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UNIVERSAL CONSTRUCTIONS FOR IDENTITY AND NUMBER

Abstract

The scheme that Gentzen designed to specify the definitions of logical connectives has a wider reach than is usually recognized, extending naturally to terms that Gentzen did not treat explicitly, including predicate symbols. After explaining (§2) how the introduction and elimination rules of natural deduction calculi provide a relational definition of a term through a universal construction, we will describe how to provide similar constructions of, first, the binary predicate symbol for identity (§3) and, second, the monadic predicate symbol for being a natural number (§4). The main result of §3 is the interderivability of the introduction rule for identity (=I) with the law of reflexivity. This result greatly simplifies the analysis of the former rule. Further, it explains why attempts either to reject =I as unsound or to justify it only relative to specific background assumptions have systematically missed its decontextualized role in inference. In

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§4 we review the realization, observed by Martin-Löf and Parsons, of the principle of mathematical induction as an elimination rule (ℕE) for the natural number predicate. Noting the universal construction underlying this realization clarifies the relationship between the induction principle and the number concept, at least when the latter is relationally specified, by resituating induction as an integral part of the very definition of the natural numbers. The common motif tying these and other results together is the fact that a term's introduction and elimination rules are a presentation of a single concept, the term's universal construction.

Keywords: proof theory, identity, Gerhard Gentzen, universal property, induction.

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1. Introduction

The scheme that Gentzen designed to specify the definitions of logical connectives has a wider reach than is usually recognized, extending naturally to terms that Gentzen did not treat explicitly, including predicate symbols. Part of the reason this extension has been elusive is that inessential features of Gentzen's scheme, including features emphasized by Gentzen himself, have been mistaken for what is central to it.

After explaining (§2) how the introduction and elimination rules of natural deduction calculi provide a relational definition of a term through a universal construction, we will describe how to provide similar constructions of, first, the binary predicate symbol for identity (§3) and, second, the monadic predicate symbol for being a natural number (§4).

The main result of §3 is the interderivability of the introduction rule for identity (=I) with the law of reflexivity. This result greatly simplifies the analysis of the former rule. Further, it explains why attempts either to reject =I as unsound or to justify it only relative to specific background assumptions have systematically missed its decontextualized role in inference.

In §4 we review the realization, first noted by Martin-Löf and Parsons, of the principle of mathematical induction as an elimination rule (ℕE) for the natural number predicate. Noting the universal construction underlying

this realization clarifies the relationship between the induction principle and the number concept, at least when the latter is relationally specified, by resituating induction as an integral part of the relational definition of the natural numbers. What appears, on standard presentations, as a synthetic principle imposed on the numbers (and therefore in need of independent justification) is revealed instead to be a structural feature of the number concept itself.

The common motif tying these and other results together is the fact that a term's introduction and elimination rules are a presentation of a single concept, the term's universal construction. Neither rule fully captures the term's meaning on its own, and neither follows, analytically or otherwise, from the other. Instead, a universal construction relationally defines a term by specifying a property or collection of properties that it has definitionally.

Relational definitions are then given expression in natural deduction by an I/E rule pair. One rule indicates that the term has the property in question, and the other rule indicates that it is extremal among all terms that do so, so that the property can be thought of as definitional for that term. Which rule (elimination or introduction) plays which role depends on the details of the particular universal construction. On this score, the identity and number predicates differ, but their difference mirrors what will be familiar from Gentzen's treatment of the standard logical connectives once that treatment is reframed in the language of universal constructions.

2. Relational definition

Implementing a suggestion originally articulated by David Hilbert,[1] Gerhard Gentzen devised in [7] a scheme for specifying the meaning of a logical particle with a pair of characteristic inference rules. With this scheme Gentzen provided definitions of the standard propositional connectives $\wedge$, $\vee$, $\neg$, and $\supset$, the standard first-order quantifiers $\forall$ and $\exists$, and the arity-0

[1] One finds the idea that rules of deductive inference can be understood as definitions in the following remark from Hilbert's 1920 lectures: "Diese Regel kann als die Definition des Seinszeichens aufgefasst werden" ([4], p. 323). Gentzen was a student at Göttingen in the early 1930s and served as Hilbert's assistant in the latter half of the decade.

connective $\bot$. In §§3 and 4 we will show how to extend Gentzen's definition scheme, and the natural deduction calculus he built on it, to include other sorts of logical particles such as predicate symbols. To prepare for that extension we first review Gentzen's original definitions.

For example, the inference rules specifying the meaning of $\supset$ are

$$\frac{A \supset B \qquad A}{B}\ \supset E \qquad\qquad \frac{\begin{array}{c}[A]\\ B\end{array}}{A \supset B}\ \supset I$$

In the first rule, $\supset$E, the E stands for "elimination." Elimination rules represent the canonical way of reasoning from a sentence whose outermost particle is the one specified. (Notice that this rule is familiar from antiquity as the inference pattern called *modus ponens*.) Rules like the second one, labeled with an I, are called "introduction" rules. These represent the canonical way of reasoning to a sentence whose outermost particle is the one specified. (The square brackets featured in the rule $\supset$I indicate that the sentence A is an assumption, and it is understood that this assumption is "discharged" by the inference—a phenomenon indicated with numerical decorations in more complex derivations but omitted in the rule templates where there is no ambiguity.)

Gentzen emphasized that the value of these inference rules lies beyond the mere fact that they are individually deductively valid, beyond even the fact that collectively they suffice to exhibit any deductively valid derivation. Other sorts of proof systems had those properties; Gentzen offered something new. The I/E pair associated with each logical particle, he claimed, can be determined from one another (the latter are "unique functions" of the former) because of how they represent the particle's definition ([7], II §5.13).

Gentzen cautioned that his description of how I/E inference rules define logical particles was still imprecise, and that only after being made precise would the way they functionally determine one another be clear. One framework for the precise articulation of this theory of definition is the concept of a universal construction, introduced by Pierre Samuel in [24] and subsequently elaborated in the development of categorical logic in

[16, 17, 12, 13, 14].[2] In what follows we will illustrate how the universal construction concept allows for a systematic development and extension of Gentzen's definition scheme, disclosing stable structural relationships that operate behind the surface of the inference rules that encode them.

In fact Gentzen's rules for $\supset$ are an excellent introductory exhibit of a universal construction. Many people already have a sense that the inference rule *modus ponens* is not just one among many patterns of valid inference involving the conditional. They think that it is *definitional* of the conditional $A \supset B$ that it licenses the inference from A to B. But if the availability of this inference is the very definition of $A \supset B$, then in fact any sentence that, together with A, allows the inference to B expresses something equivalent to or (inferentially) stronger than $A \supset B$. To illustrate this, let us rewrite *modus ponens* in the turnstile notation as

$$A \supset B, A \vdash B$$

To say that this rule is not only valid but *definitional* of the $\supset$ connective is to say that any other sentence that can legitimately fill in the blank in $_____, A \vdash B$, any sentence, that is, that stands in the same inferential relationships as $A \supset B$, must be such that $A \supset B$ is inferable from it. Defining $\supset$ with *modus ponens* determines that $A \supset B$ can be inferred from any other sentence that similarly licenses the inference from A to B, as expressed in the following rule:

$$\text{For all C, if C, A} \vdash \text{B, then C} \vdash A \supset B$$

Notice that this last rule is just Gentzen's $\supset$I rule.[3]

The I/E rules for $\supset$ are therefore a paradigm of a universal construction. In this case, $\supset$E exhibits a property that sentences of the form $A \supset B$ have. Countless other sentences might have this same property, but sentences of the form $A \supset B$ are said to have the property *universally*, so that those other

[2] A modern treatment is [1].

