} \subseteq [\mathbf{S}]^{\mathtt{m}}$. Thus $B \in [\mathbf{S}]^{\mathtt{m}}$.

For soundness, let $B \in [\mathbf{S}]^{\mathtt{m}}$. By the definition of image, there is $A \in \mathbf{S}$, such that $(A)^{\mathtt{m}} = B$. By the assumed soundness for $\mathbf{S}$, we have that for any $\langle W, R, N, v\rangle \in X$, and any $w \in W$, $w \Vdash_v A$. By Lemma 4, for any $\langle W, R, N, v\rangle \in X$ and $w \in W$, $w \vDash_v (A)^{\mathtt{m}} = B$. □

5.1. Examples of completeness results

Let us recall notation for the following formulas and logics:

$$\Box p \to p \tag{T}$$

$$\Box p \to \Box\Box p \tag{4}$$

$$\Box(p \to p) \tag{N}$$

Definition 9. 1. **E2** is the smallest regular logic containing (T).

2. **E4** is the smallest regular logic containing (T) and (4).
3. **S4** is the smallest regular (equivalently normal) logic containing (T), (4) and (N).

Each of the above logics contains **D2**. Thus, using completeness result for these logics (see e.g. [11, 13]) and Theorem 1 we directly obtain:

Corollary 2. 1. *The logic* $\mathbf{N}^+$ *is sound and complete with respect to the class of models based on frames with serial accessibility relation (or equivalently fulfilling the condition* $(\mathrm{Ser_N})$*).*

2. *The logic* $[\mathbf{E2}]^{\mathrm{u}\dot{\sim}}$ *is sound and complete with respect to models based on frames* $\langle W, R, N\rangle$ *with accessibility relation fulfilling the condition:*

$$\forall_{w\in N}\, wRw \tag{$\mathrm{Ref_N}$}$$

3. *The logic* $[\mathbf{E4}]^{\mathrm{u}\dot{\sim}}$ *is sound and complete with respect to the class of models based on frames* $\langle W, R, N\rangle$ *such that* R *fulfills the conditions* $(\mathrm{Ref_N})$, $\forall_{w\in N} R(w) \subseteq N$ *and* $\forall_{w,v\in N}\forall_{u\in W}\big(wRv \;\&\; vRu \Rightarrow wRu\big)$*.*
4. *The logic* $[\mathbf{S4}]^{\mathrm{u}\dot{\sim}}$ *is sound and complete with respect to models based on frames such that* $N = W$*, where* R *is reflexive and transitive.*

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