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PROOF TRANSLATIONS BETWEEN LABEL-FREE AND LABELED SEQUENT CALCULI IN ISCI

Abstract

In this paper we consider the Intuitionistic Sentential Calculus with Identity (ISCI). We study two main families of sequent calculi. The first one, called $\mathsf{G3}_{\mathsf{ISCI}}$, is based on a label-free multi-succedent sequent calculus that is sound and complete w.r.t. Kripke models and the second, called $\mathsf{L3}_{\mathsf{ISCI}}$, is based on a multi-succedent labeled sequent calculus that is sound and complete w.r.t. Beth models. Our goal is to investigate how the calculi, that capture distinct semantics of the logic, relate to each other through proof translations. Proof translations from $\mathsf{G3}_{\mathsf{ISCI}}$ to $\mathsf{L3}_{\mathsf{ISCI}}$ provide new results about the soundness and (cut-free) completeness of $\mathsf{G3}_{\mathsf{ISCI}}$ w.r.t. Beth models. Proof translations from

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$\mathsf{L3}_{\mathsf{ISCI}}$ to $\mathsf{G3}_{\mathsf{ISCI}}$ are more difficult and require the definition of new calculi for ISCI that provide intermediate steps in the translation process.

Keywords: non-Fregean sentential logic, sequent calculi, labeled calculi, nested sequents, proof translations.

1. Introduction

In this work we consider an extension of intuitionistic logic (IL) that arises from adding a non Fregean operator ($\approx$) called Suszko's identity. The resulting logic is known as the Intuitionistic Sentential Calculus (or Logic) with Identity (ISCI). Suszko's identity has first been investigated as an extension of classical logic called SCI [1, 10].

The motivation behind SCI is related to the ontology of situations. In classical logic, only two situations can exist, truth and falsity, that are witnessed by any true or false proposition. According to [1], this is unfortunate and could be improved with a new non Fregean operator, written $\approx$, that witnesses two identical situations. In SCI, one acknowledges the fact that there could possibly be more than two situations. Under the usual Fregean interpretation, two formulas are equivalent if they share the same logical value. Under Suszko's identity, two formulas with the same logical value might be considered non-identical if they do not describe the same situations, for instance, two formulas might be valid (and thus logically equivalent) while not having the same sets of proofs. Deduction in SCI has been thoroughly studied, resulting in a Hilbert style proof system [1], various Gentzen sequent calculi [12, 19, 20] and dual tableaux [8, 14]. One drawback of those systems is their lack of analyticity. In particular, they do not enjoy the subformula property and could therefore not provide any kind of decision procedures although SCI is known to be decidable [10]. An alternative decision procedure for SCI based on labeled tableaux has been recently proposed [9].

In the case of ISCI, we consider two main works. The first one is [4], where a Kripke semantics for ISCI is introduced along with a related (Kripke) sound and complete label-free single-succedent sequent calculus

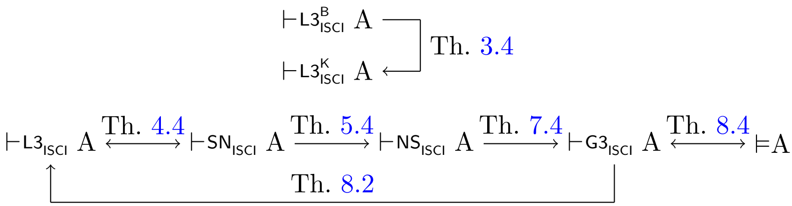

Figure 1: Proof Translation Cycle.

called $\mathsf{sG3}_{\mathsf{ISCI}}$. The second one is [6], where a new TB semantics is introduced along with two related (TB) sound and complete labeled sequent calculi called $\mathsf{L}^1_{\mathsf{ISCI}}$ and $\mathsf{L}^2_{\mathsf{ISCI}}$. Unlike the Kripke semantics, the TB semantics does not rely on partially ordered sets, but on bounded distributive lattices. Both semantics are proven sound and complete w.r.t. the Hilbert system of axioms $\mathsf{H}_{\mathsf{ISCI}}$ [2, 6]. The completeness of the Kripke semantics is achieved by the construction of a canonical model built from (Lindenbaum's idea of) equivalence classes of formulas. On the contrary, the completeness of the TB semantics relies on a canonical model built from (Beth's idea of) theories of formulas (TB standing as either "Topological Beth", or "Theory Based"). The decidability results of ISCI have been settled, from the labeled calculus $\mathsf{L}^2_{\mathsf{ISCI}}$, for a fragment involving decreasing sentential substitutions [6] and from $\mathsf{sG3}_{\mathsf{ISCI}}$ by counter-model construction, for a fragment including only implication and sentential identity [18]. Let us note that the BHK-interpretation (in terms of sets of proofs) for ISCI has been recently described in [3] by introducing a new semantics that captures the notion of identity within a constructive framework.

The main goal of the paper is to study proof translations between families of labeled and label-free sequent calculi for ISCI. We focus on two main families of sequent calculi: $\mathsf{G3}_{\mathsf{ISCI}}$, arising from the $\mathsf{sG3}_{\mathsf{ISCI}}$ calculus that is sound and complete w.r.t. Kripke models, and $\mathsf{L3}_{\mathsf{ISCI}}$, arising from the $\mathsf{L}^1_{\mathsf{ISCI}}$ and $\mathsf{L}^2_{\mathsf{ISCI}}$ calculi proven sound and complete w.r.t. Beth models. Translating proofs from a labeled to a label-free calculus is usually a more difficult problem than in the other direction. Therefore, our approach to translate $\mathsf{L3}_{\mathsf{ISCI}}$ into $\mathsf{G3}_{\mathsf{ISCI}}$ consists in defining two new families of calculi, namely $\mathsf{SN}_{\mathsf{ISCI}}$ and $\mathsf{NS}_{\mathsf{ISCI}}$, as intermediate steps in the translation process.

All of our families of calculi come with two disjoint sets of rules for sentential identities (denoted 1 and 2) and two disjoint sets of rules for disjunction and falsity (denoted K for Kripke and B for Beth). For simplicity and technical consistency, all of our calculi deal with sequents that are sets (and not multisets) of formulas, thus making the contraction rule implicit. We also consider weakening as implicit in our calculi (as it can easily be proven admissible).

The main results are depicted in Figure 1, with two translation cycles: a first one for the Beth variants and a second one for the Kripke variants, with a connection between both by showing that Beth proofs can always be turned into Kripke proofs. We deduce that the K and B proof systems are sound and cut-free complete w.r.t. the Kripke or TB semantics.

In Section 2 we recall the syntax of ISCI and the basics of its TB and Kripke semantics.

In Section 3 we introduce the family $\mathsf{L3}_{\mathsf{ISCI}}$ of labeled sequent calculi, that subsumes the $\mathsf{L}^1_{\mathsf{ISCI}}$ and $\mathsf{L}^2_{\mathsf{ISCI}}$ calculi given in [6]. $\mathsf{L3}_{\mathsf{ISCI}}$ uses sets of integers as labels and implicitly captures the labeling algebra via set union. Then we show that any Beth proof in $\mathsf{L3}^{\mathsf{B}}_{\mathsf{ISCI}}$ can be translated into a Kripke proof in $\mathsf{L3}^{\mathsf{K}}_{\mathsf{ISCI}}$.

In Section 4 we define $\mathsf{SN}_{\mathsf{ISCI}}$ that is a family of labeled sequent calculi, where labels are single letters and where the labeling algebra is captured via explicit relational atoms. In addition to the family with Beth rules for disjunction and falsity, we provide a stepwise and height-preserving translation of $\mathsf{L3}_{\mathsf{ISCI}}$-proofs into $\mathsf{SN}_{\mathsf{ISCI}}$-proofs. A complementary result is a translation in the reverse direction. In Section 5 we define $\mathsf{NS}_{\mathsf{ISCI}}$ that is a family of nested sequent calculi with both Beth and Kripke variants and we show that $\mathsf{SN}_{\mathsf{ISCI}}$-proofs can be translated into $\mathsf{NS}_{\mathsf{ISCI}}$-proofs.

In Section 6, we extend the single-succedent calculus $\mathsf{sG3}_{\mathsf{ISCI}}$ [4] to a multi-succedent calculus and also define a new Beth variant of the calculus that is sound and complete w.r.t. the TB semantics of ISCI. This gives rise to the family $\mathsf{G3}_{\mathsf{ISCI}}$ of multi-succedent label-free sequent calculi.

In Section 7 we show that $\mathsf{NS_{ISCI}}$-proofs can be translated into $\mathsf{G3_{ISCI}}$-proofs and then we deduce, from previous translations, that $\mathsf{L3_{ISCI}}$-proofs can be translated into $\mathsf{G3_{ISCI}}$-proofs.

In Section 8 we show how $\mathsf{G3_{ISCI}}$-proofs can be translated into $\mathsf{L3_{ISCI}}$-proofs. From this translation we deduce new results: $\mathsf{G3^{K}_{ISCI}}$ ($\mathsf{G3^{B}_{ISCI}}$) is sound and complete w.r.t. the Kripke (TB) semantics and all of the $\mathsf{L3_{ISCI}}$ calculi are cut-free complete.

In Section 9, we emphasize how proof translations can help us gain a better understanding of how semantics reveals itself in a calculus and how they allow us to transpose properties from one calculus to the other.

2. The Logic ISCI: Syntax and Semantics

DEFINITION 2.1. Let $\mathbf{P} = \{\, \mathrm{p}, \mathrm{q}, \dots \}$ be a countable set of propositional letters. The formulas of ISCI, the set of which is denoted by $\mathbf{F}$, are given by the grammar:

$$\mathrm{A} ::= \mathrm{p} \in \mathbf{P} \mid \bot \mid \mathrm{A} \wedge \mathrm{A} \mid \mathrm{A} \vee \mathrm{A} \mid \mathrm{A} \supset \mathrm{A} \mid \mathrm{A} \approx \mathrm{A}.$$

As usual, negation $\neg\mathrm{A}$ is defined as a shorthand $\mathrm{A} \supset \bot$ and $\top$ is then defined as $\bot \supset \bot$. We write $\mathbf{F}_{\approx}$ for the restriction of $\mathbf{F}$ to sentential identities.

ISCI admits various semantics. We first recall the TB semantics introduced in [6] as it is less widely known and more elaborated than the known Kripke semantics [4].

DEFINITION 2.2. Let $\mathbf{M}$ be a set of elements, called *worlds*, such that $\omega, \pi \in \mathbf{M}$ and $\omega \neq \pi$. A TB *frame* is a bounded distributive lattice $\mathcal{F} = (\mathbf{M}, \leqslant, \sqcup, \omega, \sqcap, \pi)$ with ω and π as least and greatest elements respectively.

DEFINITION 2.3. A TB *pre-model* is a triple $\mathcal{M} = (\mathcal{F}, [\cdot], \Vdash)$, where $\mathcal{F}$ is a TB frame, and $[\cdot]$ is a *valuation function* from $\mathbf{M}$ to $\wp(\mathbf{P} \cup \mathbf{F}_{\approx})$, such that for all worlds m and n:

($\mathcal{M}_{\pi}$) $[\pi] = \mathbf{P} \cup \mathbf{F}_{\approx}$,

($\mathcal{M}_{\mathrm{K}}$) if $\mathrm{m} \leqslant \mathrm{n}$, then $[\mathrm{m}] \subseteq [\mathrm{n}]$,

($\mathcal{M}_{\approx_1}$) $A \approx A \in [m]$,

($\mathcal{M}_{\approx_4}$) for all $\otimes \in \{\wedge, \vee, \supset, \approx\}$, if $A \approx B \in [m]$ and $C \approx D \in [m]$, then $A \otimes C \approx B \otimes D \in [m]$[1]

The *forcing relation* $\Vdash$ is inductively defined as the smallest relation on $\mathbf{M} \times \mathbf{F}$ such that:

- $m \Vdash p$ iff $p \in [m]$,
- $m \Vdash A \approx B$ iff $A \approx B \in [m]$,
- $m \Vdash \bot$ iff $\pi \leqslant m$,
- $m \Vdash A \wedge B$ iff $m \Vdash A$ and $m \Vdash B$,
- $m \Vdash A \supset B$ iff for all $n \in \mathbf{M}$, if $n \Vdash A$, then $m \sqcup n \Vdash B$,
- $m \Vdash A \vee B$ iff for some $n_1, n_2 \in \mathbf{M}$ such that $n_1 \sqcap n_2 \leqslant m$, $n_1 \Vdash A$ and $n_2 \Vdash B$.

Definition 2.4. A TB *model* is a TB pre-model satisfying the admissibility and regularity conditions:

($\mathcal{M}_{\approx_3}$) if $m \Vdash A \approx B$, then $m \Vdash B \supset A$,

($\mathcal{M}_{\mathrm{R}}$) for all $A \in \mathbf{F}$, there exists an A-*minimal* world, i.e., there exists $m_A \in \mathbf{M}$ such that $m_A \Vdash A$ and for all $n \in \mathbf{M}$, if $n \Vdash A$ then $m_A \leqslant n$.

As usual, a formula A is *true* (or *satisfied*) in a TB model $\mathcal{M}$, written $\mathcal{M} \vDash A$, iff $m \Vdash A$ for all worlds m in $\mathcal{M}$ and *valid*, written $\vDash A$, iff it is true in all models.

The Kripke semantics of ISCI is built on the simple notion of *Kripke frame (more shortly,* K *frame)* which is a partially ordered set of worlds $\mathcal{F} = (\mathbf{M}, \leqslant)$. Kripke models are obtained from Definition 2.3 by discarding conditions ($\mathcal{M}_\pi$) and ($\mathcal{M}_{\mathrm{R}}$) and replacing the forcing clause for the intuitionistic connectives with their standard interpretation. We write $\vDash_{\mathsf{K}}$ or $\vDash_{\mathsf{B}}$ instead of $\vDash$ whenever confusion may arise.

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[1] Let us note that ($\mathcal{M}_{\approx_2}$) if $A \approx B \in [m]$ then $\neg A \approx \neg B \in [m]$ can be derived from ($\mathcal{M}_{\approx_4}$).

3. The Labeled Sequent Calculi $\mathsf{L3}_{\mathsf{ISCI}}$

Let us introduce the $\mathsf{L3}_{\mathsf{ISCI}}$ family of labeled sequent calculi that subsumes the ones in [6]. A label is either a (possibly empty) finite subset of $\mathbb{N}$, or $\mathbb{N}$ itself. We write $\mathbf{L}$ for the set of labels and $\mathbf{L}^n$ for the restriction of $\mathbf{L}$ to labels of size n (sets of cardinal n). $\emptyset$ and $\mathbb{N}$ are called *label units*. We use the (possibly subscripted or primed) letters $\mathrm{a}, \mathrm{b}, \mathrm{c}$ to denote singletons and save the letters $\mathrm{x}, \mathrm{y}, \mathrm{z}$ to denote arbitrary labels. Since all of the examples in this paper use labels built from singletons $\{\, i \mid 1 \leqslant i \leqslant 9 \,\}$, we use the more concise notation 13 to unambiguously refer to the label $\{\, 1, 3 \,\}$ and not to the singleton $\{\, 13 \,\}$.

A label x is a *sublabel* of a label y if $\mathrm{x} \subseteq \mathrm{y}$. Labels are interpreted w.r.t. a labeling algebra $\mathcal{L}$ defined as the bounded lattice $(\mathbf{L}, \subseteq, \cup, \emptyset, \cap, \mathbb{N})$, where join $\cup$ and meet $\cap$ are standard set union and intersection. We consider that $\cup$ binds stronger than $\cap$ and we shall frequently write xy instead of $\mathrm{x} \cup \mathrm{y}$ ($\mathrm{xx'} \cap \mathrm{yy'}$ should therefore be read as $(\mathrm{x} \cup \mathrm{x'}) \cap (\mathrm{y} \cup \mathrm{y'})$).

A *labeled formula* is a pair (C, z), written $\mathrm{C} : \mathrm{z}$, where C is a formula and z is a label. A *labeled sequent* is a pair (Γ, Δ), written $\Gamma \vdash \Delta$, of sets of formulas. Γ and Δ are respectively called the *antecedent* and the *succedent* of the sequent.

The proof rules of $\mathsf{L3}_{\mathsf{ISCI}}$ are given in Figure 2. $\mathsf{L3}^1_{\mathsf{ISCI}}$ and $\mathsf{L3}^2_{\mathsf{ISCI}}$ are respectively defined with the sets $\{\, \mathrm{L}^1_{\approx}, \mathrm{L}^2_{\approx}, \mathrm{L}^3_{\approx}, \mathrm{L}^{3*}_{\approx} \,\}$ and $\{\, \mathrm{L}^1_{\approx}, \mathrm{aL}^2_{\approx}, \mathrm{L}^{\mathrm{r}}_{\approx} \,\}$ for sentential identity rules. For both sets of identity rules we have a K (Kripke) and a B (Beth) version of the calculus depending on the rules for disjunction and falsity. In the rule $\mathrm{L}^{\mathrm{r}}_{\approx}$ (left replacement), $\mathrm{D}^{\mathrm{A}}_{\mathrm{B}}$ denotes the result of replacing some (possibly all) occurrences of A with B in D. Given a set or multiset S of labeled formulas and a label x, the notation $\mathrm{x} \in \mathrm{S}$ is a shorthand for $(\exists(\mathrm{A} : \mathrm{xy}) \in \mathrm{S})$. Therefore, the side conditions of the rules $\mathrm{L}_{\supset}$, $\mathrm{L}^2_{\approx}$, $\mathrm{L}^3_{\approx}$ and $\mathrm{L}^{\mathrm{r}}_{\approx}$ mean that the labels introduced in their premises must already occur in the succedent of their conclusion. A sequent $\Gamma \vdash \Delta$ is *right connected* if $(\forall \mathrm{A} : \mathrm{x} \in \Gamma)(\mathrm{x} \in \Delta)$. A close inspection of the rules shows that they preserve right connectedness upwards.

Definition 3.1. Let C be a formula. An $\mathsf{L3}_{\mathsf{ISCI}}$-*proof of* C is a proof of the sequent $\vdash \mathrm{C} : \emptyset$ with the $\mathsf{L3}_{\mathsf{ISCI}}$ rules.

$$\frac{}{\Gamma, p : x \vdash \Delta, p : xy}\ \mathrm{id_p} \qquad \frac{}{\Gamma, A \approx B : x \vdash \Delta, A \approx B : xy}\ \mathrm{id}_{\approx}$$

$$\frac{}{\Gamma, \bot : x \vdash \Delta, C : xy}\ L_{\bot}^{B} \qquad \frac{}{\Gamma, \bot : x \vdash \Delta}\ L_{\bot}^{K}$$

$$\frac{\Gamma, A_1 : xa_1 \vdash \Delta, C : xya_1 \quad \Gamma, A_2 : xa_2 \vdash \Delta, C : xya_2}{\Gamma, A_1 \vee A_2 : x \vdash \Delta, C : xy}\ L_{\vee}^{B}$$

$$\frac{\Gamma, A_1 : x \vdash \Delta \quad \Gamma, A_2 : x \vdash \Delta}{\Gamma, A_1 \vee A_2 : x \vdash \Delta}\ L_{\vee}^{K} \qquad \frac{\Gamma \vdash \Delta, A_1 : y, A_2 : y}{\Gamma \vdash \Delta, A_1 \vee A_2 : y}\ R_{\vee}$$

$$\frac{\Gamma, A \supset B : x \vdash \Delta, A : xy \quad \Gamma, B : xy \vdash \Delta}{\Gamma, A \supset B : x \vdash \Delta}\ L_{\supset}(xy \in \Delta) \qquad \frac{\Gamma, A : a \vdash \Delta, B : xa}{\Gamma \vdash \Delta, A \supset B : x}\ R_{\supset}$$

$$\frac{\Gamma, A : x, B : x \vdash \Delta}{\Gamma, A \wedge B : x \vdash \Delta}\ L_{\wedge} \qquad \frac{\Gamma \vdash \Delta, A : y \quad \Gamma \vdash \Delta, B : y}{\Gamma \vdash \Delta, A \wedge B : y}\ R_{\wedge}$$

$$\frac{\Gamma, A \approx A : \emptyset \vdash \Delta}{\Gamma \vdash \Delta}\ L_{\approx}^{1} \qquad \frac{\Gamma, A \approx B : x, B \supset A : x \vdash \Delta}{\Gamma, A \approx B : x \vdash \Delta}\ aL_{\approx}^{2}$$

$$\frac{\Gamma, A \approx B : x \vdash \Delta, B : xy \quad \Gamma, A \approx B : x, A : xy \vdash \Delta}{\Gamma, A \approx B : x \vdash \Delta}\ L_{\approx}^{2}(xy \in \Delta)$$

$$\frac{\Gamma, A \approx B : x, C \approx D : y, A \otimes C \approx B \otimes D : xy \vdash \Delta}{\Gamma, A \approx B : x, C \approx D : y \vdash \Delta}\ L_{\approx}^{3}(xy \in \Delta)$$

$$\frac{\Gamma, A \otimes A \approx B \otimes B : x \vdash \Delta}{\Gamma, A \approx B : x \vdash \Delta}\ L_{\approx}^{3*} \qquad \frac{\Gamma, A \approx B : x, D : y, D_{B}^{A} : xy \vdash \Delta}{\Gamma, A \approx B : x, D : y \vdash \Delta}\ L_{\approx}^{r}(xy \in \Delta)$$

Eigenvariable conditions: In $R_{\supset}$ and $L_{\vee}^{B}$, a, a_1, a_2 are fresh singletons and $a_1 \neq a_2$.