[3] In fact the full specification of the universality involved in this definition would have it that $A \supset B$ can be inferred in any other *context* that licenses the inference from A to B, replacing the single sentence C with a set of hypotheses Γ or, in some cases, even a sequent $\Gamma \vdash C$. See [5] for examples in which this general formulation is required.

sentences have the property only by virtue of their relationship to A⊃B. The ⊃I rule is not a consequence of the ⊃E rule, but it is a consequence of the conception of that rule as definitional. Universal constructions like those provided by I/E inference rules define objects *relationally*—they fully characterize those objects in terms of the relations they bear to other objects. The concept of relational definition is a central motif in category theory, wherein "objects and arrows are determined by the role they play in the category via their relations to other objects and arrows, that is, by their position in a structure and not by what they 'are' or 'are made of' in some absolute sense" ([2, p. 29])—see [6, chapter 2] for an elaboration of the distinction between relational and essential definitions.

The I/E rules provided for ∧ and ¬ follow this paradigm exactly. But there are variations. For a fuller picture of universal constructions and of how they translate into Gentzen's natural deduction framework, let us consider the particles ∨, ∀, and ⊥ in succession.

Gentzen's introduction rules for ∨ are $\dfrac{A}{A \vee B}$ and $\dfrac{B}{A \vee B}$. One can rewrite these rules in the turnstile notation as

$$A \vdash A \vee B \text{ and } B \vdash A \vee B$$

As before, to say that these inferences are definitional of the disjunction operator is to say not only that they are valid but also that any other sentence that can legitimately fill in the blanks in $A \vdash ____$ and $B \vdash ____$, any sentence, that is, that stands in the definitional relationships of A ∨ B, must further be inferable from A ∨ B. This is expressed with the following rule:

$$\text{For all C, if } A \vdash C \text{ and } B \vdash C, \text{ then } A \vee B \vdash C$$

This "proof by cases" scheme is therefore Gentzen's ∨E rule. In the notation of natural deduction, it is written $\dfrac{A \vee B \qquad \overset{[A]}{C} \qquad \overset{[B]}{C}}{C}$. As with ⊃I, the brackets indicate assumptions, so the rule reads "C follows from the three data: A ∨ B, the derivation of C from the assumption A, and the

derivation of C from the assumption B." The assumptions are discharged by the inference.

Notice that the order at which we arrived at the rules for $\vee$ is a reversal of the procedure used for $\supset$. For $\supset$ (and $\wedge$ and $\neg$), the definitional relations are inferences from a sentence featuring the connective, and universality is invoked to arrive at the corresponding inferences leading back to that sentence. The introduction rules are inferred from the conception of the elimination rules as definitional. In the language of Bergman ([3], p. 71), these are "right-universal constructions"—so called because the turnstile points from left to right, and the $\mathrm{A} \supset \mathrm{B}$, $\mathrm{A} \wedge \mathrm{B}$, and $\neg \mathrm{A}$ expressions are the "rightmost" (or "least" or "weakest") among objects with their defining properties. For $\vee$, the situation is reversed. $\mathrm{A} \vee \mathrm{B}$ is the "leftmost" (or "strongest") object with its defining properties. All other objects standing in those defining relations must be to its "right" and therefore, by universality, inferable from it. It is a "left-universal construction."

Consider now Gentzen's elimination rule for $\forall$: $\dfrac{\forall x \mathrm{P}(x)}{\mathrm{P(a)}}$ *for any constant symbol* a. What would it mean for this rule not only to be valid but in fact to be definitional of the $\forall$ quantifier? Let us rewrite the elimination rule in the turnstile notation:

$$\forall x \mathrm{P}(x) \vdash \mathrm{P(a)} \text{ for all a}$$

To say that this rule *defines* $\forall$ is to say that any other sentence that can legitimately fill in the blank in $_____ \vdash \mathrm{P(a)}$ for all a, any sentence, that is, that stands in the same inferential relationships as $\forall x \mathrm{P}(x)$, must be such that $\forall x \mathrm{P}(x)$ is inferable from it. In the turnstile notation, we have:

$$\text{For all C, if C} \vdash \mathrm{P(a)} \text{ for all a, then C} \vdash \forall x \mathrm{P}(x)$$

As are $\supset$, $\wedge$, and $\neg$, $\forall$ is thereby seen to have right-universal construction: $\forall x \mathrm{P}(x)$ is the paradigmatic object from which all its instances can be inferred in the sense that it is in turn inferable from any other object that implies all its instances.

Some craft is needed in order to represent this universalization of the $\forall$-

Elim rule in natural deduction, for it suggests that in order to infer $\forall x\mathrm{P}(x)$ one must first establish potentially infinitely many premises, namely, all its instances. Gentzen's solution to this problem is to suppose that one can establish just one instance of $\forall x\mathrm{P}(x)$ but do so in a way that could evidently be replicated for any other instance. To formulate this supposition, Gentzen introduced the idea of deriving $\mathrm{P}(\mathfrak{e})$, where $\mathfrak{e}$ is an "eigenvariable"—a constant symbol that doesn't appear in any assumption that is undischarged at the point in the derivation tree at which $\mathrm{P}(\mathfrak{e})$ appears.[4] When this eigenvariable condition is met, one could replace every occurrence of $\mathfrak{e}$ in the derivation with any constant symbol, and the result would still be a proof. Thus the derivation of $\mathrm{P}(\mathfrak{e})$ "means" that $\mathrm{P(a)}$ is derivable for each a and puts one in a position to infer $\forall x\mathrm{P}(x)$. In the notation of natural deduction, this $\forall$-Intro rule is written $\dfrac{\mathrm{P}(\mathfrak{e})}{\forall x\mathrm{P}(x)}$, where $\mathfrak{e}$ is an eigenvariable.

Gentzen's I/E rules for $\exists$ define it with a left-universal construction. An eigenvariable condition is operative again, this time in the $\exists$E rule.

Finally, let us consider Gentzen's treatment of the arity-0 connective $\bot$. We have insisted that the relational definition of a logical particle is provided by a *pair* of rules, but for $\bot$ Gentzen provided only a single rule, $\dfrac{\bot}{\mathrm{A}}\ \bot\mathrm{E}$. The absence of a $\bot$I rule to accompany $\bot$E is puzzling at a glance. A patient look reveals what is going on, and the details are instructive.

