Bulletin of the **S**ection of **L**ogic
Volume 55/2 (2026), pp. 281–305
https://doi.org/10.18778/0138-0680.2026.09

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JM13707

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MATRIX SEMANTICS FOR CLASSICAL LOGIC: THE CASE OF THE LATTICE O6

Abstract

It is well established that classical propositional logic is Boolean. However, this view has recently been challenged. In their paper *Non-Orthomodular Models for Both Standard Quantum Logic and Standard Classical Logic: Repercussions for Quantum Computers*, Mladen Pavičić and Norman Megill present a non-distributive, non-orthomodular model for both classical and quantum logic based on lattice O6, and argue that classical propositional logic is non-distributive.

In this paper, we examine this claim. Pavičić and Megill's model is formulated within unital matrix semantics rather than as an algebraic model in the sense of Abstract Algebraic Logic. An analysis of the lattice O6 in the framework of matrix semantics reveals that the matrix $(\mathrm{O6}, \{1, a, b\})$ is adequate for $\mathcal{CL}$, but not reduced, and induces the same consequence relation as the two-element Boolean matrix B_2. Similarly, the unital matrix $(\mathrm{O6}, \{1\})$ is adequate for $\mathcal{CL}$ through reduction to the four-element Boolean matrix B_4. Furthermore, we present two

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Presented by: Hanamantagouda P. Sankappanavar
Received: January 18, 2025, **Received in revised form**: April 7, 2026,
Accepted: May 11, 2026, **Published online**: June 10, 2026

lattice constructions that yield matrix models for $\mathcal{CL}$ lacking nontrivial lattice-theoretic properties.

These results show that the adequacy of O6 is not intrinsic to its algebraic structure, but is inherited from its reducibility to Boolean matrices, and more generally that classical logic admits models with highly unconstrained lattice structure. Consequently, the existence of such non-distributive models does not undermine the distributive character of classical propositional logic.

Keywords: classical propositional logic, matrix semantics, algebraic semantics, O6 lattice, distributivity.

1. Introduction

Classical propositional logic is standardly associated with Boolean algebra. However, since the 1990s several papers have challenged this view (e.g. [5, 6, 7, 4]). Mladen Pavičić and Norman Megill claim that classical propositional logic is non-distributive and quantum logic is non-orthomodular. In their paper *Non-Orthomodular Models for Both Standard Quantum Logic and Standard Classical Logic: Repercussions for Quantum Computers* they state:

> “The following theorem holds in $\mathcal{CL}$ [classical logic]:
>
> $\vdash A \vee (B \wedge C) \equiv_i (A \vee B) \wedge (A \vee C)$, where $i = 0, ..., 5$.
>
> The theorem is usually called a distributivity law. However, when its lattice mapping: $a \cup (b \cap c) \equiv_i (a \cup b) \cap (a \cup c) = 1$ is added to an ortholattice, it does not make the ortholattice even orthomodular: it does not fail in O6. We call this property a weakly distributive one and a weakly orthomodular lattice to which the property is added a weakly distributive lattice, WDL.
>
> We see that, as with the orthomodularity in quantum logic, in the syntactical structure of classical logic there is nothing distributive. The distributivity will appear as a result of the

> way the relation of equivalence is usually defined in a proof of completeness of classical logic" [5].

Although it is true that $a \cup (b \cap c) \equiv (a \cup b) \cap (a \cup c) = 1$ holds in the non-orthomodular ortholattice O6 while $a \cup (b \cap c) = (a \cup b) \cap (a \cup c)$ does not, the conclusion that "in the syntactical structure of classical logic there is nothing distributive" does not follow from these observations. The purpose of this paper is to examine the grounds for this claim.

Tomasz Kowalski, Francesco Paoli, and Roberto Giuntini have already examined Pavičić and Megill's claims regarding orthomodular quantum logic and its alleged non-orthomodularity [2]. They have shown, using the tools of Abstract Algebraic Logic, that weakly orthomodular lattices provide only an algebraic semantics of quantum logic, not an equivalent one [2], so the claim of non-orthomodularity of quantum logic cannot be maintained.

In the present paper, we turn to the case of classical propositional logic, which has not been defended against the claims of non-distributivity yet. We argue that the reasoning of Pavičić and Megill relies on a conflation of algebraic semantics with unital matrix semantics. We analyse the lattice O6 in the framework of matrix semantics and show that its adequacy for $\mathcal{CL}$ is possible through reducibility to Boolean matrices. We point out that in O6, elements which are logically equivalent need not be identical. This separation between logical equivalence and lattice identity leads to the failure of the rule of replacement in O6. Furthermore, we present two lattice constructions that yield matrix models for $\mathcal{CL}$ lacking arbitrary lattice-theoretic properties. This demonstrates that the algebraic properties of matrix models are radically underdetermined by the logic.

The structure of the paper is as follows. In Section 2, we explain the difference between algebraic semantics and unital matrix semantics. In Section 3, we distinguish two possible meanings of distributivity for classical propositional logic and the main points of Pavičić and Megill's argument against it. Section 4 collects the necessary definitions. In Section 5 we examine the non-distributive lattice O6 as a matrix semantics and compare it with the Boolean matrix semantics. In Section 6, we further compare O6 to Boolean matrices and discuss certain undesirable features of the weakly distributive matrix, specifically the failure of the rule of replacement. Finally, in Section 7, we conclude the paper.

2. Algebraic semantics and matrix semantics

The notion of “algebraic semantics” is used in different senses in the literature and is not entirely uniform (cf. [1]). In earlier approaches, it was common to understand algebraic semantics in terms of logical matrices with a single designated element (cf. [10]). In this sense, a logic is interpreted in an algebra equipped with a distinguished set of values, and a formula is considered valid if it takes a designated value under every valuation. In contrast, in Abstract Algebraic Logic, the term “algebraic semantics” is used in a more specific sense, where logical systems are studied via their associated algebraic structures, typically by relating formulas to equations in algebras.