Figure 2: Rules for the $\mathsf{L3}_{\mathsf{ISCI}}$ Family of Labeled Calculi.

A *label substitution* is a total function $\sigma : \mathbf{L} \to \mathbf{L}$ whose restriction $\sigma^* : (\mathbf{L}^0 \cup \mathbf{L}^1) \to \mathbf{L}$ differs from the identity only for a finite number of elements in the domain and such that for all labels $z \notin (\mathbf{L}^0 \cup \mathbf{L}^1)$, $z\sigma = \bigcup\{\, x\sigma^* \mid x \in (\mathbf{L}^0 \cup \mathbf{L}^1) \text{ and } x \subseteq z \,\}$. Label substitutions extend to labeled formulas, multisets of labeled formulas and sequents as follows: $(A : x)\sigma = A : x\sigma$, $S\sigma = \{\, (A : x)\sigma \mid A : x \in S \,\}$ and $(\Gamma \vdash \Delta)\sigma = \Gamma\sigma \vdash \Delta\sigma$. We write $[x_1/y_1; \ldots ; x_n/y_n]$, where $x_i \in \mathbf{L}$ and $y_i \in (\mathbf{L}^0 \cup \mathbf{L}^1)$ for all $1 \leqslant i \leqslant n$, to denote the label substitution σ such that $z\sigma = x_i$ if $z = y_i$ for some i and $z\sigma = z$ otherwise. Hence, x/y means that x replaces y. For instance, let $\sigma = [17/\emptyset; 2/7]$, then since $347 = \bigcup\{\, \emptyset, \{\,3\,\}, \{\,4\,\}, \{\,7\,\}\,\}$, we have $(347)\sigma = \bigcup\{\, \{\,1, 7\,\}, \{\,3\,\}, \{\,4\,\}, \{\,2\,\}\,\} = 12347$. It is easy to check that $z[x/y] = (z - y) \cup x$ if $y \subseteq z$, and $z[x/y] = z$ otherwise. Thus, $(347)[\emptyset/7] = \{\,3, 4, 7\,\} - \{\,7\,\} \cup \emptyset = \{\,3, 4\,\} = 34$.

LEMMA 3.2. *Let* $s = \Gamma \vdash \Delta$. *If* $\vdash_{\mathsf{L3}_{\mathsf{ISCI}}} s$ *then* $\vdash_{\mathsf{L3}_{\mathsf{ISCI}}} s\sigma$.

PROOF: By induction on the height of the proof (see Appendix C.2). □

The fact that only labels in $\mathbf{L}^0 \cup \mathbf{L}^1$ can be replaced is essential for the soundness of Lemma 3.2. Indeed, consider the instance of id_p whose conclusion is $s = p : 12 \vdash p : 123$, we have $\vdash_{\mathsf{L3}_{\mathsf{ISCI}}} s$ but not $\vdash_{\mathsf{L3}_{\mathsf{ISCI}}} s[4/23] = p : 12 \vdash p : 14$.

Let us now show that any $\mathsf{L3}^{\mathsf{B}}_{\mathsf{ISCI}}$-proof translates to a $\mathsf{L3}^{\mathsf{K}}_{\mathsf{ISCI}}$-proof by erasing all of the singletons introduced by instances of $L^{\mathrm{B}}_{\vee}$ and globally renaming $L^{\mathrm{B}}_{\perp}$ and $L^{\mathrm{B}}_{\vee}$ as $L^{\mathrm{K}}_{\perp}$ and $L^{\mathrm{K}}_{\vee}$.

DEFINITION 3.3. Let Π be a proof of a sequent s in $\mathsf{L3}^{\mathsf{B}}_{\mathsf{ISCI}}$. B_Π is defined as the set $\{\, c_1, \ldots, c_n \,\}$ of all the fresh singletons introduced by an instance of $L^{\mathrm{B}}_{\vee}$ in Π. Moreover, σ_Π is defined as the erasing substitution $[\emptyset/c_1; \ldots ; \emptyset/c_n]$ that replaces all occurrences of c_i in B_Π with $\emptyset$.

THEOREM 3.4. *If* Π *is a proof of a sequent* s *in* $\mathsf{L3}^{\mathsf{B}}_{\mathsf{ISCI}}$, *then* $\Pi\sigma_\Pi$ *is a proof of* $s\sigma_\Pi$ *in* $\mathsf{L3}^{\mathsf{K}}_{\mathsf{ISCI}}$.

PROOF: By induction on the height of Π. We only consider the cases $L^{\mathrm{B}}_{\perp}$ and $L^{\mathrm{B}}_{\vee}$ since they are the ones that differ in $\mathsf{L3}^{\mathsf{K}}_{\mathsf{ISCI}}$.

Base case $\mathrm{L}_{\perp}^{\mathrm{B}}$: Since Π is of height 0, we have that $\mathrm{B}_{\Pi} = \emptyset$ and $\sigma_{\Pi} = \emptyset$. Hence,

$$\frac{}{\Gamma, \perp : \mathrm{x} \vdash \Delta, \mathrm{C} : \mathrm{xy}}\ \mathrm{L}_{\perp}^{\mathrm{B}} \quad \rightsquigarrow \quad \frac{}{\Gamma, \perp : \mathrm{x} \vdash \Delta, \mathrm{C} : \mathrm{xy}}\ \mathrm{L}_{\perp}^{\mathrm{K}}$$

Case $\mathrm{L}_{\vee}^{\mathrm{B}}$: We start with a proof Π:

$$\frac{\begin{matrix}\Pi_1 \\ \hdashline \Gamma, \mathrm{A}_1 : \mathrm{xa}_1 \vdash \Delta, \mathrm{C} : \mathrm{xya}_1\end{matrix} \quad \begin{matrix}\Pi_2 \\ \hdashline \Gamma, \mathrm{A}_2 : \mathrm{xa}_2 \vdash \Delta, \mathrm{C} : \mathrm{xya}_2\end{matrix}}{\Gamma, \mathrm{A}_1 \vee \mathrm{A}_2 : \mathrm{x} \vdash \Delta, \mathrm{C} : \mathrm{xy}}\ \mathrm{L}_{\vee}^{\mathrm{B}}$$

By induction hypothesis on $\Pi_{\mathrm{i}}, \mathrm{i} \in \{1, 2\}$ we get:

$$\begin{matrix}\Pi_{\mathrm{i}}\sigma_{\Pi_{\mathrm{i}}} \\ \hdashline \Gamma\sigma_{\Pi_{\mathrm{i}}}, \mathrm{A}_{\mathrm{i}} : \mathrm{xa}_{\mathrm{i}}\sigma_{\Pi_{\mathrm{i}}} \vdash \Delta\sigma_{\Pi_{\mathrm{i}}}, \mathrm{C} : \mathrm{xya}_{\mathrm{i}}\sigma_{\Pi_{\mathrm{i}}}\end{matrix}$$

Let σ be the label substitution $[\emptyset/\mathrm{a}_1; \emptyset/\mathrm{a}_2]$. After applying σ on $\Pi_{\mathrm{i}}\sigma_{\Pi_{\mathrm{i}}}$ using Lemma 3.2 we get new $\mathsf{L3}^{\mathsf{K}}_{\mathsf{ISCI}}$-proofs and since a_1 and a_2 are fresh in $\Gamma \vdash \Delta$, we have $\sigma_{\Pi_{\mathrm{i}}}\sigma = \sigma_{\Pi}$ and $\mathrm{xa}_{\mathrm{i}}\sigma_{\Pi_{\mathrm{i}}}\sigma = \mathrm{x}\sigma_{\Pi} = \mathrm{x}$, which allows us to apply an instance of $\mathrm{L}_{\vee}^{\mathrm{K}}$ as follows:

$$\frac{\begin{matrix}\Pi_1\sigma_{\Pi} \\ \hdashline \Gamma\sigma_{\Pi}, \mathrm{A}_1 : \mathrm{xa}_1\sigma_{\Pi} \vdash \Delta\sigma_{\Pi}, \mathrm{C} : \mathrm{xya}_1\sigma_{\Pi}\end{matrix} \quad \begin{matrix}\Pi_2\sigma_{\Pi} \\ \hdashline \Gamma\sigma_{\Pi}, \mathrm{A}_2 : \mathrm{xa}_2\sigma_{\Pi} \vdash \Delta\sigma_{\Pi}, \mathrm{C} : \mathrm{xya}_2\sigma_{\Pi}\end{matrix}}{\Gamma\sigma_{\Pi}, \mathrm{A}_1 \vee \mathrm{A}_2 : \mathrm{x}\sigma_{\Pi} \vdash \Delta\sigma_{\Pi}, \mathrm{C} : \mathrm{xy}\sigma_{\Pi}}\ \mathrm{L}_{\vee}^{\mathrm{K}}$$

The other cases are similar. □

4. From $\mathsf{L3}_{\mathsf{ISCI}}$ to $\mathsf{SN}_{\mathsf{ISCI}}$

We introduce the $\mathsf{SN}_{\mathsf{ISCI}}$ family of labeled calculi that extends the labeled sequent calculus $\mathsf{SN}_{\mathsf{IL}}$ for intuitionistic logic [13]. Labels in $\mathsf{SN}_{\mathsf{ISCI}}$ are not sets or multisets but single atomic symbols over a predefined alphabet which we take as the set of natural numbers in this paper. When we write 5 and 24 we actually mean the singleton $\{5\}$ and the set $\{2, 4\}$ in $\mathsf{L3}_{\mathsf{ISCI}}$. In $\mathsf{SN}_{\mathsf{ISCI}}$, 5 and 24 are the actual natural numbers five and twenty four. To avoid confusion, we use the letters $\mathrm{u}, \mathrm{v}, \mathrm{w}$ to denote labels in $\mathsf{SN}_{\mathsf{ISCI}}$ and keep $\mathrm{x}, \mathrm{y}, \mathrm{z}$ for labels in $\mathsf{L3}_{\mathsf{ISCI}}$.

$$\frac{}{\mathcal{R}, \Gamma, p : u \vdash \Delta, p : v}\ \mathrm{id}_{p}(u \overset{\mathcal{R}}{\rightsquigarrow} v) \qquad \frac{}{\mathcal{R}, \Gamma, A \approx B : u \vdash \Delta, A \approx B : v}\ \mathrm{id}_{\approx}(u \overset{\mathcal{R}}{\rightsquigarrow} v)$$

$$\frac{}{\mathcal{R}, \Gamma, \bot : u \vdash \Delta, C : v}\ L_{\bot}^{B}(u \overset{\mathcal{R}}{\rightsquigarrow} v) \qquad \frac{}{\mathcal{R}, \Gamma, \bot : u \vdash \Delta}\ L_{\bot}^{K}$$

$$\frac{\mathcal{R}, v \sqsubset u_1, \Gamma, A_1 : u_1 \vdash \Delta, C : u_1 \quad \mathcal{R}, v \sqsubset u_2, \Gamma, A_2 : u_2 \vdash \Delta, C : u_2}{\mathcal{R}, \Gamma, A_1 \vee A_2 : u \vdash \Delta, C : v}\ L_{\vee}^{B}(u \overset{\mathcal{R}}{\rightsquigarrow} v)$$

$$\frac{\mathcal{R}, \Gamma, A_1 : u \vdash \Delta \quad \mathcal{R}, \Gamma, A_2 : u \vdash \Delta}{\mathcal{R}, \Gamma, A_1 \vee A_2 : u \vdash \Delta}\ L_{\vee}^{K} \qquad \frac{\mathcal{R}, \Gamma \vdash \Delta, A_1 : u, A_2 : u}{\mathcal{R}, \Gamma \vdash \Delta, A_1 \vee A_2 : u}\ R_{\vee}$$

$$\frac{\mathcal{R}, \Gamma, A \supset B : u \vdash \Delta, A : v \quad \mathcal{R}, \Gamma, B : v \vdash \Delta}{\mathcal{R}, \Gamma, A \supset B : u \vdash \Delta}\ L_{\supset}(u \overset{\mathcal{R}}{\rightsquigarrow} v) \qquad \frac{\mathcal{R}, u \sqsubset v, \Gamma, A : v \vdash \Delta, B : v}{\mathcal{R}, \Gamma \vdash \Delta, A \supset B : u}\ R_{\supset}$$

$$\frac{\mathcal{R}, \Gamma, A : u, B : u \vdash \Delta}{\mathcal{R}, \Gamma, A \wedge B : u \vdash \Delta}\ L_{\wedge} \qquad \frac{\mathcal{R}, \Gamma \vdash \Delta, A : u \quad \mathcal{R}, \Gamma \vdash \Delta, B : u}{\mathcal{R}, \Gamma \vdash \Delta, A \wedge B : u}\ R_{\wedge}$$

$$\frac{\mathcal{R}, \Gamma, A \approx A : 0 \vdash \Delta}{\mathcal{R}, \Gamma \vdash \Delta}\ L_{\approx}^{1} \qquad \frac{\mathcal{R}, \Gamma, A \approx B : u, B \supset A : u \vdash \Delta}{\mathcal{R}, \Gamma, A \approx B : u \vdash \Delta}\ aL_{\approx}^{2}$$

$$\frac{\mathcal{R}, \Gamma, A \approx B : u \vdash \Delta, B : v \quad \mathcal{R}, \Gamma, A \approx B : u, A : v \vdash \Delta}{\mathcal{R}, \Gamma, A \approx B : u \vdash \Delta}\ L_{\approx}^{2}(u \overset{\mathcal{R}}{\rightsquigarrow} v)$$

$$\frac{\mathcal{R}, \Gamma, A \approx B : u, C \approx D : v, A \otimes C \approx B \otimes D : v \vdash \Delta}{\mathcal{R}, \Gamma, A \approx B : u, C \approx D : v \vdash \Delta}\ L_{\approx}^{3}(u \overset{\mathcal{R}}{\rightsquigarrow} v)$$

$$\frac{\mathcal{R}, \Gamma, A \approx B : u, A \otimes A \approx B \otimes B : u \vdash \Delta}{\mathcal{R}, \Gamma, A \approx B : u \vdash \Delta}\ L_{\approx}^{3*}$$

$$\frac{\mathcal{R}, \Gamma, A \approx B : u, D : v, D_{B}^{A} : v \vdash \Delta}{\mathcal{R}, \Gamma, A \approx B : u, D : v \vdash \Delta}\ L_{\approx}^{r}(u \overset{\mathcal{R}}{\rightsquigarrow} v)$$

Eigenvariable Conditions:

In $R_{\supset}$, v is fresh in $\Gamma \cup \Delta$.

In $L_{\vee}^{B}$, u_1 and u_2 are fresh in $\Gamma \cup \Delta$.

Figure 3: Rules for the $\mathsf{SN}_{\mathsf{ISCI}}$ Family of Labeled Calculi.

Labeled sequents in $\mathsf{SN}_{\mathsf{ISCI}}$ have the form $\mathcal{R}, \Gamma \vdash \Delta$, where Γ, Δ are sets of labeled formulas and $\mathcal{R}$ is the set of *relational atoms*, which are expressions of the form $\mathrm{u} \sqsubset \mathrm{v}$, where u, v are labels. The proof rules of $\mathsf{SN}_{\mathsf{ISCI}}$ are given in Figure 3. $\mathsf{SN}^1_{\mathsf{ISCI}}$ and $\mathsf{SN}^2_{\mathsf{ISCI}}$ are respectively defined with the sets $\{ \mathrm{L}^1_{\approx}, \mathrm{L}^2_{\approx}, \mathrm{L}^3_{\approx}, \mathrm{L}^{3*}_{\approx} \}$ and $\{ \mathrm{L}^1_{\approx}, \mathrm{aL}^2_{\approx}, \mathrm{L}^{\mathrm{r}}_{\approx} \}$ for sentential identity rules. For both sets of identity rules we a K (Kripke) and B (Beth) version of the calculus depending on the rules for disjunction and falsity. An $\mathsf{SN}_{\mathsf{ISCI}}$-proof of a formula C is a proof of the sequent $\vdash \mathrm{C} : 0$.

$\mathsf{SN}_{\mathsf{IL}}$ is usually formulated with rules for the reflexivity and the transitivity of $\sqsubset$ [13]. Such rules can be eliminated by introducing a *reachability predicate* $\mathrm{u} \overset{\mathcal{R}}{\rightsquigarrow} \mathrm{v}$ that can be either defined as the reflexive and transitive closure of $\sqsubset$ (denoted $\sqsubset^*$) in [7], i.e. $\mathrm{u} \overset{\mathcal{R}}{\rightsquigarrow} \mathrm{v}$ iff $(\mathrm{u} \sqsubset \mathrm{v}) \in \sqsubset^*$, or equivalently via the notion of *directed path* as in [11], defined as a chain $\mathrm{w}_1 \sqsubset \ldots \sqsubset \mathrm{w}_\mathrm{n}$ in $\mathcal{R}$ such that $\mathrm{w}_1 = \mathrm{u}$, $\mathrm{w}_\mathrm{n} = \mathrm{v}$, with the special case $\mathrm{u} \overset{\mathcal{R}}{\rightsquigarrow} \mathrm{v}$ if $\mathrm{u} = \mathrm{v}$.

Let us now explain how to translate $\mathsf{L3}_{\mathsf{ISCI}}$-proofs into $\mathsf{SN}_{\mathsf{ISCI}}$-proofs. Any non-empty (finite) label x in $\mathsf{L3}_{\mathsf{ISCI}}$ can be written as an ordered set $\{ k_1 < k_2 < \ldots < k_n \}$ of natural numbers. Let us write $\mu(\mathrm{x})$ for the singleton $\{ k_n \}$ containing the greatest element in x, with the special cases $\mu(\emptyset) = 0$ and $\mu(\mathbb{N}) = \infty$. The set $\mathcal{R}(\mathrm{x})$ of relational atoms associated with a label x is defined as the set $\{ k_i \sqsubset k_{i+1} \mid 0 \leqslant i < n \}$ where $k_0 = 0$. For example, $\mathcal{R}(\{ 1, 2, 5, 8 \}) = \{ 0 \sqsubset 1, 1 \sqsubset 2, 2 \sqsubset 5, 5 \sqsubset 8 \}$. In order to save space, let us write chains $k_1 \sqsubset k_2, k_2 \sqsubset k_3, \ldots, k_{n-1} \sqsubset k_n$ more concisely as $1 \sqsubset 2 \sqsubset \ldots \sqsubset k - 1 \sqsubset k$.

Let S be set of labeled formulas (in $\mathsf{L3}_{\mathsf{ISCI}}$ or $\mathsf{SN}_{\mathsf{ISCI}}$). The set $\lceil \mathrm{S} \rceil$ is defined as the restriction of S to the formulas whose labels are maximal w.r.t. $\subseteq$ ($\sqsubset$) in $\mathsf{L3}_{\mathsf{ISCI}}$ ($\mathsf{SN}_{\mathsf{ISCI}}$). A label x is *maximal* in S if $\mathrm{x} \in \lceil \mathrm{S} \rceil$. Given a sequent $\mathrm{s} = \Gamma \vdash \Delta$, a label x (or labeled formula $\mathrm{A} : \mathrm{x}$) is *right maximal* in s if x (or $\mathrm{A} : \mathrm{x}$) $\in \lceil \Delta \rceil$. Left maximality is defined similarly w.r.t. $\lceil \Gamma \rceil$. An instance of a rule r is *right (left) maximal* in a proof if all of its principal formulas occurring in the succedent (antecedent) are right (left) maximal. A proof is *right (left) maximal* if all of its rules are right (left) maximal.