Consider the universal construction that arises if one treats $\bot$E as definitional. First rewrite $\bot$E in turnstile notation as:

$$\bot \vdash \mathrm{A}, \text{ for all A}$$

To say that this rule is not only valid but definitional of the $\bot$ connective is to say that any other sentence that can legitimately fill in the blank in "$_____ \vdash \mathrm{A}$, for all A," any sentence, that is, that stands in the same inferential relationships as $\bot$, must itself allow inference to $\bot$:

[4] For cognitive ease and conceptual perspicuity, we use fraktur typeface for schematic constants (and later for predicate symbols and propositional letters) subject to eigenvariable conditions, reserving ordinary typeface for object-language symbols.

$$\text{For all C, if C} \vdash \text{A, for all A, then C} \vdash \bot$$

This last rule could be labeled $\bot$I, thereby providing $\bot$ with a right-universal construction. But if $\text{C} \vdash \text{A}$, for all A, then in particular, $\text{C} \vdash \bot$. The $\bot$I rule's conclusion is an instance of its premise. The rule has no content because it is self-redundant. Thus $\bot$E and its universalization $\bot$I do indeed relationally define the $\bot$ connective, but $\bot$E captures the entirety of the connective's meaning. Because the $\bot$I rule is of no use in constructing derivations, Gentzen omitted it. Such inert, or meaningless, rules are nevertheless coherent. Notice that to formulate $\bot$I in natural deduction, one again needs the concept of an eigenvariable, only now what one needs is a propositional eigenvariable $\mathfrak{E}$. The rule reads $\dfrac{\mathfrak{E}}{\bot}\ \bot\text{I}$.

Let us collect the I/E rules for each connective Gentzen considered. One can check on one's own that the rules for $\neg$, $\wedge$, and $\exists$ define these connectives as universal constructions (left-universal in the case of $\exists$).

$\supset$ rules

$$\frac{\text{A} \supset \text{B} \qquad \text{A}}{\text{B}} \qquad \frac{\begin{array}{c}[\text{A}]\\ \text{B}\end{array}}{\text{A} \supset \text{B}}$$

$\neg$ rules

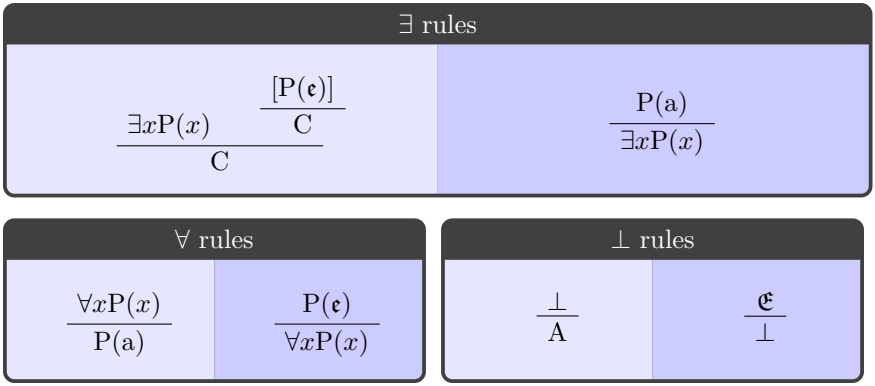
$$\frac{\neg\text{A} \qquad \text{A}}{\bot} \qquad \frac{\begin{array}{c}[\text{A}]\\ \bot\end{array}}{\neg\text{A}}$$

$\wedge$ rules

$$\frac{\text{A} \wedge \text{B}}{\text{A}} \qquad \frac{\text{A} \wedge \text{B}}{\text{B}} \qquad \frac{\text{A} \qquad \text{B}}{\text{A} \wedge \text{B}}$$

$\vee$ rules

$$\frac{\text{A} \vee \text{B} \qquad \begin{array}{c}[\text{A}]\\ \text{C}\end{array} \qquad \begin{array}{c}[\text{B}]\\ \text{C}\end{array}}{\text{C}} \qquad \frac{\text{A}}{\text{A} \vee \text{B}} \qquad \frac{\text{B}}{\text{A} \vee \text{B}}$$



Natural deduction proofs are trees constructed by iterating these rules. Applications of ∨E, ⊃I, ¬I, and ∃E are labeled with natural numbers 1, 2, ..., to indicate which, if any, assumptions (identified in brackets) are discharged, i.e., those assumptions, if any, bearing the same label as the rule. Discharge of assumptions is inessential to these rules' application so that, for example, B ⊃ A is derivable from A by one application of ⊃I with "vacuous discharge." When all the assumptions in such a proof are discharged by the application of a rule, the sentence at the tree's single root node is said to be proved. Otherwise this sentence is said to be derived from the set of assumptions that remain open at leaf nodes or, more succinctly, the inference scheme whose premises are the proof's open assumptions and whose conclusion is the sentence at the tree's root is said to be derived. The natural deduction calculus consisting solely of the I/E pairs just presented is called NI. The calculus that results by adding the additional inference rule of "indirect proof" $\dfrac{\dfrac{[\neg\mathrm{A}]}{\bot}}{\mathrm{A}}\,\mathrm{IP}$ is called NK.[5]

[5]Gentzen characterized NK in [7], not with IP, but with the rule for double negation (DN) and observed that this rule "falls outside the framework" of the I/E definitional scheme (p. 81). In [8] he wrote that DN "conflicts in fact quite categorically with the remaining forms of inference" (p. 169). The same can be said of IP or the law of excluded middle (LEM): The validity of these inference patterns is not secured by how the connectives that appear in them are relationally defined.

For an elementary template that we will refer to later, let us revisit the redundancy of $\bot$I in the context of the full calculus. Suppose π_1 is a derivation of $\mathfrak{E}$ (a propositional variable that doesn't appear in any undischarged assumptions π_1) and π_2 is a derivation whose assumptions are those of π_1 and $\bot$. Let $\pi_{1(\bot/\mathfrak{E})}$ be π_1 with every occurrence of $\mathfrak{E}$ replaced by $\bot$. Then the derivation figure

$$\begin{array}{c} \pi_1 \\ \dfrac{\mathfrak{E}}{\bot}\,\bot\text{I} \\ \pi_2 \end{array}$$

reduces to the following figure

$$\begin{array}{c} \pi_{1(\bot/\mathfrak{E})} \\ \bot \\ \pi_2 \end{array}$$

In this way we observe that the rule $\bot$I is derivable and therefore eliminable from every derivation. (Notice that the rule $\bot$E is derivable from IP and therefore can, like $\bot$I, be omitted from NK, although it has an essential role in NI.)