To avoid terminological ambiguity, in this paper we will distinguish between the two meanings of “algebraic semantics” by adopting the term “unital matrix semantics” for semantics based on logical matrices with a single designated value (cf. [3]).

The crucial difference between these approaches is that, unlike algebraic semantics in the sense of Abstract Algebraic Logic, matrix semantics does not require the underlying algebraic structure to reflect the laws of the logic. As a matter of fact, there are many nontrivial unital matrices which are sound and complete for Classical Logic, e.g. the three-valued matrix (cf. [9, ch. 3.5]), the non-orthomodular lattice O5 (cf. [2]). The non-distributive but weakly distributive lattice O6 is another example. Therefore, the existence of non-distributive matrix models does not by itself provide evidence that classical logic is non-distributive.

The proposed distinction is essential for our analysis. As we argue in Section 5, the constructions used by Pavičić and Megill are formulated in terms of matrix semantics, even though they are described as instances of algebraic semantics. This leads them to misleading conclusions about the basic properties of classical logic.

It is also well known that classical propositional logic is algebraizable and that the variety of Boolean algebras constitutes its equivalent algebraic semantics. Since the equivalent algebraic semantics of the classical logic is unique, it follows that no class of non-distributive lattices can serve as an equivalent algebraic semantics for $\mathcal{CL}$. Although this observation already settles the issue at the level of algebraic semantics, the matrix-theoretic analysis of O6 in the following sections reveals how O6 can nonetheless be adequate for $\mathcal{CL}$ despite its non-distributive structure.

3. What is distributivity of classical logic and the arguments against it

Before analysing Pavičić and Megill's argument, it is useful to clarify what it means for classical propositional logic $\mathcal{CL}$ to be distributive.

1. One possible meaning of distributivity of $\mathcal{CL}$ concerns the syntactic side of the logic. In this sense, a logic is distributive if the distributivity laws $a \wedge (b \vee c) \equiv (a \wedge b) \vee (a \wedge c)$ and $a \vee (b \wedge c) \equiv (a \vee b) \wedge (a \vee c)$ are its theorems.

2. Another possible meaning of distributivity concerns the semantics of $\mathcal{CL}$, in particular its models. In this case, distributivity is related to the fact that classical logic is closely connected with distributive lattices with complementation (i.e., Boolean algebras). This connection is expressed, for instance, by the following result [8]:

   Theorem 3.1. *A formula is provable in* $\mathcal{CL}$ *if and only if it evaluates to* 1 *under every valuation in the two-element Boolean algebra.*

   The two meanings are bridged by the theorem [8]:

   Theorem 3.2. *The Lindenbaum algebra obtained from the language of* $\mathcal{CL}$ *by quotienting with respect to deductive equivalence is a Boolean algebra.*

The second meaning admits a further refinement: a logic is distributive in a weaker semantic sense if it possesses a distributive model, and in a stronger sense if all of its models are distributive.

Pavičić and Megill do not challenge these results (cf. [7]). This raises the question of how their claim of non-distributivity of classical logic is to be understood.

In their work, distributivity appears in two contexts, both treated as properties of lattices. The first context (and, simultaneously, the first part of their argument) is connected with the notion of weak distributivity and the observation that adding the weak distributivity axiom to an ortholattice[1] does not yield a distributive lattice.

As they notice, in any ortholattice, if $a = b$, then $a \equiv b = 1$ ($\equiv$ is the classical equivalence, while $=$ is lattice identity). But only in Boolean algebras the reverse holds: if $a \equiv b = 1$, then $a = b$. Therefore Pavičić and Megill offer an alternative definition of Boolean algebras:

Definition 3.3 (Boolean algebra [7]). An ortholattice that satisfies the following condition:

$$\text{if } a \equiv b = 1, \text{ then } a = b$$

is called a Boolean algebra.

Let:

- $(p \vee (q \wedge r)) \equiv ((p \vee q) \wedge (p \vee r))$ be the logical distributivity law,
- $(a \cup (b \cap c)) = ((a \cup b) \cap (a \cup c))$ be the algebraic distributivity law,
- $(a \cup (b \cap c)) \equiv ((a \cup b) \cap (a \cup c)) = 1$ be its translation into the algebraic language via the notion of validity (in a unital matrix).

Pavičić and Megill notice that if one adds the algebraic version of the distributivity law to the ortholattice axioms, the resulting lattice is not distributive; i.e. $(a \cup (b \cap c)) \equiv ((a \cup b) \cap (a \cup c)) = 1$ is valid in the resulting lattice, while $(a \cup (b \cap c)) = ((a \cup b) \cap (a \cup c))$ is not. From this observation they conclude that the lattice model of $\mathcal{CL}$ need not be distributive. They

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[1] For the definition of an ortholattice, see Section 4.

call the property $(a \cup (b \cap c)) \equiv ((a \cup b) \cap (a \cup c)) = 1$ *weak distributivity*. Crucially, they take this notion to correspond to the logical distributive law, since they treat the latter as the appropriate translation of logical principles into lattice-theoretic terms. This leads to models in which some logical principles are represented by identities, while others are represented only by equivalence conditions. As a result, the connection between logical equivalence and algebraic identity is weakened, and the resulting models exhibit a form of "nonstandardness", which they later interpret as evidence for the "nonstandardness" of logic. On this basis they conclude that "in the syntactical structure of classical logic there is nothing distributive" [5].

