Bulletin of the **S**ection of **L**ogic
Early View, 53 pp.
https://doi.org/10.18778/0138-0680.2026.24

Esmaeil Rostami

ON SEMI-SIMPLICITY RESULTS IN RESIDUATED LATTICES

Abstract

In this paper, we develop the theory of residuated lattices by introducing and studying several new types of filters and related concepts, including semi-simple filters, essential filters, the socle of a filter, and independent families of filters. Our primary goal is to understand the inner structure of residuated lattices by analyzing these new objects. First, we establish the key properties of simple and essential filters. Next, we provide both algebraic and topological characterizations for identifying when a filter is simple or essential. Furthermore, we use the concepts of the socle and independent families of filters to consider deeper into the structure of filters and the residuated lattice itself. Also, several characterizations for semi-simple filters and semi-simple residuated lattices are provided. Using these characterizations, we establish a fundamental logical characterization of semi-simplicity, proving that a residuated lattice is semi-simple if and only if its

Presented by: Janusz Ciuciura
Received: January 3, 2026, **Received in revised form**: June 24, 2026,
Accepted: June 27, 2026, **Published online**: September 8, 2026

frame of filters satisfies the law of excluded middle which, in this setting, is equivalent to the logical principles of double negation, material implication, and the contraposition law. A central result shows that for finite residuated lattices, being semi-simple is equivalent to being hyperarchimedean, highlighting the natural connection between these concepts. Complementary results deepen our understanding of the relation between simple and maximal filters in residuated lattices, and a comparison between a ring's semi-simplicity and the semi-simplicity of the residuated lattice of its ideals are also established.

Keywords: simple filter, semi-simple filter, socle of a residuated lattice, independent family of filters.

2020 Mathematical Subject Classification: 06F35, 03G25, 03G10.

Introduction

Algebraic structures play a fundamental role in formalizing different logical systems. Boolean algebras, for instance, are the natural framework for classical logic. However, to study non-classical logics such as fuzzy logic where the logical "and" can depend on the order of propositions, we need more flexible tools. Residuated lattices serve as this primary, general framework.

Introduced by Ward and Dilworth in 1939 [23], residuated lattices generalize the ideal theory of rings. Important subclasses like MV-algebras and BL-algebras have a direct and well-established connection to fuzzy logic [7, 9].

In parallel, the concepts of simplicity and semi-simplicity are central in ring and module theory. A simple ring has no non-trivial two-sided ideals, while a semi-simple ring is, by the fundamental Wedderburn-Artin Theorem, a finite direct product of simple rings. This decomposition provides a powerful structural insight [16, 2]. The theory of semi-simple structures has since been extended beyond rings, for instance, the basic structure of semi-simple modules is detailed in [2, 24], and the concept for semi-rings, based on ideals and coideals, has been introduced and investigated for its homological properties [11, 5, 15].

However, the classical ring-theoretic perspective can sometimes obscure the underlying order-theoretic relationships. We argue that the language of residuated lattices theory can offer a clearer and more unified viewpoint. In this lens, a simple module corresponds to a two-element chain, the simplest non-trivial lattice. A semi-simple module then corresponds to a Boolean algebra, which is a direct product of these simple chains. This approach provides an intrinsic, order-based description of semi-simplicity and allows us to leverage the powerful tools of residuated lattice theory to gain a deeper understanding of these algebraic systems. This paper aims to develop this perspective systematically.

The paper is organized as follows. In Section 2, we define simple and essential filters and investigate their key properties. Sections 3 and 4 are devoted to introducing independent families of filters and the socle of a filter, complete with characterizations using the maximal spectrum. In Section 5, we define semi-simple filters and residuated lattices, and several characterizations for semi-simple filters and semi-simple residuated lattices are provided. Using these characterizations, we establish a fundamental logical characterization of semi-simplicity, proving that a residuated lattice is semi-simple if and only if its frame of filters satisfies the law of excluded middle which, in this setting, is equivalent to the logical principles of double negation, material implication, and the contraposition law. A central result we prove is that in finite residuated lattices, being semi-simple is equivalent to being hyperarchimedean, underscoring the fundamental nature of semi-simple structures. Also, a comparison between a ring's semi-simplicity and the semi-simplicity of the residuated lattice of its ideals are also established.

1. Preliminaries

In this section, we review some definitions and results which will be used throughout this paper.

Definition 1.1 ([19, 7, 21, 9]). A *residuated lattice* is an algebra $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ satisfying the following axioms:

(**RL1**) $(L, \wedge, \vee, 0, 1)$ is a bounded lattice (whose partial order is denoted by $\leq$);

(**RL2**) $(L, \odot, 1)$ is a commutative monoid;

(**RL3**) For every $x, y, z \in L$, $x \odot z \leq y$ if and only if $z \leq x \to y$ (residuation).

For $x, y \in L$ and $n \in \mathbb{N}$, we define:

$$x^* := x \to 0, x^{**} := (x^*)^*, x^0 := 1, \text{ and } x^n := x^{n-1} \odot x.$$

An element x of a residuated lattice L is called *complemented* if there is an element $y \in L$ such that $x \wedge y = 0$ and $x \vee y = 1$, if such an element y exists it is unique and it is called *the complement of* x. We denote the complement of x by x'. The set of all complemented elements in L is denoted by $\mathrm{B}(L)$ and is called *the Boolean center of* L.

In the following proposition, we collect some main and well-known properties of residuated lattices, and throughout the paper, we frequently use them without referring.

Proposition 1.2 ([19, 7, 21, 9]). Let L be a residuated lattice, $x, y, z \in L$ and $e, f \in \mathrm{B}(L)$. Then we have the following statements:

1. $x \leq y$ if and only if $x \to y = 1$;
2. If $x \leq y$, then $y^* \leq x^*$;
3. $x \odot y = 0$ if and only if $x \leq y^*$. Also, $x \odot x^* = 0$;
4. $x \odot (x \to y) \leq y$ and $y \leq x \to y$;
5. If $x \vee y = 1$, then $x \wedge y = x \odot y$ and $x^n \vee y^m = 1$ for each $m, n \in \mathbb{N}$;
6. $x \vee (y \odot z) \geq (x \vee y) \odot (x \vee z)$, $(x \vee y)^* = x^* \wedge y^*$, and $x \odot (y \vee z) = (x \odot y) \vee (x \odot z)$;
7. $e' = e^*$, $e \wedge e^* = e \odot e^* = 0$, $e^{**} = e$, $e \odot f = e \wedge f \in \mathrm{B}(L)$, and for each $n \in \mathbb{N}$, $e^n = e$;
8. $e \odot x = e \wedge x$, $e \wedge (x \odot y) = (e \wedge x) \odot (e \wedge y)$, and $e \vee (x \odot y) = (e \vee x) \odot (e \vee y)$;

9. $e \to x = e^* \vee x$, $x \to e = x^* \vee e$, $e \odot (e \to x) = e \wedge x$, and $x \odot (x \to e) = e \wedge x$;

10. If $x \vee x^* = 1$, then $x \in \mathrm{B}(L)$;

11. $e \leq x$ if and only if $e \to x = 1$ if and only if $e^* \vee x = 1$.

Definition 1.3. A non-empty subset F of a residuated lattice L is called a *filter* of L if it satisfies the following:

(F1) If $x, y \in F$, then $x \odot y \in F$;

(F2) If $x \leq y$ and $x \in F$, then $y \in F$.

Definition 1.4. A non-empty subset F of a residuated lattice L is called a *deductive system* of L if it satisfies the following:

(D1) $1 \in F$;

(D2) If $x, x \to y \in F$, then $y \in F$.

A non-empty subset F of a residuated lattice L is a filter if and only if it is a deductive system (see [19] for more details).

We denote by $\mathrm{Filt}(L)$ the set of all filters of L. A filter F is called *proper* if $F \neq L$. Clearly, a filter F of L is proper if and only if $0 \notin F$, equivalently, for each $x \in L$, either $x \notin F$ or $x^* \notin F$.

For a non-empty subset S of a residuated lattice L, we denote by $[S)$ *the filter of* L *generated by* S, defined as $[S) := \bigcap\{F \in \mathrm{Filt}(L) \mid S \subseteq F\}$. Also, we denote by $[x_1, ..., x_n)$ the filter of L generated by the set $\{x_1, ..., x_n\}$. A filter F of a residuated lattice L is called a *finitely generated* filter if $F = [x_1, ..., x_n)$ for some $x_1, ..., x_n \in F$ and $1 \leq n$, and a filter F of a residuated lattice L is called a *principal* filter if $F = [x)$ for some $x \in F$.

Definition 1.5. A *frame* (or *locale*) $\mathfrak{F}$ is a complete lattice $(\mathfrak{F}, \wedge, \vee)$ satisfying the infinite distributive law:

$$a \wedge \bigvee_{i \in \Lambda} b_i = \bigvee_{i \in \Lambda} (a \wedge b_i)$$

for each $a \in \mathfrak{F}$ and every subset $\{b_i\}_{i \in \Lambda}$ of $\mathfrak{F}$.

For any element $a \in \mathfrak{F}$, the mapping $b \mapsto a \wedge b$ from $\mathfrak{F}$ to $\mathfrak{F}$ preserves arbitrary join. This mapping therefore admits a right adjoint, denoted by $a \to \_ : \mathfrak{F} \to \mathfrak{F}$. Consequently, the following adjunction property holds:

$$a \wedge b \leq c \quad \text{if and only if} \quad b \leq a \to c \qquad \forall\, a, b, c \in \mathfrak{F}.$$

Using this implication, one defines the *pseudo-complementation* operation on $\mathfrak{F}$ by setting $a^* := a \to 0$ for each $a \in \mathfrak{F}$.

Example 1.6. It is well-known that for a residuated lattice L, the lattice $(\mathrm{Filt}(L), \subseteq)$ is complete. Actually, for a family $\{F_i\}_{i\in\Lambda}$ of filters of L, we have $\bigwedge_{i\in\Lambda} F_i = \bigcap_{i\in\Lambda} F_i$ and $\bigvee_{i\in\Lambda} F_i = [\bigcup_{i\in\Lambda} F_i)$. It is easily seen that the infinite distributive law

$$F \wedge (\bigvee_{i\in\Lambda} F_i) = \bigvee_{i\in\Lambda} (F \wedge F_i)$$

for each $F \in \mathrm{Filt}(L)$ and every subset $\{F_i\}_{i\in\Lambda}$ of $\mathrm{Filt}(L)$ holds. Thus the complete lattice $(\mathrm{Filt}(L), \subseteq)$ is a frame and for each $F \in \mathrm{Filt}(L)$, $F^* := F \to \{1\} = \{x \in L \mid F \cap [x) = \{1\}\}$ (see [19, 21]).

DEFINITION 1.7. Let $\mathfrak{F}$ be a frame. Then $\mathfrak{F}$ is said to satisfy:

1. the *law of excluded middle* if $x \vee x^* = 1$ holds for every element $x \in \mathfrak{F}$;
2. *double negation* if $x^{**} = x$ holds for every element $x \in \mathfrak{F}$;
3. *material implication* if $x \to y = x^* \vee y$ holds for every pair of elements $x, y \in \mathfrak{F}$;
4. the *contraposition law* if $x \to y = y^* \to x^*$ holds for every pair of elements $x, y \in \mathfrak{F}$.

PROPOSITION 1.8 ([19]). Let S be a non-empty subset of a residuated lattice L, $x, y \in L$, $F, G \in \mathrm{Filt}(L)$, and $\{F_i\}_{i\in\Lambda} \subseteq \mathrm{Filt}(L)$. Then

1. $[S) = \{x \in L \mid s_1 \odot \cdots \odot s_n \leq x \ \text{ for some } \ n \geqslant 1 \text{ and } s_1, ..., s_n \in S\}$;
2. $[x) = \{z \in L \mid x^n \leq z \text{ for some } n \geqslant 1\}$. In particular, if $e, f \in \mathrm{B}(L)$, then $[e) = \{z \in L \mid e \leq z\}$, and so $[e) = [f)$ if and only if $e = f$;

3. $\bigvee_{i\in\Lambda} F_i = [\bigcup_{i\in\Lambda} F_i) = \{x \in L \mid$ there are $s_1 \in F_{i_1}, ..., s_n \in F_{i_n}$ such that $s_1 \odot \cdots \odot s_n \leq x$ for some $n \geqslant 1$ and $i_1, ..., i_n \in \Lambda\}$;
4. $[x) \vee [y) = [x \wedge y) = [x \odot y)$, and $[x) \cap [y) = [x \vee y)$;
5. $F \vee G = [F \cup G) = \{x \in L \mid a \odot b \leq x$ for some $a \in F$ and $b \in G\}$;
6. $F^* \in \mathrm{Filt}(L)$, and F^* is the maximum filter of L with respect to the property that $F \cap F^* = \{1\}$, that is, for $G \in \mathrm{Filt}(L)$, $F \cap G = \{1\}$ if and only if $G \subseteq F^*$;
7. $F^* = \{x \in L \mid x \vee y = 1$ for all $y \in F\}$;
8. The lattice $(\mathrm{Filt}(L), \subseteq)$ is distributive;
9. $(\bigvee_{i\in\Lambda} F_i) \to G = \bigcap_{i\in\Lambda}(F_i \to G)$. In particular, $(F \vee G)^* = F^* \cap G^*$;
10. If $e \in \mathrm{B}(L)$, then $[e)^* = [e^*)$.

Recall that a proper filter $P \in \mathrm{Filt}(L)$ is called *prime* if for $x, y \in L$, $x \vee y \in P$ implies either $x \in P$ or $y \in P$. We denote by $\mathrm{SpecF}(L)$ the set of all prime filters of L. An easy argument shows that a proper filter P is prime if and only if for $F, G \in \mathrm{Filt}(L)$ if $F \cap G \subseteq P$ then we have either $F \subseteq P$ or $G \subseteq P$. Also, a proper filter $M \in \mathrm{Filt}(L)$ is called *maximal* if M is not strictly contained in a proper filter of L. We denote by $\mathrm{MaxF}(L)$ the set of all maximal filters of L. Clearly, every maximal filter of a residuated lattice is prime. Also, every proper filter is contained in a maximal filter, see [19, 9, 21] for more details.

Theorem 1.9 ([19]). *Let M be a proper filter of a residuated lattice L. Then the following are equivalent:*

1. *M is a maximal filter of L;*
2. *For any $x \in L$, $x \notin M$ if and only if $(x^n)^* \in M$ for some $n \in \mathbb{N}$.*

Recall that a residuated lattice L is called *local* if L has a unique maximal filter. Also, a residuated lattice L is called *semi-local* if L has only a finite number of maximal filters.

The intersection of all maximal filters of a residuated lattice L is called *the radical of* L, and will be denoted by $\mathrm{Rad}(L)$.

Theorem 1.10 ([19]). *Let L be a residuated lattice. Then*

$$\mathrm{Rad}(L) = \{x \in L \mid \textit{for any } n \in \mathbb{N}, \textit{ there is } k_n \in \mathbb{N} \textit{ such that } ((x^n)^*)^{k_n} = 0\}.$$

Recall that a filter P is called *minimal prime* if $P \in \mathrm{SpecF}(L)$ and, if $Q \in \mathrm{SpecF}(L)$ such that $Q \subseteq P$, then we have $P = Q$. We denote by $\mathrm{MinF}(L)$ the set of all minimal prime filters of L.

Theorem 1.11 ([19]). *Let P be a prime filter of a residuated lattice L. Then the following are equivalent:*

1. *P is a minimal prime filter of L;*
2. *For any $x \in P$, there is $y \in L \setminus P$ such that $x \vee y = 1$.*

Recall that a set X with a family Γ of its subsets is called a *topological space*, denoted by (X, Γ), if $\emptyset, X \in \Gamma$, the intersection of any finite number of members of Γ is in Γ, and the arbitrary union of members of Γ is in Γ. The members of Γ are called *open* subsets of X, and the complement of an open subset of X is said to be *closed*.

If Y is a subset of X, the smallest closed subset of X containing Y is called the *closure* of Y and is denoted by $\bar{Y}$, and the largest open subset of X contained in Y is called the *interior* of Y and is denoted by $Int(Y)$.

A point x of a topological space X is called an *isolated point*, if the set $\{x\}$ is open.