3. Identity

Gentzen's natural deduction calculi are often elaborated with further inference rules governing vocabulary from particular disciplines. Common examples are function symbols for arithmetical operations, relation symbols for set-membership, identity, and orderings, the predicate symbol for the property of being a natural number, the principle of mathematical induction, and constant symbols for specific numbers like 0 and 1. A natural

question is whether rules governing such vocabulary accord with Gentzen's scheme of relational definition or, alternatively, whether they are more like the rule IP in NK: required in order to ensure the derivability of certain expressions but not independently specifying the meaning of the vocabulary they facilitate reasoning about.

The treatment of identity in particular has captured the attention of several scholars. In typical natural deduction presentations of the first-order theory of identity, rules corresponding to Leibniz's "indiscernibility of identicals" law and the reflexivity property are provided:

$$\frac{a = b \qquad P(a)}{P(b)}\ \text{LL} \qquad\qquad \frac{}{a = a}\ \text{Ref}$$

It is well known, for example, that NK supplemented with these two rules is sound and complete with respect to classical first-order models in which the extension of the relation symbol $=$ is the set of all ordered pairs of objects whose first and second element are literally the same object.

But do these rules relationally define the identity predicate? Although the first rule represents a way of reasoning from an identity statement, and the second rule represents a way of reasoning to an identity statement, they do not, at a glance, seem to exhibit the sort of relationship with one another characteristic of I/E pairs. For one thing, Ref appears to be far too specific to be the canonical sort of way of establishing an identity statement worthy of being called the introduction rule from $=$. More crucially, neither rule evidently describes what it means for the property expressed by the other rule to hold universally or definitionally. In the long-standing tradition of proposing criteria for logicality, the question of the logicality of the identity predicate has received specific attention. Including rules for $=$ in a natural deduction calculus is suggestive of a logical designation, but, as discussed in [6] (p. 32) that designation might be misleading if the $=$ rules cannot be said to describe a universal construction akin to what is observed in the rules for operators that Gentzen treated.

This question, or one closely related to it, has been taken up by a school of proof theorists who ask whether these (or other) rules governing the $=$

symbol exhibit "harmony." Unfortunately, after many years of development, this line of investigation has done more to reveal that the ideal of harmony is sometimes unclear and occasionally tendentious than it has to bring us closer to an answer. In quick summary, Read ([22]) claimed that these rules obviously lack harmony and proposed an alternative pair of rules that he claimed exhibit it. Griffiths ([9]) claimed that Read's rules are no more harmonious than the original pair LL/Ref and expressed skepticism that any rules for identity are available. Read ([23]) replied that his proposed rules do in fact exhibit a special sort of "general elimination" harmony, which is what should be desired. Klev ([10]) found Read's proposed rules faulty in a number of ways and suggested that the original rules are harmonious after all. Mitsch ([19]) renewed the skeptical line but argued further that harmony is not a reasonable or desirable property for principles of reasoning about identity to have in the first place. The discussion spanning these papers has done less to converge on a shared answer than to multiply competing criteria—invertibility, normalizability, preservation of grounds, and the like—each proposed as essential to the philosophy of "inferentialism."

We hope to side-step the intricacies of this debate by pursuing the matter in the form of three precise questions: First, *Can the first-order identity predicate be characterized by a universal construction?* Second, *Is this construction representable with a pair of I/E rules in natural deduction?* Finally, after it is shown that the answer to both of these questions is "yes," *How do the I/E rules for* $=$ *relate to LL and Ref?* Whether the result exhibits harmony, and by whose lights, is left for others to consider. Perhaps, though, one can conclude that the universal construction for $=$ is a significant contribution to the theory of meaning in its own right, that it sheds light on the role principles like LL and Ref play in reasoning about identity, and that the simple terms in which the relational definition scheme casts these matters leaves little to be desired beyond the answers given.

Read in fact stumbled upon the universal construction for identity while pursuing his version of harmony. "Since the elimination-rule for '=' is the indiscernibility of identicals," he wrote, "the ground for asserting an identity must be the identity of indiscernibles." Because chiasma is provocative

but not convincing, he elaborated: "For if the introduction-rule for '=' were the identity of indiscernibles, and this was the only ground for asserting identity, Gentzen's proposal would justify as elimination rule the indiscernibility of identity" ([22, p. 115]). Read was operating from an intuitive sense that LL appears to be in order, that Ref is the rule that needs to be replaced. As he remarked at the end of his paper, "The indiscernibility of identicals, at least in extensional contexts, is usually regarded as uncontentious, even definitive of identity" (p. 118). But Read's route to discovering =I is circuitous: Instead of taking the =E rule one already has and determining from it what =I must be, he considered what =I must be in order for the =E rule one wants to be determinable from it. The culprit for this roundabout trajectory is what Read calls "Gentzen's proposal." The reference is to this line in Gentzen's thesis: "The introductions represent…the 'definitions' of the symbols concerned, and the eliminations are no more, in the final analysis, than the consequences of these definitions" ([7, II §5.13]). Taken literally, as many readers have taken him, Gentzen is espousing a uniform asymmetry between introduction and elimination rules: the former are definitions, and the latter are their consequences. We have seen that a more symmetric treatment has a natural algebraic realization: The introduction and elimination rules *together* serve as a definition—one rule specifies the defining property, and the other rule expresses the fact that the property is being thought of as defining. Which rule states the property and which rule expresses its universality depends on whether the construction is left- or right-universal. In the case of =, Read might have loosened the hold that "Gentzen's proposal" had on him in favor of the actual idea motivating him, namely, that beyond being "uncontentious," LL (and not its corresponding introduction rule) is "even definitive of identity." This leads to the following construction:

LL can be rewritten in turnstile notation as

$$a = b, P(a) \vdash P(b) \text{ for all predicates P}$$

To say that this rule is not only valid but *definitional* of the = relation is to say that any other sentence that can legitimately fill in the blank in "_____, $P(a) \vdash P(b)$ for all predicates P," any sentence, that is, that stands in the

same inferential relationships as a = b, must be such that a = b is inferable from it. Thus defining = with LL determines that a = b can be inferred from any other sentence that similarly licenses the inference, for every P, from P(a) to P(b). This is expressed by the following rule:

$$\text{For all C, if C, P(a)} \vdash \text{P(b), for all P, then C} \vdash \text{a = b}$$

Having characterized first-order identity as a right-universal construction, let us turn to the problem of presenting these rules as a I/E pair in natural deduction. =E is just LL, but to formulate =I one needs to able able to represent the idea of having available derivations from P(a) to P(b) for every possible P. This is accomplished with yet another sort of eigenvariable, one that ranges over first-order predicates. If $\mathfrak{P}$ is a predicate eigenvariable (i.e., a predicate symbol that doesn't appear in any undischarged assumption), then a derivation from $\mathfrak{P}$(a) to $\mathfrak{P}$(b) would be a template for a derivation from P(a) to P(b) for all atomic predicates.[6] The universalization of LL tells us that such a derivation suffices for an inference to a = b. Identity is therefore relationally defined by the following I/E rules: the indiscernibility of identicals and its universalization, the identity of indiscernibles:

= rules	
$\dfrac{\text{a = b} \qquad \text{P(a)}}{\text{P(b)}}$	$\dfrac{[\mathfrak{P}\text{(a)}]}{\dfrac{\mathfrak{P}\text{(b)}}{\text{a = b}}}$

Before turning to the question about these rules' relationship with the standard LL/Ref pair, let us flag an apparent inadequacy in them as a definition of identity. Identity is a symmetric relation, but the =E rule

[6]Notice that in the derivations one can construct with this template P need not be a monadic predicate symbol. One simply replaces every occurrence of $\mathfrak{P}(x)$ with an arbitrary atomic formula in which x occurs free (for example, x = c, R(x,d), or f(x)=g(d)). The eigenvariable condition guarantees that this uniform substitution yields a valid derivation.

provided is asymmetric in so far as it indicates how a = b facilitates inference from P(a) to P(b), but not vice-versa. For this reason, one could (and some have chosen to) specify two inference patterns for =E, analogous to the patterns for ∧E and ∨I: $\dfrac{\mathrm{a}=\mathrm{b} \qquad \mathrm{P(a)}}{\mathrm{P(b)}}$ as well as $\dfrac{\mathrm{a}=\mathrm{b} \qquad \mathrm{P(b)}}{\mathrm{P(a)}}$. We will return to this issue shortly.

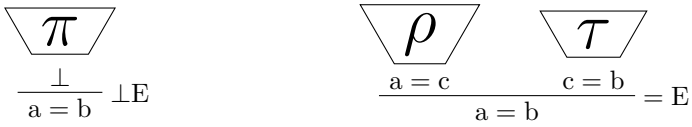
As for the relationship between =I and Ref, an immediate observation is that the latter rule is derivable from the former:

$$\dfrac{\overset{1}{[\mathfrak{P}(\mathrm{a})]}}{\mathrm{a}=\mathrm{a}}=\mathrm{I}_1$$

This is what one expects, as =I is seemingly more general than Ref. It prescribes a pattern for establishing non-trivial as well as trivial identity claims. In fact, one might reasonably worry that =I is too strong. As noted above, NK together with LL and Ref is sound and complete with respect to validity in first-order models in which = is interpreted as objectual identity. Should =I allow one to prove identity statements that Ref doesn't allow, such statements would evidently be classically invalid, and adding =I and =E to NK would result in an unsound system. To avoid this conclusion, one wants some guarantee that the *only* way one could ever implement a =I inference is to establish instances of reflexivity, that one could *never* reason from $\mathfrak{P}$(a) to $\mathfrak{P}$(b) for distinct a and b, even in the context of other assumptions, so long as the eigenvariable condition is met—not, that is, unless a = b already follows without the =I inference as in the following derivations

$$\dfrac{\dfrac{\overset{1}{[\mathfrak{P}(\mathrm{a})]} \qquad \overset{\pi}{\bot}}{\mathfrak{P}(\mathrm{b})}\bot\mathrm{E}}{\mathrm{a}=\mathrm{b}}=\mathrm{I}_1 \qquad\qquad \dfrac{\dfrac{\overset{1}{[\mathfrak{P}(\mathrm{a})]} \qquad \overset{\rho}{\mathrm{a}=\mathrm{c}}}{\mathfrak{P}(\mathrm{c})}=\mathrm{E} \qquad \overset{\tau}{\mathrm{c}=\mathrm{b}}}{\mathfrak{P}(\mathrm{b})}=\mathrm{E}}{\mathrm{a}=\mathrm{b}}=\mathrm{I}_1$$

which each obviously convert to derivations of a = b that use neither =I nor the assumption $\mathfrak{P}$(a):



This guarantee has been surprisingly elusive. Read ([22, p. 117]) concluded that =I is in fact unsound and sought to validate it nevertheless in terms of a non-standard model theory. Kremer ([11]) pointed out a flaw in Read's argument and provided an ordinary soundness proof. Kremer's proof works, but it has limited value. For one thing, validating =I by reflecting on first-order models is conceptually indirect and needlessly complex. More importantly, it leaves one with the impression that the equivalence of =I and Ref is some sort of miracle of classical quantification theory: a "miracle," because it offers no insight into why the more general sorts of inferences that =I appears to license are unavailable; "classical" (specifically), because nothing can be gleaned about whether =I might be stronger than Ref in the absence of LEM or in other non-classical contexts.

Griffiths's ([9]) arguments that "Read's rule [=I] is ... equivalent to the old identity rule [Ref]" provide more insight. They still don't fully explain the "equivalence," though. Because it appeals to first-order models, Griffiths's first proof (p. 503) is again indirect and explicitly classical. His second proof (pp. 507–509)—a standard induction on derivation length—can be extended to (some) non-classical settings. It nevertheless appears to be incomplete as an explanation for the underivability of $\mathfrak{P}(b)$ from $\mathfrak{P}(a)$ and a set X of other assumptions: The result explicitly leaves open the possibilities that (1) $\bot$ follows from $\mathfrak{P}(b) \cup X$ and that (2) the identity predicate appears in X. Griffiths remarked that "these are *precisely* the circumstances under which we can make assertions of identity with the old identity rules" (p. 503). But does either the ability to make identity assertions or the ability to derive a contradiction together with the extra assumption $\mathfrak{P}(a)$ suffice to explain how $a = b$ is supposed to be derived from X alone? Contrast the following derivation (in which the subderivation π has open assumptions X) with the above examples:

$$\dfrac{\overset{1}{[\mathfrak{P}(a)]} \quad \boxed{\pi}}{\dfrac{\dfrac{\dfrac{\bot}{\mathfrak{P}(b)}\,\bot E}{a = b}\,= I_1}}$$

Presumably these gaps can be filled in—this is what the soundness of =I guarantees, at least in the classical setting. But until they are, the relationship between =I and Ref remains unclear.

In fact, though, it is easy to show that =I is interderivable with Ref. The construction closely mirrors the exhibit of ⊥I's self-derivability from the previous section. Suppose π_1 is a derivation of $\mathfrak{P}(b)$ from $\mathfrak{P}(a)$ in which $\mathfrak{P}$ does not appear in any open assumptions other than $\mathfrak{P}(a)$ and π_2 is a derivation whose assumptions are those of π_1 (not including $\mathfrak{P}(a)$) and a = b. Consider the monadic predicate a = _ built from = by holding a fixed, and let $\pi_{1(a=_/\mathfrak{P})}$ be π_1 with every occurrence of $\mathfrak{P}$ replaced by a = _. Then the derivation figure

$$\begin{array}{c} \overset{1}{[\mathfrak{P}(a)]} \\ \boxed{\pi_1} \\ \dfrac{\mathfrak{P}(b)}{a = b}\,= I_1 \\ \boxed{\pi_2} \end{array}$$

reduces to the following figure:

$$\begin{array}{c} [a = a] \\ \boxed{\pi_{1(a=_/\mathfrak{P})}} \\ a = b \\ \boxed{\pi_2} \end{array}$$

The resulting tree has the same conclusion as the original tree, but it has one extra open assumption, a = a. To remove it, simply add a single instance of Ref:

$$\dfrac{}{\mathrm{a} = \mathrm{a}}\,\mathrm{Ref}$$

$$\pi_1(\mathrm{a}{=}_/\mathfrak{P})$$

$$\mathrm{a} = \mathrm{b}$$

$$\pi_2$$

So despite initial appearances, =I is interderivable with Ref. Its soundness with respect to standard first-order model theory is immediate: Because Ref is sound, =I is as well. But interderivability is stronger, more general, and conceptually more important than classical soundness: any natural deduction derivation featuring =I, whether or not the surrounding context is classical, can be converted to one that uses Ref instead. The conclusion is that one can never derive P(b) from P(a) for distinct a and b in a way that is genuinely arbitrary in P. Were such a derivation truly arbitrary, as the eigenvariable condition ensures, then one could construct from it a derivation of a = b from a = a.