The second context (and the second part of their argument) concerns the Lindenbaum algebra of $\mathcal{CL}$ (cf. [5, 7]). Pavičić and Megill argue that the distributivity of this algebra arises from the definition of the equivalence relation rather than from the axioms or rules of inference of $\mathcal{CL}$. To support this, they define the relation $A \approx B$ by $\Gamma \vdash A \equiv B$, and argue that the property 3.3 – namely, if $a \equiv b = 1$, then $a = b$ – "has nothing to do with any axiom or rule of inference from $\mathcal{CL}$ – it is nothing but a consequence of the definition of the relation of equivalence" [7]. They then introduce a modified equivalence relation incorporating O6-valuations, and show that the resulting quotient algebra is a weakly distributive lattice, concluding that "the syntactical structure of classical logic corresponds to (maps to) the structure of the weakly distributive lattice not the one of the Boolean algebra" [7].

This line of reasoning raises several issues. First, the standard Lindenbaum equivalence is defined as $\vdash A \equiv B$, independently of any Γ. The Γ-dependent relation they use yields a different quotient and does not correspond to the usual Lindenbaum algebra of $\mathcal{CL}$. Furthermore, in the genuine Lindenbaum algebra, elements are cosets $[A]$ rather than formulas, and the condition $a \equiv b = 1$ properly reads $[A] \equiv [B] = [C \vee \neg C]$, which reduces to $[A] = [B]$, i.e. $\vdash A \equiv B$. The property they treat as characteristic of distributive lattices is therefore trivial in the standard construction: it follows directly from the meaning of coset equality, rather than from any independent algebraic constraint.

Even within their nonstandard setting, however, the conclusion does not follow in the intended sense. Since the distributivity law is a theorem of $\mathcal{CL}$, it follows from every set of premises Γ. Hence, according to their definition, for any formulas A, B such that $A \equiv B$ is a theorem of $\mathcal{CL}$, $A \approx B$, and thus $[A] = [B]$. The distributive character of this quotient algebra is therefore not an arbitrary artifact of definition, but a direct reflection of the inferential structure of the logic.

A further issue concerns the terminology: not every quotient of the formula algebra $\mathcal{F}$ deserves to be called a Lindenbaum algebra. Since $\mathcal{F}$ is an absolutely free algebra, every algebra of the same signature is its homomorphic image. Hence any such algebra arises as a quotient by some congruence, making constructions of this kind trivial, unless the congruence is determined by the consequence relation of the logic itself.

Most fundamentally, their modified construction incorporating O6-valuations (cf. [7]) is circular in the following sense: the congruence relation is defined from the outset to respect identifications induced by O6-valuations, so the resulting quotient inherits the structure of O6 by design. This does not show that the syntactic structure of $\mathcal{CL}$ naturally maps into O6. It shows only that imposing O6-based identifications on formulas produces a quotient with O6-like structure. That is a property of the chosen equivalence relation, not of classical logic.

In what follows, we will show that the lattice O6 can indeed serve as an adequate semantics for $\mathcal{CL}$, but only within the framework of matrix semantics. In particular, its comparison with Boolean semantics does not support the claim that it provides a more faithful representation of the logical structure of classical logic than Boolean semantics.

4. Preliminaries

Definition 4.1 (Ortholattice). An ortholattice $\langle L, \cap, \cup, ', 1, 0\rangle$ consists of a nonempty set L, two binary operations $\cap$ and $\cup$ called the lattice meet and join, respectively, a unary operation $'$ of orthocomplementation, and constants $1, 0 \in L$ (called the top and bottom elements), such that for every $a, b \in L$:

a) $a \cap a = a; \quad a \cup a = a,$

b) $a \cap b = b \cap a; \quad a \cup b = b \cup a,$

c) $a \cap (b \cap c) = (a \cap b) \cap c; \quad a \cup (b \cup c) = (a \cup b) \cup c,$

d) $a \cap (a \cup b) = a; \quad a \cup (a \cap b) = a,$

e) $a \cap 1 = a; \quad a \cup 1 = 1,$

f) $a \cap 0 = 0; \quad a \cup 0 = a,$

g) $a \cup a' = 1; \quad a \cap a' = 0,$

h) $a = (a')',$

i) if $a \leq b$, then $b' \leq a'$.

Definition 4.2 (Weakly orthomodular lattice WOML). A weakly orthomodular lattice is an ortholattice which satisfies the condition: $(a' \cap (a \cup b)) \cup b' \cup (a \cap b) = 1$.

Definition 4.3 (Distributive ortholattice). A distributive ortholattice is an ortholattice that satisfies the distributivity identity: $a \cup (b \cap c) = (a \cup b) \cap (a \cup c)$.

Definition 4.4 (Weakly distributive lattice WDL). A weakly distributive lattice WDL is a weakly orthomodular lattice that satisfies the condition:

$$a \cup (b \cap c) \equiv (a \cup b) \cap (a \cup c) = 1,$$

where $a \equiv b := (a' \cup b) \cap (b' \cup a)$ is the lattice-theoretic biconditional, and $=$ is the lattice identity.

Definition 4.5 (Boolean algebra). A Boolean algebra $\mathcal{B} = \langle B, \cap, \cup, ', 1, 0 \rangle$ is a distributive ortholattice.

In the presence of orthocomplementation, distributivity characterizes Boolean algebras.

Since in ortholattices $\cap$ is definable by $\cup$ and $'$ via de Morgan laws, Boolean algebras may also be stated as $\mathcal{B} = \langle B, \cup, ', 1, 0 \rangle$.

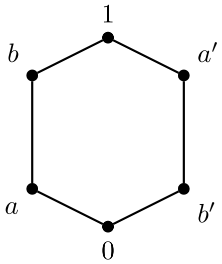

Figure 1: Lattice O6.

Definition 4.6 (O6 lattice). The lattice $O6 = \langle \{1, a, b, a', b', 0\}, \cap, \cup, ', 1, 0\rangle$ is the ortholattice shown in figure 1.

O6 is a non-orthomodular, non-distributive, weakly distributive lattice. It is called the benzene ring or the hexagon.