For every subset A of a residuated lattice L, we set $U(A) := \{P \in \mathrm{SpecF}(L) \mid A \nsubseteq P\}$, and for each $a \in L$, we set $U(a) := U(\{a\})$. The family $\{U(A)\}_{A \subseteq L}$ satisfies the axioms for open subsets for a topology over $\mathrm{SpecF}(L)$. This topology is called *the Stone topology*. Since every maximal filter is prime, we can consider $\mathrm{MaxF}(L)$ as a subspace of $\mathrm{SpecF}(L)$. For each $A \subseteq L$, we define $U_{Max}(A) := U(A) \cap \mathrm{MaxF}(L)$. Then the family $\{U_{Max}(A)\}_{A \subseteq L}$ satisfies the axioms for open subsets for the subspace topology over $\mathrm{MaxF}(L)$. For every subset A of a residuated lattice L, we set $V(A) := \mathrm{SpecF}(L) \setminus U(A)$ and $V_{Max}(A) := \mathrm{MaxF}(L) \setminus U_{Max}(A)$.

Then the family $\{V(A)\}_{A \subseteq L}$ satisfies the axioms for closed subsets for the Stone topology over $\mathrm{SpecF}(L)$, and the family $\{V_{Max}(A)\}_{A \subseteq L}$ satisfies the axioms for closed sets for the subspace topology over $\mathrm{MaxF}(L)$, for more information see [8].

Proposition 1.12 ([19]). *Let L be a residuated lattice, $F \in \mathrm{Filt}(L)$ and $a \in L \setminus F$. Then we have the following statements:*

1. *There exists a prime filter P of L containing F such that $a \notin P$;*
2. *F is the intersection of all prime filters which contain F, that is, $F = \bigcap V(F)$;*
3. *$\bigcap \mathrm{SpecF}(L) = \bigcap \mathrm{MinF}(L) = \{1\}$.*

Proposition 1.13 ([8]). *Let L be a residuated lattice, $F, G \in \mathrm{Filt}(L)$ and $x, y \in L$. Then the following statements hold:*

1. *$F = L$ if and only if $V_{Max}(F) = \emptyset$ if and only if $V(F) = \emptyset$;*
2. *$V(F) \subseteq V(G)$ if and only if $U(G) \subseteq U(F)$ if and only if $G \subseteq F$. In particular, $V(F) = V(G)$ if and only if $U(F) = U(G)$ if and only if $F = G$;*
3. *If $G \subseteq F$, then $V_{Max}(F) \subseteq V_{Max}(G)$ and $U_{Max}(G) \subseteq U_{Max}(F)$. In particular, if $\mathrm{Rad}(L) = \{1\}$, then $V(F) = V(G)$ if and only if $V_{Max}(F) = V_{Max}(G)$ if and only if $F = G$;*
4. *$U(x) \cap U(y) = U(x \vee y)$ and $U_{Max}(x) \cap U_{Max}(y) = U_{Max}(x \vee y)$;*
5. *$V(x) \cup V(y) = V(x \vee y)$ and $V_{Max}(x) \cup V_{Max}(y) = V_{Max}(x \vee y)$;*
6. *If $\{X_k\}_{k \in K}$ is a family of subsets of L, then $U(\bigcup_{k \in K} X_k) = \bigcup_{k \in K} U(X_k)$ and $U_{Max}(\bigcup_{k \in K} X_k) = \bigcup_{k \in K} U_{Max}(X_k)$;*
7. *If $\{X_k\}_{k \in K}$ is a family of subsets of L, then $V(\bigcup_{k \in K} X_k) = \bigcap_{k \in K} V(X_k)$ and $V_{Max}(\bigcup_{k \in K} X_k) = \bigcap_{k \in K} V_{Max}(X_k)$.*

An element x of a residuated lattice L is called *archimedean* if there is $n \in \mathbb{N}$ such that $x^n \in \mathrm{B}(L)$. Also, a residuated lattice L is called *hyperarchimedean* if all its elements are archimedean.

Theorem 1.14 ([19]). *For a residuated lattice L, the following are equivalent:*

1. *L is hyperarchimedean;*
2. $\mathrm{SpecF}(L) = \mathrm{MaxF}(L)$*;*
3. $\mathrm{SpecF}(L) = \mathrm{MinF}(L)$*.*

Theorem 1.15 ([19]). *If L is a hyperarchimedean residuated lattice, then* $\mathrm{Rad}(L) = \{1\}$*.*

Recall that an element a in a Boolean algebra is called a *co-atom* if $a \neq 1$ and if $a \leq b$, then we have either $a = b$ or $b = 1$. Also, a Boolean algebra is said to be *co-atom-less*, if it contains no co-atoms.

2. Simple and Essential Filters

We begin this section by recalling the definition of a simple filter in a residuated lattice.

Definition 2.1. A filter F of a residuated lattice L is called *simple*, whenever $F \neq \{1\}$ and if $G \subseteq F$ for some $G \in \mathrm{Filt}(L)$, then we have either $G = \{1\}$ or $G = F$. We denote by $\mathrm{SimF}(L)$ the set of all simple filters of a residuated lattice L.

Remark 2.2. Let F be a simple filter of a residuated lattice L. If $1 \neq x \in F$, then since $\{1\} \neq [x) \subseteq F$, we have $F = [x)$. Hence, every simple filter is principal and is generated by each $x \in F$ with $x \neq 1$.

Remark 2.3. Let L be a non-trivial residuated lattice such that $\mathrm{Filt}(L)$ is a finite set (e.g., finite residuated lattices). Then clearly every non-trivial filter of L contains a simple filter. But in general this fact is not true. For example, assume that L is a co-atom-less Boolean algebra (e.g., the Boolean algebra of clopen subsets of the Cantor Set, the Boolean algebra of Lindenbaum-Tarski Algebra of Propositional Logic, and the Boolean algebra of measure algebra on $[0,1]$). Now, if F is a non-trivial filter of L, then $F \neq \{1\}$. Hence, there exists $1 \neq a \in F$. Since a is not a co-atom, there exists $b \in L$ such that $a \lneq b \lneq 1$. Hence, we have $\{1\} \neq [b) \subset [a) \subseteq F$ by Proposition 1.8(2). Thus, F is not a simple filter of L, and so L has no simple filters.

In Theorem 4.6, we provide some characterizations of residuated lattices L such that every non-trivial filter of L contains a simple filter.

According to the definition of a simple filter, it appears that this concept is dual to the notion of a maximal filter. In the following proposition, we consider the relation between simple filters and maximal filters that are generated by Boolean elements.

PROPOSITION 2.4. Let $e \in \mathrm{B}(L)$ for a residuated lattice L. Then $[e)$ is a maximal filter of L if and only if $[e^*)$ is a simple filter of L.

PROOF: $\Rightarrow$). Let $[e)$ be a maximal filter of L. If $[e^*) = \{1\}$, then $e^* = 1$, and so $e = 0$. Thus, $[e) = L$, which contradicts the maximality of $[e)$. Now, if $\{1\} \subset F \subseteq [e^*)$ for some filter F of L, then choose $1 \neq x \in F$. If $x \in [e)$, then $x \in [e) \cap [e^*) = [e \vee e^*) = \{1\}$, and so $x = 1$, a contradiction. So $x \notin [e)$ and hence $[x) \nsubseteq [e)$. Thus by Proposition 1.8(4) and the maximality of $[e)$, we have $[e \odot x) = [e) \vee [x) = L$. Hence by Proposition 1.8(1), there exists $n \in \mathbb{N}$ such that $e \odot x^n = e^n \odot x^n = (e \odot x)^n = 0$. This shows that $x^n \leq e^*$, and so $e^* \in [x)$. Hence $[e^*) \subseteq [x)$. Now since $[x) \subseteq F$ and $F \subseteq [e^*)$, we have $F = [e^*)$. Therefore, $[e^*)$ is a simple filter of L.

$\Leftarrow$). Let $[e^*)$ be a simple filter of L. If $[e) = L$, then by Proposition 1.8(1) there exists $n \in \mathbb{N}$ such that $e^n = 0$, and so $e = 0$. Thus $e^* = 1$, and so $[e^*) = \{1\}$, which is impossible. Now, if $[e) \subset F$ for some filter F of L, then choose $x \in F \setminus [e)$. Since $[x) \cap [e^*) \subseteq [e^*)$, we have either $[x) \cap [e^*) = [e^*)$ or $[x) \cap [e^*) = \{1\}$.

If $[x) \cap [e^*) = [e^*)$, then $[e^*) \subseteq [x) \subseteq F$. Thus $e, e^* \in F$. This shows that $0 = e \odot e^* \in F$, and so $F = L$, in this case. If $[x) \cap [e^*) = \{1\}$, then $[x \vee e^*) = [x) \cap [e^*) = \{1\}$ by Proposition 1.8(5). This shows that $x \vee e^* = 1$. Hence by Proposition 1.2, we have

$$e = e \odot 1 = e \odot (x \vee e^*) = (e \odot x) \vee (e \odot e^*) = e \odot x \leq x.$$

Thus $e \leq x$, and so $x \in [e)$, a contradiction. Therefore, $[e)$ is a maximal filter of L. $\square$

The following result establishes some properties of simple filters.

Proposition 2.5. Let T be a simple filter of a residuated lattice L. Then one of the following cases holds:

1. There exists $x \in \mathrm{B}(L)$ such that $T = [x)$. In this case, $[x^*)$ is the only maximal filter of L not containing T;

2. For all $x \in T$, we have $[x^*) = L$. In this case, T is contained in all maximal filters of L, that is, $T \subseteq \mathrm{Rad}(L)$.

Proof: Let $1 \neq x \in T$ be arbitrary. Since T is simple and $\{1\} \neq [x) \subseteq F$, we have $T = [x)$. By Proposition 1.8(5), $[x \vee x^*) = [x) \cap [x^*) \subseteq T$. Hence one of the following cases happens:

Case 1. If $[x \vee x^*) = \{1\}$ for some $1 \neq x \in T$, then $x \vee x^* = 1$, and so $x \in \mathrm{B}(L)$ by Proposition 1.2. Now, since T is a simple filter of L, by Proposition 2.4, $M := [x^*)$ is a maximal filter of L. It is obvious that $T \nsubseteq M$. Now, if N is a maximal filter of L with $M \neq N$, then since $x \vee x^* = 1 \in N$, we have either $x \in N$ or $x^* \in N$. If $x^* \in N$, then $M \subseteq N$ and since M is maximal, we have $M = N$, a contradiction. Hence, for all maximal filters N of L with $N \neq M$, we have $x \in N$, and so $T = [x) \subseteq N$. Therefore, in this case, $[x^*)$ is the only maximal filter of L not containing T.

Case 2. If $[x) \cap [x^*) = [x \vee x^*) = T = [x)$ for all $1 \neq x \in T$, then $[x) \subseteq [x^*)$ for all $1 \neq x \in T$, and so by Proposition 1.8(2) for each $x \in T$ there exists $n_x \in \mathbb{N}$ such that $(x^*)^{n_x} \leq x$. Hence $(x^*)^{n_x+1} = 0$ for all $x \in T$, and so for all $1 \neq x \in T$, we have $[x^*) = L$ (note that clearly $[1^*) = L$). Now, if $a \in T$, then for all $n \in \mathbb{N}$ we have $a^n \in T$. Hence $[(a^n)^*) = L$, and so by Proposition 1.8(2) there exists $k_n \in \mathbb{N}$ such that $((a^n)^*)^{k_n} = 0$. Thus by Theorem 1.10, $a \in \mathrm{Rad}(L)$, that is, a is contained in all maximal filters of L. Hence, T is contained in all maximal filters of L in this case. □

As a direct consequence of Proposition 2.5, we have the following corollaries.

Corollary 2.6. Let T be a non-trivial filter of a residuated lattice L with $\mathrm{Rad}(L) = \{1\}$. Then T is simple if and only if there exists $e \in \mathrm{B}(L)$ such that $T = [e)$ and $[e^*)$ is a maximal filter of L.

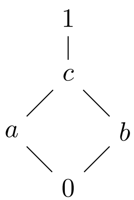
Proof: Since $\mathrm{Rad}(L) = \{1\}$ and $T \neq \{1\}$, we have $T \nsubseteq \mathrm{Rad}(L)$. Hence by Proposition 2.5, if T is a simple filter of L, then there exists $e \in \mathrm{B}(L)$ such that $T = [e)$ and $[e^*)$ is the only maximal filter of L not containing T. The converse is true by Proposition 2.4. □

Corollary 2.7. Let T be a simple filter of a residuated lattice L and $x \in T$. Then $[x^*) \neq L$ if and only if $x \in \mathrm{B}(L) \setminus \{1\}$ and $T = [x)$.

Proof: $\Rightarrow$). By the proof of Proposition 2.5, if $[x^*) \neq L$ for some $x \in T$, then $[x \vee x^*) = \{1\}$, $x \in \mathrm{B}(L)$ and $T = [x)$. In this case, if $x = 1$, then $x^* = 0$ and so $[x^*) = L$, a contradiction. Hence $x \in \mathrm{B}(L) \setminus \{1\}$ and $T = [x)$.

$\Leftarrow$). Let $x \in \mathrm{B}(L) \setminus \{1\}$ and $T = [x)$. If $[x^*) = L$, then by Proposition 1.8, there exists $n \in \mathbb{N}$ such that $(x^*)^n = 0$. Now since $x \in \mathrm{B}(L)$, we have $x^* \in \mathrm{B}(L)$, and so $x^* = (x^*)^n = 0$ by Proposition 1.2. Thus, $x = x^{**} = 1$, which is impossible. Therefore, $[x^*) \neq L$. □

Example 2.8. Let $L = \{0, a, b, c, 1\}$ denote the residuated lattice with Hasse diagram depicted in the following figure, and define the operations $\odot$ and $\rightarrow$ given by the accompanying tables, see [13, Example 4.4]:



$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	0	a	a
b	0	0	b	b	b
c	0	a	b	c	c
1	0	a	b	c	1

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	b	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

The filter $[c) = \{c, 1\}$ is simple, but c is not a Boolean element of L, and the filter $[c^*) = [0) = L$ is not maximal in L. Also, the filters $[a)$ and $[a^*) = [b)$ are maximal filters of L. Hence, Proposition 2.4 dose not hold in general, and the assumption $\mathrm{Rad}(L) = \{1\}$ in Corollary 2.6 cannot be omitted. Also, note that in this example, the filter $[c)$ is simple, but it is not prime or maximal filter, and the filter $[a)$ is maximal (and so it is prime), but it is not simple. Hence there are no general logical relationships between prime and simple filters.

Definition 2.9. For a filter F of a residuated lattice L, we set

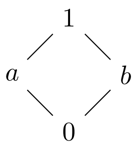
$$E_F := \{H \in \mathrm{Filt}(L) \mid H \subseteq F \text{ and for each } G \in \mathrm{Filt}(L), \\ \text{if } H \cap G = \{1\}, \text{ then } F \cap G = \{1\}\}.$$

If $H \in E_F$, we say that H is an essential filter in F. In particular case, if $H \in E_L$, we say that H is an essential filter in L.

Remark 2.10. Clearly, for each filter F of a residuated lattice L, we have $F \in E_F$. Hence $E_F \neq \emptyset$ for each filter F of a residuated lattice L.

Remark 2.11. Let $F, G, H \in \mathrm{Filt}(L)$ for a residuated lattice L such that $H \subseteq G \subseteq F$. Then, clearly, if $H \in E_F$, we have $H \in E_G$ and $G \in E_F$. Also, note that if $H \in E_G$ and $G \in E_F$, then $H \in E_F$.

Example 2.12. Let $L = \{0, a, b, 1\}$ denote a residuated lattice with Hasse diagram depicted in the following figure, and define the operations $\odot$ and $\rightarrow$ given by the accompanying tables:



$\odot$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	b	1	b	1
b	a	a	1	1
1	0	a	b	c

The filter $[a) = \{a, 1\}$ is not essential in L since $[b) \cap [a) = \{1\}$ but $[b) \cap L \neq \{1\}$. Nevertheless, it is a maximal (hence prime) and simple filter. Also, the filter L is essential in L, but it is neither prime (and it is not maximal) nor simple. Hence, there are no general logical relationships between prime filters and essential filters. Also, there are no general logical relationships between simple filters and essential filters

Example 2.13. In Example 2.8, the filter $[c)$ is simple and essential, but it is not prime.

Example 2.14. In any non-trivial local residuated lattice, since every proper filter is contained in the unique maximal filter, we deduce that the unique maximal filter is essential.

The following proposition is a direct consequence of the definition of F^*, Proposition 1.8, and Remark 2.10.

PROPOSITION 2.15. Let F be a filter of a residuated lattice L. Then the following are equivalent:

1. F is an essential filter in L;
2. $F^* = \{1\}$;
3. For $x \in L$, if $x \vee y = 1$ for all $y \in F$, then $x = 1$;
4. For each $1 \neq x \in L$, there exists $y \in F$ such that $x \vee y = 1$.