The apparent generality of =I compared to Ref is therefore illusory. Further, the interderivability of =I and Ref explains the adequacey of the pair Ref/LL: It explains how those rules fully capture the relational meaning of = despite appearing not to exhibit the characteristic features of Gentzen's Intro/Elim scheme. For it reveals that the Ref/LL pair is in fact adequate to a universal construction, one that can be made explicit in a higher-order, but equivalent formulation.

This suggests that the same universal construction should admit a lower-order formulation. Such a formulation can in fact be given. If one takes Ref to express the defining property of the identity predicate, i.e., the fact that t=t for all terms t, then one can characterize identity as the extremal binary predicate with this property. Formally, for all binary predicates C, if $\vdash \mathrm{C}(\mathrm{t},\mathrm{t})$ for all therms t, then $\mathrm{a} = \mathrm{b} \vdash \mathrm{C}(\mathrm{a},\mathrm{b})$. In the format of natural deduction, this universalization of Ref reads as follows:

$$\dfrac{\mathrm{a} = \mathrm{b} \qquad \mathrm{C}(\mathfrak{e}, \mathfrak{e})}{\mathrm{C}(\mathrm{a}, \mathrm{b})}\ \text{M-L}$$

where $\mathfrak{e}$ is an eigenvariable. This last rule, which Martin-Löf proposed as the elimination rule for $=$ in [18, §3.8.2], thus provides together with Ref a left-universal construction of identity. Martin-Löf called the pair Ref/M-L "one way of formulating the standard rules for equality," and it is worth comparing it with the canonical pair Ref/LL. It is easy to see that M-L is derivable from LL: Suppose π is a derivation of $\mathrm{C}(\mathfrak{e}, \mathfrak{e})$ in which $\mathfrak{e}$ does not appear in any open assumptions and τ is a derivation of $\mathrm{a} = \mathrm{b}$. If $\pi_{(\mathrm{a}/\mathfrak{e})}$ is the derivation of $\mathrm{C}(\mathrm{a}, \mathrm{a})$ that results from replacing every occurence of $\mathfrak{e}$ in π with a, then the derivation figure

$$\dfrac{\begin{array}{cc}\tau & \pi \\ \mathrm{a} = \mathrm{b} & \mathrm{C}(\mathfrak{e}, \mathfrak{e})\end{array}}{\mathrm{C}(\mathrm{a}, \mathrm{b})}\ \text{M-L}$$

can be replaced by the following figure:

$$\dfrac{\begin{array}{cc}\tau & \pi_{(\mathrm{a}/\mathfrak{e})} \\ \mathrm{a} = \mathrm{b} & \mathrm{C}(\mathrm{a}, \mathrm{a})\end{array}}{\mathrm{C}(\mathrm{a}, \mathrm{b})}\ \text{LL}$$

LL is not so straightforwardly derivable from M-L, but in the presence of rules for $\supset$ adequate to its relational definition, one may, as Martin-Löf observed, instantiate $\mathrm{C}(x, y)$ with $\mathrm{P}(x) \supset \mathrm{P}(y)$. Then given a derivation π of $\mathrm{P}(\mathrm{a})$ and a derivation τ of $\mathrm{a} = \mathrm{b}$, the figure

$$\dfrac{\begin{array}{cc}\tau & \pi \\ \mathrm{a} = \mathrm{b} & \mathrm{P}(\mathrm{a})\end{array}}{\mathrm{P}(\mathrm{b})}\ \text{LL}$$

can be replaced by the following figure:

$$\dfrac{\pi \qquad \dfrac{\tau \qquad \dfrac{\overset{1}{[\mathrm{P}(\mathfrak{e})]}}{\mathrm{P}(\mathfrak{e}) \supset \mathrm{P}(\mathfrak{e})} \supset \mathrm{I}_1}{\mathrm{a} = \mathrm{b} \qquad \mathrm{P}(\mathrm{a}) \supset \mathrm{P}(\mathrm{b})} \text{M-L}}{\mathrm{P}(\mathrm{a}) \qquad \mathrm{P}(\mathrm{b})} \supset \mathrm{E}$$

Thus the rule pair Ref/LL is seen to be adequate both to the higher-order right-universal construction described by the rules =E/=I as well as to the lower-order left-universal construction described by the rules Ref/M-L, provided only that one has a relational definition of $\supset$ at hand.

Returning now to the question of the symmetry of the =E rule, we see its apparent asymmetry vanish. According to the relational definition scheme we are expounding, =E and =I together define identity. The first rule specifies a property that identity statements have, and the second rule indicates that they are the rightmost statements with that property. This universal construction implies the symmetry of identity, as is evident in this derivation of b = a from a = b:

$$\dfrac{[\mathrm{a} = \mathrm{b}] \qquad \dfrac{\overset{1}{[\mathfrak{P}(\mathrm{a})]}}{\mathrm{a} = \mathrm{a}} = \mathrm{I}_1}{\mathrm{b} = \mathrm{a}} = \mathrm{E}$$

The proof that the "other half" of Leibniz's law is derivable is then an elementary extension of this same proof tree:

$$\dfrac{[\mathrm{P}(\mathrm{b})] \qquad \dfrac{[\mathrm{a} = \mathrm{b}] \qquad \dfrac{\overset{1}{[\mathfrak{P}(\mathrm{a})]}}{\mathrm{a} = \mathrm{a}} = \mathrm{I}_1}{\mathrm{b} = \mathrm{a}} = \mathrm{E}}{\mathrm{P}(\mathrm{a})} = \mathrm{E}$$

The point is that, like reflexivity, the symmetry of identity is a consequence of its universal construction. That construction defines identity as the relation such that everything predicated of an object so related to another

object is predicated of that other object as well. =E articulates the property that this definition says identity has. =I articulates the fact that the property is definitional for identity, that it is *the* relation with that property in the sense that it can be immediately inferred from any relation with the same property. All the familiar properties of identity are determined by this one defining concept. It so happens that the transitivity of identity follows from =E alone and that its reflexivity is interderivable with =I. Symmetry is just as much a feature of this same relational definition, even though it is implied by neither inference rule individually, but instead emerges from their joint adequacy to the universal construction. There is no reason to expect all the construction's properties to fall neatly onto either the "introduction" or "elimination" side of how it is presented in natural deduction. An I/E pair of rules is adequate solely according to whether it accurately presents the underlying universal construction. Insisting that the =E rule explicitly convey symmetry only overburdens what is meant to be a matter of presentation with a task of direct expression.