Definition 4.7 (Matrix). A logical matrix for a propositional language $\mathcal{F}$ is a pair $M = (\mathcal{A}, D)$, where $\mathcal{A}$ is an algebra of the same signature as $\mathcal{F}$ and $D \subseteq A$ is a set of designated elements.

Definition 4.8 (Matrix homomorphism). Let $M = (\mathcal{A}, D_M)$ and $N = (\mathcal{B}, D_N)$ be matrices, where $\mathcal{A}$ and $\mathcal{B}$ are algebras of the same signature. A mapping $h : \mathcal{A} \to \mathcal{B}$ is a homomorphism if:

1. $h(a \vee b) = h(a) \vee h(b)$,
2. $h(\neg a) = \neg h(a)$,
3. $h(D_M) \subseteq D_N$.

Definition 4.9 (Valuation). A valuation in a matrix $M = (\mathcal{A}, D)$ is a homomorphism $v : \mathcal{F} \to \mathcal{A}$.

Definition 4.10 (Satisfaction). A formula $A \in \mathcal{F}$ is satisfied in a matrix M under a valuation v if $v(A) \in D$.

Definition 4.11 (Validity in a matrix). Let $M = (\mathcal{A}, D)$ be a matrix. A formula $A \in \mathcal{F}$ is valid in M, written $\models_M A$, if for every valuation v, $v(A) \in D$.

A formula A is a consequence of $\Gamma \subseteq \mathcal{F}$ relative to M, written $\Gamma \models_M A$, if for every valuation v, whenever $v(X) \in D$ for all $X \in \Gamma$, then $v(A) \in D$.

Definition 4.12 (Rule validity in a matrix). An inference rule Γ/A is valid in a matrix $M = (\mathcal{A}, D)$ if for every valuation v, whenever $v(X) \in D$ for all $X \in \Gamma$, then $v(A) \in D$.

Definition 4.13 (Weakly adequate matrix). A matrix M is weakly adequate for a logic $\mathcal{L}$ if a formula A is a theorem of $\mathcal{L}$ if and only if it is valid in M.

Definition 4.14 (Adequate matrix). A matrix M is adequate for a logic $\mathcal{L}$ if it is weakly adequate and every rule of inference of $\mathcal{L}$ is valid in M.

Definition 4.15 (Matrix B_2). The matrix B_2 is the two-element Boolean algebra with $\{1\}$ as the set of designated elements: $\mathrm{B}_2 = ((\{1, 0\}, \vee, \neg), \{1\})$.

Matrix B_2 is an adequate matrix for $\mathcal{CL}$.

Throughout the paper, we use the logical symbols $\vee, \wedge$ and $\neg$ for matrix operations, while $\cup, \cap$, and $'$ denote their lattice-theoretic counterparts.

5. Lattice O6 as matrix semantics for classical logic

Pavičić and Megill showed that weakly distributive models, in particular the lattice O6, are sound and complete for $\mathcal{CL}$. Although they describe their approach in terms of algebraic semantics, their definitions of model and validity have the form typical of matrix semantics. Let us examine their definitions.

Definition 5.1 (Model [7]). We call $\mathcal{M} = \langle \mathcal{A}, h \rangle$ a model if $\mathcal{A}$ is an algebra and $h : \mathcal{F} \to \mathcal{A}$, called a valuation, is a morphism of formulas $\mathcal{F}$ into $\mathcal{A}$, preserving the operations $\neg, \vee$ while turning them into $', \cup$.

Definition 5.2 (Validity in model [7]). We call a formula $A \in \mathcal{F}$ valid in the model $\mathcal{M}$, and write $\models_{\mathcal{M}} A$, if $h(A) = 1$ for all valuations h on the model, i.e. for all h associated with the base set $\mathcal{A}$ of the model. We call a formula $A \in \mathcal{F}$ a consequence of $\Gamma \subseteq \mathcal{F}$ in the model $\mathcal{M}$ and write $\Gamma \models_{\mathcal{M}} A$ if $h(X) = 1$ for all $X \in \Gamma$ implies $h(A) = 1$, for all valuations h.

Note that these definitions correspond, in substance, to the standard definitions of validity in a (unital) matrix (Definitions 4.11, 4.12).

These definitions blur the distinction between satisfaction under a single valuation and validity across all valuations. Pavičić and Megill partially address this by allowing the term "model" to refer either to a specific pair $\langle \mathcal{A}, h \rangle$ or to the class of all such pairs based on a fixed algebra $\mathcal{A}$ (cf. [7, sec. 3]).

However, once this ambiguity is resolved, the underlying structure becomes clear: a model is determined by an algebra together with homomorphisms from the algebra of formulas, and validity is defined by requiring that formulas take the value 1 under all such homomorphisms. In other words, their notion of validity coincides with validity in a unital matrix $(\mathcal{A}, \{1\})$.

This differs fundamentally from algebraic semantics in the sense of Abstract Algebraic Logic, where formulas are related to equations in algebras.

Thus, despite the terminology used by Pavičić and Megill, their framework is most naturally understood as a unital matrix semantics. In what follows, we adopt this matrix-theoretic perspective and analyse the lattice O6 accordingly.

5.1. O6 as a matrix semantics of classical logic

We first introduce the matrix O6 and its operations. We defined the lattice $\mathrm{O6} = (\{1, a, b, a', b', 0\}, \cap, \cup, ', 1, 0)$ in Section 4. Interpreting the logical connective $\vee$ by the lattice operation $\cup$ and the logical negation $\neg$ by orthocomplementation $'$, we obtain the matrix O6.

Definition 5.3 (Matrix O6). A matrix O6 for $\mathcal{CL}$ is the matrix $\mathrm{O6} = ((\{1, a, b, a', b', 0\}, \vee, \neg), D)$ with the operations $\vee, \neg$ defined in Table 1 and with D being the set of designated elements.

We will consider O6 matrices with various choices of subset D. We will

Table 1: Operations $\vee$ and $\neg$ in O6.