Definition 4.1. Let $\Gamma \vdash \Delta$ be a labeled sequent in $\mathsf{L3}_{\mathsf{ISCI}}$. Let $\mathrm{A}_1 : \mathrm{x}_1, \ldots,$ $\mathrm{A}_\mathrm{m} : \mathrm{x}_\mathrm{m}$ and $\mathrm{B}_1 : \mathrm{y}_1$, ..., $\mathrm{B}_\mathrm{n} : \mathrm{y}_\mathrm{n}$ be enumerations of Γ and Δ respectively.

The translation $\mathsf{LS}(\Gamma \vdash \Delta)$ is the $\mathsf{SN}_{\mathsf{ISCI}}$ sequent $\mathcal{R}', \Gamma' \vdash \Delta'$ where $\Gamma' = \{ A_i : \mu(x_i) \mid 1 \leqslant i \leqslant m \}$, $\Delta' = \{ B_j : \mu(y_j) \mid 1 \leqslant j \leqslant n \}$ and $\mathcal{R}' = \bigcup \{ \mathcal{R}(z) \mid z \text{ is maximal in } \Gamma \cup \Delta \}$.

For instance, the translation of the sequent $A : 1, B : 34 \vdash C : 2345, D : 18$ in $\mathsf{L3}_{\mathsf{ISCI}}$ is the sequent $0 \sqsubset 2 \sqsubset 3 \sqsubset 4 \sqsubset 5, 0 \sqsubset 1 \sqsubset 8, A : 1, B : 4 \vdash C : 5, D : 8$ in $\mathsf{SN}_{\mathsf{ISCI}}$, where the set of relational atoms is given by the two maximal labels 2345 and 18.

Definition 4.2. A proof Π in $\mathsf{L3}_{\mathsf{ISCI}}$ is *standard* if it does not contain any occurrence of $\mathbb{N}$ and all instances of $R_{\supset}$ and $L_{\vee}^{B}$ introduce fresh labels that are maximal in their premises (for instance, by setting $a = \{ 3k + 1 \}$ and $a_i = \{ 3k + i + 1 \}$ for the smallest suitable k).

Theorem 4.3. *Any* $\mathsf{L3}_{\mathsf{ISCI}}$*-proof can be transformed into a right maximal standard proof.*

Proof: Any $\mathsf{L3}_{\mathsf{ISCI}}$-proof of formula C can be turned into a standard proof by successive applications of Lemma 3.2 since no rule in $\mathsf{L3}_{\mathsf{ISCI}}$ can introduce $\mathbb{N}$ (which can therefore only be present in arbitrarily defined sequents). Showing that a standard proof Π can be turned into a right maximal proof follows from a routine induction on the height of Π. ☐

Theorem 4.4. *Any right maximal standard* $\mathsf{L3}_{\mathsf{ISCI}}$*-proof can be translated into a (right maximal standard)* $\mathsf{SN}_{\mathsf{ISCI}}$*-proof.*

Proof: By induction on the height of the proof in $\mathsf{L3}_{\mathsf{ISCI}}$ (see Appendix B). ☐

Example 4.5. As an illustration of the translation, let us consider an $\mathsf{L3}^{\mathsf{B}}_{\mathsf{ISCI}}$-proof of the formula $((p \supset r) \wedge (q \supset r)) \supset ((p \vee q) \supset r)$.

$$\Pi_1 \left\{ \dfrac{\dfrac{}{p \supset r : 1, q \supset r : 1, p : 45 \vdash r : 145, p : 145}\,\mathrm{id}_p \qquad \dfrac{}{q \supset r : 1, p : 45, r : 145 \vdash r : 145}\,\mathrm{id}_p}{p \supset r : 1, q \supset r : 1, p : 45 \vdash r : 145}\,L_{\supset} \right.$$

$$\Pi_2 \left\{ \dfrac{\dfrac{}{p \supset r : 1, q \supset r : 1, q : 46 \vdash r : 146, q : 146}\,\mathrm{id}_p \qquad \dfrac{}{p \supset r : 1, q : 46, r : 146 \vdash r : 146}\,\mathrm{id}_p}{p \supset r : 1, q \supset r : 1, q : 46 \vdash r : 146}\,L_{\supset} \right.$$

$$\Pi\left\{\dfrac{\dfrac{\dfrac{\dfrac{\Pi_1 \qquad \Pi_2}{\mathrm{p} \supset \mathrm{r} : 1, \mathrm{q} \supset \mathrm{r} : 1, \mathrm{p} \vee \mathrm{q} : 4 \vdash \mathrm{r} : 14}\,\mathrm{L}^{\mathrm{B}}_{\vee}}{\mathrm{p} \supset \mathrm{r} : 1, \mathrm{q} \supset \mathrm{r} : 1 \vdash (\mathrm{p} \vee \mathrm{q}) \supset \mathrm{r} : 1}\,\mathrm{R}_{\supset}}{(\mathrm{p} \supset \mathrm{r}) \wedge (\mathrm{q} \supset \mathrm{r}) : 1 \vdash (\mathrm{p} \vee \mathrm{q}) \supset \mathrm{r} : 1}\,\mathrm{L}_{\wedge}}{\vdash ((\mathrm{p} \supset \mathrm{r}) \wedge (\mathrm{q} \supset \mathrm{r})) \supset ((\mathrm{p} \vee \mathrm{q}) \supset \mathrm{r}) : \emptyset}\,\mathrm{R}_{\supset}\right.$$

The translation of Π in $\mathsf{SN}^{\mathsf{B}}_{\mathsf{ISCI}}$ is given below.

$$\mathsf{LS}(\Pi_1)\left\{\dfrac{\dfrac{}{\begin{matrix}0 \sqsubset 1 \sqsubset 4 \sqsubset 5, \\ \mathrm{p} \supset \mathrm{r} : 1, \mathrm{q} \supset \mathrm{r} : 1, \mathrm{p} : 5\end{matrix} \vdash \begin{matrix}\mathrm{r} : 5, \\ \mathrm{p} : 5\end{matrix}}\,\mathrm{id_p} \qquad \dfrac{}{\begin{matrix}0 \sqsubset 1 \sqsubset 4 \sqsubset 5, \\ \mathrm{q} \supset \mathrm{r} : 1, \mathrm{p} : 5, \mathrm{r} : 5\end{matrix} \vdash \mathrm{r} : 5}\,\mathrm{id_p}}{0 \sqsubset 1 \sqsubset 4 \sqsubset 5, \mathrm{p} \supset \mathrm{r} : 1, \mathrm{q} \supset \mathrm{r} : 1, \mathrm{p} : 5 \vdash \mathrm{r} : 5}\,\mathrm{L}_{\supset}\right.$$

$$\mathsf{LS}(\Pi_2)\left\{\dfrac{\dfrac{}{\begin{matrix}0 \sqsubset 1 \sqsubset 4 \sqsubset 6, \\ \mathrm{p} \supset \mathrm{r} : 1, \mathrm{q} \supset \mathrm{r} : 1, \mathrm{q} : 6\end{matrix} \vdash \begin{matrix}\mathrm{r} : 6, \\ \mathrm{q} : 6\end{matrix}}\,\mathrm{id_p} \qquad \dfrac{}{\begin{matrix}0 \sqsubset 1 \sqsubset 4 \sqsubset 6, \\ \mathrm{p} \supset \mathrm{r} : 1, \mathrm{q} : 6, \mathrm{r} : 6\end{matrix} \vdash \mathrm{r} : 6}\,\mathrm{id_p}}{0 \sqsubset 1 \sqsubset 4 \sqsubset 6, \mathrm{p} \supset \mathrm{r} : 1, \mathrm{q} \supset \mathrm{r} : 1, \mathrm{q} : 6 \vdash \mathrm{r} : 6}\,\mathrm{L}_{\supset}\right.$$

$$\mathsf{LS}(\Pi)\left\{\dfrac{\dfrac{\dfrac{\dfrac{\mathsf{LS}(\Pi_1) \qquad \mathsf{LS}(\Pi_2)}{0 \sqsubset 1 \sqsubset 4, \mathrm{p} \supset \mathrm{r} : 1, \mathrm{q} \supset \mathrm{r} : 1, \mathrm{p} \vee \mathrm{q} : 4 \vdash \mathrm{r} : 4}\,\mathrm{L}^{\mathrm{B}}_{\vee}}{0 \sqsubset 1, \mathrm{p} \supset \mathrm{r} : 1, \mathrm{q} \supset \mathrm{r} : 1, \vdash (\mathrm{p} \vee \mathrm{q}) \supset \mathrm{r} : 1}\,\mathrm{R}_{\supset}}{0 \sqsubset 1, (\mathrm{p} \supset \mathrm{r}) \wedge (\mathrm{q} \supset \mathrm{r}) : 1 \vdash (\mathrm{p} \vee \mathrm{q}) \supset \mathrm{r} : 1}\,\mathrm{L}_{\wedge}}{\vdash ((\mathrm{p} \supset \mathrm{r}) \wedge (\mathrm{q} \supset \mathrm{r})) \supset ((\mathrm{p} \vee \mathrm{q}) \supset \mathrm{r}) : 0}\,\mathrm{R}_{\supset}\right.$$

Since $\mathsf{L3}_{\mathsf{ISCI}}$ does not contain any transitivity or reflexivity rules, the completeness of $\mathsf{SN}_{\mathsf{ISCI}}$ without such rules is a corollary of Theorem 4.4. An important consequence of this result is that for all sequents s in a structural free $\mathsf{SN}_{\mathsf{ISCI}}$-proof Π of a formula C, the set $\mathcal{R}$ of relational atoms describes a tree structure such that if $\mathrm{u} \sqsubset \mathrm{v} \in \mathcal{R}$, then the node corresponding to v is an immediate successor of the node corresponding to u.

Definition 4.6. A *labeled tree sequent* is a labeled sequent $\tau = \mathcal{R}, \Gamma \vdash \Delta$ such that $\mathcal{R}$ forms a (minimal) tree and all labels in $\Gamma \cup \Delta$ occur in $\mathcal{R}$ (unless $\mathcal{R}$ is empty, in which case every labeled formula in $\Gamma \cup \Delta$ must share the same label). A *labeled tree proof* is a proof containing only labeled tree sequents. A labeled tree proof has the *fixed root property* iff every labeled sequent in the proof has the same root, in which case it is called *standard proof*.

Let us remark that Definition 4.1 maps $\mathsf{L3_{ISCI}}$-proofs to labeled tree proofs in $\mathsf{SN_{ISCI}}$ with the fixed root property (the root being 0) since $\emptyset \subseteq x$ for any label x in $\mathsf{L3_{ISCI}}$.

Definition 4.7. Let $\mathcal{R}, \Gamma \vdash \Delta$ be a labeled sequent in $\mathsf{SN_{ISCI}}$. Let $A_1 : u_1$, ..., $A_m : u_m$ and $B_1 : v_1$, ..., $B_n : v_n$ be enumerations of Γ and Δ respectively. Let $\mathcal{R}$ be set of relational atoms and u be a label occurring in $\mathcal{R}$, $\mathcal{R}(u) = \{ v \mid v \neq 0 \text{ and } v \overset{\mathcal{R}}{\rightsquigarrow} u \}$. The translation $\mathsf{SL}(\Gamma \vdash \Delta)$ is the $\mathsf{L3_{ISCI}}$ sequent $\Gamma' \vdash \Delta'$ where $\Gamma' = \{ A_i : \mathcal{R}(x_i) \mid 1 \leqslant i \leqslant m \}$, $\Delta' = \{ B_j : \mathcal{R}(y_j) \mid 1 \leqslant j \leqslant n \}$.

For instance, the sequent $0 \sqsubset 2 \sqsubset 3 \sqsubset 4 \sqsubset 5, 0 \sqsubset 1 \sqsubset 8, A : 1, B : 4 \vdash C : 5, D : 8$ in $\mathsf{SN_{ISCI}}$ translates into the sequent $A : 1, B : 234 \vdash C : 2345, D : 18$ in $\mathsf{L3_{ISCI}}$.

Theorem 4.8. *Any standard* $\mathsf{SN_{ISCI}}$*-proof can be translated into an* $\mathsf{L3_{ISCI}}$*-proof.*

Proof: A direct consequence of the translation cycle depicted in Figure 1 or more directly proven by induction on the height of the $\mathsf{SN_{ISCI}}$-proof. □

5. From $\mathsf{SN_{ISCI}}$ to $\mathsf{NS_{ISCI}}$

We introduce the new (family of) nested sequent calculi $\mathsf{NS_{ISCI}}$ as extensions of $\mathsf{NS_{mLJ}}$, the nested sequent calculus given for IL in [15].

Definition 5.1. A *nested sequent* is inductively defined as follows:

1. if $s = \Gamma \vdash \Delta$ is a sequent, where Γ, Δ sets of formulas, then s is a nested sequent;

2. if s is a sequent and $\nu_1, \ldots, \nu_n$ are nested sequents then $s, [\nu_1], \ldots, [\nu_n]$ is a nested sequent.

We use the letters ν and Λ (possibly primed or subscripted) to denote nested sequents and sets of nestings respectively. A nested sequent can more conveniently be written as an expression $\Gamma \vdash \Delta, \Lambda$, where all members of Λ are expressions $[\Gamma' \vdash \Delta', \Lambda']$. As usual, we introduce the standard notion of nested-holed contexts [15].

$$\frac{}{\mathcal{S}\{\Gamma, \mathrm{p} \vdash \Delta, \mathrm{p}, \Lambda\}}\,\mathrm{Nid_p} \qquad \frac{}{\mathcal{S}\{\Gamma, \mathrm{A} \approx \mathrm{B} \vdash \Delta, \mathrm{A} \approx \mathrm{B}, \Lambda\}}\,\mathrm{Nid_\approx}$$

$$\frac{}{\mathcal{S}\{\Gamma, \bot \vdash \Delta, \mathrm{C}, \Lambda\}}\,\mathrm{NL}^{\mathrm{B}}_{\bot} \qquad \frac{}{\mathcal{S}\{\Gamma, \bot \vdash \Delta, \Lambda\}}\,\mathrm{NL}^{\mathrm{K}}_{\bot}$$

$$\frac{\mathcal{S}\{\Gamma \vdash \Delta, \Lambda, [\mathrm{A} \vdash \mathrm{C}]\} \quad \mathcal{S}\{\Gamma \vdash \Delta, \Lambda, [\mathrm{B} \vdash \mathrm{C}]\}}{\mathcal{S}\{\Gamma, \mathrm{A} \vee \mathrm{B} \vdash \Delta, \mathrm{C}, \Lambda\}}\,\mathrm{NL}^{\mathrm{B}}_{\vee}$$

$$\frac{\mathcal{S}\{\Gamma, \mathrm{A} \vdash \Delta, \Lambda\} \quad \mathcal{S}\{\Gamma, \mathrm{B} \vdash \Delta, \Lambda\}}{\mathcal{S}\{\Gamma, \mathrm{A} \vee \mathrm{B} \vdash \Delta, \Lambda\}}\,\mathrm{NL}^{\mathrm{K}}_{\vee} \qquad \frac{\mathcal{S}\{\Gamma \vdash \Delta, \mathrm{A}, \mathrm{B}, \Lambda\}}{\mathcal{S}\{\Gamma \vdash \Delta, \mathrm{A} \vee \mathrm{B}, \Lambda\}}\,\mathrm{NR}_{\vee}$$

$$\frac{\mathcal{S}\{\Gamma, \mathrm{A} \supset \mathrm{B} \vdash \Delta, \mathrm{A}, \Lambda\} \quad \mathcal{S}\{\Gamma, \mathrm{B} \vdash \Delta, \Lambda\}}{\mathcal{S}\{\Gamma, \mathrm{A} \supset \mathrm{B} \vdash \Delta, \Lambda\}}\,\mathrm{NL}_{\supset} \qquad \frac{\mathcal{S}\{\Gamma \vdash \Delta, \Lambda, [\mathrm{A} \vdash \mathrm{B}]\}}{\mathcal{S}\{\Gamma \vdash \Delta, \mathrm{A} \supset \mathrm{B}, \Lambda\}}\,\mathrm{NR}_{\supset}$$

$$\frac{\mathcal{S}\{\Gamma, \mathrm{A}, \mathrm{B} \vdash \Delta, \Lambda\}}{\mathcal{S}\{\Gamma, \mathrm{A} \wedge \mathrm{B} \vdash \Delta, \Lambda\}}\,\mathrm{NL}_{\wedge} \qquad \frac{\mathcal{S}\{\Gamma \vdash \Delta, \mathrm{A}, \Lambda\} \quad \mathcal{S}\{\Gamma \vdash \Delta, \mathrm{B}, \Lambda\}}{\mathcal{S}\{\Gamma \vdash \Delta, \mathrm{A} \wedge \mathrm{B}, \Lambda\}}\,\mathrm{NR}_{\wedge}$$

$$\frac{\mathcal{S}\{\Gamma \vdash \Delta, \Lambda, [\Gamma', \mathrm{A} \vdash \Delta', \Lambda']\}}{\mathcal{S}\{\Gamma, \mathrm{A} \vdash \Delta, \Lambda, [\Gamma' \vdash \Delta', \Lambda']\}}\,\mathrm{lift}$$

$$\frac{\mathcal{S}\{\Gamma, \mathrm{A} \approx \mathrm{A} \vdash \Delta, \Lambda\}}{\mathcal{S}\{\Gamma \vdash \Delta, \Lambda\}}\,\mathrm{NL}^{1}_{\approx} \qquad \frac{\mathcal{S}\{\Gamma, \mathrm{B} \supset \mathrm{A} \vdash \Delta, \Lambda\}}{\mathcal{S}\{\Gamma, \mathrm{A} \approx \mathrm{B} \vdash \Delta, \Lambda\}}\,\mathrm{NaL}^{2}_{\approx}$$

$$\frac{\mathcal{S}\{\Gamma \vdash \Delta, \mathrm{B}, \Lambda\} \quad \mathcal{S}\{\Gamma, \mathrm{A} \vdash \Delta, \Lambda\}}{\mathcal{S}\{\Gamma, \mathrm{A} \approx \mathrm{B} \vdash \Delta, \Lambda\}}\,\mathrm{NL}^{2}_{\approx} \qquad \frac{\mathcal{S}\{\Gamma, \mathrm{A} \otimes \mathrm{C} \approx \mathrm{B} \otimes \mathrm{D} \vdash \Delta, \Lambda\}}{\mathcal{S}\{\Gamma, \mathrm{A} \approx \mathrm{B}, \mathrm{C} \approx \mathrm{D} \vdash \Delta, \Lambda\}}\,\mathrm{NL}^{3}_{\approx}$$

$$\frac{\mathcal{S}\{\Gamma, \mathrm{A} \otimes \mathrm{A} \approx \mathrm{B} \otimes \mathrm{B} \vdash \Delta, \Lambda\}}{\mathcal{S}\{\Gamma, \mathrm{A} \approx \mathrm{B} \vdash \Delta, \Lambda\}}\,\mathrm{NL}^{3*}_{\approx} \qquad \frac{\mathcal{S}\{\Gamma, \mathrm{D}^{\mathrm{A}}_{\mathrm{B}} \vdash \Delta, \Lambda\}}{\mathcal{S}\{\Gamma, \mathrm{A} \approx \mathrm{B}, \mathrm{D} \vdash \Delta, \Lambda\}}\,\mathrm{NL}^{\mathrm{r}}_{\approx}$$

Figure 4: Rules for the $\mathsf{NS_{ISCI}}$ Family of Nested Sequent Calculi.

DEFINITION 5.2. A *nested-holed context* is a nested sequent that contains a hole of the form $\{\}$ in place of nestings. Such a context is denoted by $\mathcal{S}\{\}$. Given a nested-holed context and a nested sequent ν, $\mathcal{S}\{\nu\}$ denotes the nested sequent where the hole $\{\}$ has been replaced with $[\nu]$, assuming that it is removed if ν is empty and if $\mathcal{S}$ is empty then $\mathcal{S}\{\nu\} = \nu$.