The indiscernibility of identicals is not typically thought to be so closely related to the identity of indiscernibles. It is of course wrong to say (as a literal reading of Gentzen suggests) that one is a *consequence* of the other. What the universal construction confirms is that the identity of indiscernibles follows, not from the indiscernibility of identicals as a principle, but from *the conception of this principle as definitional*. One can coherently believe in the indiscernibilty of identicals while denying the identity of indiscernibiles. What one cannot do is regard the former as definitional of = without thereby accepting the latter as well. Any incentive to do so surely vanishes, however, once one observes that the identity of indiscernibles is interderivable with reflexivity!

4. Number

The closest Gentzen ever came to specifying what he meant by "logic" is a remark in the introduction of his [7] thesis. "Logic," he wrote, is simply what "comprises the types of inference that are continually used in all parts of mathematics." Reversing the traditional scheme of first defining logical

validity and then identifying inference principles that fit that definition, Gentzen just observed that some patterns of reasoning abound and began empirically tallying the techniques that show up everywhere in mathematical proofs. These are evidently the logical inferences, in contrast to others that are unique to geometry, arithmetic, and other specific branches of mathematics. As he wrote in his continuation of the above remark, "What remains to be added to these are axioms and forms of inference that may be considered as being proper to the particular branches of mathematics, e.g., in elementary number theory the axioms of the natural numbers, of addition, multiplication, and exponentiation, as well as the inference of complete induction; in geometry the geometric axioms" (p. 68).

One could be excused for questioning whether the principle of mathematical induction really is used less continuously or in fewer branches of mathematics than, say, inference patterns like $\vee$I or $\bot$I. There is evidence that Gentzen himself considered including it as a basic part of his logical calculi before ultimately deciding to designate it as an axiom to be added when investigating particular topics: The natural deduction calculus as formulated in Gentzen's handwritten draft included an induction rule; he abandoned it after his ambition to prove the consistency of Heyting arithmetic with a normalization procedure was dashed by the extended calculus's lack of a subformula property ([25, p. 241]). But although Gentzen did not present induction as a rule of inference in his published work, many of his followers have done so.

The induction principle is most commonly formulated as an axiom:

$$(\mathrm{P}(0) \wedge \forall x(\mathrm{P}(x) \supset \mathrm{P}(\mathsf{s}(x)))) \supset \forall x \mathrm{P}(x)$$

The principle thus depends on a prior characterization of a constant symbol 0 and an "immediate successor" function $\mathsf{s}(x)$ satisfying such defining conditions as $\forall x \neg(\mathsf{s}(x) = 0)$, $\forall x \neg(\mathsf{s}(x) = x)$, and $\forall x \forall y(\mathsf{s}(x) = \mathsf{s}(y) \supset x = y)$. We will assume that such a characterization has been provided. But even assuming this, the principle obviously fails in interpretations with objects that are not successors of 0 if these objects are not in the extension of P. Its validity further depends on the assumption that one is quantifying only over natural numbers. This, in addition to the failure of the subformula

property for Heyting arithmetic, might be among the considerations that led Gentzen to conclude that induction is "proper" to "elementary number theory": It is not less ubiquitous in mathematical practice than $\vee$I or $\bot$I; but unlike those inference patterns it is invalid unless it is understood that one is reasoning specifically about natural numbers.

Representations of induction as a rule of inference are simpler and more direct than axiomatic representations, though. For example, in sequent calculi the principle can be formulated without the conjunction and conditional operator as follows:

$$\frac{\Gamma, \mathrm{P}(\mathfrak{e}) \vdash \mathrm{P}(\mathfrak{s}(\mathfrak{e})), \Delta}{\Gamma, \mathrm{P}(0) \vdash \forall x \mathrm{P}(x), \Delta}$$

The following, equivalent natural deduction presentation features the same simplification:

$$\frac{\mathrm{P}(0) \qquad \begin{array}{c}[\mathrm{P}(\mathfrak{e})] \\ \mathrm{P}(\mathfrak{s}(\mathfrak{e}))\end{array}}{\forall x \mathrm{P}(x)}$$

In both of these rules, $\mathfrak{e}$ is understood to be subject to eigenvariable conditions. (Such conditions in sequent calculi are "local": $\mathfrak{e}$ does not appear in the "side formulas" Γ, Δ.) As with the axiom, both rules' validity depends on an understanding that one is reasoning solely about natural numbers, that the quantifiers do not range over other sorts of objects.

The question we want to ask about the induction rule is whether it accords in any meaningful way with Gentzen's relational definition scheme. At first sight this rule, despite being written in the format of natural deduction, appears to "fall outside the framework" of the I/E definitional scheme in just the same way that IP and DN do: Neither does the induction rule appear to express a universal property, nor is it clear what concept induction could be a universal construction of.

To see around this impasse, one need only reconsider induction's invalidity as a general principle of reasoning. Instead of thinking of induction as an inference principle available only when reasoning about a specific

subject matter or datatype (the natural numbers), one can think of it, following Martin-Löf ([18], §3.8.3), as a pattern of reasoning that becomes deductively valid in general when another premise is added. The extra premise specifies that an object is a natural number, and the conclusion is rewritten so that it specifies that the same object satisfies the predicate P:

$$\frac{\mathbb{N}(\mathrm{a}) \qquad \mathrm{P}(0) \qquad \begin{array}{c}[\mathrm{P}(\mathfrak{e})]\\ \mathrm{P}(\mathsf{s}(\mathfrak{e}))\end{array}}{\mathrm{P}(\mathrm{a})}$$

Thus reformulated, induction reclaims the right to be thought of as deductively valid. It is also more direct even than the previous natural deduction style formulation because of the absence of the universal quantifier. But now, in addition to the special constant symbol 0 and the special function symbol s, the rule features a special unary predicate symbol $\mathbb{N}$. This symbol needs its own definition if the reformulated induction scheme is to be meaningful.