LEMMA 2.16. *Let $F, H \in \mathrm{Filt}(L)$ for a residuated lattice L such that $H \subseteq F$. Then $H \vee (H^* \cap F) \in E_F$.*

PROOF: Since $H \subseteq F$, we have $H \vee (H^* \cap F) \subseteq F$. Now if $(H \vee (H^* \cap F)) \cap G = \{1\}$ for some $G \in \mathrm{Filt}(L)$, then we have $(H \cap G) \vee ((H^* \cap F) \cap G) = \{1\}$ by Proposition 1.8(8). Hence, we have $H \cap G = \{1\}$ and $(H^* \cap F) \cap G = \{1\}$. From $H \cap G = \{1\}$, we conclude that $G \subseteq H^*$ by Proposition 1.8(6). Thus from $(H^* \cap F) \cap G = \{1\}$, we have $F \cap G = \{1\}$. Therefore, $H \vee (H^* \cap F) \in E_F$. □

Recall that a subset A of a topological space X is called *nowhere dense* if its closure has empty interior.

The following theorem states a topological characterization for a filter to be essential in a residuated lattice L with $\mathrm{Rad}(L) = \{1\}$.

THEOREM 2.17. *Let F be a filter of a residuated lattice L with $\mathrm{Rad}(L) = \{1\}$. Then $F \in E_L$ if and only if $V_{Max}(F)$ is a nowhere dense subset of $\mathrm{MaxF}(L)$.*

PROOF: $\Rightarrow$). Assume that $F \in E_L$. Since $Int_{Max}(V_{Max}(F))$ is an open subset of $\mathrm{MaxF}(L)$, there exists $G \in \mathrm{Filt}(L)$ such that $Int_{Max}(V_{Max}(F)) = U_{Max}(G)$. Assume that $U_{Max}(G) \neq \emptyset$. Choose $M \in U_{Max}(G)$. Hence $G \nsubseteq M$. Now if $\bigcap V_{Max}(G) \subseteq M$, then $G \subseteq M$, a contradiction. Hence, $\bigcap V_{Max}(G) \nsubseteq M$. Choose $x \in (\bigcap V_{Max}(G)) \setminus M$, and hence $x \in N$ for all $N \in V_{Max}(G)$.

Now assume that $y \in F$. Then $x \vee y \in N$ for all $N \in V_{Max}(G)$. If $N \in \mathrm{MaxF}(L) \setminus V_{Max}(G)$, then $N \in U_{Max}(G) = Int_{Max}(V_{Max}(F)) \subseteq V_{Max}(F)$, hence $F \subseteq N$, and so $y \in N$. Thus $x \vee y \in N$. Therefore, $x \vee y \in N$ for all

$N \in \mathrm{MaxF}(L)$, that is, $x \vee y \in \mathrm{Rad}(L) = \{1\}$, and so $x \vee y = 1$. From $x \notin M$, we have $x \neq 1$. Thus, $F \notin E_F$ by Proposition 2.15, which contradicts the assumption. Therefore, $Int_{Max}(V_{Max}(F)) = U_{Max}(G) = \emptyset$, that is, the closed subset $V_{Max}(F)$ of $\mathrm{MaxF}(L)$ is nowhere dense in $\mathrm{MaxF}(L)$.

$\Leftarrow$). Assume that $V_{Max}(F)$ is a nowhere dense subset of $\mathrm{MaxF}(L)$. Hence $Int_{Max}(V_{Max}(F)) = \emptyset$. Let $1 \neq x \in L$. Since $\mathrm{Rad}(L) = \{1\}$, we have $x \notin \mathrm{Rad}(L)$. Hence, $U_{Max}(x) \neq \emptyset$. By assumption, $U_{Max}(x) \nsubseteq V_{Max}(F)$, and so $U_{Max}(x) \cap U_{Max}(F) \neq \emptyset$. Choose $M \in U_{Max}(x) \cap U_{Max}(F)$. Then $x \notin M$ and $F \nsubseteq M$. Choose $y \in F \setminus M$. If $x \vee y = 1$, then since M is prime, we have either $x \in M$ or $y \in M$, which contradicts $x \notin M$ and $y \notin M$. Hence, $x \vee y \neq 1$. Therefor for each $1 \neq x \in L$, there exists $y \in F$ such that $x \vee y \neq 1$. Hence $F \in E_L$ by Proposition 2.15. □

Remark 2.18. In Example 2.8, $V_{Max}([c)) = \mathrm{MaxF}(L)$ is not a nowhere dense subset of $\mathrm{MaxF}(L)$, but $[c) \in E_L$. Hence the hypothesis $\mathrm{Rad}(L) = \{1\}$ in Theorem 2.17 is necessary.

PROPOSITION 2.19. Let L be a residuated lattice and $\mathfrak{A} \subseteq \mathrm{SpecF}(L)$. Then the closure of $\mathfrak{A}$ in $\mathrm{SpecF}(L)$ is equal to $V(\bigcap \mathfrak{A})$. In particular, if $\mathfrak{A} \subseteq \mathrm{MaxF}(L)$, then the closure of $\mathfrak{A}$ in $\mathrm{MaxF}(L)$ is equal to $V(\bigcap \mathfrak{A}) \cap \mathrm{MaxF}(L) = V_{Max}(\bigcap \mathfrak{A})$.

PROOF: Since $\mathfrak{A} \subseteq V(\bigcap \mathfrak{A})$ and $V(\bigcap \mathfrak{A})$ is a closed subset of $\mathrm{SpecF}(L)$, we have $\bar{\mathfrak{A}} \subseteq V(\bigcap \mathfrak{A})$. Now, let $\mathfrak{C}$ be a closed subset of $\mathrm{SpecF}(L)$ such that $\mathfrak{A} \subseteq \mathfrak{C}$. From the definition, there exists $F \in \mathrm{Filt}(L)$ such that $\mathfrak{C} = V(F)$. Hence $\mathfrak{A} \subseteq V(F)$, and so for each $P \in \mathfrak{A}$, we have $F \subseteq P$. Thus $F \subseteq \bigcap \mathfrak{A}$, and so $V(\bigcap \mathfrak{A}) \subseteq V(F) = \mathfrak{C}$ by Proposition 1.13(2). This shows that $V(\bigcap \mathfrak{A})$ is the smallest closed subset of $\mathrm{SpecF}(L)$ that contains $\mathfrak{A}$. Therefore, $\bar{\mathfrak{A}} = V(\bigcap \mathfrak{A})$. The last part of the proposition follows from the fact that if X is a topological space and Y is a subspace of X, then for a subset A of Y the closure of A in Y is equal to the intersection of Y and the closure of A in X, see [6, Proposition 2.1.1]. □

Recall that a subset Y of a topolgival space X is called *dense*, if $\bar{Y} = X$.

COROLLARY 2.20. Let L be a residuated lattice and $\mathfrak{A} \subseteq \mathrm{SpecF}(L)$. Then

$\mathfrak{A}$ is a dense subset of $\mathrm{SpecF}(L)$ if and only if $\bigcap \mathfrak{A} = \{1\}$.

PROOF: By the definition and Propositions 1.12, 1.13, and 2.19, we have

$$\begin{aligned}
\mathfrak{A} \text{ is dense in } \mathrm{SpecF}(L) &\Leftrightarrow \bar{\mathfrak{A}} = \mathrm{SpecF}(L)\\
&\Leftrightarrow V(\bigcap \mathfrak{A}) = \mathrm{SpecF}(L)\\
&\Leftrightarrow V(\bigcap \mathfrak{A}) = V(\{1\})\\
&\Leftrightarrow \bigcap \mathfrak{A} = \{1\}.
\end{aligned}$$

□

COROLLARY 2.21. Let L be a residuated lattice such that $\mathrm{Rad}(L) = \{1\}$ and $\mathfrak{B} \subseteq \mathrm{MaxF}(L)$. Then $\mathfrak{B}$ is a dense subset of $\mathrm{MaxF}(L)$ if and only if $\bigcap \mathfrak{B} = \{1\}$.

PROOF: Since $\mathrm{Rad}(L) = \{1\}$, for a filter F of L we have $V_{Max}(F) = \mathrm{MaxF}(L)$ if and only if $V_{Max}(F) = V_{Max}(\{1\})$ if and only if $F = \{1\}$, see Proposition 1.13(3). Hence by Proposition 2.19, we have

$$\begin{aligned}
\mathfrak{B} \text{ is dense in } \mathrm{MaxF}(L) &\Leftrightarrow \text{The closure of } \mathfrak{B} \text{ in } \mathrm{MaxF}(L) \text{ is equal to } \mathrm{MaxF}(L)\\
&\Leftrightarrow V(\bigcap \mathfrak{B}) \cap \mathrm{MaxF}(L) = \mathrm{MaxF}(L)\\
&\Leftrightarrow V_{Max}(\bigcap \mathfrak{B}) = \mathrm{MaxF}(L)\\
&\Leftrightarrow V_{Max}(\bigcap \mathfrak{B}) = V_{Max}(\{1\})\\
&\Leftrightarrow \bigcap \mathfrak{B} = \{1\}.
\end{aligned}$$

□

THEOREM 2.22. *Let $\mathfrak{A}$ be a non-empty family of filters of a residuated lattice L such that $\mathfrak{A} \subseteq \mathrm{SpecF}(L)$ and $\mathfrak{A} \cap E_L = \emptyset$, that is, $\mathfrak{A}$ is a non-empty family of prime filters that are not essential in L. Then the following are equivalent:*

1. *$\mathfrak{A}$ is a dense subset of $\mathrm{SpecF}(L)$;*
2. *A filter F of L is essential in L if and only if $F \nsubseteq P$ for each $P \in \mathfrak{A}$.*

PROOF: $1 \Rightarrow 2$). Let F be an essential filter in L. Since every filter in $\mathfrak{A}$ is not essential in L, we have $F \nsubseteq P$ for each $P \in \mathfrak{A}$ by Remark 2.11. Conversely, assume that $F \nsubseteq P$ for each $P \in \mathfrak{A}$. Choose $x \in F^*$. Then by the definition, $[x) \cap F = \{1\}$. Hence, $[x) \cap F \subseteq P$ for each $P \in \mathfrak{A}$. Now since $F \nsubseteq P$ for each $P \in \mathfrak{A}$, we have $[x) \subseteq P$ for each $P \in \mathfrak{A}$, that is, $[x) \subseteq \bigcap_{P\in\mathfrak{A}} P$. By assumption and Corollary 2.20, $\bigcap_{P\in\mathfrak{A}} P = \{1\}$. Hence $x = 1$, and so $F^* = \{1\}$. Thus F is an essential filter in L by Proposition 2.15.

$2 \Rightarrow 1$). For each $P \in \mathfrak{A}$, $P^* \neq \{1\}$ by Proposition 2.15. Hence, we can choose $1 \neq x_P \in P^*$ for each $P \in \mathfrak{A}$. For each $P \in \mathfrak{A}$, if $x_P \in P$, then $x_P = x_P \vee x_P = 1$, a contradiction. Hence, $x_P \notin P$ for each $P \in \mathfrak{A}$. Consider the $F := [\{x_P \mid P \in \mathfrak{A}\})$. For each $P \in \mathfrak{A}$ since $x_P \notin P$, we have $F \nsubseteq P$ for each $P \in \mathfrak{A}$. Thus by assumption, the filter F of L is essential, that is, $F^* = \{1\}$ by Proposition 2.15. Now let $x \in \bigcap \mathfrak{A}$. Hence $x \vee x_P = 1$ for all $P \in \mathfrak{A}$. Thus $[x) \cap F = \{1\}$, and so we must have $[x) \subseteq F^* = \{1\}$. This shows that $x = 1$, and hence $\bigcap \mathfrak{A} = \{1\}$. Therefore, $\mathfrak{A}$ is a dense subset of $\mathrm{SpecF}(L)$ by Corollary 2.20. □

Proposition 2.23. *Let P be a prime filter of a residuated lattice L. If P is not essential in L, then it is a minimal prime filter of L.*

Proof: Assume that P is a prime filter of a residuated lattice L such that it is not essential. By Proposition 2.15, $P^* \neq \{1\}$. Choose $1 \neq x \in P^*$. If $x \in P$, then $x = x \vee x = 1$, a contradiction. Hence, $x \notin P$. Now assume that $Q \subseteq P$ for some $Q \in \mathrm{SpecF}(L)$. If $y \in P$, then $x \vee y = 1$. Hence $x \vee y \in Q$. Since $x \notin P$, we have $x \notin Q$, and so $y \in P$. Thus $P \subseteq Q$. Therefore, $P = Q$ and so P is a minimal prime filter of L. □

Corollary 2.24. *Let P be a prime and finitely generated filter of a residuated lattice L. Then P is not essential in L if and only of it is a minimal prime filter of L.*

Proof: $\Rightarrow$). It follows from Proposition 2.23.

$\Leftarrow$). Let P be a minimal prime and finitely generated filter of a residuated lattice L. Hence, there exist $x_1, ..., x_n \in P$ such that $P = [x_1, ..., x_n)$. By assumption and Theorem 1.11 for each $i \in \{1, ..., n\}$, there exists $y_i \in L \setminus P$ such that $x_i \vee y_i = 1$. Set $y := y_1 \vee \cdots \vee y_n$. Since P is a prime filter, we have $y \in L \setminus P$ and $x_i \vee y = 1$ for each $i \in \{1, ..., n\}$. Now if $a \in P$, then by Proposition 1.8 there exist $m_1, ..., m_n \in \mathbb{N}$ such that $x_1^{m_1} \odot \cdots \odot x_n^{m_n} \leq a$. By Proposition 1.2, we have $x_i^{m_i} \vee y = 1$ for each

$i \in \{1, ..., n\}$. Hence we have:

$$1 \odot \cdots \odot 1 = (x_1^{m_1} \vee y) \odot \cdots \odot (x_n^{m_n} \vee y) \leq (x_1^{m_1} \odot \cdots \odot x_n^{m_n}) \vee y \leq a \vee y.$$

Therefore, $a \vee y = 1$ and so $y \in P^*$. Now since $y \in L \setminus P$, we have $y \neq 1$, and so $P^* \neq \{1\}$. Thus, P is not essential in L by Proposition 2.15. □

THEOREM 2.25. *Let F be a filter of a residuated lattice L such that each minimal prime filter of L is finitely generated (e.g., finite residuated latices). Then F is an essential filter of L if and only if $F \nsubseteq P$ for each minimal prime filter P of L.*

PROOF: By assumption, every minimal prime filter of L is finitely generated. Hence $P \notin E_L$ for all minimal prime filters P of L by Corollary 2.24. On the other hand, by Proposition 1.12, $\bigcap_{P \in \mathrm{MinF}(L)} P = \{1\}$. Hence the result follows from Corollary 2.20 and Theorem 2.22. □

3. Independence of a family of filters

We begin this section with the definition of an independent family of filters.

DEFINITION 3.1. A non-empty family $\{F_i\}_{i \in \Lambda}$ of filters of a residuated lattice L is called *independent*, whenever $F_j \cap (\bigvee_{j \neq i \in \Lambda} F_i) = \{1\}$ for each $j \in \Lambda$.

DEFINITION 3.2. Let $F \in \mathrm{Filt}(L)$ and $\{F_i\}_{i \in \Lambda}$ be a non-empty family of filters of a residuated lattice L. Then we say that *F is the direct sum of the family* $\{F_i\}_{i \in \Lambda}$, denoted by $F = \bigoplus_{i \in \Lambda} F_i$, whenever $\{F_i\}_{i \in \Lambda}$ is an independent family of filters of L and $F = \bigvee_{i \in \Lambda} F_i$. In this case, F_i is called *a direct summand of F*.

Remark 3.3. If $\mathfrak{F}$ is an empty family of filters, then $\bigvee \mathfrak{F} = \{1\}$. Thus, any family of filters containing exactly one element is independent. So each filter is a direct summand of itself.

Example 3.4. Let $e \in \mathrm{B}(L)$ for a residuated lattice L. Then by Proposition 1.8, since $[e) \cap [e^*) = [e \vee e^*) = \{1\}$ and $[e) \vee [e^*) = [e \odot e^*) = [0) = L$, we have $L = [e) \oplus [e^*)$.

The following proposition gives some useful characterizations for a two elements family of filters to be independent.

PROPOSITION 3.5. Let $F, G \in \mathrm{Filt}(L)$ for a residuated lattice L. Then the following are equivalent:

1. $\{F, G\}$ is an independent family of filters of L;
2. For $f, f' \in F$ and $g, g' \in G$, if $f \odot g \leq f' \odot g'$, then $f \leq f'$ and $g \leq g'$;
3. For $f, f' \in F$ and $g, g' \in G$, if $f \odot g = f' \odot g'$, then $f = f'$ and $g = g'$.