The translation is similar in principle to the one given for IL in [5] for proofs with prefixes and more thoroughly studied in [11] in the context of labeled systems with relational atoms. Given a fixed root labeled tree sequent $\mathcal{R}, \Gamma \vdash \Delta$ in $\mathsf{SN}_{\mathsf{ISCI}}$, the main idea is to use the tree structure described by the relational atoms in $\mathcal{R}$ to determine the depth of the nested sequents: if $\mathrm{u} \sqsubset \mathrm{v} \in \mathcal{R}$ then all of the formulas labeled with u should be nested one level deeper than the ones labeled with v.

DEFINITION 5.3. Let $\tau = \mathcal{R}, \Gamma \vdash \Delta$ be a labeled tree sequent with root u. Let $\mathrm{w}_1, \ldots, \mathrm{w}_\mathrm{n}$ be all of the labels such that $\mathrm{u} \sqsubset \mathrm{w}_\mathrm{i} \in \mathcal{R}$. S_u is the restriction of a set S of labeled formulas to the formulas labeled with u, i.e. $\mathrm{S}_\mathrm{u} = \{\, A \;/\; \mathrm{A} : \mathrm{u} \in \mathrm{S} \,\}$. $\mathsf{N}(\tau) = \mathsf{N}_\mathrm{u}(\tau)$ is recursively defined on the tree structure of $\mathcal{R}$ as follows: $\mathsf{N}_\mathrm{v}(\tau) = \Gamma_\mathrm{v} \vdash \Delta_\mathrm{v}, \Lambda_\mathrm{v}$ with $\Lambda_\mathrm{v} = [\mathsf{N}_{\mathrm{w}_1}(\tau)], \ldots, [\mathsf{N}_{\mathrm{w}_\mathrm{n}}(\tau)]$.

The rules of the nested sequent calculi $\mathsf{NS}_{\mathsf{ISCI}}$ are given in Figure 4. $\mathsf{NS}^1_{\mathsf{ISCI}}$ and $\mathsf{NS}^2_{\mathsf{ISCI}}$ are defined as having the sets $\{\, \mathrm{NL}^1_{\approx}, \mathrm{NL}^2_{\approx}, \mathrm{NL}^3_{\approx}, \mathrm{NL}^{3*}_{\approx} \,\}$ and $\{\, \mathrm{NL}^1_{\approx}, \mathrm{NaL}^2_{\approx}, \mathrm{NL}^\mathrm{r}_{\approx} \,\}$ for sentential identity rules respectively. For both sets we have two K (Kripke) and B (Beth) variants depending on the rules for disjunction and falsity. Following the standard terminology for nested systems, we distinguish *creation rules* that introduce new nestings in their premises from *upgrade rules* that only move information between nestings without creating new ones. For example, $\mathrm{NR}_{\supset}$ is a creation rule, while lift is an upgrade rule. One noticeable difference between the Beth and the Kripke variants of the rules is that the former gives rise to a nested-like creation rule for left disjunction, while the latter only gives rise to a sequent-like rule. We now state the following translation result.

THEOREM 5.4. *Any standard* $\mathsf{SN}_{\mathsf{ISCI}}$*-proof can be translated into an* $\mathsf{NS}_{\mathsf{ISCI}}$*-proof.*

Proof: By induction on the height of the $\mathsf{SN}_{\mathsf{ISCI}}$-proof Π (see Appendix A). □

Example 5.5.

The translation of the proof given in Example 4.5 after the erasure of the Beth labels 5 and 6 (to get an $\mathsf{SN}^{\mathsf{K}}_{\mathsf{ISCI}}$-proof) is given below:

$$\Pi_3^1 \left\{ \dfrac{\dfrac{}{\vdash [\vdash [\mathrm{p} \supset \mathrm{r}, \mathrm{q} \supset \mathrm{r}, \mathrm{p} \vdash \mathrm{r}, \mathrm{p}]]}\,\mathrm{Nid_p} \qquad \dfrac{}{\vdash [\vdash [\mathrm{p} \supset \mathrm{r}, \mathrm{q} \supset \mathrm{r}, \mathrm{p}, \mathrm{r} \vdash \mathrm{r}]]}\,\mathrm{Nid_p}}{\vdash [\vdash [\mathrm{p} \supset \mathrm{r}, \mathrm{q} \supset \mathrm{r}, \mathrm{p} \vdash \mathrm{r}]]}\,\mathrm{NL}_{\supset} \right.$$

$$\Pi_3^2 \left\{ \dfrac{\dfrac{}{\vdash [\vdash [\mathrm{p} \supset \mathrm{r}, \mathrm{q} \supset \mathrm{r}, \mathrm{q} \vdash \mathrm{r}, \mathrm{q}]]}\,\mathrm{Nid_p} \qquad \dfrac{}{\vdash [\vdash [\mathrm{p} \supset \mathrm{r}, \mathrm{q} \supset \mathrm{r}, \mathrm{q}, \mathrm{r} \vdash \mathrm{r}]]}\,\mathrm{Nid_p}}{\vdash [\vdash [\mathrm{p} \supset \mathrm{r}, \mathrm{q} \supset \mathrm{r}, \mathrm{q} \vdash \mathrm{r}]]}\,\mathrm{NL}_{\supset} \right.$$

$$\Pi_3 \left\{ \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\Pi_3^1 \\ \vdash [\vdash [\mathrm{p} \supset \mathrm{r}, \mathrm{q} \supset \mathrm{r}, \mathrm{p} \vdash \mathrm{r}]]\end{array} \qquad \begin{array}{c}\Pi_3^2 \\ \vdash [\vdash [\mathrm{p} \supset \mathrm{r}, \mathrm{q} \supset \mathrm{r}, \mathrm{q} \vdash \mathrm{r}]]\end{array}}{\vdash [\vdash [\mathrm{p} \supset \mathrm{r}, \mathrm{q} \supset \mathrm{r}, \mathrm{p} \vee \mathrm{q} \vdash \mathrm{r}]]}\,\mathrm{NL}^{\mathrm{K}}_{\vee}}{\vdash [\mathrm{p} \supset \mathrm{r} \vdash [\mathrm{q} \supset \mathrm{r}, \mathrm{p} \vee \mathrm{q} \vdash \mathrm{r}]]}\,\mathrm{lift}}{\vdash [\mathrm{p} \supset \mathrm{r}, \mathrm{q} \supset \mathrm{r} \vdash [\mathrm{p} \vee \mathrm{q} \vdash \mathrm{r}]]}\,\mathrm{lift}}{\vdash [\mathrm{p} \supset \mathrm{r}, \mathrm{q} \supset \mathrm{r} \vdash (\mathrm{p} \vee \mathrm{q}) \supset \mathrm{r}]}\,\mathrm{NR}_{\supset}}{\vdash [(\mathrm{p} \supset \mathrm{r}) \wedge (\mathrm{q} \supset \mathrm{r}) \vdash (\mathrm{p} \vee \mathrm{q}) \supset \mathrm{r}]}\,\mathrm{NL}_{\wedge}}{\vdash ((\mathrm{p} \supset \mathrm{r}) \wedge (\mathrm{q} \supset \mathrm{r})) \supset ((\mathrm{p} \vee \mathrm{q}) \supset \mathrm{r})}\,\mathrm{NR}_{\supset} \right.$$

6. The Label-Free Sequent Calculi $\mathsf{G3}_{\mathsf{ISCI}}$

In this section we start from the single-succedent label-free sequent calculus $\mathsf{sG3}_{\mathsf{ISCI}}$ that deals with multisets and for which the weakening, contraction and cut rules are proved admissible [4]. We first extend it to a multi-succedent calculus. Then we devise new label-free disjunction and falsity rules that are sound and complete w.r.t. the TB semantics of ISCI to achieve Beth variants of the calculi. The $\mathsf{G3}_{\mathsf{ISCI}}$ proof rules are given in Figure 5, with sequents $\Gamma \vdash \Delta$, where Γ and Δ are sets of formulas. $\mathsf{G3}^1_{\mathsf{ISCI}}$ and $\mathsf{G3}^2_{\mathsf{ISCI}}$ are respectively defined with the sets $\{\, \mathrm{L}^1_{\approx}, \mathrm{L}^2_{\approx}, \mathrm{L}^3_{\approx}, \mathrm{L}^{3*}_{\approx} \,\}$ and $\{\, \mathrm{L}^1_{\approx}, \mathrm{aL}^2_{\approx}, \mathrm{L}^{\mathrm{r}}_{\approx} \,\}$) for sentential identity rules. For both sets of identity rules, we have a K version of the calculus if we consider the rules $L_\perp^K$ and $L_\vee^K$ respectively for falsity and disjunction. We also have a B (Beth) version of the calculus if we consider the rules $L_\perp^B$ and $L_\vee^B$ respectively for falsity and disjunction.

$$\frac{}{\Gamma, p \vdash \Delta, p}\,\mathrm{id_p} \qquad \frac{}{\Gamma, A \approx B \vdash \Delta, A \approx B}\,\mathrm{id}_\approx$$

$$\frac{}{\Gamma, \perp \vdash \Delta, C}\,L_\perp^B \qquad \frac{}{\Gamma, \perp \vdash \Delta}\,L_\perp^K$$

$$\frac{\Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma, A \vee B \vdash \Delta, C}\,L_\vee^B \qquad \frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}\,L_\vee^K \qquad \frac{\Gamma \vdash \Delta, A_1, A_2}{\Gamma \vdash \Delta, A_1 \vee A_2}\,R_\vee$$

$$\frac{\Gamma, A \supset B \vdash \Delta, A \quad \Gamma, B \vdash \Delta}{\Gamma, A \supset B \vdash \Delta}\,L_\supset \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash \Delta, A \supset B}\,R_\supset$$

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\,L_\wedge \qquad \frac{\Gamma \vdash \Delta, A \quad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B}\,R_\wedge$$

$$\frac{\Gamma, A \approx A \vdash \Delta}{\Gamma \vdash \Delta}\,L_\approx^1 \qquad \frac{\Gamma, A \approx B, B \supset A \vdash \Delta}{\Gamma, A \approx B \vdash \Delta}\,aL_\approx^2$$

$$\frac{\Gamma, A \approx B \vdash \Delta, B \quad \Gamma, A \approx B, A \vdash \Delta}{\Gamma, A \approx B \vdash \Delta}\,L_\approx^2 \qquad \frac{\Gamma, A \approx B, C \approx D, (A \otimes C) \approx (B \otimes D) \vdash \Delta}{\Gamma, A \approx B, C \approx D \vdash \Delta}\,L_\approx^3$$

$$\frac{\Gamma, A \approx B, (A \otimes A) \approx (B \otimes B) \vdash \Delta}{\Gamma, A \approx B \vdash \Delta}\,L_\approx^{3*} \qquad \frac{\Gamma, A \approx B, D, D_B^A \vdash \Delta}{\Gamma, A \approx B, D \vdash \Delta}\,L_\approx^r$$

Figure 5: Rules for the $\mathsf{G3}_{\mathsf{ISCI}}$ Family of Sequent Calculi.

We note that in the rule $L_\perp^B$, the presence of C on the right-hand side prevents the succedent from being empty. In the rule $L_\vee^B$, the context Δ gets discarded, as in $R_\supset$, from the conclusion to the premises. Moreover one can observe also the rule $L_\approx^r$ is not explicit in [4] but can be introduced in some variants of the calculus. Here we propose to consider a similar rule extended to multisets.

Example 6.1. A $\mathsf{G3}^1_{\mathsf{ISCI}}$-proof that sentential identity is commutative is given below:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{p \approx q, q \approx q, (q \approx p) \approx (q \approx q) \vdash q \approx p, q \approx q}\,\mathrm{id}_{\approx} \qquad \dfrac{}{p \approx q, q \approx q, (q \approx p) \approx (q \approx q), q \approx p \vdash q \approx p}\,\mathrm{id}_{\approx}}{p \approx q, q \approx q, (q \approx p) \approx (q \approx q) \vdash q \approx p}\,\mathrm{L}^2_{\approx}}{p \approx q, q \approx q \vdash q \approx p}\,\mathrm{L}^3_{\approx}}{p \approx q \vdash q \approx p}\,\mathrm{L}^1_{\approx}$$

Theorem 6.2. $\mathsf{G3}^1_{\mathsf{ISCI}}$*-proofs translate into* $\mathsf{G3}^2_{\mathsf{ISCI}}$*-proofs.*

Proof: We show that the sentential identity rules of $\mathsf{G3}^2_{\mathsf{ISCI}}$ can simulate those of $\mathsf{G3}^1_{\mathsf{ISCI}}$.

Case $\mathrm{L}^2_{\approx}$:

$$\dfrac{\dfrac{\Gamma, A \approx B, B \supset A \vdash \Delta, B \qquad \Gamma, A \approx B, B \supset A, A \vdash \Delta}{\Gamma, A \approx B, B \supset A \vdash \Delta}\,\mathrm{L}_{\supset}}{\Gamma, A \approx B \vdash \Delta}\,\mathrm{aL}^2_{\approx}$$

Case $\mathrm{L}^3_{\approx}$:

$$\dfrac{\dfrac{\dfrac{\Gamma, A \approx B, C \approx D, (A \otimes C) \approx (A \otimes C), (A \otimes C) \approx (B \otimes C), (A \otimes C) \approx (B \otimes D) \vdash \Delta}{\Gamma, A \approx B, C \approx D, (A \otimes C) \approx (A \otimes C), (A \otimes C) \approx (B \otimes C) \vdash \Delta}\,\mathrm{L}^{\mathrm{r}}_{\approx}}{\Gamma, A \approx B, C \approx D, (A \otimes C) \approx (A \otimes C) \vdash \Delta}\,\mathrm{L}^{\mathrm{r}}_{\approx}}{\Gamma, A \approx B, C \approx D \vdash \Delta}\,\mathrm{L}^1_{\approx}$$

Case $\mathrm{L}^{3*}_{\approx}$:

$$\dfrac{\dfrac{\Gamma, A \approx B, (A \otimes A) \approx (A \otimes A), (A \otimes A) \approx (B \otimes B) \vdash \Delta}{\Gamma, A \approx B, (A \otimes A) \approx (A \otimes A) \vdash \Delta}\,\mathrm{L}^{\mathrm{r}}_{\approx}}{\Gamma, A \approx B \vdash \Delta}\,\mathrm{L}^1_{\approx}$$

□

Theorem 6.3. $\mathsf{G3}^1_{\mathsf{ISCI}}$ *and* $\mathsf{G3}^2_{\mathsf{ISCI}}$ *are cut-free complete.*

Proof: The single-succedent $\mathsf{sG3}^1_{\mathsf{ISCI}}$ is proven cut-free complete in [4], which implies the result for (multi-succedent) $\mathsf{G3}^1_{\mathsf{ISCI}}$. As the translation of Theorem 6.2 does not require the cut rule we conclude that $\mathsf{G3}^2_{\mathsf{ISCI}}$ is cut-free complete. □

7. From $\mathsf{NS}_{\mathsf{ISCI}}$ to $\mathsf{G3}_{\mathsf{ISCI}}$

An important result proven in [15] is that any basic nested calculus can be sequentialized: if NS is a basic nested system which sequentialises to a sequent system SC, then the sequent $\Gamma \vdash \Delta$ is provable in NS iff it is provable in SC. The result is too technical to be described here in full details but it relies on the fact that any proof in a basic nested system can be turned into what we will call here a *standard proof*.

DEFINITION 7.1. The *depth of an application of a rule* in a derivation is the depth of its principal formula. A *sequential* $\mathcal{B}^s$ (nested block $\mathcal{B}^n$) in a proof Π is a maximal bottom-up sequence of applications of sequent-like (nested-like) rules in a branch of Π having the same depth d. The depth of such a sequential (nested) block is defined by $\mathsf{dp}(\mathcal{B}^s) = d$ ($\mathsf{dp}(\mathcal{B}^n) = d$).

DEFINITION 7.2. A proof Π in a basic nested system NS is *standard* iff:

1. axioms are applied eagerly;
2. Π is *end-active*, i.e. all rules are applied only in the deepest nestings of a sequent;
3. if a sequential block $\mathcal{B}^s$ immediately follows a nested block $\mathcal{B}^n$ then $\mathsf{dp}(\mathcal{B}^s) = \mathsf{dp}(\mathcal{B}^n) + 1$;
4. if a nested block $\mathcal{B}^n$ immediately follows a sequential block $\mathcal{B}^s$ then $\mathsf{dp}(\mathcal{B}^n) = \mathsf{dp}(\mathcal{B}^s)$;
5. nested blocks have exactly one occurrence of a creation rule.

In order to be able to translate $\mathsf{NS}_{\mathsf{ISCI}}$-proofs to $\mathsf{G3}_{\mathsf{ISCI}}$-proofs, all we have to do is to show that $\mathsf{NS}_{\mathsf{ISCI}}$ calculi are basic nested systems in the sense of [15].

DEFINITION 7.3. A nested system NS is *basic* if it satisfies the following conditions:

1. nested-like rules must have exactly one nesting in the premises or conclusion;

2. nested-like rules always move information deeper inside nestings;
3. upgrade rules must have exactly one principal and auxiliary formula;
4. upgrade rules move only one piece of information at a time.

THEOREM 7.4. *Any standard* $\mathsf{NS}_{\mathsf{ISCI}}$*-proof can be translated into a* $\mathsf{G3}_{\mathsf{ISCI}}$*-proof.*

PROOF: Since the Kripke nested calculi $\mathsf{NS}^{\mathsf{K}}_{\mathsf{ISCI}}$ only add sequent-like rules to the basic nested system $\mathsf{NS}_{\mathsf{mLJ}}$ [15], they all qualify as basic nested systems. For the Beth nested calculi $\mathsf{NS}^{\mathsf{B}}_{\mathsf{ISCI}}$, we observe that the new creation rule $\mathrm{NL}^{\mathsf{B}}_{\vee}$ satisfies the conditions of Definition 7.3. Therefore, the result follows as an immediate consequence of Theorem 29 in [15]. □

From [15] we know that nested blocks in a standard proof Π of a formula C should be thought of as macros turning nested sequents into sequents, which actually means that we only need to replay in $\mathsf{G3}_{\mathsf{ISCI}}$, bottom-up from the end sequent ⊢ C, the same sequent-like rule application order encoded in Π, to obtain the corresponding $\mathsf{G3}_{\mathsf{ISCI}}$-proof of C.

Example 7.5. Since the $\mathsf{NS}_{\mathsf{ISCI}}$-proof given in Example 5.5 is standard, one can translate it into a $\mathsf{G3}_{\mathsf{ISCI}}$-proof using the same rule application order as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p \supset r, q \supset r, p \vdash r, p}\,\mathrm{id_p} \quad \dfrac{}{q \supset r, p, r \vdash r}\,\mathrm{id_p}}{p \supset r, q \supset r, p \vdash r}\,\mathrm{L}_{\supset} \quad \dfrac{\dfrac{}{p \supset r, q \supset r, q \vdash r, q}\,\mathrm{id_p} \quad \dfrac{}{p \supset r, q, r \vdash r}\,\mathrm{id_p}}{p \supset r, q \supset r, q \vdash r}\,\mathrm{L}_{\supset}}{p \supset r, q \supset r, p \vee q \vdash r}\,\mathrm{L}^{\mathsf{B}}_{\vee}}{p \supset r, q \supset r \vdash (p \vee q) \supset r}\,\mathrm{R}_{\supset}}{(p \supset r) \wedge (q \supset r) \vdash (p \vee q) \supset r}\,\mathrm{L}_{\wedge}}{\vdash ((p \supset r) \wedge (q \supset r)) \supset ((p \vee q) \supset r)}\,\mathrm{R}_{\supset}$$

THEOREM 7.6. $\mathsf{L3}_{\mathsf{ISCI}}$*-proofs can be translated into* $\mathsf{G3}_{\mathsf{ISCI}}$*-proofs*

PROOF: The result follows from Theorems 4.4, 5.4 and 7.4. □

8. From $\mathsf{G3}_{\mathsf{ISCI}}$ to $\mathsf{L3}_{\mathsf{ISCI}}$

In this section we consider proof translations from the label-free $\mathsf{G3}_{\mathsf{ISCI}}$ sequent calculi to the labeled sequent calculi $\mathsf{L3}_{\mathsf{ISCI}}$. The main problem is to turn label-free sequents into labeled sequents. Let $\Gamma = C_1, \ldots, C_m$ be a sequence of m formulas. A label vector $\vec{c}$ is a non-empty sequence of labels $c_1, \ldots, c_n$ such that for any i such that $1 \leqslant i \leqslant n$ we have $c_i \in \mathbf{L}^1$ and for any i, j such that $1 \leqslant i, j \leqslant n$, if $i \neq j$ then $c_i \neq c_j$.