$\vee$	1	a	b	a’	b’	0		$\neg$
1	1	1	1	1	1	1		0
a	1	a	b	1	1	a		a’
b	1	b	b	1	1	b		b’
a’	1	1	1	a’	a’	a’		a
b’	1	1	1	a’	b’	b’		b
0	1	a	b	a’	b’	0		1

also use O6 to denote the algebra $(\{1, a, b, a', b', 0\}, \vee, \neg)$, e.g. in $(\mathrm{O6}, \{1\})$ to denote the unital matrix O6 or in $(\mathrm{O6}, \{1, a, b\})$ to denote an O6 matrix with $D = \{1, a, b\}$.

Conjunction is defined in accordance with the lattice infimum (meet) and presented in Table 2. Implication is defined in Table 3 in accordance with the definition $a \to b := \neg a \vee b$. Equivalence is defined in Table 4 in accordance with the standard definition $a \equiv b := (a \to b) \wedge (b \to a)$.

From Table 4 we observe that elements a and b are equivalent ($a \equiv b = 1$), although non-identical. The same holds for a' and b'. This fact has a direct consequence for the distributivity law $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$. Consider a, b, and b'. Then $a \vee (b \wedge b') = a \vee 0 = a$, while $(a \vee b) \wedge (a \vee b') = b \wedge 1 = b$. Thus, the equality fails, while the equivalence of both sides is still 1. So the matrix is weakly distributive, but not distributive.

5.2. O6 and the two-element Boolean matrix B_2

Recall the following theorem [9]:

Theorem 5.4. *If there is a homomorphism f from matrix M to N that maps undesignated elements of M into undesignated elements on N, then all the propositions that are valid in N are valid in M.*

Proposition 5.5. O6 is an adequate matrix for classical propositional logic $\mathcal{CL}$.

Table 2: Operation ∧ in O6.

∧	1	a	b	a’	b’	0
1	1	a	b	a’	b’	0
a	a	a	a	0	0	0
b	b	a	b	0	0	0
a’	a’	0	0	a’	b’	0
b’	b	0	0	b’	b’	0
0	0	0	0	0	0	0

Table 3: Operation → in O6.

→	1	a	b	a’	b’	0
1	1	a	b	a’	b’	0
a	1	a	1	a’	a’	a’
b	1	1	1	a’	b’	0
a’	1	a	b	1	1	a
b’	1	b	b	1	1	b
0	1	1	1	1	1	1

Table 4: Operation ≡ in O6.

≡	1	a	b	a’	b’	0
1	1	a	b	a’	b’	0
a	a	1	1	0	0	a’
b	b	1	1	0	0	b’
a’	a’	0	0	1	1	a
b’	b’	0	0	1	1	b
0	0	a’	b’	a	b	1

Proof: Define a mapping $f : \mathrm{B}_2 \to \mathrm{O6}$ by:

$$\begin{gathered} f(1) = 1 \\ f(0) = 0. \end{gathered}$$

Then f is a homomorphism and maps undesignated elements of B_2 into undesignated elements of O6. Therefore, by Theorem 5.4, all propositions valid in O6 are valid in B_2. Since B_2 is an adequate matrix for $\mathcal{CL}$, it follows that all propositions valid in B_2 are derivable in $\mathcal{CL}$. Thus, O6 is complete for $\mathcal{CL}$.

Conversely, define a mapping $g : \mathrm{O6} \to \mathrm{B}_2$ as:

$$\begin{gathered} g(1) = 1 \\ g(a) = 1 \\ g(b) = 1 \\ g(a') = 0 \\ g(b') = 0 \\ g(0) = 0. \end{gathered}$$

It is straightforward to verify in the truth tables for O6 that g is a homomorphism. We take the set of designated elements to be $D = \{1, a, b\}$. Then, by Theorem 5.4, all propositions valid in B_2 are valid in O6. Since B_2 is an adequate matrix for $\mathcal{CL}$, all propositions derivable in $\mathcal{CL}$ are valid in B_2. It follows that every theorem of $\mathcal{CL}$ is valid in O6. Thus, every theorem of $\mathcal{CL}$ is valid in O6, i.e. O6 is sound for $\mathcal{CL}$.

Finally, for rules of inference, if we consider Modus Ponens, it is straightforward to verify (see Table 3) that whenever A and $(A \to B)$ are designated, B is also designated. Hence, the rules of inference of $\mathcal{CL}$ are valid in O6.

Therefore, O6 is an adequate matrix for $\mathcal{CL}$. □

Remark 5.6. The choice of designated elements in O6 is essential in this proof of adequacy. The only possible choices of D for which there exists a matrix homomorphism $g : \mathrm{O6} \to \mathrm{B}_2$ are $D = \{1, a, b\}$ and $D = \{1, a', b'\}$ (and these sets are the maximal filters in the lattice O6). Note that the map $f : \mathrm{O6} \to \mathrm{O6}$ such that $f(1) = 1, f(a) = a', f(b) = b', f(a') = a, f(b') =$

$b, f(0) = 0$, is an automorphism, and the two choices of designated elements differ only by a symmetry of the lattice O6.

If $D = \{1\}$, then any mapping $g_1 : \mathrm{O6} \to \mathrm{B}_2$ must satisfy $g_1(1) = 1$ and $g_1(x) = 0$ for all other elements. However, such a mapping fails to preserve negation, since

$$g_1(\neg b) = g_1(b') = 0 \neq 1 = \ \neg 0 = \ \neg g_1(b).$$

Hence, no such homomorphism exists.

A similar argument shows that for any other choice of D containing 1 (e.g. $D = \{1, a\}$, $D = \{1, a, a'\}$, $D = \{1, a, b, a', b'\}$, etc.), either $\vee$ or $\neg$ is not preserved. Therefore, no homomorphism with B_2 exists in these cases.