PROOF: $1 \Rightarrow 2$). Assume that $f \odot g \leq f' \odot g'$ for some $f, f' \in F$ and $g, g' \in G$. Then from $f \odot g \leq f' \odot g'$, we deduce that $f \odot g \leq f'$ and $f \odot g \leq g'$. Thus $g \leq f \to f'$ and $f \leq g \to g'$. Hence, $f \to f', g \to g' \in F \cap G = \{1\}$. Thus $f \to f' = 1$ and $g \to g' = 1$, and so $f \leq f'$ and $g \leq g'$.

$2 \Rightarrow 3$). The proof is straightforward .

$3 \Rightarrow 1$). Assume $x \in F \cap G$. Then since $x \odot 1 = 1 \odot x$ and $x, 1 \in F \cap G$, we have $x = 1$, and so $\{F, G\}$ is an independent family of filters of L. □

COROLLARY 3.6. Let $F, F', G, G' \in \mathrm{Filt}(L)$ for a residuated lattice L such that $F \subseteq F'$ and $G \subseteq G'$. If $F \oplus G = F' \oplus G'$, then $F = F'$ and $G = G'$.

PROOF: Let $a \in F'$ and $b \in G'$. Then $a \odot b \in F' \oplus G'$, and hence $a \odot b \in F \oplus G$. Thus by Proposition 1.8(5), there exist $f \in F$ and $g \in G$ such that $f \odot g \leq a \odot b$. Now since $a, f \in F'$ and $b, g \in G'$, we have $f \leq a$ and $g \leq b$ by Proposition 3.5. From $f \in F$ and $g \in G$, we have $a \in F$ and $b \in G$. Therefore, $F' \subseteq F$ and $G' \subseteq G$, and so $F = F'$ and $G = G'$. □

PROPOSITION 3.7. Let $F, G \in \mathrm{Filt}(L)$ for a residuated lattice L. Then $L = F \oplus G$ if and only if there exists $e \in \mathrm{B}(L)$ such that $F = [e)$ and $G = [e^*)$. In particular, if $e, g \in \mathrm{B}(L)$ and $L = [e) \oplus [g)$, then $g = e^*$.

PROOF: $\Rightarrow$). Since $L = F \oplus G$, we have $L = F \vee G$ and $F \cap G = \{1\}$. Thus, there exist $e \in F$ and $g \in G$ such that $e \odot g = 0$. From $e \vee g \in F \cap G = \{1\}$, we have $e \vee g = 1$. Hence by Proposition 1.2(5), $e \wedge g = e \odot g = 0$. Hence $e \in \mathrm{B}(L)$ and $e^* = g$.

Now since $e \in F$ and $e^* \in G$, we have $[e) \subseteq F$ and $[e^*) \subseteq G$. If $x \in F$ and $y \in G$, then $0 = e \odot e^* \leq x \odot y$. Hence by Proposition 3.5, we have

$e \leq x$ and $e^* \leq y$. This shows that $x \in [e)$ and $y \in [e^*)$. Therefore, $F = [e)$ and $G = [e^*)$.

$\Leftarrow$). The result now follows from Example 3.4.

The particular case follows immediately from above argument, and the proof is complete. $\square$

Proposition 3.8. Let $F, G \in \mathrm{Filt}(L)$ for a residuated lattice L such that $L = H \oplus G$. Then F is maximal if and only if G is simple.

Proof: By Proposition 3.7, there exists $e \in \mathrm{B}(L)$ such that $F = [e)$ and $G = [e^*)$. Hence the result follows from Proposition 2.4. $\square$

Proposition 3.9. Let $\{F_i\}_{i\in\Lambda}$ be a non-empty family of filters of a residuated lattice L. Then the following are equivalent:

1. $\{F_i\}_{i\in\Lambda}$ is an independent family of filters of L;
2. Every finite subfamily $\{F_i\}_{i\in\Lambda'}$ of $\{F_i\}_{i\in\Lambda}$ is an independent family of filters of L;
3. If A and B are two non-empty subsets of Λ such that $A \cap B = \emptyset$, then $(\bigvee_{i\in A} F_i) \cap (\bigvee_{j\in B} F_j) = \{1\}$;
4. Let $\{F_i\}_{i\in\Lambda'}$ be a finite subfamily of $\{F_i\}_{i\in\Lambda}$ and $f_i, f_i' \in F_i$ for each $i \in \Lambda'$. If $\odot_{i\in\Lambda'} f_i \leq \odot_{i\in\Lambda'} f_i'$, then $f_i \leq f_i'$ for each $i \in \Lambda'$;
5. Let $\{F_i\}_{i\in\Lambda'}$ be a finite subfamily of $\{F_i\}_{i\in\Lambda}$ and $f_i, f_i' \in F_i$ for each $i \in \Lambda'$. If $\odot_{i\in\Lambda'} f_i = \odot_{i\in\Lambda'} f_i'$, then $f_i = f_i'$ for each $i \in \Lambda'$.

Proof: $1 \Rightarrow 2$). It is clear from the definition.

$2 \Rightarrow 3$). Let A and B be two non-empty subsets of Λ such that $A \cap B = \emptyset$. Choose $x \in (\bigvee_{i\in A} F_i) \cap (\bigvee_{j\in B} F_j)$. Then by Proposition 1.8(3), there exist finite sets $A' \subseteq A$ and $B' \subseteq B$ such that $x \in (\bigvee_{i\in A'} F_i) \cap (\bigvee_{j\in B'} F_j)$. Hence, $x \in \bigvee_{i\in A'} F_i$ and $x \in \bigvee_{j\in B'} F_j$. Thus, there exist $a_i \in F_i$ for each $i \in A'$ and $b_j \in F_j$ for each $j \in B'$ such that $\odot_{i\in A'} a_i \leq x$ and $\odot_{j\in B'} b_j \leq x$. For each $i \in A'$, $a_i \vee (\odot_{j\in B'} b_j) \in F_i \cap (\bigvee_{j\in B'} F_j)$. By assumption each $i \in A'$, we have $F_i \cap (\bigvee_{j\in B'} F_j) = \{1\}$. Thus for each $i \in A'$, $a_i \vee (\odot_{j\in B'} b_j) = 1$. Hence by Proposition 1.2, we have:

$$\begin{aligned} 1 = \odot_{i\in A'} 1 = \odot_{i\in A'} \left(a_i \vee (\odot_{j\in B'} b_j)\right) \\ \leq (\odot_{i\in A'} a_i) \vee (\odot_{j\in B'} b_j) \\ \leq x. \end{aligned}$$

Hence $x = 1$. Therefore, $(\bigvee_{i\in A} F_i) \cap (\bigvee_{j\in B} F_j) = \{1\}$.

$3 \Rightarrow 1$). It is clear from the definition, when for each $i \in \Lambda$, we set $A := \{i\}$ and $B := \{j \in \Lambda \mid i \neq j\}$.

$2 \Leftrightarrow 4 \Leftrightarrow 5$). It follows by using an inductive argument and Proposition 3.5. □

Proposition 3.10. Let T be a simple filter of a residuated lattice L. Then we have the following:

1. There exists a prime filter P of L such that $\{T, P\}$ is a family of independent filters of L;
2. If $\mathrm{Rad}(L) = \{1\}$, then T is a direct summand of L.

Proof: 1). By Proposition 1.12, $\bigcap_{P\in \mathrm{SpecF}(L)} P = \{1\}$. Since $T \neq \{1\}$, we have $T \nsubseteq \bigcap_{P\in \mathrm{SpecF}(L)} P$. Hence, there exists $P \in \mathrm{SpecF}(L)$ such that $T \nsubseteq P$. Thus $T \cap P \neq T$. From the fact that T is simple, we have $T \cap P = \{1\}$. Therefore, $\{T, P\}$ is a family of independent filters of L.

2). Since $\mathrm{Rad}(L) = \{1\}$ and $T \neq \{1\}$, we deduce that $T \nsubseteq \mathrm{Rad}(L)$. Hence, there exists $M \in \mathrm{MaxF}(L)$ such that $T \nsubseteq M$. Thus $T \cap M \neq T$. From the fact that T is simple, we conclude that $T \cap M \subseteq T$ and $T \nsubseteq M$. Hence, $T \cap M = \{1\}$ and $T \vee M = L$, that is, $L = T \oplus M$. Therefore, T is a direct summand of L. □

We end this section with two lemmas that will be used in the following sections.

Lemma 3.11. *Let $F, G, H \in \mathrm{Filt}(L)$ for a residuated lattice L such that $F \subseteq G$. Then*

$$G \cap (F \vee H) = F \vee (G \cap H).$$

Proof: Let $x \in G \cap (F \vee H)$. Then $x \in G$ and $x \in F \vee H$. Hence, there exist $f \in F$ and $h \in H$ such that $f \odot h \leq x$. Thus $h \leq f \to x$. Now since $h \leq f \to x$ and $x \leq f \to x$, we have $f \to x \in G \cap H$. Hence from $f \odot (f \to x) \leq x$, we have $x \in F \vee (G \cap H)$ by Proposition 1.8(5). Therefore, $G \cap (F \vee H) \subseteq F \vee (G \cap H)$. Conversely, from $F \subseteq G$, we have $F \vee (G \cap H) \subseteq G$. Also, clearly $F \vee (G \cap H) \subseteq F \vee H$. Hence, $F \vee (G \cap H) \subseteq G \cap (F \vee H)$. Therefore $G \cap (F \vee H) = F \vee (G \cap H)$. $\square$

Lemma 3.12. *Let $F \in \mathrm{Filt}(L)$ for a residuated lattice L such that for each $H \in \mathrm{Filt}(L)$ with $H \subseteq F$, there exists $G \in \mathrm{Filt}(L)$ such that $F = H \oplus G$. Then for each $\{1\} \neq H \in \mathrm{Filt}(L)$ with $H \subseteq F$, there exists $T \in \mathrm{SimF}(L)$ such that $T \subseteq H$.*

Proof: Let $\{1\} \neq H \in \mathrm{Filt}(L)$ with $H \subseteq F$. Choose $1 \neq x \in H$ and set $\Sigma := \{A \in \mathrm{Filt}(L) \mid A \subset [x)\}$. Since $\{1\} \in \Sigma$, we have $\Sigma \neq \emptyset$. Clearly Σ is a partially ordered set by set inclusion and any chain of Σ has an upper bound in Σ (clearly, the union of any chain in Σ is an upper bound for it). Thus, by Zorn's lemma, Σ has a maximal element, say B. Since $B \subseteq F$, our assumption follows that $F = B \oplus B'$ for some $B' \in \mathrm{Filt}(L)$. So $[x) = [x) \cap F = [x) \cap (B \oplus B') = B \vee ([x) \cap B')$ by Lemma 3.11. Now, if we show that $[x) \cap B' \in \mathrm{SimF}(L)$, we are done since $[x) \cap B' \subseteq [x) \subseteq H$.

If $[x) \cap B' = \{1\}$, then from $[x) = B \vee ([x) \cap B')$, we obtain $B = [x)$, a contradiction. Hence $[x) \cap B' \neq \{1\}$. Now, let $G \subseteq [x) \cap B'$ for some $G \in \mathrm{Filt}(L)$. We have two cases:

Case 1. If $G \subseteq B$, then $G \subseteq B \cap B' = \{1\}$, and so $G = \{1\}$ in this case.

Case 2. If $G \nsubseteq B$, then $B \subset B \vee G \subseteq [x)$. By the maximality of B, we must have $B \vee G = [x)$. Choose $a \in [x) \cap B'$. Since $[x) \cap B' \subseteq [x) = B \vee G$, there exist $b \in B$ and $g \in G$ such that $b \odot g \leq a$ by Proposition 1.8. Hence $b \leq g \to a$. From $a \leq g \to a$ and $b \leq g \to a$, we have $g \to a \in B \cap [x) \cap B' = \{1\}$. Thus $g \to a = 1$, and so $g \leq a$. Now since $g \in G$, we have $a \in G$. Therefore, $[x) \cap B' \subseteq G$, and so $[x) \cap B' = G$ in this case.

Therefore, $[x) \cap B' \in \mathrm{SimF}(L)$. $\square$

4. The Socle of a filter

In this section, the socle of a filter is introduced, and then we consider its relation with essential filters.

Definition 4.1. The *socle* of a filter F of a residuated lattice L, denoted by $\mathrm{Soc}(F)$, is defined as the join $\bigvee\{T \mid T \in \mathrm{SimF}(L), T \subseteq F\}$ if this set is non-empty; otherwise, it is define to be $\{1\}$.

The next theorem states the relation between the socle of a filter and essential filters.

Theorem 4.2. *Let F be a filter of a residuated lattice L. Then*

$$\mathrm{Soc}(F) = \bigcap E_F.$$

Proof: If $\{T \in \mathrm{SimF}(L) \mid T \subseteq F\} = \emptyset$, then $\mathrm{Soc}(F) = \{1\}$ and we have $\mathrm{Soc}(F) \subseteq \bigcap E_F$ in this case.

Now assume that $\{T \in \mathrm{SimF}(L) \mid T \subseteq F\} \neq \emptyset$. If $A \in \{T \in \mathrm{SimF}(L) \mid T \subseteq F\}$ and $B \in E_F$, by the definition of E_F, if $A \cap B = \{1\}$, then $A \cap F = \{1\}$. Now since $A \subseteq F$, we have $A = \{1\}$, a contradiction. Hence for each $A \in \{T \in \mathrm{SimF}(L) \mid T \subseteq F\}$ and $B \in E_F$, we have $A \cap B \neq \{1\}$, and since A is a simple filter of L, we have $A \cap B = A$. Thus, for each $A \in \{T \in \mathrm{SimF}(L) \mid T \subseteq F\}$ and $B \in E_F$, we have $A \subseteq B$. Hence, $\mathrm{Soc}(F) \subseteq \bigcap E_F$.

Let $H \in \mathrm{Filt}(L)$ such that $H \subseteq \bigcap E_F$. Now since $F \in E_F$, we conclude that $H \subseteq F$, hence that $H \vee (H^* \cap F) \in E_F$ by Lemma 2.16, and finally that $H \subseteq \bigcap E_F \subseteq H \vee (H^* \cap F)$. Thus, by Lemma 3.11, we have

$$\bigcap E_F = \bigcap E_F \cap (H \vee (H^* \cap F)) = H \vee ((\bigcap E_F) \cap H^* \cap F).$$

From $H \cap ((\bigcap E_F) \cap H^* \cap F) \subseteq H \cap H^* = \{1\}$, we have $H \cap ((\bigcap E_F) \cap H^* \cap F) = \{1\}$. Hence

$$\bigcap E_F = H \oplus ((\bigcap E_F) \cap H^* \cap F).$$

Thus the condition of Lemma 3.12 for the filter $\bigcap E_F$ is satisfied. From

$\mathrm{Soc}(F) \subseteq \bigcap E_F$, we have $\bigcap E_F = \mathrm{Soc}(F) \oplus ((\bigcap E_F) \cap \mathrm{Soc}(F)^* \cap F)$.

Set $A := (\bigcap E_F) \cap \mathrm{Soc}(F)^* \cap F$. If $A = \{1\}$, then $\mathrm{Soc}(F) = \bigcap E_F$ and we are done. Now assume that $A \neq \{1\}$. Then by Lemma 3.12, there exists $T \in \mathrm{SimF}(L)$ such that $T \subseteq A$. Hence,

$$\{1\} \neq T \subseteq \mathrm{Soc}(F) \cap ((\bigcap E_F) \cap \mathrm{Soc}(F)^* \cap F) = \{1\},$$

which is impossible. □

Theorem 4.3. *Let F be a filter of a residuated lattice L. Then we have the following:*

1. $\mathrm{Soc}(F) = F \cap \mathrm{Soc}(L)$;
2. $\mathrm{Soc}(\mathrm{Soc}(F)) = \mathrm{Soc}(F)$.

Proof: 1). If $\mathrm{Soc}(L) = \{1\}$, then we have nothing to prove, hence assume that $\mathrm{Soc}(L) \neq \{1\}$. Since $\mathrm{Soc}(F) \subseteq F$ and $\mathrm{Soc}(F) \subseteq \mathrm{Soc}(L)$, we have $\mathrm{Soc}(F) \subseteq F \cap \mathrm{Soc}(L)$. Now let $x \in F \cap \mathrm{Soc}(L)$. Hence $x \in \mathrm{Soc}(L)$. Then by Proposition 1.8(3), there exist $T_1, ..., T_n \in \mathrm{SimF}(L)$ and elements $x_1 \in T_1, ..., x_n \in T_n$ such that $x_1 \odot \cdots \odot x_n \leq x$. Now, define $I := \{i \in \{1, ..., n\} \mid x \vee x_i = 1\}$. We consider the following cases:

Case 1. If $I = \emptyset$, then $x \vee x_i \neq 1$ for all $i \in \{1, ..., n\}$. Hence, $1 \neq x \vee x_i \in F \cap T_i$, and so $F \cap T_i \neq \{1\}$. From $F \cap T_i \subseteq T_i$ and the fact that T_i is a simple filter, we have $F \cap T_i = T_i$. Thus $T_i \subseteq F$ for all $i \in \{1, ..., n\}$, that is, $T_i \subseteq \mathrm{Soc}(F)$ for all $i \in \{1, ..., n\}$. Therefore, $x \in \mathrm{Soc}(F)$ in this case.