In other words, a label vector is a non-empty finite sequence of pairwise distinct singleton labels. We define $\check{c}$ as $c_1 \cup \ldots \cup c_n$ In particular, given a strictly positive integer n, $\vec{n}$ is defined as the sequence of singletons $\{1\}, \{2\}, \ldots, \{n\}$ and $\check{n}$ is the label $\{1, 2, \ldots, n\}$. As a special case we set $\vec{0} = \emptyset$. Finally, we define $\Gamma : \vec{c}$ as $C_1 : d_1, \ldots, C_n : d_n$, where $d_i = c_i$ if $i \leqslant n$ and $d_i = c_n$ otherwise.

Definition 8.1. Let $\Gamma \vdash \Delta$ be a $\mathsf{G3}_{\mathsf{ISCI}}$ label-free sequent. Given a label vector $\vec{c}$ for Γ, the translation L of $\Gamma \vdash \Delta$ under $\vec{c}$, written $\mathsf{L}(\Gamma \vdash \Delta, \vec{c})$ is defined as the $\mathsf{L3}_{\mathsf{ISCI}}$ labeled sequent $\Gamma : \vec{c} \vdash \Delta : \check{c}$. In particular, $\mathsf{L}(\Gamma \vdash \Delta) = \mathsf{L}(\Gamma \vdash \Delta, \vec{n})$, where $n = |\Gamma|$.

Theorem 8.2. *$\mathsf{G3}_{\mathsf{ISCI}}$-proofs translate into $\mathsf{L3}_{\mathsf{ISCI}}$-proofs.*

Proof: By induction on the height of $\mathsf{G3}_{\mathsf{ISCI}}$-proofs (see Appendices C and C.1). □

Example 8.3. Let us translate the $\mathsf{G3}^1_{\mathsf{ISCI}}$-proof given in Example 6.1 into a right maximal $\mathsf{L3}^1_{\mathsf{ISCI}}$-proof. The $\mathsf{G3}^1_{\mathsf{ISCI}}$-proof starts with two axioms $\mathrm{id}_\approx$. Applying Definition 8.1 and the translation pattern for $\mathrm{id}_\approx$, we get the following two $\mathsf{L3}^1_{\mathsf{ISCI}}$-proofs:

$$\Pi^0_1 \left\{ \frac{}{p \approx q : 1, q \approx q : 2, (q \approx p) \approx (q \approx q) : 3 \vdash q \approx q : 123} \mathrm{id}_\approx (2 \subseteq 123) \right.$$

$$\Pi^0_2 \left\{ \frac{}{p \approx q : 1, q \approx q : 2, (q \approx p) \approx (q \approx q) : 3, q \approx p : 4 \vdash q \approx p : 1234} \mathrm{id}_\approx (4 \subseteq 1234) \right.$$

The next rule is $\mathrm{L}^2_\approx$ and the translation of its conclusion yields:

$$p \approx q : 1, q \approx q : 2, (q \approx p) \approx (q \approx q) : 3 \vdash q \approx p : 123$$

The restriction $xy \in \lceil\Delta\rceil$ then leads to the replacement of 4 with 123 in Π_2^0 since the active formula of $L_\approx^2$ is $q \approx p$, which is labeled with 4 in Π_2^0, while the succedent formula in the conclusion of $L_\approx^2$ is labeled with 123:

$$\Pi_2^1 \left\{ \frac{}{p \approx q : 1, q \approx q : 2, (q \approx p) \approx (q \approx q) : 3, q \approx p : 123 \vdash q \approx p : 123} \mathrm{id}_\approx (123 \subseteq 123) \right.$$

We combine Π_1^0 and Π_2^1 using $L_\approx^2$:

$$\Pi^1 \left\{ \frac{\Pi_1^0 \qquad \Pi_2^1}{p \approx q : 1, q \approx q : 2, (q \approx p) \approx (q \approx q) : 3 \vdash q \approx p : 123} L_\approx^2 \right.$$

The next rule is $L_\approx^3$. The conclusion translation is $p \approx q : 1, q \approx q : 2 \vdash q \approx p : 12$ Before translating $L_\approx^3$ in $\mathsf{G3}^1_{\mathsf{ISCI}}$ to $L_\approx^3$ in $\mathsf{L3}^1_{\mathsf{ISCI}}$, we need to replace 3 with 12 in Π^1 since the active formula of $L_\approx^3$ is $(q \approx p) \approx (q \approx q) : 3$.

$$\Pi^2 \left\{ \frac{\dfrac{\Pi^1[12/3]}{p \approx q : 1, q \approx q : 2, (q \approx p) \approx (q \approx q) : 12 \vdash q \approx p : 12}}{p \approx q : 1, q \approx q : 2 \vdash q \approx p : 12} L_\approx^3 \right.$$

The next rule is $L_\approx^1$. The conclusion translation yields: $p \approx q : 1 \vdash q \approx p : 1$ The restriction $x \in \lceil\Delta\rceil$ for $L_\approx^1$ leads to the replacement of 2 with 1 in Π^2:

$$\Pi^3 \left\{ \frac{\dfrac{\Pi^2[1/2]}{p \approx q : 1, q \approx q : 1 \vdash q \approx p : 1}}{p \approx q : 1 \vdash q \approx p : 1} L_\approx^1 \right.$$

The final result is given below:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\begin{array}{l}p \approx q : 1, q \approx q : 1,\\ (q \approx p) \approx (q \approx q) : 1\end{array} \vdash q \approx q : 1} \mathrm{id}_\approx \qquad \dfrac{}{\begin{array}{l}p \approx q : 1, q \approx q : 1, q \approx p : 1\\ (q \approx p) \approx (q \approx q) : 1\end{array} \vdash q \approx p : 1} \mathrm{id}_\approx}{p \approx q : 1, q \approx q : 1, (q \approx p) \approx (q \approx q) : 1 \vdash q \approx p : 1} L_\approx^2}{p \approx q : 1, q \approx q : 1 \vdash q \approx p : 1} L_\approx^3}{p \approx q : 1 \vdash q \approx p : 1} L_\approx^1$$

THEOREM 8.4. $\mathsf{G3}^{\mathsf{K}}_{\mathsf{ISCI}}$ *(*$\mathsf{G3}^{\mathsf{B}}_{\mathsf{ISCI}}$*) is sound and complete w.r.t. Kripke (*TB*) semantics.*

PROOF: For the soundness, we have $\vdash_{\mathsf{G3}^{\mathsf{K}}_{\mathsf{ISCI}}} A \Rightarrow \models_{\mathsf{K}} A$, that is proven in Appendix D and $\vdash_{\mathsf{G3}^{\mathsf{B}}_{\mathsf{ISCI}}} A \Rightarrow \models_{\mathsf{B}} A$ proven as follows: $\vdash_{\mathsf{G3}^{\mathsf{B}}_{\mathsf{ISCI}}} A$ implies $\vdash_{\mathsf{L3}^{\mathsf{B}}_{\mathsf{ISCI}}}$

A by Theorem 8.2 and the implication $\vdash_{\mathsf{L3}^{\mathsf{B}}_{\mathsf{ISCI}}} A \Rightarrow \models_{\mathsf{B}} A$ is proven in [6]. The cut-free completeness of the single-succedent calculus $\mathsf{sG3}^{\mathsf{K}}_{\mathsf{ISCI}}$ w.r.t. the Kripke semantics is proven in [4]. Since $\vdash_{\mathsf{sG3}^{\mathsf{K}}_{\mathsf{ISCI}}} A$ implies $\vdash_{\mathsf{G3}^{\mathsf{K}}_{\mathsf{ISCI}}} A$, we get $\models_{\mathsf{K}} A \Rightarrow \vdash_{\mathsf{G3}^{\mathsf{K}}_{\mathsf{ISCI}}} A$. Now since $\models_{\mathsf{B}} A \Rightarrow \vdash_{\mathsf{L3}^{\mathsf{B}}_{\mathsf{ISCI}}} A$ is proven in [6], it follows from the translation cycle in Figure 1 that $\vdash_{\mathsf{L3}^{\mathsf{B}}_{\mathsf{ISCI}}} A$ implies $\vdash_{\mathsf{G3}^{\mathsf{B}}_{\mathsf{ISCI}}} A$, hence $\models_{\mathsf{B}} A \Rightarrow \vdash_{\mathsf{G3}^{\mathsf{B}}_{\mathsf{ISCI}}} A$. □

Corollary 8.5. All of the $\mathsf{L3}_{\mathsf{ISCI}}$ calculi are cut-free complete.

Proof: Since the translation in Definition 8.1 does not require the cut rule, the cut-free completeness of $\mathsf{G3}^1_{\mathsf{ISCI}}$ and $\mathsf{G3}^2_{\mathsf{ISCI}}$ (see Theorem 6.3) entails the cut-free completeness of both $\mathsf{L3}^1_{\mathsf{ISCI}}$ and $\mathsf{L3}^2_{\mathsf{ISCI}}$ by Theorem 8.2. □

9. Conclusion and Perspectives

In this paper we studied how families of labeled and label-free sequent calculi, that capture distinct semantics of the logic ISCI, relate to each other. We considered a syntactical approach based on proof translations between calculi rather than working directly within the semantics. Although the long translation cycle depicted in Figure 1 might at first sight appear as an unnecessary hassle, we now point out some of the merits of this approach.

Firstly, while most labeled calculi indeed reflect the main properties of a given semantics in their labeling algebras, they do not necessarily fully and faithfully capture all of such properties. For example, the family $\mathsf{L3}_{\mathsf{ISCI}}$ of labeled calculi was carefully crafted so that the Beth variants would differ as little as possible from the Kripke variants while still enabling one of the most important and interesting feature of the TB semantics: remaining sound when the eigenvariable conditions are dropped to allow the reuse of the singletons introduced by a previous instance of a $\mathrm{L}^{\mathsf{B}}_{\vee}$ or $\mathrm{R}_{\supset}$ rule with the same principal formula. The soundness of the $\mathsf{L3}^{\mathsf{B}}_{\mathsf{ISCI}}$ family without the eigenvariable conditions comes from the regularity property of the TB semantics (condition of Definition 2.3) although the minimality of (the world realizing) a reused singleton (in a liberalized soundness proof as the one given in [6]) cannot be syntactically witnessed inside the calculus (as comparing labels via set inclusion alone is too weak). Let us recall that the

Kripke variants $\mathsf{L3}^{\mathsf{K}}_{\mathsf{ISCI}}$ are unsound without the eigenvariable conditions. The TB semantics of ISCI and its associated Beth labeled calculi therefore play a central role in the decidability arguments given in [6].

Secondly, we argue that proof translations can help us gain a better understanding of how the semantics reveals itself in a calculus. Indeed, the proof translation approach allows us to depart from the traditional way of devising labeled proof systems, which consists in turning the forcing clauses of the underlying semantics into logical proof rules while capturing its properties inside a labeling algebra. For instance, to devise a Beth $\mathsf{SN}_{\mathsf{ISCI}}$ labeled system, we would have introduced a syntactic operator "|" to reflect the lattice meet of the TB semantics, which would have led to extended relational atoms of the form $(\mathrm{v} \mid \mathrm{w})\,\mathrm{R}\,\mathrm{u}$. The rule for left and right disjunction would have respectively taken the following forms:

$$\frac{\mathcal{R}, (\mathrm{v} \mid \mathrm{w})\ \mathrm{R}\ \mathrm{u}, \Gamma, \mathrm{A} : \mathrm{v}, \mathrm{B} : \mathrm{w} \vdash \Delta}{\mathcal{R}, \Gamma, \mathrm{A} \vee \mathrm{B} : \mathrm{u} \vdash \Delta}\ \mathrm{v}, \mathrm{w}\ \text{fresh}$$

$$\frac{\mathcal{R}, (\mathrm{v} \mid \mathrm{w})\ \mathrm{R}\ \mathrm{u}, \Gamma \vdash \Delta, \mathrm{A} : \mathrm{v} \qquad \mathcal{R}, (\mathrm{v} \mid \mathrm{w})\ \mathrm{R}\ \mathrm{u}, \Gamma \vdash \Delta, \mathrm{B} : \mathrm{w}}{\mathcal{R}, (\mathrm{v} \mid \mathrm{w})\ \mathrm{R}\ \mathrm{u}, \Gamma \vdash \Delta, \mathrm{A} \vee \mathrm{B} : \mathrm{u}}$$

Proceeding by translation from $\mathsf{L3}^{\mathsf{B}}_{\mathsf{ISCI}}$ enabled a simpler and more concise account of the TB semantics in the $\mathsf{SN}^{\mathsf{B}}_{\mathsf{ISCI}}$ calculi since they do not require any of the extra machinery described previously (no special meet operator "|", no extended relational atoms). Guessing such calculi directly from the semantics would not have been that obvious. Guessing the rules for Beth disjunction and falsity in $\mathsf{NS}^{\mathsf{B}}_{\mathsf{ISCI}}$ and $\mathsf{G3}^{\mathsf{B}}_{\mathsf{ISCI}}$ would have been even more difficult. The shape of the rule $\mathrm{NL}^{\mathrm{B}}_{\vee}$ as depicted in Figure 4 and the fact that it should behave as a creation rule in $\mathsf{NS}_{\mathsf{ISCI}}$ just like $\mathrm{NR}_{\supset}$ actually came from the translation described in Section 5 after noticing the similarity of $\mathrm{L}^{\mathrm{B}}_{\vee}$ with $\mathrm{R}_{\supset}$ in $\mathsf{SN}^{\mathsf{B}}_{\mathsf{ISCI}}$. In turn, the similarity of $\mathrm{NL}^{\mathrm{B}}_{\vee}$ with $\mathrm{NR}_{\supset}$ gave rise to the idea of a discarding context Δ from the conclusion to the premises for the Beth disjunction rule $\mathrm{L}^{\mathrm{B}}_{\vee}$ in $\mathsf{G3}^{\mathsf{B}}_{\mathsf{ISCI}}$.

Thirdly, the translation approach makes it easier to transpose results from one proof system to another one. For example, the decidability result proven in [18] relies on the fact that the proof search space for a formula

A in (single-succedent) $\mathsf{sG3}_{\mathsf{ISCI}}$ can be restricted to a bounded set of formulas generated from A. Such a result can directly be exported to $\mathsf{L3}_{\mathsf{ISCI}}$ via the translation of Theorem 8.2. We argue that it is not the technical complexity (of the soundness proof) of a translation that matters, but what we can learn from it and what we can do with it. Our translations are all proven sound by standard inductive proofs that can convincingly be checked or rebuilt by a human reader. For us, this is a feature. Moreover our translations provide cut-free completeness for all the calculi in the paper, knowing that cut-free completeness via cut-elimination involves much harder proofs, with a significantly higher number of cases, thus making them more error-prone.

Lastly, another benefit of proof translations is to give people proofs in the formalism they understand better. For example, Theorem 3.4 is a key contribution for labeled systems (but can be transposed to all of our calculi). Since Beth proofs remain sound without the eigenvariable, Beth calculi are better suited for giving decidability arguments as explained previously. With Kripke proofs, one would have to devise additional mechanisms to mitigate the introduction of fresh labels. On the other hand, Kripke proofs are more easily understood because Kripke disjunction seem more "natural" to grasp for most people. Well, use a Beth-like calculus under the hood, then use our results to provide a Kripke proof.

In future works we expect to find direct translations from $\mathsf{L3}_{\mathsf{ISCI}}$ and $\mathsf{SN}_{\mathsf{ISCI}}$ to $\mathsf{G3}_{\mathsf{ISCI}}$, without the intermediate step via proofs in $\mathsf{NS}_{\mathsf{ISCI}}$. We conjecture that the heavy machinery of end-active nested proofs described in Section 6 can be mimicked in $\mathsf{L3}_{\mathsf{ISCI}}$ and $\mathsf{SN}_{\mathsf{ISCI}}$ by following a rule application strategy that always expands formulas with labels that are maximal. Moreover we want to tackle the challenging problem of translating multi-succedent $\mathsf{G3}_{\mathsf{ISCI}}$-proofs into single-succedent $\mathsf{sG3}_{\mathsf{ISCI}}$-proofs. Achieving direct translations from multi-succedent to single-succedent calculi is a notoriously difficult task. For propositional IL, although there are indirect translations requiring intermediate steps (one in [16] involving nested sequents and a detour through bi-intuitionistic logic), the only actual direct translation we are aware of is the one by proof reconstruction from the connection method [17]. Unfortunately, such a method fails in our case as it very strongly depends on the subformula property to calculate atomic paths, a property that current calculi for ISCI fail to enjoy.

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A. Appendix: From $\mathsf{SN}_{\mathsf{ISCI}}$ to $\mathsf{NS}_{\mathsf{ISCI}}$

THEOREM A.1. *Any standard* $\mathsf{SN}_{\mathsf{ISCI}}$*-proof can be translated into an* $\mathsf{NS}_{\mathsf{ISCI}}$*-proof.*

PROOF: By induction on the height of the $\mathsf{SN}_{\mathsf{ISCI}}$-proof Π mapping each rule in Π to its corresponding rule in $\mathsf{NS}_{\mathsf{ISCI}}$ (with possible additional steps involving the rule lift) and stepwise translating each sequent in Π into an $\mathsf{NS}_{\mathsf{ISCI}}$ nested sequent using Definition 5.3. To avoid confusion, we add a superscript i to the objects Γ, Δ, Λ described in Definition 5.3 when translating the i-th premiss of a rule and we keep the original non-superscripted notation for the translation of its conclusion.

Base case id: This case subsumes the base cases $\mathrm{id_p}$ and $\mathrm{id}_{\approx}$. We start with an axiom

$$\dfrac{}{\mathcal{R}, \Gamma, \mathrm{A} : \mathrm{u} \vdash \Delta, \mathrm{A} : \mathrm{v}}\ \mathrm{id}(\mathrm{u} \overset{\mathcal{R}}{\rightsquigarrow} \mathrm{v})$$

If $\mathrm{u} = \mathrm{v}$ then formulas have the same depth and we have an axiom in $\mathsf{NS}_{\mathsf{ISCI}}$ directly. Otherwise, $\mathrm{u} \overset{\mathcal{R}}{\rightsquigarrow} \mathrm{v}$ derives from the transitive closure of a chain $\mathrm{w}_1 \sqsubset \ldots \sqsubset \mathrm{w}_\mathrm{n}$ with $\mathrm{w}_1 = \mathrm{u}$ and $\mathrm{w}_\mathrm{n} = \mathrm{v}$. We then get an axiom in $\mathsf{NS}_{\mathsf{ISCI}}$ after n applications of the rule lift.

Base cases $\mathrm{L}^{\mathrm{B}}_{\perp}$ and $\mathrm{L}^{\mathrm{K}}_{\perp}$: Similar to Base case id.