Consequently, this proof can only establish the adequacy of matrix $(\mathrm{O6}, \{1, a, b\})$ for $\mathcal{CL}$, and not of the unital matrix $(\mathrm{O6}, \{1\})$.

For completeness, we briefly recall the standard notions of congruence and quotient constructions, which will be used in the proof below.

Definition 5.7 (Congruence). An equivalence relation θ on an algebra $\mathcal{A}$ is a congruence if it is preserved by all operations of $\mathcal{A}$.

Every homomorphism determines a congruence relation given by $a\,\theta\,b$ iff $h(a) = h(b)$.

Definition 5.8 (Quotient algebra). Let $\mathcal{A}$ be an algebra with operations $\{O_i\}$, let θ be a congruence relation on $\mathcal{A}$, and A/θ the collection of equivalence classes determined by θ (where $[a]_\theta = \{b \in A : a\,\theta\,b\}$ is the equivalence class of a). The quotient algebra determined by θ is the algebra $\mathcal{A}/\theta = \langle A/\theta, \{Q_i\}\rangle$, where operations Q_i on equivalence classes are defined as:

$Q_i([a_1], ..., [a_n]) = [O_i(a_1, ..., a_n)]$.

Definition 5.9 (Matrix congruence). Let $M = (\mathcal{A}, D)$ be a logical matrix. A congruence θ in $\mathcal{A}$ is a matrix congruence if $a \in D$ implies $[a]_\theta \subseteq D$.

Definition 5.10 (Quotient matrix). The matrix $(\mathcal{A}/\theta, D/\theta)$, where $\mathcal{A}/\theta$ is the quotient of $\mathcal{A}$ and $D/\theta = \{[a]_\theta : a \in D\}$ is called the matrix quotient of M by θ, M/θ.

THEOREM 5.11 (Homomorphism Theorem). *Every homomorphic image of an algebra $\mathcal{A}$ is isomorphic to a quotient of $\mathcal{A}$, and vice versa.*

THEOREM 5.12 ([10]). *Let $M = (\mathcal{A}, D)$ be a logical matrix, and let θ be a matrix congruence in M. Then for every language interpreted in M:*
$M^{\models} = (M/\theta)^{\models}$.

Where $M^{\models}$ is the matrix consequence of matrix M.

PROPOSITION 5.13. Let O6 be the matrix $(\mathrm{O6}, \{1, a, b\})$, and B_2 be the two-valued Boolean matrix. Then:
$\mathrm{O6}^{\models} = \mathrm{B}_2^{\models}$.

PROOF: By Theorem 5.11, B_2 is the homomorphic image of O6 under the homomorphism g, and hence is isomorphic to the quotient $\mathrm{O6}/G$, where G is the congruence defined by: aGb iff $g(a) = g(b)$.

$D = \{1, a, b\}$. Then $[1]_G = [a]_G = [b]_G = \{1, a, b\} \subseteq D$, so G is a matrix congruence. Therefore, $\mathrm{O6}/G$ is a quotient matrix.

By Theorem 5.12, $\mathrm{O6}^{\models} = (\mathrm{O6}/G)^{\models}$, and since $\mathrm{O6}/G$ is isomorphic to B_2, it follows that $\mathrm{O6}^{\models} = \mathrm{B}_2^{\models}$. □

The weakly distributive matrix $(\mathrm{O6}, \{1, a, b\})$ and the Boolean matrix B_2 have the same matrix consequence.

DEFINITION 5.14 (Reduced matrix). A matrix M is called reduced if it admits no nontrivial matrix congruences.

As the proof shows, under congruence G, the matrix $(\mathrm{O6}, \{1, a, b\})$ reduces to B_2.

5.3. O6 and the unital four-element Boolean matrix B_4

Let $B_4 = ((\{1, a, b, 0\}, \vee, \neg), \{1\})$ be the unital four-element Boolean matrix, and let $(\mathrm{O6}, \{1\})$ be the unital matrix O6.

PROPOSITION 5.15. $(\mathrm{O6}, \{1\})$ is an adequate (unital) matrix for classical propositional logic $\mathcal{CL}$.

PROOF: Define a mapping $f : B_4 \to \mathrm{O6}$ as:

$$\begin{aligned} f(1) &= 1 \\ f(a) &= a \\ f(b) &= a' \\ f(0) &= 0. \end{aligned}$$

Then f is a homomorphism and maps undesignated elements of B_4 to undesignated elements of O6.

Conversely, define a mapping $g : \mathrm{O6} \to \mathrm{B}_4$ as:

$$\begin{aligned} g(1) &= 1 \\ g(a) &= a \\ g(b) &= a \\ g(a') &= b \\ g(b') &= b \\ g(0) &= 0. \end{aligned}$$

Then g is a homomorphism and maps undesignated elements of O6 to undesignated elements of B_4.

Since there exist matrix homomorphisms in both directions between $(\mathrm{O6}, \{1\})$ and B_4, it follows by Theorem 5.4 that both matrices validate the same formulas. Since B_4 is sound and complete for $\mathcal{CL}$, the same holds for $(\mathrm{O6}, \{1\})$. □

Observe that B_4 is isomorphic to the quotient matrix $\mathrm{O6}/\theta$, where θ is the congruence with equivalence classes $\{1\}$, $\{a, b\}$, $\{a', b'\}$, and $\{0\}$. The mapping g corresponds to the canonical quotient map. Since $g^{-1}(1) = \{1\}$, the matrix $(\mathrm{O6}, \{1\})$ reduces to B_4 via a nontrivial matrix congruence. Thus, the adequacy of $(\mathrm{O6}, \{1\})$ is not grounded in its internal algebraic structure, but is inherited via its reduction to a Boolean matrix.