Case 2. If $I = \{1, ..., n\}$, then $x \vee x_i = 1$ for all $i \in \{1, ..., n\}$. By Proposition 1.2(6), we have

$$1 = (x \vee x_1) \odot \cdots \odot (x \vee x_n) \leq x \vee (x_1 \odot \cdots \odot x_n).$$

Since $x_1 \odot \cdots \odot x_n \leq x$, it follows that $x \vee (x_1 \odot \cdots \odot x_n) = x$. Therefore, $1 \leq x$, which implies $x = 1$, and so $x \in \mathrm{Soc}(F)$ in this case.

Case 3. If $I \neq \emptyset$ and $I \neq \{1, ..., n\}$. Set $J := \{1, ..., n\} \setminus I$. Hence by Proposition 1.2(6), we have:

$$\begin{aligned}(\odot_{j\in J}(x\vee x_j)) =& (\odot_{i\in I}(x\vee x_i))\odot(\odot_{j\in J}(x\vee x_j))\\ &= (x\vee x_1)\odot\cdots\odot(x\vee x_n)\\ &\le x\vee(x_1\odot\cdots\odot x_n)=x.\end{aligned}$$

Hence

$$(\odot_{j\in J}(x\vee x_j))\le x\vee(x_1\odot\cdots\odot x_n)=x. \tag{*}$$

Now since $x\vee x_j\ne 1$ for each $j\in J$. Hence, $1\ne x\vee x_j\in F\cap T_j$, and so $F\cap T_j\ne\{1\}$ for each $j\in J$. From $F\cap T_j\subseteq T_j$ and the fact that T_j is a simple filter, we have $F\cap T_j=T_j$. Thus $T_j\subseteq F$ for each $j\in J$, that is, $x\vee x_j\in \mathrm{Soc}(F)$ for each $j\in J$. Thus $(\odot_{j\in J}(x\vee x_j))\in\mathrm{Soc}(F)$. Therefore, $x\in\mathrm{Soc}(F)$ by $(*)$ in this case.

2). By part (1), Since $\mathrm{Soc}(F)\subseteq\mathrm{Soc}(L)$ we have

$$\mathrm{Soc}(\mathrm{Soc}(F))=\mathrm{Soc}(F)\cap\mathrm{Soc}(L)=\mathrm{Soc}(F).$$

□

THEOREM 4.4. *Let M be a maximal filter of a residuated lattice L with $\mathrm{Rad}(L)=\{1\}$. Then the following are equivalent:*

1. *M is an isolated point of $\mathrm{MaxF}(L)$;*
2. *There exists $e\in\mathrm{B}(L)$ such that $M=[e^*)$;*
3. *M is a direct summand of L.*

PROOF: $1\Rightarrow 2$). Let M be an isolated point of $\mathrm{MaxF}(L)$. Hence there exists $a\in L$ such that $\{M\}=U_{Max}(a)$, that is, $a\in(\bigcap_{M\ne N\in\mathrm{MaxF}(L)}N)\setminus M$. Since $a\notin M$, there exists $n\in\mathbb{N}$ such that $(a^n)^*\in M$ by Theorem 1.9. Set $e:=a^n$. Hence, $e\in(\bigcap_{M\ne N\in\mathrm{MaxF}(L)}N)\setminus M$ and $e^*\in M$. Thus $e\vee e^*\in\bigcap_{N\in\mathrm{MaxF}(L)}N=\mathrm{Rad}(L)=\{1\}$, that is, $e\vee e^*=1$, and so $e\in\mathrm{B}(L)$ by Proposition 1.2(10). Now since $e^*\in M$, we have $[e^*)\subseteq M$. If $x\in M$, then $x\vee e\in\mathrm{Rad}(L)=\{1\}$. Hence $x\vee e=1$. Thus by Proposition 1.2, we have $e^*\to x=e^{**}\vee x=e\vee x=1$. Hence, $e^*\le x$, and so $x\in[e^*)$. Therefore, $M=[e^*)$.

$2 \Rightarrow 1$). It is clear that if there exists $e \in \mathrm{B}(L)$ such that $M = [e^*)$, then we have $\{M\} = U_{Max}(e)$, and so M is an isolated point of $\mathrm{MaxF}(L)$.

$2 \Rightarrow 3$). If there exists $e \in \mathrm{B}(L)$ such that $M = [e^*)$, then by Example 3.4, we have $L = [e) \oplus [e^*)$, and so M is a direct summand of L.

$3 \Rightarrow 2$). It follows from Proposition 3.7. $\square$

THEOREM 4.5. *Let L be a residuated lattice with* $\mathrm{Rad}(L) = \{1\}$. *Then*

$$\mathrm{Soc}(L) = \{a \in L \mid U_{Max}(a) \text{ is a finite set}\}.$$

PROOF: If $\mathrm{Soc}(L) = \{1\}$, then $\mathrm{Soc}(L) \subseteq \{a \in L \mid U_{Max}(a) \text{ is a finite set}\}$. Now, assume that $\mathrm{Soc}(L) \neq \{1\}$ and let $1 \neq a \in \mathrm{Soc}(L) = \bigvee_{T \in \mathrm{SimF}(L)} T$. Then there exist $T_1, ..., T_n \in \mathrm{SimF}(L)$ and $a_1 \in T_1$,..., $a_n \in T_n$ such that $a_1 \odot \cdots \odot a_n \leq a$. Hence, $a \in T_1 \vee \cdots \vee T_n$ and so by Proposition 1.13(2), we have $U_{Max}(a) \subseteq U_{Max}(T_1 \vee \cdots \vee T_n)$. Also, by Proposition 1.13, $U_{Max}(T_1 \vee \cdots \vee T_n) = U_{Max}(T_1) \cup \cdots \cup U_{Max}(T_n)$. From Proposition 2.5, $U_{Max}(T_i)$ is a finite set for each i, hence $U_{Max}(a)$ is a finite set, and finally that $\mathrm{Soc}(L) \subseteq \{a \in L \mid U_{Max}(a) \text{ is a finite set}\}$.

Conversely, for $a \in L$, if $U_{Max}(a) = \emptyset$, then $[a) = \{1\}$ by Proposition 1.12(2), and so $a \in \mathrm{Soc}(L)$. Now, assume that $U_{Max}(a)$ is a non-empty and finite set for $a \in L$, say $U_{Max}(a) = \{M_1, ...M_n\}$. Hence for each $N \in \mathrm{MaxF}(L) \setminus U_{Max}(a)$, we have $a \in N$. If $n = 1$, then M_1 is an isolated point of $\mathrm{MaxF}(L)$, and so there exists $e \in \mathrm{B}(L)$ such that $M = [e)$ by Theorem 4.4. Now since $a \vee e \in \mathrm{Rad}(L) = \{1\}$, we have $a \vee e = 1$. Thus by Proposition 1.2, $e^* \leq a$ and hence $a \in [e^*)$. Form Proposition 2.4 and the fact that $[e)$ is a maximal filter, $[e^*)$ is a simple filter and so $a \in \mathrm{Soc}(L)$ in this case. Now assume that $2 \leq n$. Let $j \in \{1, ..., n\}$ be fixed. Since $M_1, ..., M_n$ are maximal filters, there exists $a_i \in M_i \setminus M_j$ for each $i \neq j$ and $i \in \{1, ..., n\}$. Set $b_j := a \vee a_1 \vee \cdots \vee \hat{a_j} \vee \cdots \vee a_n$, where the hat means that term is omitted. Clearly $b_j \in (\bigcap_{M_j \neq N \in \mathrm{MaxF}(L)} N) \setminus M_j$. Hence $U_{Max}(b_j) = \{M_j\}$ for each $j \in \{1, ..., n\}$. Thus for each $j \in \{1, ..., n\}$, M_j is an isolated point of $\mathrm{MaxF}(L)$, and so there exists $e_j \in \mathrm{B}(L)$ such that $M_j = [e_j)$ by Theorem 4.4. Now since $a \vee e_1 \vee \cdots \vee e_n \in \mathrm{Rad}(L) = \{1\}$, we have $a \vee e_1 \vee \cdots \vee e_n = 1$. Thus by Proposition 1.2, $(e_1 \vee \cdots \vee e_n)^* \leq a$, that is, $(e_1^* \odot \cdots \odot e_n^*) \leq a$ and hence $a \in [e_1^*) \vee \cdots \vee [e_n^*)$. From Proposition

2.4 and the fact that $[e_j)$ is a maximal filter, $[e_j^*)$ is a simple filter and so $a \in \mathrm{Soc}(L)$ in this case. $\square$

THEOREM 4.6. *Let L be a residuated lattice. Then the following are equivalent:*

1. $\mathrm{Soc}(L) \in E_L$*, that is,* $\mathrm{Soc}(L)$ *is an essential filter of* L*;*
2. *For each* $\{1\} \neq F \in \mathrm{Filt}(L)$*, we have* $\mathrm{Soc}(F) \neq \{1\}$*;*
3. *Every non-trivial filter of* L *contains a simple filter;*
4. *The intersection of any non-empty subfamily of essential filters is an essential filter, that is, the intersection of any non-empty subfamily of* E_L *is a member of* E_L*.*

PROOF: $1 \Rightarrow 2$). By Theorem 4.3, $\mathrm{Soc}(F) = F \cap \mathrm{Soc}(L)$. If $\mathrm{Soc}(F) = \{1\}$, then $F \cap \mathrm{Soc}(L) = \{1\}$. Now by assumption since $\mathrm{Soc}(L)$ is an essential filter of L, we have $F = \{1\}$.

$2 \Rightarrow 3$). It is clear by the definition of the socle of a filter.

$3 \Rightarrow 1$). If F is a non-trivial filter of L, then F contains a simple filter by our assumption. Hence, $F \cap \mathrm{Soc}(L) \neq \{1\}$, that is, $(\mathrm{Soc}(L))^* = \{1\}$, and so $\mathrm{Soc}(L) \in E_L$ by Proposition 2.15.

$1 \Rightarrow 4$). Let $\mathrm{Soc}(L) \in E_L$. If $\mathfrak{A}$ is a non-empty subfamily of E_L, then $\mathrm{Soc}(L) \subseteq \bigcap \mathfrak{A}$. Thus, $\bigcap \mathfrak{A}$ is an essential filter of L by Remark 2.11.

$4 \Rightarrow 1$). By Theorem 4.2, $\mathrm{Soc}(L) = \bigcap E_L$. Hence $\mathrm{Soc}(L) \in E_L$ by our assumption. $\square$

Recall that a residuated lattice L is called *Artinian*, whenever every decreasing chain of filters of L is stationary, see [1, 18].

COROLLARY 4.7. $\mathrm{Soc}(L) \in E_L$ for any non-trivial Artinian (e.g., finite residuated lattice) residuated lattice L.

PROOF: It is clear that any non-trivial filter of a non-trivial Artinian residuated lattice contains a simple filter. Hence the result follows from Theorem 4.6. $\square$

For a residuated lattice L, we set
$M_0(L) := \{M \in \mathrm{MaxF}(L) \mid M \text{ is an isolated point of } \mathrm{MaxF}(L)\}$.

THEOREM 4.8. *Let L be a residuated lattice with* $\mathrm{Rad}(L) = \{1\}$. *Then* $M_0(L)$ *is a dense subset of* $\mathrm{MaxF}(L)$ *if and only if one of the (equivalent) conditions of Theorem 4.6 holds.*

PROOF: Assume that $M_0(L)$ is a dense subset of $\mathrm{MaxF}(L)$, hence that $\bigcap M_0(L) = \{1\}$ by Corollary 2.21. If $M \in M_0(L)$, then there exists $e_M \in \mathrm{B}(L)$ such that $M = [e_M^*)$ by Theorem 4.4. Now by Proposition 2.4, the filter $[e_M)$ is simple. Hence if $x \in (\mathrm{Soc}(L))^*$, then $x \vee e_M = 1$ for each $M \in M_0(L)$. Thus $e_M^* \leq x$ for each $M \in M_0(L)$, that is, $x \in [e_M^*)$ for each $M \in M_0(L)$. Thus $x \in \bigcap M_0(L) = \{1\}$. This shows that $x = 1$, and so $\mathrm{Soc}(L) \in E_L$ by Proposition 2.15.

Conversely, assume that $\mathrm{Soc}(L) \in E_L$ and $x \in \bigcap M_0(L)$. Assume that $y \in \mathrm{Soc}(L)$. If $\mathrm{Soc}(L) = \{1\}$, then $x \vee y = 1$. Now assume that $\mathrm{Soc}(L) \neq \{1\}$. Then by Proposition 1.8 and Corollary 2.6, there there exist $e_1, \ldots, e_n \in \mathrm{B}(L)$ such that $[e_1), \ldots, [e_n) \in \mathrm{SimF}(L)$ and $e_1 \odot \cdots \odot e_n \leq y$. By Proposition 2.4 and Theorem 4.4 for each i, $[e_i^*)$ is a maximal filter and is an isolated point of $\mathrm{MaxF}(L)$. Hence $x \in [e_i^*)$, that is, $e_i^* \leq x$. So $x \vee e_i = 1$ for each $i \in \{1, \ldots, n\}$. Hence by Proposition 1.2, we have

$$1 = (x \vee e_1) \odot \cdots \odot (x \vee x_n) \leq x \vee (e_1 \odot \cdots \odot e_n) \leq x \vee y,$$

and so $x \vee y = 1$.
Therefore, $x \vee y = 1$ for each $y \in \mathrm{Soc}(L)$. Thus by our assumption and Proposition 2.15, we have $x = 1$, and so $\bigcap M_0(L) = \{1\}$. Therefore, $M_0(L)$ is a dense subset of $\mathrm{MaxF}(L)$ by Corollary 2.21.

□

THEOREM 4.9. *Let L be a residuated lattice with* $\mathrm{Rad}(L) = \{1\}$. *Then* $\mathrm{Soc}(L)$ *is a principal filter of L if and only if* $M_0(L)$ *is a finite subset of* $\mathrm{MaxF}(L)$.

PROOF: $\Rightarrow$). Assume that $\mathrm{Soc}(L)$ is a principal filter of L. Hence, there exists $a \in L$ such that $\mathrm{Soc}(L) = [a)$. First assume that $\mathrm{Soc}(L) = \{1\}$. Now if $M \in M_0(L)$, then there exists $e_M \in \mathrm{B}(L)$ such that $M = [e_M^*)$ by Theorem 4.4. By Proposition 2.4, the filter $[e_M)$ is simple, and so $\mathrm{Soc}(L) \neq \{1\}$ by the definition, a contradiction. Hence $M_0(L) = \emptyset$ in this case. Now assume that $\mathrm{Soc}(L) \neq \{1\}$. By Theorem 4.4 and our assumption, every member of $M_0(L)$ is generated by a Boolean element, hence assume that $M_0 = \{[e_i)\}_{i\in\Lambda}$, where $\{e_i\}_{i\in\Lambda} \subseteq \mathrm{B}(L)$. By the definition, $\mathrm{Soc}(L) = \bigvee_{T\in\mathrm{SimF}(L)} T$. From Corollary 2.6 since $a \in \mathrm{Soc}(L)$, there exist $T_1, ..., T_n \in \mathrm{SimF}(L)$ and $f_1, ..., f_n \in \mathrm{B}(L)$ such that $T_j = [f_j)$ for each $j \in \{1, ..., n\}$ and $f_1 \odot \cdots \odot f_n \leq a$. Hence for each $i \in \Lambda$ since $[e_i)$ is a maximal filter, $[e_i^*)$ is a minimal filter, and so $e_i^* \in \mathrm{Soc}(L)$ for each $i \in \Lambda$. Now assume that $i \in \Lambda$, then $e_i^* \in \mathrm{Soc}(L) = [a)$. Thus there exists $m \in \mathbb{N}$ such that $a^m \leq e_i^*$. Hence,

$$\begin{aligned}
f_1 \wedge \cdots \wedge f_n =& f_1 \odot \cdots \odot f_n = (f_1 \odot \cdots \odot f_n)^m \leq a^m \leq e_i^* \\
&\Rightarrow f_1 \wedge \cdots \wedge f_n \leq e_i^* \\
&\Rightarrow e_i \leq (f_1 \wedge \cdots \wedge f_n)^* \\
&\Rightarrow e_i \leq f_1^* \vee \cdots \vee f_n^* \\
&\Rightarrow [f_1^* \vee \cdots \vee f_n^*) \subseteq [e_i) \\
&\Rightarrow [f_1^*) \cap \cdots \cap [f_n^*) \subseteq [e_i).
\end{aligned}$$

Now since $[e_i)$ is maximal, it is prime, and so there exists $j \in \{1, ..., n\}$ such that $[f_j^*) \subseteq [e_i)$. Now since $[f_j)$ is simple, by Proposition 2.4, $[f_j^*)$ is maximal. Also since $[e_i)$ is a maximal filter, we have $[f_j^*) = [e_i)$. So $e_i = f_j^*$ by Proposition 1.8. This shows that $M_0(L) = \{[f_1^*), ..., [f_n^*)\}$, and so it is a finite subset of $\mathrm{MaxF}(L)$.