Case $\mathrm{R}_{\supset}$: We start with

$$\dfrac{\begin{array}{c}\Pi \\ \hdashline \mathcal{R}, \mathrm{u} \sqsubset \mathrm{v}, \Gamma, \mathrm{A} : \mathrm{v} \vdash \Delta, \mathrm{B} : \mathrm{v}\end{array}}{\mathcal{R}, \Gamma \vdash \Delta, \mathrm{A} \supset \mathrm{B} : \mathrm{u}}\ \mathrm{R}_{\supset}$$

The translation of the conclusion is a nested sequent of the form $\mathcal{S}\{\Gamma_\mathrm{u} \vdash \Delta_\mathrm{u}, \Lambda_\mathrm{u}\}$. The translation of the premiss is a nested sequent of the form $\mathcal{S}\{\Gamma^1_\mathrm{u} \vdash \Delta^1_\mathrm{u}, \Lambda^1_\mathrm{u}\}$ for which, by induction hypothesis, we have a proof Π'. Since v is fresh and $\mathrm{u} \sqsubset \mathrm{v} \in \mathcal{R}$, v represents a nesting level one deep w.r.t. u. Therefore, by Definition 5.3, we have:

$$\Gamma_\mathrm{u} = \Gamma^1_\mathrm{u} \qquad \Delta_\mathrm{u} = \Delta^1_\mathrm{u}, \mathrm{A} \supset \mathrm{B} \qquad \Lambda^1_\mathrm{u} = \Lambda_\mathrm{u}, [\mathrm{A} \vdash \mathrm{B}].$$

We can then conclude with an instance of $\mathrm{NR}_\supset$ as follows:

$$\begin{array}{c}
\Pi' \\
\hdashline
\mathcal{S}\{\Gamma^1_u \vdash \Delta^1_u, \Lambda^1_u\} \\
\hdashline
\mathcal{S}\{\Gamma^1_u \vdash \Delta^1_u, \Lambda_u, [A \vdash B]\} \\
\hline
\mathcal{S}\{\Gamma^1_u \vdash \Delta^1_u, A \supset B, \Lambda_u\} \\
\hdashline
\mathcal{S}\{\Gamma_u \vdash \Delta_u \Lambda_u\}
\end{array}
\begin{array}{l}
\\ \\ (\Lambda^1_u = \Lambda_u, [A \vdash B]) \\ \mathrm{NR}_\supset \\ (\Gamma_u = \Gamma^1_u,\ \Delta_u = \Delta^1_u, A \supset B)
\end{array}$$

Case $\mathrm{L}_\supset$**:** We start with

$$\dfrac{\begin{array}{c}\Pi_1 \\ \hdashline \mathcal{R}, \Gamma, A \supset B : u \vdash \Delta, A : v\end{array} \quad \begin{array}{c}\Pi_2 \\ \hdashline \mathcal{R}u, \Gamma, B : v \vdash \Delta\end{array}}{\mathcal{R}, \Gamma, A \supset B : u \vdash \Delta}\ \mathrm{L}_\supset(u \overset{\mathcal{R}}{\rightsquigarrow} v)$$

The translation of the conclusion is a nested sequent $\mathcal{S}\{\Gamma_u \vdash \Delta_u, \Lambda_u\}$. The translations of the premises are nested sequents $\mathcal{S}\{\Gamma^1_u \vdash \Delta^1_u, \Lambda^1_u\}$ and $\mathcal{S}\{\Gamma^2_u \vdash \Delta^2_u, \Lambda^2_u\}$ for which we have, by induction hypothesis, proofs $\Pi_1{}'$ and $\Pi_2{}'$.

Since $u \overset{\mathcal{R}}{\rightsquigarrow} v$ we know that v represents a deeper nesting level than u, by Definition 5.3:

$$\begin{array}{c}
\Gamma_u = \Gamma^1_u, A \supset B \qquad \Delta_u = \Delta^1_u = \Delta^2_u \\
\Lambda^1_u = \Lambda_u(\Gamma^1_v \vdash \Delta^1_v, A, \Lambda^1_v)) \\
\Gamma^2_u = \Gamma^1_u \qquad \Lambda^2_u = \Lambda_u(\Gamma^2_v, B \vdash \Delta^2_v, \Lambda^2_v)) \\
\Delta^2_v = \Delta^1_v = \Delta_v \qquad \Lambda^2_v = \Lambda^1_v = \Lambda_v \qquad \Gamma^2_v = \Gamma^1_v = \Gamma_v.
\end{array}$$

After an application of $\mathrm{NL}_\supset$ and some applications of lift rule we can conclude as follows:

$$\begin{array}{c}
\dfrac{\begin{array}{c}\Pi_1{}' \\ \hdashline \mathcal{S}\{\Gamma^1_u \vdash \Delta^1_u, \Lambda^1_u\} \\ \hdashline \mathcal{S}\{\Gamma_u, A \supset B \vdash \Delta_u, \Lambda_u(\Gamma_v \vdash \Delta_v, A, \Lambda_v)\}\end{array} \quad \begin{array}{c}\Pi_2{}' \\ \hdashline \mathcal{S}\{\Gamma^2_u \vdash \Delta^2_u, \Lambda^2_u\} \\ \hdashline \mathcal{S}\{\Gamma_u \vdash \Delta_u, \Lambda_u(\Gamma_v, B \vdash \Delta_v, \Lambda_v)\}\end{array}}{\mathcal{S}\{\Gamma_u, A \supset B \vdash \Delta_u, \Lambda_u(\Gamma_v, A \supset B \vdash \Delta_v, \Lambda_v)\}}\ \mathrm{NL}_\supset \\
\hdashline
\mathcal{S}\{\Gamma_u, A \supset B \vdash \Delta_u, \Lambda_u(\Gamma_v \vdash \Delta_v, \Lambda_v)\}\ \ \mathrm{lift}*
\end{array}$$

Case $L_\wedge$ Since there is no label modification in the sequents we have

$$\dfrac{\begin{array}{c}\Pi\\ \hdashline \mathcal{R}, \Gamma, A : u, B : u \vdash \Delta\end{array}}{\mathcal{R}, \Gamma, A \wedge B : u \vdash \Delta}\, L_\wedge$$

$$\downarrow$$

$$\dfrac{\begin{array}{c}\Pi'\\ \hdashline \mathcal{S}\{\Gamma_u, A, B \vdash \Delta_u, \Lambda_u\}\end{array}}{\mathcal{S}\{\Gamma_u, A \wedge B \vdash \Delta_u, \Lambda_u\}}\, NL_\wedge$$

Case $R_\wedge$ Since there is no label modification in the sequents we have

$$\dfrac{\begin{array}{c}\Pi_1\\ \hdashline \mathcal{R}, \Gamma \vdash \Delta, A : u\end{array} \quad \begin{array}{c}\Pi_2\\ \hdashline \mathcal{R}, \Gamma \vdash \Delta, B : u\end{array}}{\mathcal{R}, \Gamma \vdash \Delta, A \wedge B : u}\, R_\wedge$$

$$\downarrow$$

$$\dfrac{\begin{array}{c}\Pi_1{}'\\ \hdashline \mathcal{S}\{\Gamma_u \vdash \Delta_u, A, \Lambda_u\}\end{array} \quad \begin{array}{c}\Pi_2{}'\\ \hdashline \mathcal{S}\{\Gamma_u \vdash \Delta_u, B, \Lambda_u\}\end{array}}{\mathcal{S}\{\Gamma_u \vdash \Delta_u, A \wedge B, \Lambda_u\}}\, NR_\wedge$$

Case $R_\vee$ Since there is no label modification in the sequents we have

$$\dfrac{\begin{array}{c}\Pi\\ \hdashline \mathcal{R}, \Gamma \vdash \Delta, A : u, B : u\end{array}}{\mathcal{R}, \Gamma \vdash \Delta, A \vee B : u}\, R_\vee$$

$$\downarrow$$

$$\dfrac{\begin{array}{c}\Pi'\\ \hdashline \mathcal{S}\{\Gamma_u \vdash \Delta_u, A, B, \Lambda_u\}\end{array}}{\mathcal{S}\{\Gamma_u \vdash \Delta_u, A \vee B, \Lambda_u\}}\, NR_\vee$$

Case $L_\vee^K$ Since there is no label modification we have

$$\dfrac{\begin{array}{c}\Pi_1\\ \hdashline \mathcal{R}, \Gamma, A : u \vdash \Delta\end{array} \quad \begin{array}{c}\Pi_2\\ \hdashline \mathcal{R}, \Gamma, B : u \vdash \Delta\end{array}}{\mathcal{R}, \Gamma, A \vee B : u \vdash \Delta}\, R_\wedge$$

$$\downarrow$$

$$\dfrac{\dfrac{\Pi_1{}'}{\mathcal{S}\{\Gamma_u, A \vdash \Delta_u, \Lambda_u\}} \quad \dfrac{\Pi_2{}'}{\mathcal{S}\{\Gamma_u, B \vdash \Delta_u, \Lambda_u\}}}{\mathcal{S}\{\Gamma_u, A \vee B \vdash \Delta_u, \Lambda_u\}}\ NL^K_\vee$$

Case $L^B_\vee$ This case is similar to case $R_\supset$, but with two premises and two fresh labels. We start with :

$$\dfrac{\dfrac{\Pi_1}{\begin{matrix}\mathcal{R}, v \sqsubset u_1 \\ \Gamma, A_1 \vee A_2 : u, A_1 : u_1\end{matrix} \vdash \Delta, C : u_1} \quad \dfrac{\Pi_2}{\begin{matrix}\mathcal{R}, v \sqsubset u_2 \\ \Gamma, A_1 \vee A_2 : u, A_2 : u_2\end{matrix} \vdash \Delta, C : u_2}}{\mathcal{R}, \Gamma, A_1 \vee A_2 : u \vdash \Delta, C : v}\ R_\wedge$$

The translation of the conclusion is the following nested sequent $\mathcal{S}\{\Gamma_u \vdash \Delta_u, \Lambda_u(\Gamma_v \vdash \Delta_v, \Lambda_v)\}$. The translation of the premises are nested sequents $\mathcal{S}\{\Gamma^1_u \vdash \Delta^1_u, \Lambda^1_u\}$ and $\mathcal{S}\{\Gamma^2_u \vdash \Delta^2_u, \Lambda^2_u\}$ for which, by induction hypothesis, we have proofs Π'_1 and Π'_1. Since u_1 and u_2 are fresh and $v \sqsubset u_1 \in \mathcal{R}$ and $v \sqsubset u_2 \in \mathcal{R}$, u_1 and u_2 both represent a nesting level one deep w.r.t. v. Also, since $u \overset{\mathcal{R}}{\rightsquigarrow} v$ for the application of the rule, v represents a nesting level deeper than u (they could be equal).

Therefore, by Definition 5.3, we have:

$$\Gamma_u = \Gamma^1_u, A_1 \vee A_2 \quad \Delta_u = \Delta^1_u = \Delta^2_u \quad \Lambda^1_u = \Lambda_u(\Gamma^1_v \vdash \Delta^1_v, \Lambda^1_v([A_1 \vdash C])$$
$$\Gamma^2_u = \Gamma^1_u$$
$$\Lambda^2_u = \Lambda_u(\Gamma^2_v \vdash \Delta^2_v, \Lambda^2_v([A_2 \vdash C]) \quad \Lambda_v = \Lambda^1_v = \Lambda^2_v \quad \Delta_v = \Delta^1_v, C = \Delta^2_v, C.$$

We can then conclude with an instance of $NL^B_\vee$ and possible multiple lift rule application as follows:

$$\dfrac{\dfrac{\dfrac{\Pi'_1}{\mathcal{S}\{\Gamma^1_u \vdash \Delta^1_u, \Lambda^1_u\}}}{\mathcal{S}\{\Gamma^1_u \vdash \Delta^1_u, \Lambda_u(\Gamma^1_v \vdash \Delta^1_v, \Lambda^1_v([A_1 \vdash C])\}} \quad \dfrac{\dfrac{\Pi'_2}{\mathcal{S}\{\Gamma^2_u \vdash \Delta^2_u, \Lambda^2_u\}}}{\mathcal{S}\{\Gamma^2_u \vdash \Delta^2_u, \Lambda_u(\Gamma^2_v \vdash \Delta^2_v, \Lambda^2_v([A \vdash C])\}}}{\dfrac{\dfrac{\mathcal{S}\{\Gamma^1_u \vdash \Delta^2_u, \Lambda_u(\Gamma^2_v, A_1 \vee A_2 \vdash \Delta^2_v, C, \Lambda^2_v)\}}{\mathcal{S}\{\Gamma^1_u, A_1 \vee A_2 \vdash \Delta^2_u, \Lambda_u(\Gamma^2_v \vdash \Delta^2_v, C, \Lambda^2_v)\}}\ lift*}{\mathcal{S}\{\Gamma_u \vdash \Delta_u, \Lambda_u(\Gamma_v \vdash \Delta_v, \Lambda_v)\}}}\ NL^B_\vee$$

If $u = v$ then there is no lift applied.

Case $L^1_{\approx}$ Since there is no label modification we have

$$\dfrac{\dfrac{\Pi_1}{\mathcal{R}, \Gamma, A \approx A : 0 \vdash \Delta}}{\mathcal{R}, \Gamma \vdash \Delta} L^1_{\approx}$$

$$\downarrow$$

$$\dfrac{\dfrac{\Pi_1{}'}{\mathcal{S}\{\Gamma_0, A \approx A \vdash \Delta_0, \Lambda_0\}}}{\mathcal{S}\{\Gamma_0 \vdash \Delta_0, \Lambda_0\}} NL^1_{\approx}$$

Case $aL^2_{\approx}$ Since there is no label modification we have

$$\dfrac{\dfrac{\Pi_1}{\mathcal{R}, \Gamma, A \approx B : u, B \approx A : u \vdash \Delta}}{\mathcal{R}, \Gamma, A \approx B : u \vdash \Delta} L^1_{\approx}$$

$$\downarrow$$

$$\dfrac{\dfrac{\Pi_1{}'}{\mathcal{S}\{\Gamma_u, A \approx B, B \approx A \vdash \Delta_u, \Lambda_u\}}}{\mathcal{S}\{\Gamma_u, A \approx B \vdash \Delta_u, \Lambda_u\}} NL^1_{\approx}$$

Case $L^2_{\approx}$ This case is very similar to case $L_{\supset}$.

$$\dfrac{\dfrac{\Pi_1}{\mathcal{R}, \Gamma \vdash \Delta, B : v} \quad \dfrac{\Pi_2}{\mathcal{R}u, \Gamma, A : v \vdash \Delta}}{\mathcal{R}, \Gamma, A \approx B : u \vdash \Delta} L^2_{\approx}(u \overset{\mathcal{R}}{\rightsquigarrow} v)$$

$$\downarrow$$

$$\dfrac{\dfrac{\Pi_1{}'}{\mathcal{S}\{\Gamma_u \vdash \Delta_u, \Lambda_u(\Gamma_v \vdash \Delta_v, \Lambda_v, B)\}} \quad \dfrac{\Pi_2{}'}{\mathcal{S}\{\Gamma_u \vdash \Delta_u, \Lambda_u(\Gamma_v, A \vdash \Delta_v, \Lambda_v)\}}}{\dfrac{\mathcal{S}\{\Gamma_u \vdash \Delta_u, \Lambda_u(\Gamma_v, A \approx B \vdash \Delta_v, \Lambda_v)\}}{\mathcal{S}\{\Gamma_u, A \approx B \vdash \Delta_u, \Lambda_u(\Gamma_v \vdash \Delta_v, \Lambda_v)\}}\,\text{lift}}\,\text{NL}^2_{\approx}$$

Case $\text{L}^3_{\approx}$ This case is very similar to case $\text{L}_{\supset}$.

$$\dfrac{\dfrac{\Pi}{\mathcal{R}u, \Gamma, A \approx B : u, C \approx D : v, A \otimes C \approx B \otimes D : v \vdash \Delta}}{\mathcal{R}, \Gamma, A \approx B : u, C \approx D : v \vdash \Delta}\,\text{L}^3_{\approx}(u \overset{\mathcal{R}}{\rightsquigarrow} v)$$

$$\downarrow$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\Pi_1{}'}{\mathcal{S}\{\Gamma_u, A \approx B \vdash \Delta_u, \Lambda_u(\Gamma_v, C \approx D, A \otimes C \approx B \otimes D \vdash \Delta_v, \Lambda_v)\}}}{\mathcal{S}\{\Gamma_u, A \approx B, C \approx D \vdash \Delta_u, \Lambda_u(\Gamma_v, C \approx D, A \otimes C \approx B \otimes D \vdash \Delta_v, \Lambda_v)\}}\,\text{lift}}{\mathcal{S}\{\Gamma_u, A \approx B, C \approx D, A \otimes C \approx B \otimes D \vdash \Delta_u, \Lambda_u(\Gamma_v, C \approx D \vdash \Delta_v, \Lambda_v)\}}\,\text{lift}}{\mathcal{S}\{\Gamma_u, A \approx B, C \approx D \vdash \Delta_u, \Lambda_u(\Gamma_v, C \approx D \vdash \Delta_v, \Lambda_v)\}}\,\text{NL}^3_{\approx}$$

Since we work with sets of formulas, the duplication of formulas is implicit in the rules.

Case $\text{L}^3_{\approx}$ Similar to the previous case, and we do not need to manage the labels.

$$\dfrac{\dfrac{\Pi}{\mathcal{R}u, \Gamma, A \approx B : u, A \otimes A \approx B \otimes B : u \vdash \Delta}}{\mathcal{R}, \Gamma, A \approx B : u \vdash \Delta}\,\text{L}^3_{\approx}$$

$$\downarrow$$

$$\dfrac{\dfrac{\Pi_1{}'}{\mathcal{S}\{\Gamma_u, A \approx B, A \otimes A \approx B \otimes B \vdash \Delta_u, \Lambda_u\}}}{\mathcal{S}\{\Gamma_u, A \approx B \vdash \Delta_u, \Lambda_u\}}\,\text{NL}^{3*}_{\approx}$$

Case $\text{L}^{r}_{\approx}$ This case is very similar to the case of $\text{L}_{\supset}$.

$$\dfrac{\dfrac{\Pi}{\mathcal{R}u, \Gamma, A \approx B : u, D : v, D^A_B : v \vdash \Delta}}{\mathcal{R}, \Gamma, A \approx B : u, D : v \vdash \Delta}\ L^r_{\approx}(u \overset{\mathcal{R}}{\rightsquigarrow} v)$$

$$\downarrow$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\Pi_1{}'}{\mathcal{S}\{\Gamma_u, A \approx B \vdash \Delta_u, \Lambda_u(\Gamma_v, D, D^A_B \vdash \Delta_v, \Lambda_v)\}}}{\mathcal{S}\{\Gamma_u, A \approx B, D \vdash \Delta_u, \Lambda_u(\Gamma_v, D, D^A_B \vdash \Delta_v, \Lambda_v)\}}\ \text{lift}}{\mathcal{S}\{\Gamma_u, A \approx B, C \approx D, D^A_B \vdash \Delta_u, \Lambda_u(\Gamma_v, C \approx D \vdash \Delta_v, \Lambda_v)\}}\ \text{lift}}{\mathcal{S}\{\Gamma_u, A \approx B, D \vdash \Delta_u, \Lambda_u(\Gamma_v, D \vdash \Delta_v, \Lambda_v)\}}\ NL^r_{\approx}$$

Since we work with sets of formulas, the duplication of formulas is implicit in the rules. □

B. Appendix: From L3$_{ISCI}$ to SN$_{ISCI}$

LEMMA B.1. *Let* s *be a sequent in* L3$_{ISCI}$ *and let* x *and* y *be two labels in* s *such that* $x \subseteq y$*, then* $\mu(x) \overset{\mathcal{R}}{\rightsquigarrow} \mu(y)$ *in* LS(s)*.*

PROOF: If y is maximal in s, then by Definition 4.1 we have a chain $0 \sqsubset \ldots \sqsubset \mu(x) \sqsubset \ldots \sqsubset \mu(y)$. Hence, $\mu(x) \overset{\mathcal{R}}{\rightsquigarrow} \mu(y)$. Otherwise, there is some maximal $z \in s$ such that $x \subseteq y \subseteq z$ and by Definition 4.1 we have a chain $0 \sqsubset \ldots \sqsubset \mu(x) \sqsubset \ldots \sqsubset \mu(y) \sqsubset \ldots \sqsubset \mu(z)$. Hence, $\mu(x) \overset{\mathcal{R}}{\rightsquigarrow} \mu(y)$. □

THEOREM B.2. *Any standard* L3$_{ISCI}$*-proof can be translated into an* SN$_{ISCI}$*-proof.*

PROOF: The proof is by induction on the height of the proof in L3$_{ISCI}$, with a case analysis on the last rule r applied in the L3$_{ISCI}$ proof. We write $\mathcal{R}$ and $\mathcal{R}_i$ for the sets of relational atoms obtained from the translation of the conclusion and of the i^{th} premiss of r respectively.