As we have shown, the matrix O6 is an adequate matrix for classical logic, with D being either $\{1\}$ or $\{1, a, b\}$. The unital matrix O6 provides an adequate semantics for $\mathcal{CL}$ not because of its internal algebraic structure, but because it admits a reduction to the Boolean matrix B_4.

Furthermore, it is also a special case of the following construction, which allows us to obtain from B_4 various unital matrices that are sound and complete for classical logic but which do not have any specific nontrivial lattice properties (such as, e.g., distributivity)[2].

The construction goes as follows:

Let $B4 = (\{1, a, b, 0\}, \vee, \neg)$ be the 4-element Boolean algebra and $B_4 = (B4, \{1\})$ be the unital Boolean matrix. For any two disjoint bounded lattices L_0 and L_1, we can construct a new unital matrix M by "replacing" the element a with L_0 and b with L_1. The join in the extended lattice is defined in an obvious way, while for the complement $'$ we require:

$$\begin{gathered} 1' = 0, 0' = 1, \\ a' \in L_1 \text{ for all } a \in L_0, \\ b' \in L_0 \text{ for all } b \in L_1. \end{gathered}$$

Then we can define a map $h : M \to B_4$ as:

$$\begin{gathered} h(1) = 1, \\ h(0) = 0, \\ h(x) = a, \text{ for all } x \text{ in } L_0, \\ h(y) = b, \text{ for all } y \text{ in } L_1. \end{gathered}$$

The map h is a matrix homomorphism and maps undesignated elements of M into undesignated elements of B_4. A mapping $h^{-1} : B_4 \to M$ can be defined as follows:

$$\begin{gathered} h^{-1}(1) = 1, \\ h^{-1}(0) = 0, \\ h^{-1}(a) = x, \text{ where } x \text{ is some element of } L_0, \\ h^{-1}(b) = y, \text{ where } y \text{ is some element of } L_1, \\ x' = y \text{ and } y' = x. \end{gathered}$$

The map h^{-1} is a matrix homomorphism and maps undesignated elements of B_4 into undesignated elements of M. Therefore, M and B_4 validate the same formulas, and M is a sound and complete semantics for classical logic. Consequently, the existence of such matrices does not reflect any intrinsic logical properties of their underlying lattices, but rather

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[2] I would like to thank the anonymous referee for suggesting this construction.

the fact that matrix semantics allows arbitrary algebraic structure to be combined with a fixed Boolean core.

In particular, the lattice O6 appears as just one instance of a much broader class of constructions.

Another construction can be formulated in a more explicitly lattice-theoretic manner[3]. Let $\mathrm{L} = (L, \cap, \cup)$ be a bounded lattice and L^* be its dual lattice, obtained by reversing the partial order on L. Assume that the set L is disjoint from the universe of the algebra $B4$. For each $x \in L$, let x^* denote its corresponding element in L^*, and for each $y \in L^*$, let y^* denote its counterpart in L.

Define a new algebra $\mathcal{A} = (A, \cap, \cup, ')$ such that $A = \{0\} \cup L \cup L^* \cup \{1\}$. The lattice operations $\cap$ and $\cup$ are determined by the natural partial order extending those of L and L^*, while the unary operation $'$ is defined by:

$$\begin{gathered} 1' = 0, 0' = 1, \\ x' = x^* \text{ for all } x \in L \cup L^*. \end{gathered}$$

The resulting lattice $\mathcal{A}$ forms an ortholattice. As in the previous construction, one can define matrix homomorphisms $h : \mathcal{A} \to B_4$ and $k : B_4 \to \mathcal{A}$ such that designated elements are preserved, which shows that $\mathcal{A}$ is sound and complete for classical propositional logic.

Moreover, the partition of A into the sets $\{0\}$, L, L^*, and $\{1\}$ determines a matrix congruence, and the corresponding quotient algebra is isomorphic to $B4$.

Since L was an arbitrary bounded lattice, it follows that models of classical logic may contain sublattices with completely unrestricted lattice-theoretic properties, and in particular need not satisfy any nontrivial lattice identities.

These two constructions share a common feature: in each case, the resulting matrix admits a congruence whose quotient is isomorphic to B_4. Thus, their adequacy for $\mathcal{CL}$ does not stem from their internal lattice structure, which may lack arbitrary properties, but from their reducibility to Boolean matrices.

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[3] I would like to thank the anonymous referee for suggesting this construction.

6. Lattice O6 and Boolean algebras

The homomorphism $g : \mathrm{O6} \to \mathrm{B}_4$ from the matrix $(\mathrm{O6}, \{1, a, b\})$ into the unital Boolean matrix $(\mathrm{B4}, \{1\})$ suggests an interesting relationship between weakly distributive lattices and distributive lattices. First, notice that O6 is similar to B_4, but with the elements a and b split into distinct nodes. In distributive lattices, complementation is unique (meaning that for each element a there is only one element a' such that $a \cap a' = 0$ and $a \cup a' = 1$), while in the weakly distributive lattice O6 it is not. For example, for the element a both a' and b' serve as complements, for a' both a and b serve as complements, etc. In the Boolean case, if a and b were complements to a', a would be identified with b, and a' with b'. At the same time, Table 4 shows that a and b are equivalent in O6, and so are a' and b' (but e.g. a and 1 are not). From the point of view of the logical operation of equivalence, a and b "act" like one element, while from the point of view of lattice identity, they are not identical. This suggests that O6 can be viewed as a "broken" Boolean lattice in which identity has been separated from equivalence.

This feature reveals a fundamental limitation of O6 as a model of $\mathcal{CL}$.

First of all, for B_2 we have the feature [8]:

Theorem 6.1. *If the formula $\Phi \equiv \Psi$ is a theorem of $\mathcal{CL}$, then for every valuation $v : \mathcal{F} \to B_2$ we have $v(\Phi) = v(\Psi)$.*

Formulas Φ and Ψ, such that for every $v : \mathcal{F} \to \mathrm{B}_2$ we have $v(\Phi) = v(\Psi)$, are called semantically equivalent in matrix B_2. An equivalence relation that has the property described in Theorem 6.1. is said to have a normal interpretation in matrix B_2.