$\Leftarrow$). Assume that $M_0(L)$ is a finite subset of $\mathrm{MaxF}(L)$. First assume that $M_0(L) = \emptyset$. If $T \in \mathrm{SimF}(L)$, then by Corollary 2.6 there exists $e \in \mathrm{B}(L)$ such that $T = [e)$. By Proposition 2.4 and Theoerm 4.4, the filter $[e^*)$ is an isolated point of $\mathrm{MaxF}(L)$, a contradiction. Hence $\mathrm{SimF}(L) = \emptyset$, and so $\mathrm{Soc}(L) = \{1\}$ is a principal filter of L in this case. Now assume that $M_0(L) \neq \emptyset$. Since $\mathrm{Rad}(L) = \{1\}$, every simple filter of L is of the form $[e)$ for some $e \in \mathrm{B}(L)$ such that $[e^*) \in M_0(L)$ by Proposition 2.4 and Theorem 4.4. Hence there exists a one to one correspondence between the sets $\mathrm{SimF}(L)$ and $M_0(L)$ by Proposition 1.8(2). Now since $M_0(L)$ is a finite set, $\mathrm{SimF}(L)$ is also finite. Assume that $\mathrm{SimF}(L) = \{[e_1), ..., [e_n)\}$. Then

$$\mathrm{Soc}(L) = [e_1) \vee \cdots \vee [e_n) = [e_1 \odot \cdots \odot e_n).$$

Hence $\mathrm{Soc}(L)$ a principal filter of L. □

5. Semi-simple filters

In this section, we consider semi-simple filters of a residuated lattice. We begin with the following definition.

DEFINITION 5.1. A filter F of a residuated lattice L is called *semi-simple*, if $F \neq \{1\}$ and $F = \bigvee_{i\in\Lambda} T_i$ for some non-empty family $\{T_i\}_{i\in\Lambda}$ of simple filters.

In the next theorem, we state some equivalent conditions for a filter F of a residuated lattice L being semi-simple.

THEOREM 5.2. *Let F be a filter of a residuated lattice L. Then the following are equivalent:*

1. *$F = \bigvee_{i\in\Lambda} T_i$ for some non-empty family $\{T_i\}_{i\in\Lambda}$ of simple filters, that is, F is semi-simple;*
2. *$F = \bigoplus_{j\in\Omega} T_j$ for some non-empty family $\{T_j\}_{j\in\Omega}$ of simple filters;*
3. *If $H \in \mathrm{Filt}(L)$ and $H \subseteq F$, then there exists $G \in \mathrm{Filt}(L)$ such that $F = H \oplus G$;*
4. *If $H \in \mathrm{Filt}(L)$ and $H \subseteq F$, then $F = H \oplus (H^* \cap F)$;*
5. *$E_F = \{F\}$.*

PROOF: $1 \Rightarrow 2$). Assume that $F = \bigvee_{i\in\Lambda} T_i$ for some non-empty family $\{T_i\}_{i\in\Lambda}$ of simple filters. Set $\Sigma := \{K \subseteq \Lambda \mid \bigvee_{k\in K} T_k = \bigoplus_{k\in K} T_k\}$. Clearly Σ is a partially ordered set by set inclusion, and for each $i \in \Lambda$, $\{i\} \in \Sigma$, and so $\Sigma \neq \emptyset$. Let $\{\Gamma_\alpha\}$ be a chain in Σ. Set $\Gamma := \bigcup_\alpha \Gamma_\alpha$. For $\beta \in \Gamma$, if $x \in T_\beta \bigcap(\bigvee_{\beta\neq\alpha\in\Gamma} T_\alpha)$, then there exists $\alpha_1, ..., \alpha_n \in \Gamma$ such that $x \in T_\beta \bigcap(T_{\alpha_1} \vee \cdots \vee T_{\alpha_n})$. Hence, there exists γ such that $\beta, \alpha_1, ..., \alpha_n \in \Gamma_\gamma$, and so $x \in T_\beta \bigcap(T_{\alpha_1} \vee \cdots \vee T_{\alpha_n}) \subseteq T_\beta \bigcap(\bigvee_{\beta\neq\alpha\in\Gamma_\gamma} T_\alpha) = \{1\}$, and so $x = 1$. Thus $\Gamma \in \Sigma$. Hence, every chain in Σ has an upper bound in Σ. Thus, by Zorn's Lemma, Σ has a maximal element, say Ω. Set

$G := \bigvee_{a\in\Omega} T_a = \bigoplus_{a\in\Omega} T_a$. If $F \neq G$, then there exists $i \in \Lambda$ such that $T_i \nsubseteq G$. Since T_i is simple and $T_i \nsubseteq G$, we have $T_i \bigcap(\bigvee_{a\in\Omega} T_a) = \{1\}$. If $x \in T_{a_*} \bigcap(T_i \vee (\bigvee_{a_*\neq a\in\Omega} T_a))$ for some $a_* \in \Omega$, then $x \in T_i \vee (\bigvee_{a_*\neq a\in\Omega} T_a)$ and so there exist $t \in T_i$ and $l \in \bigvee_{a_*\neq a\in\Omega} T_a$ such that $t \odot l \leq x$, hence $t \leq l \to x$. From $t \leq l \to x$ and $x \leq l \to x$, we have $l \to x \in T_{a_*} \cap T_i$. Now, since T_i and T_{a_*} are two distinc simple filters, we have $T_{a_*} \cap T_i = \{1\}$. This shows that $l \to x = 1$, and so $l \leq x$. Thus $x \in T_{a_*} \cap (\bigvee_{a_*\neq a\in\Omega} T_a) = \{1\}$. Hence $x = 1$ and so $\Omega \cup \{i\} \in \Sigma$, a contradiction. Therefore $G = F$, and so $F = \bigoplus_{j\in\Omega} T_j$, where $\{T_j\}_{j\in\Omega} \subseteq \mathrm{SimF}(L)$.

$2 \Rightarrow 1$). The proof is straightforward .

$2 \Rightarrow 3$). Let $F = \bigoplus_{j\in\Omega} T_j$ for some non-empty family $\{T_j\}_{j\in\Omega}$ of simple filters. Choose $H \in \mathrm{Filt}(L)$ with $H \subseteq F$. If $H = F$, then we have nothing to prove. So assume that $H \neq F$. Set $\Sigma := \{K \subseteq \Omega \mid H \cap (\bigvee_{k\in K} T_k) = \{1\}\}$. Clearly Γ is a partially ordered set by set inclusion and since $H \neq F$, there exists $j \in \Omega$ such that $T_j \nsubseteq H$, and so $\{j\} \in \Gamma$, hence that $\Sigma \neq \emptyset$.

Let $\{\Gamma_\alpha\}$ be a chain in Σ. Set $\Gamma := \bigcup_\alpha \Gamma_\alpha$. If $x \in H \cap (\bigvee_{j\in\Gamma} T_j)$, then there exist $j_1, ..., j_n \in \Gamma$ such that $x \in H \cap (T_{j_1} \vee \cdots \vee T_{j_n})$. Hence there exists $j_* \in \Gamma$ such that $j_1, ..., j_n \in \Gamma_{j_*}$, and so $x \in H \cap (\bigvee_{j\in\Gamma_{j_*}} T_j) = \{1\}$. Thus $x = 1$, and hence every chain in Σ has an upper bound in Σ. Thus, by Zorn's lemma, Σ has a maximal element, say A. Now set $G := \bigvee_{a\in A} T_a$. Hence $H \cap G = \{1\}$. By a similar argument in ($1 \Rightarrow 2$) $F = H \vee G$. Therefore, $F = H \oplus G$, and we are done.

$3 \Rightarrow 1$). By Lemme 3.12, for each $\{1\} \neq H \in \mathrm{Filt}(L)$ with $H \subseteq F$ there exists $T \in \mathrm{SimF}(L)$ such that $T \subseteq H$. Set $G := \bigvee_{T\in\mathrm{SimF}(L), T\subseteq F} T$. By assumption, $F = G \oplus G'$ for some $G' \in \mathrm{Filt}(L)$. If $G' \neq \{1\}$, then by Lemma 3.12 there exists $T \in \mathrm{SimF}(L)$ such that $T \subseteq G$. Hence $T \subseteq G \cap G' = \{1\}$, and so $T = \{1\}$, a contradiction. Thus, $G' = \{1\}$, and so $F = \bigvee_{T\in\mathrm{SimF}(L), T\subseteq F} T$.

$3 \Rightarrow 5$). By definition $F \in E_F$. Let $H \in E_F$. By assumption then there exists $G \in \mathrm{Filt}(L)$ such that $F = H \oplus G$. Since $H \cap G = \{1\}$, we deduce that $F \cap G = \{1\}$. From $G \subseteq F$, we have $G = \{1\}$, and so $F = H$. Hence $E_F = \{F\}$.

$5 \Rightarrow 4$). Let $H \in \mathrm{Filt}(L)$ and $H \subseteq F$. By Lemma 2.16, $H \vee (H^* \cap F) \in E_F$. So $H \vee (H^* \cap F) = F$. Now since $H \cap (H^* \cap F) = \{1\}$. we have

$F = H \oplus (H^* \cap F)$, and we are done.

$4 \Rightarrow 3)$. The proof is straightforward. $\square$

Corollary 5.3. Let $F, G \in \mathrm{Filt}(L)$ such that $G \subseteq F$. Then if F is semi-simple, then G is also semi-simple.

Proof: Assume that $H \in \mathrm{Filt}(L)$ such that $H \subseteq G$. Then $H \subseteq F$. Hence by Theorem 5.2, there exists $H' \in \mathrm{Filt}(L)$ such that $F = H \oplus H'$. Thus we have $H \subseteq G \subseteq H \oplus H' = H \vee H'$, and so by Lemma 3.11 $G = G \cap (H \vee H') = H \vee (G \cap H')$. Now since $H \cap (G \cap H') = \{1\}$, we have $G = H \oplus (G \cap H')$. Therefore, G is semi-simple by Theorem 5.2. $\square$

Lemma 5.4. *Let $e, f \in \mathrm{B}(L)$. Then $[e) \to [f) = \{x \in L \mid f \leq e \vee x\}$. In particular, $[e) \to [f) = [f)^* \to [e)^*$.*

Proof: By definition we have

$$x \in [e) \to [f) \Leftrightarrow [e) \cap [x) \subseteq [f) \Leftrightarrow [e \vee x) \subseteq [f) \Leftrightarrow f \leq e \vee x.$$

Hence, $[e) \to [f) = \{x \in L \mid f \leq e \vee x\}$. Now, using Proposition 1.2, we deduce

$$\begin{aligned} x \in [e) \to [f) &\Rightarrow f \leq e \vee x \\ &\Rightarrow 1 = f \vee f^* \leq e \vee x \vee f^* \\ &\Rightarrow e \vee x \vee f^* = 1 \\ &\Rightarrow e^* \leq x \vee f^* \\ &\Rightarrow x \in [f^*) \to [e^*) \\ &\Rightarrow x \in [f)^* \to [e)^*. \end{aligned}$$

Hence, $[e) \to [f) \subseteq [f)^* \to [e)^*$. A symmetric argument yields the reverse inclusion $[e) \to [f) \supseteq [f)^* \to [e)^*$. Therefore, we have $[e) \to [f) = [f)^* \to [e)^*$. $\square$

In the following theorem, we state some equivalent conditions for the filter L of a residuated lattice L being semi-simple.

Theorem 5.5. *Let L be a non-trivial residuated lattice L. Then the following are equivalent:*

1. *$L = \bigvee_{i\in\Lambda} T_i$ for some non-empty family $\{T_i\}_{i\in\Lambda}$ of simple filters, that is, L is semi-simple as a filter of L;*
2. *$L = \bigoplus_{j\in\Omega} T_j$ for some non-empty family $\{T_j\}_{j\in\Omega}$ of simple filters;*
3. *If $H \in \mathrm{Filt}(L)$, then there exists $G \in \mathrm{Filt}(L)$ such that $L = H \oplus G$;*
4. *The frame $\mathrm{Filt}(L)$ satisfies the law of excluded middle, equivalently, for each $H \in \mathrm{Filt}(L)$, we have $L = H \oplus H^*$;*
5. *Every filter of L is principal and generated by a Boolean element;*
6. *$E_L = \{L\}$;*
7. *$L = T_1 \oplus \cdots \oplus T_n$ for some $T_1, ..., T_n \in \mathrm{SimF}(L)$;*
8. *L is hyperarchimedean and semi-local;*
9. *$\mathrm{Rad}(L) = \{1\}$ and L semi-local;*
10. *Every proper filter of L can be written uniquely, up to order, as a finite intersection of distinct maximal filters;*
11. *The filter $\{1\}$ can be written as a finite intersection of distinct maximal filters;*
12. *Every non-trivial filter of L is semi-simple;*
13. *The frame $\mathrm{Filt}(L)$ satisfies satisfies double negation;*
14. *The frame $\mathrm{Filt}(L)$ satisfies satisfies material implication;*
15. *The frame $\mathrm{Filt}(L)$ satisfies satisfies the contraposition law.*

PROOF:

By Theorem 5.2, we have $1 \Leftrightarrow 2 \Leftrightarrow 3 \Leftrightarrow 4 \Leftrightarrow 6$.

$5 \Leftrightarrow 6$). It follows from Example 3.4 and Proposition 3.7.

$2 \Rightarrow 7$). Assume that $L = \bigoplus_{j\in\Omega} T_j$ for some non-empty family $\{T_j\}_{j\in\Omega}$ of simple filters. Since $0 \in L$, there exist $j_1, ..., j_n \in \Omega$ such that $0 \in T_{j_1} \vee \cdots \vee T_{j_n}$, that is, $L = T_{j_1} \vee \cdots \vee T_{j_n}$. Now since $\{T_j\}_{j\in\Omega}$ is an independent family of filters of L, $\{T_{j_1}, ..., T_{j_n}\}$ is also an independent family of filters of L by Proposition 3.9. Hence $L = T_{j_1} \oplus \cdots \oplus T_{j_n}$.

$7 \Rightarrow 2$). It is clear.

$3 \Rightarrow 8$). Let $P \in \mathrm{SpecF}(L)$. Choose $M \in \mathrm{MaxF}(L)$ such that $P \subseteq M$. By assumption, there exists $G \in \mathrm{Filt}(L)$ such that $L = M \oplus G$. Now if $P \neq M$, then there exists $x \in M \setminus P$. If $y \in G$, then $x \vee y \in M \cap G = \{1\}$, and so $x \vee y = 1 \in P$. Since $x \notin P$, we have $y \in P$. Hence $G \subseteq P$. From $G \cap M = \{1\}$ and $G \subseteq P \subseteq M$, we have $G = \{1\}$. Thus $L = M \vee G = M \vee \{1\} = M$, a contradiction. Hence, $P = M$, and so L is hyperarchimedean by Theorem 1.14.