Base case id**:**

This case subsumes both id_p and $id_{\approx}$. By Lemma B.1, since $x \subseteq xy$, we have $\mu(x) \overset{\mathcal{R}}{\rightsquigarrow} \mu(y)$. Hence,

$$\frac{}{\Gamma, \mathrm{A} : \mathrm{x} \vdash \Delta, \mathrm{A} : \mathrm{xy}}\,\mathrm{id} \quad \rightsquigarrow \quad \frac{}{\mathcal{R}, \mathsf{LS}(\Gamma), \mathrm{A} : \mu(\mathrm{x}) \vdash \mathsf{LS}(\Delta), \mathrm{A} : \mu(\mathrm{xy})}\,\mathrm{id}$$

Base case $\mathrm{L}_\perp^{\mathrm{B}}$:

Similar to the base case id.

$$\frac{}{\Gamma, \perp : \mathrm{x} \vdash \Delta, \mathrm{A} : \mathrm{xy}}\,\mathrm{L}_\perp^{\mathrm{B}} \quad \rightsquigarrow \quad \frac{}{\mathcal{R}, \mathsf{LS}(\Gamma), \perp : \mu(\mathrm{x}) \vdash \mathsf{LS}(\Delta), \mathrm{A} : \mu(\mathrm{xy})}\,\mathrm{id}$$

Base case $\mathrm{L}_\perp^{\mathrm{K}}$:

$$\frac{}{\Gamma, \perp : \mathrm{x} \vdash \Delta}\,\mathrm{L}_\perp^{\mathrm{K}} \quad \rightsquigarrow \quad \frac{}{\mathcal{R}, \mathsf{LS}(\Gamma), \mathrm{A} : \mu(\mathrm{x}) \vdash \mathsf{LS}(\Delta)}\,\mathrm{L}_\perp^{\mathrm{K}}$$

Case $\mathrm{R}_\supset$:

Since the $\mathsf{L3}_{\mathsf{ISCI}}$ proof is standard (Definition 4.2) , a must be a fresh singleton $\{ i \}$ (with $i = 3k$ for some k) such that i is the greatest natural number in the premiss. Therefore, $\mu(\mathrm{a}) = \mu(\mathrm{xa}) = i$. We now show that $\mathcal{R}_1 = \mathcal{R} \cup \{ \mu(\mathrm{x}) \sqsubset \mu(\mathrm{xa}) \}$. Since x is right maximal, by right connectedness we have that x is maximal in the whole conclusion, which by the freshness of a implies that xa is maximal in the whole premiss. Therefore, $\mathcal{R}_1 = \mathcal{R} - \mathcal{R}(\mathrm{x}) \cup \mathcal{R}(\mathrm{xa})$. Let us write a as the ordered set $\{ i_1 < i_2 < \ldots < i_n \}$. Then, $\mathrm{xa} = \{ i_1 < i_2 < \ldots < i_n < i \}$. Thus, we get $\mathcal{R}(\mathrm{x}) = \{ 0 \sqsubset i_1, i_1 \sqsubset i_2, \ldots, i_{n-1} \sqsubset i_n \}$ and $\mathcal{R}(\mathrm{xa}) = \{ 0 \sqsubset i_1, i_1 \sqsubset i_2, \ldots, i_{n-1} \sqsubset i_n, i_n \sqsubset i \}$, where $i_n = \mu(\mathrm{x})$ and $i = \mu(\mathrm{xa})$. Hence, $\mathcal{R}_1 = \mathcal{R} \cup \{ \mu(\mathrm{x}) \sqsubset \mu(\mathrm{xa}) \}$ and we can conclude the translated proof $\mathsf{LS}(\Pi_1)$ obtained by induction hypothesis with an instance of $\mathrm{R}_\supset$ in $\mathsf{SN}_{\mathsf{ISCI}}$ as follows:

$$\dfrac{\dfrac{\Pi_1}{\Gamma, \mathrm{A} : \mathrm{a} \vdash \Delta, \mathrm{A} \supset \mathrm{B} : \mathrm{xa}}}{\Gamma \vdash \Delta, \mathrm{A} \supset \mathrm{B} : \mathrm{x}}\,\mathrm{R}_\supset \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\mathsf{LS}(\Pi_1)}{\mathcal{R}, \mu(\mathrm{x}) \sqsubset \mu(\mathrm{xa}), \mathsf{LS}(\Gamma), \mathrm{A} : \mu(\mathrm{a}) \vdash \mathsf{LS}(\Delta), \mathrm{B} : \mu(\mathrm{xa})}\,(\mathcal{R}_1)}{\mathcal{R}_1, \mathsf{LS}(\Gamma), \mathrm{A} : \mu(\mathrm{a}) \vdash \mathsf{LS}(\Delta), \mathrm{B} : \mu(\mathrm{xa})}}{\mathcal{R}, \mathsf{LS}(\Gamma) \vdash \mathsf{LS}(\Delta), \mathrm{A} \supset \mathrm{B} : \mu(\mathrm{x})}\,\mathrm{R}_\supset$$

Case $\mathrm{L}_\supset$:

Since no new label is created then both premises share the same label relations ($\mathcal{R}_1 = \mathcal{R}_2 = \mathcal{R}$). As $\mathrm{x} \subseteq \mathrm{xy}$, by application of Lemma B.1, we know that $\mu(x) \sqsubset \mu(xy)$. Therefore the following transformation is valid.

$$\dfrac{\dfrac{\Pi_1}{\Gamma, A \supset B : x \vdash \Delta, A : xy} \quad \dfrac{\Pi_2}{\Gamma, B : xy \vdash \Delta}}{\Gamma, A \supset B : x \vdash \Delta} L_\supset$$

$$\downarrow$$

$$\dfrac{\dfrac{\mathsf{LS}(\Pi_1)}{\mathcal{R}_1, \mathsf{LS}(\Gamma), A \supset B : \mu(x) \vdash \mathsf{LS}(\Delta), A : \mu(xy)} \quad \dfrac{\mathsf{LS}(\Pi_2)}{\mathcal{R}_2, \mathsf{LS}(\Gamma), B : \mu(xy) \vdash \mathsf{LS}(\Delta)}}{\mathcal{R}, \mathsf{LS}(\Gamma), A \supset B : \mu(x) \vdash \mathsf{LS}(\Delta)} L_\supset$$

Case $L_\wedge$**:**

Since no new label is created the translation is the following:

$$\dfrac{\dfrac{\Pi_1}{\Gamma, A : x, B : x \vdash \Delta}}{\Gamma, A \wedge B : x \vdash \Delta} L_\wedge \quad \rightsquigarrow \quad \dfrac{\dfrac{\mathsf{LS}(\Pi_1)}{\mathcal{R}_1, \mathsf{LS}(\Gamma), A : \mu(x), B : \mu(x) \vdash \mathsf{LS}(\Delta)}}{\mathcal{R}, \mathsf{LS}(\Gamma), A \wedge B : \mu(x) \vdash \mathsf{LS}(\Delta)} L_\wedge(\mathcal{R}_1 = \mathcal{R})$$

Case $R_\wedge$**:**

The translation is straightforward since we do not worry about the labels.

$$\dfrac{\dfrac{\Pi_1}{\Gamma \vdash \Delta, A : x} \quad \dfrac{\Pi_2}{\Gamma \vdash \Delta, B : x}}{\Gamma \vdash \Delta, A \wedge B : x} R_\wedge$$

$$\downarrow$$

$$\dfrac{\dfrac{\mathsf{LS}(\Pi_1)}{\mathcal{R}_1, \mathsf{LS}(\Gamma) \vdash \mathsf{LS}(\Delta), A : \mu(x)} \quad \dfrac{\mathsf{LS}(\Pi_2)}{\mathcal{R}_2, \mathsf{LS}(\Gamma) \vdash \mathsf{LS}(\Delta), B : \mu(x)}}{\mathcal{R}, \mathsf{LS}(\Gamma) \vdash \mathsf{LS}(\Delta), A \wedge B : \mu(x)} R_\wedge(\mathcal{R}_1 = \mathcal{R}_2 = \mathcal{R})$$

Case $R_\vee$**:**

Similar to case $L_\wedge$.

$$\dfrac{\begin{array}{c}\Pi_1\\ \hdashline \Gamma \vdash \Delta, \mathrm{A} : \mathrm{x}, \mathrm{B} : \mathrm{x}\end{array}}{\Gamma \vdash \Delta, \mathrm{A} \vee \mathrm{B} : \mathrm{x}}\,\mathrm{R}_{\vee} \quad \rightsquigarrow \quad \dfrac{\begin{array}{c}\mathsf{LS}(\Pi_1)\\ \hdashline \mathcal{R}_1, \mathsf{LS}(\Gamma) \vdash \mathsf{LS}(\Delta), \mathrm{A} : \mu(\mathrm{x}), \mathrm{B} : \mu(\mathrm{x})\end{array}}{\mathcal{R}, \mathsf{LS}(\Gamma) \vdash \mathsf{LS}(\Delta), \mathrm{A} \vee \mathrm{B} : \mu(\mathrm{x})}\,\mathrm{R}_{\vee}(\mathcal{R}_1 = \mathcal{R})$$

Case $\mathrm{L}_{\vee}^{\mathrm{K}}$:

Similar to case $\mathrm{R}_{\wedge}$.

$$\dfrac{\begin{array}{c}\Pi_1\\ \hdashline \Gamma, \mathrm{A} : \mathrm{x} \vdash \Delta\end{array} \quad \begin{array}{c}\Pi_2\\ \hdashline \Gamma, \mathrm{B} : \mathrm{x} \vdash \Delta\end{array}}{\Gamma \mathrm{A} \vee \mathrm{B} : \mathrm{x} \vdash \Delta}\,\mathrm{L}_{\vee}^{\mathrm{K}}$$

$$\downarrow$$

$$\dfrac{\begin{array}{c}\mathsf{LS}(\Pi_1)\\ \hdashline \mathcal{R}_1, \mathsf{LS}(\Gamma), \mathrm{A} : \mu(\mathrm{x}) \vdash \mathsf{LS}(\Delta)\end{array} \quad \begin{array}{c}\mathsf{LS}(\Pi_2)\\ \hdashline \mathcal{R}_2, \mathsf{LS}(\Gamma), \mathrm{B} : \mu(\mathrm{x}) \vdash \mathsf{LS}(\Delta)\end{array}}{\mathcal{R}, \mathsf{LS}(\Gamma), \mathrm{A} \vee \mathrm{B} : \mu(\mathrm{x}) \vdash \mathsf{LS}(\Delta)}\,\mathrm{L}_{\vee}^{\mathrm{K}}(\mathcal{R}_1 = \mathcal{R}_2 = \mathcal{R})$$

Case $\mathrm{L}_{\vee}^{\mathrm{B}}$:

Similar to case $\mathrm{R}_{\supset}$, except with two premises.
Since the $\mathsf{L3}_{\mathsf{ISCI}}$ proof is standard (Definition 4.2), a_1 and a_2 must be fresh singletons, respectively $\{\, i \,\}$ and $\{\, j \,\}$ (with $i = 3k + 1$ and $j = 3k + 2$ for some k), such that i and j are the greatest natural numbers in their respective premises. Therefore, $\mu(\mathrm{a}_1) = \mu(\mathrm{xa}_1) = \mu(\mathrm{xya}_1) = i$ and $\mu(\mathrm{a}_2) = \mu(\mathrm{xa}_2) = \mu(\mathrm{xa}_2) = i$. We now show that $\mathcal{R}_1 = \mathcal{R} \cup \{\, \mu(\mathrm{xy}) \sqsubset \mu(\mathrm{xya}_1) \,\}$ and $\mathcal{R}_2 = \mathcal{R} \cup \{\, \mu(\mathrm{xy}) \sqsubset \mu(\mathrm{xya}_2) \,\}$. Since xy is right maximal, by right connectedness we have that xy is maximal in the whole conclusion, which by the freshness of a_1 and a_2 implies that xya_1 and xya_1 are maximal in their respective premises. Therefore, $\mathcal{R}_1 = \mathcal{R} - \mathcal{R}(\mathrm{xy}) \cup \mathcal{R}(\mathrm{xya})$ and $\mathcal{R}_2 = \mathcal{R} - \mathcal{R}(\mathrm{xy}) \cup \mathcal{R}(\mathrm{xya}_2)$.

Let us write a_1 and a_2 as the ordered set $\{\, i_1 < i_2 < \ldots < i_n \,\}$. Then, $\mathrm{xya}_1 = \{\, i_1 < i_2 < \ldots < i_n < i \,\}$ and $\mathrm{xya}_2 = \{\, i_1 < i_2 < \ldots < i_n < j \,\}$. Thus, we get $\mathcal{R}(\mathrm{xy}) = \{\, 0 \sqsubset i_1, i_1 \sqsubset i_2, \ldots, i_{n-1} \sqsubset i_n \,\}$ and $\mathcal{R}(\mathrm{xya}_1) = \{\, 0 \sqsubset i_1, i_1 \sqsubset i_2, \ldots, i_{n-1} \sqsubset i_n, i_n \sqsubset i \,\}$, where $i_n = \mu(\mathrm{xy})$ and $i = \mu(\mathrm{xa})$ and $\mathcal{R}(\mathrm{xya}_2) = \{\, 0 \sqsubset i_1, i_1 \sqsubset i_2, \ldots, i_{n-1} \sqsubset i_n, i_n \sqsubset j \,\}$, where $i_n = \mu(\mathrm{xy})$ and $j = \mu(\mathrm{xa})$. Hence, $\mathcal{R}_1 = \mathcal{R} \cup \{\, \mu(\mathrm{xy}) \sqsubset \mu(\mathrm{xya}_1) \,\}$ and $\mathcal{R}_2 = \mathcal{R} \cup \{\, \mu(\mathrm{xy}) \sqsubset \mu(\mathrm{xya}_2) \,\}$. Now since by applying

Lemma B.1 we have $\mu(\mathrm{x}) \sqsubset \mu(\mathrm{xy})$ we can consider the translated proofs $\mathsf{LS}(\Pi_1)$ and $\mathsf{LS}(\Pi_1)$ obtained by induction hypothesis with an instance of $\mathrm{L}^{\mathrm{B}}_{\vee}$. in $\mathsf{SN}_{\mathsf{ISCI}}$ as follows:

$$\dfrac{\begin{array}{c}\Pi_1\\ \hdashline \Gamma, \mathrm{A}_1 : \mathrm{xa}_1 \vdash \Delta, \mathrm{C} : \mathrm{xya}_1\end{array} \quad \begin{array}{c}\Pi_2\\ \hdashline \Gamma, \mathrm{A}_2 : \mathrm{xa}_2 \vdash \Delta, \mathrm{C} : \mathrm{xya}_2\end{array}}{\Gamma \mathrm{A} \vee \mathrm{B} : \mathrm{x} \vdash \Delta, \mathrm{C} : \mathrm{xy}}\ \mathrm{L}^{\mathrm{B}}_{\vee}$$

$$\downarrow$$

$$\Pi_3 \left\{ \begin{array}{c}\mathsf{LS}(\Pi_1)\\ \hdashline \mathcal{R}, \mu(\mathrm{xy}) \sqsubset \mu(\mathrm{xya}_1), \mathsf{LS}(\Gamma), \mathrm{A}_1 : \mu(\mathrm{xa}_1) \vdash \mathsf{LS}(\Delta), \mathrm{C} : \mu(\mathrm{xya}_1)\\ \hdashline \mathcal{R}_1, \mathsf{LS}(\Gamma), \mathrm{A}_1 : \mu(\mathrm{xa}_1) \vdash \mathsf{LS}(\Delta), \mathrm{C} : \mu(\mathrm{xya}_1)\end{array}\right.$$

$$\Pi_4 \left\{ \begin{array}{c}\mathsf{LS}(\Pi_2)\\ \hdashline \mathcal{R}, \mu(\mathrm{xy}) \sqsubset \mu(\mathrm{xya}_2), \mathsf{LS}(\Gamma), \mathrm{A}_2 : \mu(\mathrm{xa}_2) \vdash \mathsf{LS}(\Delta), \mathrm{C} : \mu(\mathrm{xya}_2)\\ \hdashline \mathcal{R}_2, \mathsf{LS}(\Gamma), \mathrm{A}_2 : \mu(\mathrm{xa}_2) \vdash \mathsf{LS}(\Delta), \mathrm{C} : \mu(\mathrm{xya}_2)\end{array}\right.$$

$$\dfrac{\Pi_3 \qquad \Pi_4}{\mathcal{R}, \mathsf{LS}(\Gamma), \mathrm{A} \vee \mathrm{B} : \mu(\mathrm{x}) \vdash \mathsf{LS}(\Delta), \mathrm{C} : \mu(\mathrm{xya}_1)}\ \mathrm{L}^{\mathrm{B}}_{\vee}$$

Case $\mathrm{L}^1_{\approx}$:

Since $\mu(\emptyset) = 0$ we have the following translation

$$\dfrac{\begin{array}{c}\Pi_1\\ \hdashline \Gamma, \mathrm{A} \approx \mathrm{A} : \emptyset \vdash \Delta\end{array}}{\Gamma \vdash \Delta}\ \mathrm{L}^1_{\approx} \quad \rightsquigarrow \quad \dfrac{\begin{array}{c}\mathsf{LS}(\Pi_1)\\ \hdashline \mathcal{R}_1, \mathsf{LS}(\Gamma), \mathrm{A} \approx \mathrm{A} : 0 \vdash \mathsf{LS}(\Delta)\end{array}}{\mathcal{R}, \mathsf{LS}(\Gamma) \vdash \mathsf{LS}(\Delta)}\ \mathrm{L}^1_{\approx}(\mathcal{R}_1 = \mathcal{R})$$

Case $\mathrm{aL}^2_{\approx}$:

Since we do not change any label we have

$$\dfrac{\begin{array}{c}\Pi_1\\ \hdashline \Gamma, \mathrm{A} \approx \mathrm{B} : \mathrm{x}, \mathrm{B} \approx \mathrm{A} : \mathrm{x} \vdash \Delta\end{array}}{\Gamma, \mathrm{A} \approx \mathrm{B} : \mathrm{x} \vdash \Delta}\ \mathrm{aL}^2_{\approx}$$

$$\downarrow$$

$$\rightsquigarrow \quad \dfrac{\begin{array}{c}\Pi_1\\ \hdashline \mathcal{R}_1, \mathsf{LS}(\Gamma), \mathrm{A} \approx \mathrm{B} : \mu(\mathrm{x}), \mathrm{B} \approx \mathrm{A} : \mu(\mathrm{x}) \vdash \mathsf{LS}(\Delta)\end{array}}{\mathcal{R}, \mathsf{LS}(\Gamma), \mathrm{A} \approx \mathrm{B} : \mu(\mathrm{x}) \vdash \mathsf{LS}(\Delta)}\ \mathrm{aL}^2_{\approx}(\mathcal{R}_1 = \mathcal{R})$$

Case $L^2_\approx$:

Similar to case $L_\supset$.

$$\dfrac{\dfrac{\Pi_1}{\Gamma, A \approx B : x \vdash \Delta, B : xy} \quad \dfrac{\Pi_2}{\Gamma, A \approx B : x, A : xy \vdash \Delta}}{\Gamma, A \approx B : x \vdash \Delta}\, L^2_\approx$$

$$\downarrow$$

$$\dfrac{\dfrac{\mathsf{LS}(\Pi_1)}{\begin{array}{l}\mathcal{R}_1, \mathsf{LS}(\Gamma),\\ A \approx B : \mu(x)\end{array} \vdash \mathsf{LS}(\Delta), B : \mu(xy)} \quad \dfrac{\mathsf{LS}(\Pi_2)}{\begin{array}{l}\mathcal{R}_2, \mathsf{LS}(\Gamma),\\ A \approx B : \mu(x), A : \mu(xy)\end{array} \vdash \mathsf{LS}(\Delta) :}}{\mathcal{R}, \mathsf{LS}(\Gamma), A \approx B : \mu(x) \vdash \mathsf{LS}(\Delta)}\, L^2_\approx$$

with $\mathcal{R}_1 = \mathcal{R}_2 = \mathcal{R}$.