In the case of O6, as we have seen, this feature does not hold. Furthermore, formulas that are logically equivalent in $\mathcal{CL}$ are not semantically equivalent in matrix O6, and equivalence does not have a normal interpretation in matrix O6.

Secondly, another problem is that in the lattice O6 the rule of replacement of equivalents does not hold.

The rule of replacement of logical equivalents is a derivable rule of $\mathcal{CL}$:

$$\frac{\Phi \equiv \Psi}{\Omega \equiv \Omega(\Phi/\Psi)}$$

which states that if a formula Φ is logically equivalent to Ψ, then Ω is logically equivalent to a formula obtained from Ω through replacement of one or more occurrences of Φ in Ω by Ψ [8].

If we take Pavičić and Megill's interpretation of the language of $\mathcal{CL}$ in ortholattices, the rule will take the form:

$$\frac{\Phi \equiv \Psi = 1}{\Omega \equiv \Omega(\Phi/\Psi) = 1}.$$

Let Φ be the formula $a \vee (b \wedge c)$, Ψ the formula $((a \vee b) \wedge (a \vee c))$, and Ω the formula $(a \vee (b \wedge c)) \to (a \vee (b \wedge c))$. If we interpret these formulas in O6 and replace the first occurrence of Φ (the antecedent of Ω) with Ψ, we obtain:

$$\frac{(a \cup (b \cap c)) \equiv ((a \cup b) \cap (a \cup c)) = 1}{((a \cup (b \cap c)) \to (a \cup (b \cap c))) \equiv (((a \cup b) \cap (a \cup c)) \to (a \cup (b \cap c))) = 1}.$$

Consider the valuation assigning a to a, b to b, and b' to c. Then the weak distributivity as the premise holds. In the conclusion, however, the left side has the value: $a \cup (b \cap b') \to (a \cup (b \cap b')) = (a \cup 0) \to (a \cup 0) = a \to a = a$, while the right side has value: $((a \cup b) \cap (a \cup b')) \to (a \cup (b \cap b')) = (b \cap 1) \to (a \cup 0) = b \to a = 1$. Hence $a \equiv 1 = a \neq 1$. Therefore, the rule fails in O6.

This failure shows that O6 does not preserve one of the fundamental structural properties of classical logic, namely the substitutivity of logically equivalent formulas. Consequently, although O6 may reproduce the set of valid formulas, it does not fully preserve the inferential structure of $\mathcal{CL}$.

7. Conclusion

The aim of this paper was to analyse the lattice O6 as a matrix semantics for classical propositional logic $\mathcal{CL}$, and to examine what the existence of such non-distributive matrix models reveals about the distributive character of classical logic.

The matrix-theoretic analysis yields the following results:

- The matrix $(\mathrm{O6}, \{1, a, b\})$ is adequate for $\mathcal{CL}$ but not reduced, and it reduces via a matrix homomorphism to the Boolean matrix B_2.
- The unital matrix $(\mathrm{O6}, \{1\})$ is also adequate for $\mathcal{CL}$, and reduces via a matrix homomorphism to the Boolean matrix B_4.
- More generally, the class of adequate matrix semantics for $\mathcal{CL}$ is very broad: as shown in Sections 5 and 6, it includes structures lacking any nontrivial lattice-theoretic properties, including distributivity, double negation, and the law of excluded middle.

These results can be situated within the framework introduced in Section 3. Syntactic distributivity of $\mathcal{CL}$ is not affected: distributivity remains a theorem. Weak semantic distributivity trivially holds, as Boolean matrices are distributive models of $\mathcal{CL}$. Strong semantic distributivity fails, as witnessed by the constructions of Sections 5 and 6 – but this failure was never in doubt, and does not bear on the logical character of $\mathcal{CL}$. Pavičić and Megill's result, charitably interpreted, bears only on this last sense.

The analysis reveals not only that O6 is adequate for $\mathcal{CL}$, but also why: its adequacy is mediated by matrix homomorphisms to Boolean matrices, and is therefore inherited rather than intrinsic. This illustrates a general phenomenon: the flexibility of matrix semantics allows many algebraically diverse structures to validate the same set of formulas without preserving the underlying logical structure. In particular, O6 fails to preserve essential structural features of $\mathcal{CL}$, such as the substitutivity of logically equivalent formulas, further undermining its status as a faithful model.

The existence of non-distributive matrix models therefore does not constitute evidence that classical logic itself lacks distributivity. Distributivity is a theorem of $\mathcal{CL}$, and its presence in the equivalent algebraic semantics follows from the uniqueness guaranteed by Abstract Algebraic Logic. The claim that weakly distributive lattices better correspond to the syntactic structure of $\mathcal{CL}$ than Boolean algebras rests on a conflation of matrix semantics with algebraic semantics in the sense of Abstract Algebraic Logic. The definitions of model and validity employed by Pavičić and Megill are matrix-theoretic in character, and their construction yields a weakly distributive quotient only by incorporating O6-valuations into the equivalence relation from the outset. Consequently, the distributive character of classical propositional logic remains intact.

Acknowledgements. I am grateful to the anonymous referees for their detailed comments and valuable suggestions, which significantly improved the paper.

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Funding information: This research was supported by the National Science Centre, Poland, under grant no. 2021/41/N/HS1/01319.

Conflict of interests: None.

Ethical considerations: The Author assures of no violations of publication ethics and takes full responsibility for the content of the publication.

Declaration regarding the use of GAI tools: The Author declares that they have used ChatGPT 5 during preparation of the manuscript for the purpose of language proofing and ensuring appropriate academic style of the paper. The Author declares that they have thoroughly verified the text and take full responsibility for its contents.