Now since $(3 \Leftrightarrow 7)$, $L = T_1 \oplus \cdots \oplus T_n$ for some $T_1, ..., T_n \in \mathrm{SimF}(L)$. By Theorem 1.15 since L is hyperarchimedean, we have $\mathrm{Rad}(L) = \{1\}$, and so by Corollary 2.6 there exist $e_1, ..., e_n \in \mathrm{B}(L)$ such that $T_i = [e_i)$ for each $i \in \{1, ..., n\}$. Hence for each $i \in \{1, ..., n\}$ we have

$$\begin{aligned} L = T_1 \oplus \cdots \oplus T_n =& [e_1) \oplus \cdots \oplus [e_n) \\ =& [e_i) \oplus [e_1) \oplus \cdots \oplus \widehat{[e_i)} \oplus \cdots \oplus [e_n) \\ =& [e_i) \oplus [e_1) \vee \cdots \vee \widehat{[e_i)} \vee \cdots \vee [e_n) \\ =& [e_i) \oplus [e_i \odot \cdots \odot \hat{e_i} \odot \cdots \odot e_n), \end{aligned}$$

where the hat means that term is omitted. Set $f_i := e_i \odot \cdots \odot \hat{e_i} \odot \cdots \odot e_n$ for each $i \in \{1, ..., n\}$. Hence $L = [e_i) \oplus [f_i)$ for each $i \in \{1, ..., n\}$. By Propositions 3.7 and 3.8, $[f_i)$ is a maximal filter for each $i \in \{1, ..., n\}$ and $f_i = e_i^*$. We show that $\mathrm{MaxF}(L) = \{[f_1), ..., [f_n)\}$. By above argument, $\{[f_1), ..., [f_n)\} \subseteq \mathrm{MaxF}(L)$. Now let $N \in \mathrm{MaxF}(L)$. For each $i \in \{1, ..., n\}$, we have $[e_i) \cap [f_i) = \{1\} \subseteq N$. Hence for each $i \in \{1, ..., n\}$, we have either $[e_i) \subseteq N$ or $[f_i) \subseteq N$. If there exists $i \in \{1, ..., n\}$ such that $[f_i) \subseteq N$, then $[f_i) = N$ since $[f_i)$ is a maximal filter of L, and so $N \in \{[f_1), ..., [f_n)\}$. Now assume that for all $i \in \{1, ..., n\}$, $[e_i) \subseteq N$, then $L = [e_1) \vee \cdots \vee [e_n) \subseteq N$. Thus $L = N$, a contradiction. Therefore, $\mathrm{MaxF}(L) = \{[f_1), ..., [f_n)\}$ and so L is semi-local.

$8 \Rightarrow 1$). Assume that $\mathrm{MaxF}(L) = \{M_1, ..., M_n\}$. Since L is hyperarchimedean, we have $\mathrm{Rad}(L) = \{1\}$ by Theorem 1.15. Now if $n = 1$, then $\mathrm{MaxF}(L) = \{M_1\}$ and $M_1 = \mathrm{Rad}(L) = \{1\}$, and so L is simple and we are done. Now assume that $n \geq 2$. For each $i \in \{1, ..., n\}$ set $N_i := M_1 \cap \cdots \cap \hat{M_i} \cap \cdots \cap M_n$, where the hat means that term is omitted. Hence for each $i \in \{1, ..., n\}$, we have $M_i \cap N_i = \mathrm{Rad}(L) = \{1\}$ and $M_i \vee N_i = M_i \vee (M_1 \cap \cdots \cap \hat{M_i} \cap \cdots \cap M_n) = \bigcap_{j \neq i}(M_i \vee M_j) = \bigcap_{j \neq i} L = L$. Hence $L = M_i \oplus N_i$ for each i. Thus by Proposition 3.8, N_i is a simple filter of L for each $i \in \{1, ..., n\}$. So for each $i \in \{1, ..., n\}$ there exists $e_i \in \mathrm{B}(L)$ such that $N_i = [e_i)$ and $M_i = [e_i^*)$. Since $e_1^* \vee \cdots \vee e_n^* \in \bigcap_{i=1}^n M_i = \mathrm{Rad}(L) = \{1\}$, we have $e_i^* \vee \cdots \vee e_n^* = 1$. Set $f := e_i \odot \cdots \odot e_n$, hence we have:

$$\begin{aligned} f = f \odot 1 =& f \odot (e_1^* \vee \cdots \vee e_n^*) \\ =& (f \odot e_1^*) \vee \cdots \vee (f \odot e_n^*) \\ \leq& (e_1 \odot e_1^*) \vee \cdots \vee (e_n \odot e_n^*) = 0. \end{aligned}$$

Hence $f = e_i \odot \cdots \odot e_n = 0$. Thus $L = N_1 \vee \cdots \vee N_n$ and we are done.

$8 \Rightarrow 9$). It follows from Theorem 1.15.

$9 \Rightarrow 8$). Assume that $\mathrm{MaxF}(L) = \{M_1, ..., M_n\}$. Hence $\mathrm{Rad}(L) = M_1 \cap \cdots \cap M_n = \{1\}$. Now if $P \in \mathrm{SpecF}(L)$, then $M_1 \cap \cdots \cap M_n = \{1\} \subseteq P$. Hence, there exists $i \in \{1, ..., n\}$ such that $M_i \subseteq P$. Since $M_i \in \mathrm{MaxF}(L)$, we have $M_i = P$. Thus, $P \in \mathrm{MaxF}(L)$, and so $\mathrm{SpecF}(L) = \mathrm{MaxF}(L)$. Therefore, L is hyperarchimedean by Theorem 1.14.

$8 \Rightarrow 10$). Assume that L is hyperarchimedean and semi-local. By Theorem 1.14, $\mathrm{SpecF}(L) = \mathrm{MaxF}(L)$, and so by Proposition 1.12(2), every proper filter of L can be written as a finite intersection of distinct maximal filters. Now let $P_1 \cap \cdots \cap P_n$ and $Q_1 \cap \cdots \cap Q_m$ be two finite intersections of distinct maximal filters that are equal to a proper filter F of L. For each $i \in \{1, ..., n\}$, since $Q_1 \cap \cdots \cap Q_m \subseteq P_i$, there exists a unique $j_i \in \{1, ..., m\}$ such that $Q_{j_i} \subseteq P_i$. Now since $P_i, Q_{j_i} \in \mathrm{MaxF}(L)$, we have $Q_{j_i} = P_i$. Similarly, for each $j \in \{1, ..., m\}$ since $P_1 \cap \cdots \cap P_n \subseteq Q_j$, there exists a unique $i_j \in \{1, ..., n\}$ such that $P_{i_j} \subseteq Q_j$. Now since $P_{i_j}, Q_j \in \mathrm{MaxF}(L)$, we have $Q_j = P_{i_j}$. Hence, there exists a one to one correspondence between two sets $\{P_1, ..., P_n\}$ and $\{Q_1, ..., Q_m\}$ and correspondence elements are equal. Hence, every proper filter of L can be written uniquely, up to order, as a finite intersection of distinct maximal filters.

$10 \Rightarrow 11$). It is clear.

$11 \Rightarrow 8$). Assume that $\{1\} = P_1 \cap \cdots \cap P_n$ be a finite intersection of distinct maximal filters for the filter $\{1\}$ of L. Now if $P \in \mathrm{SpecF}(L)$, then from $P_1 \cap \cdots \cap P_n = \{1\} \subseteq P$, there exists $i \in \{1, ..., n\}$ such that $P_i \subseteq P$. Now since $P_i \in \mathrm{MaxF}(L)$, we have $P_i = P$. Hence $\mathrm{MaxF}(L) \subseteq \mathrm{SpecF}(L) \subseteq \{P_1, ..., P_n\} \subseteq \mathrm{MaxF}(L)$. This shows that $\mathrm{MaxF}(L) = \mathrm{SpecF}(L) = \{P_1, ..., P_n\}$. Hence L is hyperarchimedean and semi-local by Theorem 1.15.

$1 \Leftrightarrow 12$). It is clear by Corollary 5.3.

$5 \Rightarrow 13$). Let $F \in \mathrm{Filt}(L)$. Then there exits $e \in \mathrm{B}(L)$ such that $F = [e)$. Now, by Proposition 1.8, we have

$$F^{**} = [e)^{**} = [e^{**}) = [e) = F.$$

Hence, the frame $\mathrm{Filt}(L)$ satisfies satisfies double negation.

$13 \Rightarrow 4$). Let $H \in \mathrm{Filt}(L)$. By assumption and Proposition 1.8, we have:

$$H \vee H^{*} = (H \vee H^{*})^{**} = (H^{*} \cap H^{**})^{*} = (H^{*} \cap H)^{*} = \{1\}^{*} = L.$$

Now since $H \cap H^{*} = \{1\}$, we have $H \oplus H^{*} = L$, and so the frame $\mathrm{Filt}(L)$ satisfies the law of excluded middle.

$5 \Rightarrow 14$). Let $F \in \mathrm{Filt}(L)$. Then there exits $e \in \mathrm{B}(L)$ such that $F = [e)$. Now, by Proposition 1.8, we have

$$F \vee F^{*} = [e) \vee [e)^{*} = [e) \vee [e^{*}) = [e \odot e^{*}) = [0) = L.$$

Hence, the frame $\mathrm{Filt}(L)$ satisfies satisfies material implication.

$14 \Rightarrow 4$). Let $H \in \mathrm{Filt}(L)$. By assumption and Proposition 1.8, we have:

$$H \vee H^{*} = H^{*} \rightarrow H^{*} = L.$$

Now since $H \cap H^{*} = \{1\}$, we have $H \oplus H^{*} = L$, and so the frame $\mathrm{Filt}(L)$ satisfies the law of excluded middle.

$5 \Rightarrow 15$). Let $F, G \in \mathrm{Filt}(L)$. Then there exits $e, f \in \mathrm{B}(L)$ such that $F = [e)$ and $G = [f)$. Now, by Lemma 5.4, we have

$$F \rightarrow G = [e) \rightarrow [f) = [f)^{*} \rightarrow [e)^{*} = G^{*} \rightarrow F^{*}.$$

Hence, the frame Filt(L) satisfies satisfies the the contraposition law.

15 $\Rightarrow$ 13). Let $F \in$ Filt(L). By assumption and Proposition 1.8, we have

$$F^{**} \rightarrow F = F^{*} \rightarrow F^{***} = F^{*} \rightarrow F^{*} = L.$$

Hence, $F^{**} = F$, and so the frame Filt(L) satisfies satisfies double negation. □

Example 5.6. Consider the unit interval $L := [0,1]$ with the usual order. Define the operations as follows:

$$x \wedge y := min\{x,y\}, x \vee y := max\{x,y\}, x \odot y := max\{0, x+y-1\},$$

$$and$$

$$x \rightarrow y := min\{1, 1-x+y\}.$$

Then $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is a residuated lattice, called *Lukasiewicz* structure. An easy argument shows for any $x \in [0,1)$ there exists $n \in \mathbb{N}$ such that $x^n = 0$. Hence Filt$(L) = \{\{1\}, L\}$, and so L is a simple (and so semi-simple) residuated lattice. Thus, simple residuated lattices and semi-simple residuated lattices are not finite in general.

In the following result, we show that in the finite case semi-simple residuated lattices and hyperarchimedean residuated lattices are the same.

Corollary 5.7. Let L be a semi-local (e.g., finite residuated lattices) residuated lattice. Then the following are equivalent:

1. L is semi-simple;
2. L is hyperarchimedean;
3. Rad$(L) = \{1\}$.

Proof: The corollary follows from Theorem 5.5. □

In the following example, we show that the condition "L is semi-local" in Corollary 5.7 is necessary.

Example 5.8. Let $\{0,1\}$ be the two-point Boolean algebra. Set $A_i := \{0,1\}$ for each $i \in \mathbb{N}$ and consider the infinite direct products of residuated lattices $A_i's$, that is, the residuated lattice $L = \prod_{i\in\mathbb{N}} A_i$. Then clearly $\mathrm{Rad}(L) = \{(1,1,...)\} = \{1_L\}$, but L is not semi-local. Hence L is not semi-simple by Theorem 5.5. Thus, Corollary 5.7 is not true in general without the hypothesis that L is semi-local.

We end this section with a comparison between a ring's semi-simplicity and the semi-simplicity of the residuated lattice of its ideals.

In the literature, there are some equivalent definitions for semi-simple rings, here we recall one of them. Recall that a commutative unitary ring R is called *semi-simple*, whenever it is a direct product of finitely many fields, see [22, Theorem 3.5.19].

For a commutative unitary ring R, it is known that the lattice of ideals $(Id(R), \cap, +, \odot, \rightarrow, 0 = \{0\}, 1 = R)$ is a residuated lattice and the order relation is $\subseteq$, see [20]. Actually, for every $I, J \in Id(R)$:

$$I + J := \{i + j \mid i \in I, j \in J\},\ I \odot J := \{\sum_{k=1}^{n} i_k j_k \mid \ i_k \in I, j_k \in J\},$$

and

$$I \rightarrow J := \{x \in A \mid x \cdot I \subseteq J\}.$$

THEOREM 5.9. *Let R be a semi-simple commutative unitary ring. Then the residuated lattice $(Id(R), \cap, +, \odot, \rightarrow, 0 = \{0\}, 1 = R)$ is semi-simple.*

PROOF: Let $R := R_1 \times \cdots \times R_n$ be a semi-simple ring, where $R_i's$ are fields. It is well-known that $Id(R) = \{I_1 \times \cdots \times I_n \mid I_i \in Id(R_i)$ for each $i = \{1, ..., n\}$. Now since R_i is a field, we have $Id(R_i) = \{\{0_{R_i}\}, R_i\}$. Hence, $(Id(R), \cap, +, \odot, \rightarrow, 0 = \{0\}, 1 = R)$ is a finite residuated lattice. Clearly, for each $I_1 \times \cdots \times I_n \in R$ and $m \in \mathbb{N}$, we have $(I_1 \times \cdots \times I_n)^m = I_1 \times \cdots \times I_n$, and $(I_1 \times \cdots \times I_n)^* = J_1 \times \cdots \times J_n$, where for each i

$$J_i = \begin{cases} \{0_{R_i}\} & \text{if } I_i = R_i, \\ R_i & \text{if } I_i = \{0_{R_i}\}. \end{cases}$$

Hence for $I_1 \times \cdots \times I_n \in Id(R)$, if there is $k \in \mathbb{N}$ such that $((I_1 \times \cdots \times I_n)^*)^k = 0_{Id(R)} = \{0_R\} = \{(0_{R_1}, ..., 0_{R_n})\}$, then $I_1 \times \cdots \times I_n = R_1 \times \cdots \times R_n = R = 1_{Id(R)}$. Thus, $\mathrm{Rad}(Id(R)) = \{1_{Id(R)}\}$ by Theorem 1.10. Therefore, the result follows from Corollary 5.7. □

The converse of Theorem 5.9 is not true in general.

Example 5.10. Let K be a field, and consider the ring $R := K[[x]]/\langle x^2\rangle$, the quotient of the power series ring $K[[x]]$ over the field K by the ideal generated by x^2. An easy ring theory argument shows that $Id(R) = \{\{0_R\} := \langle x^2\rangle/\langle x^2\rangle, a := \langle x\rangle/\langle x^2\rangle, 1 := R\}$ is a residuated lattice with Hasse diagram depicted in the following figure, and the operations $\odot$ and $\rightarrow$ given by the accompanying tables:

$$\begin{array}{c} 1 \\ | \\ a \\ | \\ 0 \end{array}$$

$\odot$	0	a	1
0	0	0	0
a	0	0	a
1	0	a	1

$\rightarrow$	0	a	1
0	1	1	1
a	a	1	1
1	0	a	1

Now since R is a local ring that is not a field, it is not a semi-simple ring, but since $(a^2)^* = 1$, we have for each $n \in \mathbb{N}$, $((a^2)^*)^n = 1^n = 1 \neq 0$, and hence $\mathrm{Rad}(Id(R)) = \{1\}$. Therefore, by Corollary 5.7, $Id(R)$ is a semi-simple residuated lattice.

6. A Comparative Study of Filter Notions

Let F be a filter of a residuated lattice L. For arbitrary elements $x, y, z \in L$, we recall the following standard list of filter conditions, which are widely used in the algebraic study of substructural logics (see [4] for more information):

(F$_1$) For every $x \in L$, we have $x \vee x^* \in F$;

(F$_2$) If $x \rightarrow (z^* \rightarrow y) \in F$ and $y \rightarrow z \in F$, then $x \rightarrow z \in F$;

(F$_3$) If $x \rightarrow (y \rightarrow z) \in F$ and $x \rightarrow y \in F$, then $x \rightarrow z \in F$;

(F_4) If $(x \to y) \to x \in F$, then $x \in F$;

(F_5) For every $x \in L$, $x \to x^2 \in F$;

(F_6) If $x \to y \in F$, then $((y \to x) \to x) \to y \in F$;

(F_7) If $z^{**} \to (x \to y) \in F$ and $z^{**} \to x \in F$, then $z^{**} \to y \in F$;

(F_8) For every $x \in L$, we have $x^{**} \to x \in F$ holds;

(F_9) For all $x, y \in L$, $(x \to y) \vee (y \to x) \in F$;

(F_{10}) For every $x \in L$, at least one of $x \in F$ or $x^* \in F$ is true.

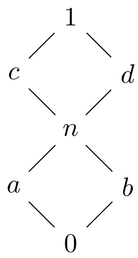
Each of these properties characterizes a specific well-known class of filters. The formal definitions are summarized as follows.