If, keeping with the program of relational definition, one asks if there are characteristic inferences involving the concept of natural number that could be thought of as definitional of this concept, two inferences (inspired by the idea that the natural numbers are 0 and its successors) come immediately to mind:

$$\frac{}{\mathbb{N}(0)} \qquad\qquad \frac{\mathbb{N}(\mathrm{a})}{\mathbb{N}(\mathsf{s}(\mathrm{a}))}$$

Notice that these inferences are both introduction-style, in the usual sense that they are canonical ways of reasoning *to* a statement characterized by the predicate we aim to define. To say that these inferences are valid is to say that 0 and its successors are natural numbers. But we wanted to say something much stronger, namely, that the natural numbers just *are* 0 and its successors. That is to say that these inferences are not only valid for the predicate symbol $\mathbb{N}$, but definitional for it. This requires reading the introduction clauses not merely as sound conditions on $\mathbb{N}$, but

as constitutive constraints that determine its inferential role. As Charles Parsons explained:

> [T]hese rules are understood as the canonical way of arriving at statements to the effect that something is a natural number; it is only by virtue of them that something is a natural number. Thus a common way of stating inductive definitions is by giving rules like these and then saying something like, "Nothing is a number except by these rules." That amounts to taking $\mathbb{N}$ to be minimal so that the introduction rules hold. ([20, p. 142])

In Gentzen's relational definition scheme, this is expressed by saying any predicate P that stands in the inferential roles that these inferences specify for $\mathbb{N}$ must be such that one can legitimately infer P(a) from $\mathbb{N}$(a) (not only are 0 and its successors natural numbers, but anything true of 0 and its successors is true of any natural number, because the natural numbers just *are* 0 and its successors). In the turnstile notation, one has the defining inferences,

$$\vdash \mathbb{N}(0) \text{ and } \mathbb{N}(a) \vdash \mathbb{N}(\mathsf{s}(a)) \text{ for all a}$$

and a corresponding rule, expressing that these inferences are definitional,

$$\text{For all P, if } \vdash \mathrm{P}(0) \text{ and } \mathrm{P}(a) \vdash \mathrm{P}(\mathsf{s}(a)) \text{ for all a, then } \mathbb{N}(a) \vdash \mathrm{P}(a)$$

This left-universal construction is the relational definition of $\mathbb{N}$ we sought.[7]

But notice that the natural deduction presentation of the universalizing rule just presented is the induction rule itself. As Parsons observed, the minimality of $\mathbb{N}$ implies that P, because the introduction rules hold for it as well, "must be true of any natural number. But that is just the induction principle. It can be given the form of an 'elimination rule' for $\mathbb{N}$, that is,

[7] To say that these introduction and elimination inferences for the natural number predicate define the number concept is, of course, not to say that with them one is able in first-order logic to uniquely characterize the natural numbers. The number concept relationally defined in this way is not, as no first-order definition of the number concept can be, categorical. But categoricity is not an essential feature of meaning given relationally.

a rule of inference that from the above two premises with the additional premise $\mathbb{N}(a)$ allows inference" of $P(a)$ ([20, p. 142]). The problem of defining the concept of natural number, so that the reformulated induction rule it appears in is meaningful, has led to the observation that induction is actually part of that very definition, the elimination rule for $\mathbb{N}$:[8]

$\mathbb{N}$ rules

$$\frac{\mathbb{N}(a) \qquad P(0) \qquad \begin{matrix}[P(\mathfrak{e})] \\ P(\mathsf{s}(\mathfrak{e}))\end{matrix}}{P(a)} \qquad \frac{}{\mathbb{N}(0)} \qquad \frac{\mathbb{N}(a)}{\mathbb{N}(\mathsf{s}(a))}$$

Induction, which at first sight reads like an additional principle governing the natural numbers, is thus revealed to be the elimination rule that fixes what it is to be a natural number in the first place.

There is a long tradition of pointing to induction as a synthetic principle of reasoning, something that doesn't follow simply from how the numbers are formally defined. A prominent exponent of this tradition is Henri Poincaré, who in the third installment of his "Mathematics and Logic" essay series wrote:

> An integer is a cardinal number; zero is an integer—every integer has a successor different from itself which is also an integer—every integer but zero has a predecessor different from itself which is also an integer. There would perhaps be some advantage in modifying this definition, but it was the one I had in mind when I spoke of definition by recurrence; what matters to me is that it does not *analytically* imply the principle of induction. ([21, §IV])

[8] This left-universal construction of the natural number predicate was first described by Lawvere in [15] under the label "Peano categories." The presentation of the construction in the natural deduction format, with induction as the elimination rule for $\mathbb{N}$, first appeared in §3.8.3 of [18].

Of course Poincaré was right that mathematical induction does not follow from 0 and its successors all being natural numbers. We have seen, however, that if one just *means* "0 and its successors" by "natural number," then induction *does* follow from this definition. $\mathbb{N}$E and $\mathbb{N}$I together are a left-universal construction of $\mathbb{N}$: The latter rule specifies the defining properties of $\mathbb{N}$, and the former rule, which we recognize as the general formulation of the induction principle, expresses the fact that those properties are definitional for $\mathbb{N}$. If induction is indeed synthetic, it appears that this is because the concept of number itself is synthetic—not, as Poincaré suggested, because induction extends beyond what that concept determines.

5. Conclusion

The Göttingen theory of definition, as envisioned by Hilbert and first actualized by Gentzen, is a versatile and powerful framework for explicating meaning. Its versatility is evident in the wide range of terms it applies to, from propositional connectives and quantifiers to concepts like identity and even the natural numbers. Its power is manifest in the features of these concepts it unearths. An example is the fact that the law of reflexivity is interderivable with the principle of identity of indiscernibles, the latter being the universalization of the indiscernibility of identicals. Even more profoundly, the principle of mathematical induction has long been thought to stand in need of justification in order to be applicable to the natural numbers as traditionally defined. A fair portion of the philosophy of arithmetic has addressed whether such justification was inherently circular, or whether the induction principle could be derived from the definition. On one precisification of the Göttingen framework, the number concept is a left-universal construction, and the induction principle is an immanent feature of that very construction, the elimination rule. Accepting the validity of mathematical induction is part of what it is to understand what the natural numbers are.

To see either its versatility or its power, though, one must understand the relational definition scheme precisely. There is a great difference between a term having a property and that same property being definitional

for that term. Poincaré was right that the principle of induction does not follow merely from the defining properties of the number concept. Gentzen misstated the relationship between introduction and elimination rules when he said that the latter are consequences of the former. What Poincaré overlooked, what Gentzen understood but did not have the vocabulary to express, is that elimination rules, although they do not follow from their corresponding introduction rules, can nevertheless be consequences of *the conception of those rules as definitional*. One way of articulating this possibility, we've seen, is to distinguish natural deduction's introduction and elimination rules from the universal construction they represent. Mathematical induction ($\mathbb{N}$E) is in this way an immediate consequence of understanding the number concept's defining inferences ($\mathbb{N}$I) *as defining*.

An exception proves the rule. The *ex falso quodlibet* inference ($\bot$E) captures the full meaning of the $\bot$ connective. But this is because the right-universal construction it derives from is degenerate: The corresponding $\bot$I rule is content-less, a triviality, lending nothing to the meaning of the connective. The distinction between a property and the understanding of that property as definitional collapses when a single rule of inference expresses the full content of the universal construction. Typical universal constructions are robust: Their full meaning is captured by an I/E pair of rules. One rule indicates that the term has the property in question, and the other rule indicates that it is extremal among all terms that do so, so that the property can be thought of as definitional for that term. Identity and number are typical concepts in this sense.

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