Definition 6.1. A filter F of L is said to be:

1. a *Boolean filter* if it satisfies (F_1);
2. an *implicative deductive system* if it satisfies (F_2);
3. an *implicative filter* if it satisfies (F_3);
4. a *positive implicative filter* if it satisfies (F_4);
5. a *Heyting filter* (also called a G-filter) if it satisfies (F_5);
6. a *fantastic filter* if it satisfies (F_6);
7. an *easy filter* if it satisfies (F_7);
8. an *involution filter* or a *regular filter* if it satisfies (F_8);
9. an *MTL-filter* if it satisfies (F_9);
10. an *obstinate filter* if it satisfies (F_{10}).

By [17, Theorem 3.9], implicative deductive systems and Boolean filters coincide; hence, we consider only Boolean filters in what follows. We now proceed to compare the key concepts of this paper with the notions listed above. To this end, we begin with the following examples of residuated lattices and subsequently provide a general comparison.

Example 6.2. Let $L = \{0, a, b, n, c, d, 1\}$ denote the residuated lattice with Hasse diagram depicted in the following figure, and define the operations $\odot$ and $\to$ via the accompanying tables, see [14, Example, Page 190]:



$\odot$	0	a	b	n	c	d	1
0	0	0	0	0	0	0	0
a	0	a	0	a	a	a	a
b	0	0	b	b	b	b	b
n	0	a	b	n	n	n	n
c	0	a	b	n	c	n	c
d	0	a	b	n	n	d	d
1	0	a	b	n	c	d	1

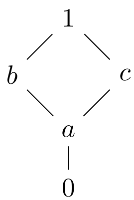
$\to$	0	a	b	n	c	d	1
0	1	1	1	1	1	1	1
a	b	1	b	1	1	1	1
b	a	a	1	1	1	1	1
n	0	a	b	1	1	1	1
c	0	a	b	d	1	d	1
d	0	a	b	c	c	1	1
1	0	a	b	n	c	d	1

Example 6.3. Let $L = \{0, a, b, 1\}$ denote the residuated lattice with Hasse diagram depicted in the following figure, and define the operations $\odot$ and $\to$ via the accompanying tables, see [10, Example 3.3]:

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

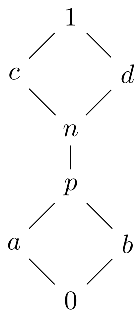
Example 6.4. Let $L = \{0, a, b, c, 1\}$ denote the residuated lattice with Hasse diagram depicted in the following figure, and define the operations $\odot$ and $\to$ via the accompanying tables, see [10, Example 3.22]:



$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	a	a	a
b	0	a	b	a	b
c	0	a	a	c	c
1	0	a	b	c	1

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	1	1	1
b	0	c	1	c	1
c	0	b	b	1	1
1	0	a	b	c	1

Example 6.5. Let $L = \{0, a, b, p, n, c, d, 1\}$ denote the residuated lattice with Hasse diagram depicted in the following figure, and define the operations $\odot$ and $\to$ via the accompanying tables, see [14, Example, Page 193]:



$\odot$	0	a	b	p	n	c	d	1
0	0	0	0	0	0	0	0	0
a	0	a	0	a	a	a	a	a
b	0	0	b	b	b	b	b	b
p	0	a	b	p	p	p	p	p
n	0	a	b	p	n	n	n	n
c	0	a	b	p	n	c	n	c
d	0	a	b	p	n	n	d	d
1	0	a	b	p	n	c	d	1

$\to$	0	a	b	p	n	c	d	1
0	1	1	1	1	1	1	1	1
a	b	1	b	1	1	1	1	1
b	a	a	1	1	1	1	1	1
p	0	a	b	1	1	1	1	1
n	0	a	b	p	1	1	1	1
c	0	a	b	p	d	1	d	1
d	0	a	b	p	c	c	1	1
1	0	a	b	p	n	c	d	1

Example 6.6. Let $L = \{0, a, b, c, d, e, f, 1\}$ be a chain with the usual total order

$$0 < a < b < c < d < e < f < 1.$$

The lattice operations are the minimum and maximum, i.e., $x \wedge y = \min\{x, y\}$ and $x \vee y = \max\{x, y\}$ for all $x, y \in L$.

We define the multiplication $\odot$ as the minimum:

$$x \odot y := \min\{x, y\}, \text{ for all } x, y \in L.$$

This operation is clearly commutative, associative, and has 1 as identity element.

The residuum (implication) $\to$ is defined by the Gödel implication:

$$x \to y := \begin{cases} 1, & \text{if } x \leq y, \\ y, & \text{if } x > y. \end{cases}$$

An easy argument shows that $(L, \wedge, \vee, \odot, \to, 0, 1)$ is a residuated lattice.

6.1. Simple or semi-simple filters need not satisfy the standard properties

For each of the ten classical types, there exists a simple (and so a semi-simple) filter that fails to be of that type.

1. **Non-Boolean (and hence non-obstinate).** In Example 6.2, the simple (semi-simple) filter $[c) = \{1, c\}$ satisfies

$$a \vee a^* = a \vee b = n \notin [c),$$

so it is not Boolean. By [3, Theorem 3.12], since every obstinate filter is Boolean, the filter $[c)$ is not obstinate either.

2. **Non-implicative and non-positive implicative.** In Example 6.3, the unique simple (and so the unique semi-simple) filter is $F = [b) = \{1, b\}$. We have

$$a \to (a \to 0) = a \to a = 1 \in F, \qquad a \to a = 1 \in F,$$

but $a \to 0 = a \notin F$. Hence F is not implicative; since every positive implicative filter is implicative, it is not positive implicative.

3. **Non-Heyting.** Again in Example 6.3, for the simple (semi-simple) filter $F = [b) = \{1, b\}$, we get

$$a \to a^2 = a \to 0 = a \notin F,$$

so F is not Heyting.

4. **Non-fantastic.** In Example 6.4, the filter $[b) = \{1, b\}$ is simple, and so it is semi-simple. We have $0 \to a = 1 \in [b)$, but

$$((a \to 0) \to 0) \to a = (0 \to 0) \to a = 1 \to a = a \notin [b),$$

so it is not fantastic.

5. **Non-easy.** In Example 6.3, take $F = [b) = \{1, b\}$. Then F is a simple filter, and so it is a semi-simple filter. But it is not easy, because $a^{**} \to (a \to 0) = a^{**} \to a^* = a \to a = 1 \in F, a^{**} \to a = a \to a = 1 \in F$, but $a^{**} \to 0 = a^{***} = a^* = a \notin F$.

6. **Non-involution.** In Example 6.2, the simple (semi-simple) filter $[c) = \{1, c\}$ fails (F_8) since

$$n^{**} \to n = 0^* \to n = 1 \to n = n \notin [c).$$

7. **Non-MTL.** In Example 6.2, the filters $[c) = \{1, c\}$ and $[d) = \{1, d\}$ are simple, and so are semi-simple. However,

$$(a \to b) \vee (b \to a) = b \vee a = n \notin [c), [d),$$

so they are not MTL-filters.

6.2. A filter satisfying all classical properties but not simple nor semi-simple

Let L be the finite Gödel chain of Example 6.6, i.e.,

$$L = \{0, a, b, c, d, e, f, 1\}, \qquad 0 < a < b < c < d < e < f < 1,$$

with the usual lattice operations and the Gödel implication. In such a chain, the only simple filter is $[f) = \{1, f\}$, and hence the only semi-simple filter is also $[f)$.

Now consider the filter

$$F := [a) = \{x \in L \mid x \geq a\} = \{1, a, b, c, d, e, f\}.$$

We verify that F satisfies each of the conditions (F_1)–(F_{10}).

(F_1) For every $x \in L$, since L is a chain, $x \vee x^* \in F$ because either $x \geq a$ or $x^* \geq a$ (indeed, $x^* = 0$ for $x > 0$, so $x \vee 0 = x \in F$ for $x \geq a$; for $x = 0$, $0^* = 1 \in F$). Thus F is Boolean.

(F_3) Suppose $x \to (y \to z) \in F$ and $x \to y \in F$. We must show $x \to z \in F$. Equivalently, if $x \to z = 0$, then either $x \to (y \to z) \notin F$ or $x \to y \notin F$. If $x \to z = 0$, then $x > z$ and $z = 0$. Hence $z = 0$ and $x \neq 0$. If $y \neq 0$, then $y \to z = y \to 0 = 0$, so $x \to (y \to z) = x \to 0 = 0 \notin F$; if $y = 0$, then $x \to y = x \to 0 = 0 \notin F$. In either case, the premise fails. Therefore F is implicative.

(F_4) Assume $(x \to y) \to x \in F$. If $x = 0$, then $(0 \to y) \to 0 = 1 \to 0 = 0 \notin F$, contradiction. Hence $x \neq 0$, and since F contains all nonzero elements, $x \in F$. Thus F is positive implicative.

(F_5) In a Gödel chain, $x^2 = x \odot x = \min\{x, x\} = x$ for all x. Hence $x \to x^2 = x \to x = 1 \in F$, so F is Heyting.

(F_6) Suppose $x \to y \in F$. We need $((y \to x) \to x) \to y \in F$. If this element were 0, then its consequent y would be 0 and its antecedent $(y \to x) \to x \neq 0$. But then $((0 \to x) \to x) \to 0 = (1 \to x) \to 0 = x \to 0 = 0$ would imply $x \neq 0$, so $x \to y = x \to 0 = 0 \in F$, a contradiction. Thus the element is in F. So F is fantastic.

(F_7) That F satisfies F_7 follows from a simple computation.

(F_8) A direct computation confirms that F satisfies F_8.

(F_9) For all $x, y \in L$, since L is a chain, either $x \leq y$ or $y \leq x$. If $x \leq y$, then $x \to y = 1 \in F$; otherwise, $y \to x = 1 \in F$. Hence $(x \to y) \vee (y \to x) = 1 \in F$, so F is an MTL-filter.

(F_{10}) For every $x \in L$, either $x \in F$ (if $x \neq 0$) or $x^* = 1 \in F$ (if $x = 0$). Thus F is obstinate.

Thus F is Boolean, implicative, positive implicative, Heyting, fantastic, involutive, MTL, obstinate, and also an implicative deductive system and an easy filter. However, F is not simple and not semi-simple (because the only semi-simple filter is $[f)$). This proves that none of the classical properties implies simplicity or semi-simplicity.

6.3. Essential filters need not satisfy the classical properties

For each of the ten classical types, there exists an essential filter that fails to be of that type.

1. **Non-Boolean, non-implicative, non-positive implicative, non-fantastic, non-involution, non-obstinate.** In the Gödel chain of Example 6.6, consider the filter

$$F := [f) = \{1, f\}.$$

First note that F is essential. Indeed, if $x \vee f = 1$ for some $x \in L$, we must have $x = 1$. Thus we get $F^* = \{1\}$, and so F is essential by Proposition 2.15.

We now verify that F fails each of the conditions (F_1), (F_2), (F_4), (F_6), (F_8), and (F_{10}).

- **Non-Boolean:** Take $x = a$. Since $a > 0$, we have $a^* = a \to 0 = 0$. Hence
$$a \vee a^* = a \vee 0 = a \notin F.$$
Thus (F_1) fails, and so (F_2) fails.
- **Non-positive implicative:** Take $x = a$, $y = 0$. Then

$$x \to y = a \to 0 = 0,$$

so

$$(x \to y) \to x = 0 \to a = 1 \in F,$$

but

$$x = a \notin F.$$

Thus (F_4) fails.

- **Non-fantastic:** Take $x = 0$, $y = a$. Then $x \to y = 0 \to a = 1 \in F$. Now

$$y \to x = a \to 0 = 0, \qquad (y \to x) \to x = 0 \to 0 = 1,$$

and hence

$$((y \to x) \to x) \to y = 1 \to a = a \notin F.$$

So (F_6) fails.

- **Non-involutive:** Take $x = a$. Since $a > 0$, $a^* = 0$, so $a^{**} = 0^* = 0 \to 0 = 1$. Therefore

$$x^{**} \to x = 1 \to a = a \notin F,$$

so (F_8) fails.

- **Non-obstinate:** Clearly $a \notin F$. Also,

$$a^* = a \to 0 = 0 \notin F.$$

Thus neither a nor a^* belongs to F, so (F_{10}) fails.

2. **Non-implicative, Non-Heyting, Non-easy.**

In Example 6.3, consider the filter $F := [b) = \{1, b\}$. We have $F^* = \{1\}$ because if $x \vee b = 1$, then necessarily $x = 1$, and hence it is essential by Proposition 2.15.

We now verify that F fails each of the conditions (F_3), (F_5), and (F_7).

- **Non-implicative:** F is not an implicative filter since $a \to (a \to 0) \in F$, $a \to a \in F$, and $a \to 0 \notin F$.
- **Non-Heyting:** By taking $x = a$, we have

$$a \to a^2 = a \to 0 = a \notin F,$$

so (F_5) fails. Thus F is not a Heyting filter.
- **Non-easy:** F is not easy, because $a^{**} \to (a \to 0) = a^{**} \to a^* = a \to a = 1 \in F, a^{**} \to a = a \to a = 1 \in F$, but $a^{**} \to 0 = a^{***} = a^* = a \notin F$.

3. **Non-MTL.** In Example 6.5, consider the filter $F := [n)$. We have $F^* = \{1\}$ because if $x \vee n = 1$, then necessarily $x = 1$. Hence F is essential by Proposition 2.15. However,

$$(a \to b) \vee (b \to a) = b \vee a = p \notin F,$$

so (F_9) fails. Thus F is not an MTL-filter.

These examples collectively show that an essential filter may fail any of the conditions (F_1)–(F_{10}).

6.4. A classical filter need not be essential

In Example 2.12, one easily checks that the filter $F = \{1, a\}$ satisfies all of (F_1)–(F_{10}), and hence it is Boolean, implicative, positive implicative, Heyting, fantastic, easy, involutive, MTL, and obstinate, but by Proposition 2.15, it is not essential since $F^* = \{1, b\} \neq \{1\}$.

Thus, none of the classical properties implies essentiality.

7. Conclusions

In this paper, we have considered a systematic theory of semi-simplicity in residuated lattices by introducing and thoroughly investigating the notions of simple filters, essential filters, the socle of a filter, independent families of filters, and semi-simple filters. The central result that, for finite residuated lattices, semi-simplicity is equivalent to being hyperarchimedean reveals a deep and natural connection between order-theoretic, algebraic, and topological aspects of these structures.

For future research, we propose:

1. Extend the Wedderburn–Artin-type structure theorem to infinite semi-simple residuated lattices: characterise exactly when a (possibly infinite) semi-simple residuated lattice can be decomposed as a direct product (or subdirect product) of simple residuated lattices.

2. Study the socle Soc(L) in greater depth: determine when Soc(L) is non-trivial, atomic, or coincides with the Boolean center $\mathrm{B}(L)$.

3. Develop a Krull–Gabriel dimension theory for residuated lattices using the length of chains of essential filters or the transcendence degree of independent families of simple filters.

4. Investigate categorical aspects: define and study semi-simple objects in the categories RL of residuated lattices, MTL, BL, MV, and Heyting algebras, and characterize when these categories have enough projective or injective objects in terms of semi-simplicity conditions.

5. Explore topological consequences: characterize compactness, Hausdorffness, or sobriety of $\mathrm{SpecF}(L)$ and $\mathrm{MaxF}(L)$ in terms of semi-simplicity or the existence of sufficiently many simple filters.

6. Apply the theory to algebraisable logics: determine which substructural logics have algebraic semantics whose equivalent algebraic category is semi-simple (e.g., classical logic, Lukasiewicz logic, Gödel logic) and derive new metalogical properties (standard completeness, strong completeness, finite strong completeness) from the other way around.

7. Examine the behavior of semi-simplicity under common constructions such as ordinal sums, and nucleus-based extensions of residuated lattices.

8. Since simple filters are upsets, the topological framework of [12] can be naturally applied to further study semi-simple residuated lattices, which offers a promising direction for future research.

We believe that pursuing these directions will not only enrich the structure theory of residuated lattices, but also provide new bridges between universal algebra, non-classical logic, and lattice-based algebra.

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Esmaeil Rostami

Mahani Math Center
Afzalipour Research Institute
Shahid Bahonar University of Kerman
Kerman, Iran

Department of Pure Mathematics
Faculty of Mathematics and Computer
Shahid Bahonar University of Kerman
Kerman, Iran

e-mail: e_rostami@uk.ac.ir

Funding information: This research was conducted using funds from Afzalipour Research Institute, Shahid Bahonar University of Kerman.

Conflict of interests: None.

Ethical considerations: The Author assures of no violations of publication ethics and takes full responsibility for the content of the publication.

Declaration regarding the use of GAI tools: Not used.