Case $L^3_\approx$:

Since $x \subseteq xy$ and $y \subseteq xy$, by application of Lemma B.1 we have $\mu(x) \sqsubset \mu(xy)$ and $\mu(y) \sqsubset \mu(xy)$, and we have the following translation

$$\dfrac{\dfrac{\Pi_1}{\Gamma, A \approx B : x, C \approx D : y, A \otimes C \approx B \otimes D : xy \vdash \Delta}}{\Gamma, A \approx B : x, C \approx D : y \vdash \Delta}\, L^3_\approx$$

$$\downarrow$$

$$\dfrac{\dfrac{\mathsf{LS}(\Pi_1)}{\mathcal{R}_1, \mathsf{LS}(\Gamma), A \approx B : \mu(x), C \approx D : \mu(x), A \otimes C \approx B \otimes D : \mu(xy) \vdash \mathsf{LS}(\Delta)}}{\mathcal{R}, \mathsf{LS}(\Gamma), A \approx B : \mu(x), C \approx D : \mu(x) \vdash \mathsf{LS}(\Delta)}\, L^3_\approx(\mathcal{R}_1 = \mathcal{R})$$

Case $L^{3*}_\approx$:

Similar to the previous case.

$$\dfrac{\dfrac{\Pi_1}{\Gamma, A \approx B : x, A \otimes A \approx B \otimes B : x \vdash \Delta}}{\Gamma, A \approx B : x \vdash \Delta}\, L^{3*}_\approx$$

$$\dfrac{\dfrac{\downarrow \quad \mathsf{LS}(\Pi_1)}{\mathcal{R}_1, \mathsf{LS}(\Gamma), A \approx B : \mu(x), A \otimes A \approx B \otimes B : \mu(x) \vdash \mathsf{LS}(\Delta)}}{\mathcal{R}, \mathsf{LS}(\Gamma), A \approx B : \mu(x) \vdash \mathsf{LS}(\Delta)} \; L^{3*}_{\approx}(\mathcal{R}_1 = \mathcal{R})$$

Case $L^{r}_{\approx}$:

Since $x \subseteq xy$ and $y \subseteq xy$, by application of Lemma B.1 we have $\mu(x) \sqsubset \mu(xy)$ and $\mu(y) \sqsubset \mu(xy)$, and we have the following translation

$$\dfrac{\dfrac{\Pi_1}{\Gamma, A \approx B : x, D : y, D^{A}_{B} : xy \vdash \Delta}}{\Gamma, A \approx B : x, D : y \vdash \Delta} \; L^{3*}_{\approx}$$

$$\downarrow$$

$$\dfrac{\dfrac{\mathsf{LS}(\Pi_1)}{\mathcal{R}_1, \mathsf{LS}(\Gamma), A \approx B : \mu(x), D : \mu(y), D^{A}_{B} : \mu(xy) \vdash \mathsf{LS}(\Delta)}}{\mathcal{R}, \mathsf{LS}(\Gamma), A \approx B : \mu(x), D : \mu(y) \vdash \mathsf{LS}(\Delta)} \; L^{3*}_{\approx}(\mathcal{R}_1 = \mathcal{R}) \qquad \square$$

C. Appendix: From G3$_{\mathsf{ISCI}}$ to L3$_{\mathsf{ISCI}}$

In this appendix we give the full proof of Theorem 8.2 for the translation of G3$_{\mathsf{ISCI}}$-proofs into L3$_{\mathsf{ISCI}}$-proofs. In fact we show a more general result that also includes the translation cases for the maximal variants of the rules given in Figure 6. Such rules only introduce (in their premises) active formulas whose label is right maximal and therefore help reduce the number of choices for the labels to introduce in a bottom-up application of the rules. An immediate corollary of such translations is that restricting the proof-search process to the class of right maximality preserving proofs does not change the set of provable formulas, i.e. replacing the original rules with the right maximal ones still yields a calculus that is complete w.r.t. the Kripke and TB semantics of ISCI.

THEOREM C.1. G3$_{\mathsf{ISCI}}$*-proofs translate into* L3$_{\mathsf{ISCI}}$*-proofs.*

PROOF: By induction on the height of G3$_{\mathsf{ISCI}}$-proofs. We start with a G3$_{\mathsf{ISCI}}$-proof Π of a sequent $\Gamma \vdash \Delta$. We transform Π into a L3$_{\mathsf{ISCI}}$-proof of

$$\frac{\Gamma, A_1 : xa_1 \vdash \Delta, C : xya_1 \quad \Gamma, A_2 : xa_2 \vdash \Delta, C : xya_2}{\Gamma, A_1 \vee A_2 : x \vdash \Delta, C : xy} \nu L^{B}_{\vee}(xy \in \lceil \Delta, C : xy \rceil)$$

$$\frac{\Gamma \vdash \Delta, A : xy \quad \Gamma, B : xy \vdash \Delta}{\Gamma, A \supset B : x \vdash \Delta} \nu L_{\supset}(xy \in \lceil \Delta \rceil) \quad \frac{\Gamma, A \approx A : x \vdash \Delta}{\Gamma \vdash \Delta} \nu L^{1}_{\approx}(x \in \lceil \Delta \rceil)$$

$$\frac{\Gamma, A \approx B : x \vdash \Delta, B : xy \quad \Gamma, A \approx B : x, A : xy \vdash \Delta}{\Gamma, A \approx B : x \vdash \Delta} \nu L^{2}_{\approx}(xy \in \lceil \Delta \rceil)$$

$$\frac{\Gamma, A \approx B : x, C \approx D : y, A \otimes C \approx B \otimes D : xyz \vdash \Delta}{\Gamma, A \approx B : x, C \approx D : y \vdash \Delta} \nu L^{3}_{\approx}(xyz \in \lceil \Delta \rceil)$$

$$\frac{\Gamma, A \approx B : x, D : y, D^{A}_{B} : xyz \vdash \Delta}{\Gamma, A \approx B : x, D : y \vdash \Delta} \nu L^{r}_{\approx}(xyz \in \lceil \Delta \rceil)$$

Figure 6: Maximal Rules for $\mathsf{L3}_{\mathsf{ISCI}}$.

$\mathsf{L}(\Gamma \vdash \Delta, \vec{n})$, where $n = |\Gamma|$. For convenience, we write a_i as a shorthand for the singleton $\{ n + i \}$. In the base case, we give a direct translation of the axioms of $\mathsf{G3}_{\mathsf{ISCI}}$. In the inductive case, we suppose that Π has height $h + 1$ and that it ends with a rule r of arity k. We first apply the induction hypothesis to all of the $\mathsf{G3}_{\mathsf{ISCI}}$-subproofs Π_j $(1 \leqslant j \leqslant k)$ to get the corresponding $\mathsf{L3}_{\mathsf{ISCI}}$-proofs Π'_j. If necessary, we perform some label substitutions σ_j in the subproofs Π_j using Lemma 3.2 to get the new subproofs $\Pi'_j\sigma_j$ and further extend them to new conclusions s_j with weakening steps r_j when required. Finally, the resulting subproofs are combined into a $\mathsf{L3}_{\mathsf{ISCI}}$-proof of $\mathsf{L}(\Gamma \vdash \Delta, \vec{n})$ that ends with the rule r.

The proof principle is depicted below:

$$\frac{\Pi_1 \quad \cdots \quad \Pi_k}{\Gamma \vdash \Delta} r \quad \rightsquigarrow \quad \frac{\dfrac{\Pi'_1\sigma_1}{s_1} r_1 \quad \cdots \quad \dfrac{\Pi'_k\sigma_k}{s_k} r_k}{\Gamma : \vec{n} \vdash \Delta : \check{n}} r$$

The soundness of label substitutions in $\mathsf{L3}_{\mathsf{ISCI}}$ is proven in Lemma C.13. The weakening admissibility property is proven in Lemma C.14. Given a proof Π and a labeled sequent $\Gamma \vdash \Delta$, we write $\Pi + \Gamma \vdash \Delta$ for the proof obtained from Π by appending Γ and Δ respectively to left-hand and right-hand side of all of the sequents occurring in Π.

Base case $\mathrm{id_p}$**:**

$$\dfrac{}{\Gamma, \mathrm{p} \vdash \Delta, \mathrm{p}}\,\mathrm{id_p} \quad \rightsquigarrow \quad \dfrac{}{\Gamma : \vec{\mathrm{n}}, \mathrm{p} : \mathrm{a}_1 \vdash \Delta : \check{\mathrm{n}}\mathrm{a}_1, \mathrm{p} : \check{\mathrm{n}}\mathrm{a}_1}\,\mathrm{id_p}$$

Base case $\mathrm{id}_{\approx}$**:**

$$\dfrac{}{\Gamma, \mathrm{A} \approx \mathrm{B} \vdash \Delta, \mathrm{A} \approx \mathrm{B}}\,\mathrm{id}_{\approx} \quad \rightsquigarrow \quad \dfrac{}{\Gamma : \vec{\mathrm{n}}, \mathrm{A} \approx \mathrm{B} : \mathrm{a}_1 \vdash \Delta : \check{\mathrm{n}}\mathrm{a}_1, \mathrm{A} \approx \mathrm{B} : \check{\mathrm{n}}\mathrm{a}_1}\,\mathrm{id}_{\approx}$$

Base case $\mathrm{L}_{\bot}^{\mathrm{K}}$**:**

$$\dfrac{}{\Gamma, \bot \vdash \Delta}\,\mathrm{L}_{\bot}^{\mathrm{K}} \quad \rightsquigarrow \quad \dfrac{}{\Gamma : \vec{\mathrm{n}}, \bot : \mathrm{a}_1 \vdash \Delta : \check{\mathrm{n}}\mathrm{a}_1}\,\mathrm{L}_{\bot}^{\mathrm{K}}$$

Base case $\mathrm{L}_{\bot}^{\mathrm{B}}$**:**

$$\dfrac{}{\Gamma, \bot \vdash \Delta, \mathrm{C}}\,\mathrm{L}_{\bot}^{\mathrm{B}} \quad \rightsquigarrow \quad \dfrac{}{\Gamma : \vec{\mathrm{n}}, \bot : \mathrm{a}_1 \vdash \Delta : \check{\mathrm{n}}\mathrm{a}_1, \mathrm{C} : \check{\mathrm{n}}\mathrm{a}_1}\,\mathrm{L}_{\bot}^{\mathrm{B}}$$

Case $\mathrm{L}_{\wedge}$**:** We start with a proof Π whose height is $h+1$ and apply the induction hypothesis on the subproof Π_1 which has height h to get Π_1'. We then use Lemma 3.2 to replace a_2 with a_1.

$$\dfrac{\begin{array}{c}\Pi_1 \\ \hdashline \Gamma, \mathrm{A}, \mathrm{B} \vdash \Delta\end{array}}{\Gamma, \mathrm{A} \wedge \mathrm{B} \vdash \Delta}\,\mathrm{L}_{\wedge} \quad \rightsquigarrow \quad \begin{array}{c}\Pi_1' \\ \hdashline \Gamma : \vec{\mathrm{n}}, \mathrm{A} : \mathrm{a}_1, \mathrm{B} : \mathrm{a}_2 \vdash \Delta : \check{\mathrm{n}}\mathrm{a}_1\mathrm{a}_2\end{array}$$

$$\rightsquigarrow \quad \begin{array}{c}\Pi_1'[\mathrm{a}_1/\mathrm{a}_2] \\ \hdashline \Gamma : \vec{\mathrm{n}}, \mathrm{A} : \mathrm{a}_1, \mathrm{B} : \mathrm{a}_1 \vdash \Delta : \check{\mathrm{n}}\mathrm{a}_1\end{array}$$

We finally conclude with an application of $\mathrm{L}_{\wedge}$ as follows:

$$\dfrac{\begin{array}{c}\Pi_1'[\mathrm{a}_1/\mathrm{a}_2] \\ \hdashline \Gamma : \vec{\mathrm{n}}, \mathrm{A} : \mathrm{a}_1, \mathrm{B} : \mathrm{a}_1 \vdash \Delta : \check{\mathrm{n}}\mathrm{a}_1\end{array}}{\Gamma : \vec{\mathrm{n}}, \mathrm{A} \wedge \mathrm{B} : \mathrm{a}_1 \vdash \Delta : \check{\mathrm{n}}\mathrm{a}_1}\,\mathrm{L}_{\wedge}$$

Case $\mathrm{R}_{\wedge}$**:** We apply the induction hypothesis on the subproofs Π_1 and Π_2 to get Π_1' and Π_2'. Then we conclude with $\mathrm{R}_{\wedge}$ as depicted below:

$$\dfrac{\dfrac{\Pi_1}{\Gamma \vdash \Delta, A} \quad \dfrac{\Pi_2}{\Gamma \vdash \Delta, B}}{\Gamma \vdash \Delta, A \wedge B}\, R_\wedge \quad \rightsquigarrow \quad \dfrac{\dfrac{\Pi'_1}{\Gamma : \vec{n} \vdash \Delta : \check{n}, A : \check{n}} \quad \dfrac{\Pi'_2}{\Gamma : \vec{n} \vdash \Delta : \check{n}, B : \check{n}}}{\Gamma : \vec{n} \vdash \Delta : \check{n}, A \wedge B : \check{n}}\, R_\wedge$$

Case $L_\vee^K$: We start with the following proof and apply the induction hypothesis to the subproofs Π_1 and Π_2 to get Π'_1 and Π'_2:

$$\dfrac{\dfrac{\Pi_1}{\Gamma, A \vdash \Delta} \quad \dfrac{\Pi_2}{\Gamma, B \vdash \Delta}}{\Gamma, A \vee B \vdash \Delta}\, L_\vee^K \quad \rightsquigarrow \quad \dfrac{\Pi'_1}{\Gamma : \vec{n}, A : a_1 \vdash \Delta : \check{n}a_1} \quad \dfrac{\Pi'_2}{\Gamma : \vec{n}, B : a_1 \vdash \Delta : \check{n}a_1}$$

We use Lemma 3.2 to replace a_1 with $\check{n}$ in both Π'_1 and Π'_2. Finally, we conclude with $L_\vee^K$:

$$\dfrac{\dfrac{\Pi'_1[\check{n}/a_1]}{\Gamma : \vec{n}, A : \check{n} \vdash \Delta : \check{n}} \quad \dfrac{\Pi'_2[\check{n}/a_1]}{\Gamma : \vec{n}, B : \check{n} \vdash \Delta : \check{n}}}{\Gamma : \vec{n}, A \vee B : \check{n} \vdash \Delta : \check{n}}\, L_\vee^K$$

Case $L_\vee^B$: We start with the following proof and apply the induction hypothesis to the subproofs Π_1 and Π_2 to get Π'_1 and Π'_2:

$$\dfrac{\dfrac{\Pi_1}{\Gamma, A \vdash C} \quad \dfrac{\Pi_2}{\Gamma, B \vdash C}}{\Gamma, A \vee B \vdash \Delta, C}\, L_\vee^B \quad \rightsquigarrow \quad \dfrac{\Pi'_1}{\Gamma : \vec{n}, A : a_1 \vdash C : \check{n}a_1} \quad \dfrac{\Pi'_2}{\Gamma : \vec{n}, B : a_1 \vdash C : \check{n}a_1}$$

We use Lemma 3.2 to replace a_1 with a_1a_2 in Π_1 and with a_1a_3 in Π'_2 to obtain the new proofs $\Pi'_1[a_1a_2/a_1]$ and $\Pi'_2[a_1a_3/a_1]$. We then use weakening admissibility (Lemma C.14) on $\Pi'_1[a_1a_2/a_1]$ and $\Pi'_2[a_1a_3/a_1]$ to add $\Delta : \check{n}a_1$ on the right-hand side of all the sequents occurring in $\Pi'_1[a_1a_2/a_1]$ and $\Pi'_2[a_1a_3/a_1]$. Finally, since a_2 and a_3 are fresh in the conclusion, we can conclude with $L_\vee^B$:

$$\dfrac{\dfrac{\Pi'_1[a_1a_2/a_1] + \vdash \Delta : \check{n}a_1}{\Gamma : \vec{n}, A : a_1a_2 \vdash \Delta : \check{n}a_1, C : \check{n}a_1a_2} \quad \dfrac{\Pi'_2[a_1a_3/a_1] + \vdash \Delta : \check{n}a_1}{\Gamma : \vec{n}, B : a_1a_3 \vdash \Delta : \check{n}a_1, C : \check{n}a_1a_3}}{\Gamma : \vec{n}, A \vee B : a_1 \vdash \Delta : \check{n}a_1, C : \check{n}a_1}\, L_\vee^B$$

with $\check{n}a_1 \in \lceil \Delta : \check{n}a_1 C : \check{n}a_1 \rceil$.

Remark C.2. The instance of $L^B_\vee$ preserves right connectedness and $\check{n}a_1 \in \lceil \Delta : \check{n}a_1, C : \check{n}a_1 \rceil$ implies $\check{n}a_1 \in \Delta : \check{n}a_1, C : \check{n}a_1$.

Case $R_\vee$**:** This case is similar to Case $R_\wedge$.

$$\dfrac{\dfrac{\Pi_1}{\Gamma \vdash \Delta, A_1, A_2}}{\Gamma \vdash \Delta, A_1 \vee A_2} R_\vee \quad \rightsquigarrow \quad \dfrac{\dfrac{\Pi'_1}{\Gamma : \vec{n} \vdash \Delta : \check{n}, A_1 : \check{n}, A_2 : \check{n}}}{\Gamma : \vec{n} \vdash \Delta : \check{n}, A_1 \vee A_2 : \check{n}} R_\vee$$

Case $L_\supset$**:** We apply the induction hypothesis on the subproofs Π_1 and Π_2 to obtain the proofs Π'_1 and Π'_2.

$$\dfrac{\dfrac{\Pi_1}{\Gamma, A \supset B \vdash \Delta, A} \quad \dfrac{\Pi_2}{\Gamma, B \vdash \Delta}}{\Gamma, A \supset B \vdash \Delta} L_\supset$$

$$\rightsquigarrow \quad \dfrac{\Pi'_1}{\Gamma : \vec{n}, A \supset B : a_1 \vdash \Delta : \check{n}a_1, A : \check{n}a_1} \quad \dfrac{\Pi'_2}{\Gamma : \vec{n}, B : a_1 \vdash \Delta : \check{n}a_1}$$

We use Lemma 3.2 to replace a_1 with $\check{n}a_1$ in Π'_2. Finally, since $\check{n}a_1$ is right maximal in the conclusion, we can apply $L_\supset$.

$$\dfrac{\dfrac{\Pi'_1}{\Gamma : \vec{n}, A \supset B : a_1 \vdash \Delta : \check{n}a_1, A : \check{n}a_1} \quad \dfrac{\Pi'_2[\check{n}a_1/a_1]}{\Gamma : \vec{n}, B : \check{n}a_1 \vdash \Delta : \check{n}a_1}}{\Gamma : \vec{n}, A \supset B : a_1 \vdash \Delta : \check{n}a_1} L_\supset(\check{n}a_1 \subseteq \Delta)$$

Remark C.3. The instance of $L_\supset$ preserves right connectedness in $\Pi'_2[\check{n}a_1/a_1]$. It also preserves right connectedness in Π'_1, even in the single-succedent restriction.

Case $R_\supset$**:** We apply the induction hypothesis on the subproof Π_1 to obtain the proof Π'_1 and we use weakening admissibility (Lemma C.14) to add $\Delta : \check{n}$ to the right-hand side of all of the sequents occurring in Π'_1. Since a_1 is fresh in the conclusion, we can apply $R_\supset$.

$$\dfrac{\dfrac{\Pi_1}{\Gamma, A \vdash B}}{\Gamma \vdash \Delta, A \supset B} R_\supset \quad \rightsquigarrow \quad \dfrac{\dfrac{\Pi_1' + \vdash \Delta : \check{n}}{\Gamma : \vec{n}, A : a_1 \vdash \Delta : \check{n}, B : \check{n}a_1}}{\Gamma : \vec{n} \vdash \Delta : \check{n}, A \supset B : \check{n}} R_\supset$$

Case $L^1_\approx$: We get Π_1' from Π_1 by induction hypothesis and replace a_1 with $\emptyset$ ((Lemma 3.2).

$$\dfrac{\dfrac{\Pi_1}{\Gamma, A \approx A \vdash \Delta}}{\Gamma \vdash \Delta} L^1_\approx \quad \rightsquigarrow \quad \dfrac{\Pi_1'}{\Gamma : \vec{n}, A \approx A : a_1 \vdash \Delta : \check{n}a_1} \quad \rightsquigarrow \quad \dfrac{\dfrac{\Pi_1'[\emptyset / a_1]}{\Gamma : \vec{n}, A \approx A : \emptyset \vdash \Delta : \check{n}}}{\Gamma : \vec{n} \vdash \Delta : \check{n}} L^1_\approx$$

Remark C.4. The instance of $L^1_\approx$ preserves right connectedness. The active formula is left minimal.

Case $\nu L^1_\approx$: We get Π_1' from Π_1 by induction hypothesis and replace a_1 with $\check{n}$ ((Lemma 3.2).

$$\dfrac{\dfrac{\Pi_1}{\Gamma, A \approx A \vdash \Delta}}{\Gamma \vdash \Delta} L^1_\approx \quad \rightsquigarrow \quad \dfrac{\Pi_1'}{\Gamma : \vec{n}, A \approx A : a_1 \vdash \Delta : \check{n}a_1}$$

$$\rightsquigarrow \quad \dfrac{\dfrac{\Pi_1'[\check{n} / a_1]}{\Gamma : \vec{n}, A \approx A : \check{n} \vdash \Delta : \check{n}}}{\Gamma : \vec{n} \vdash \Delta : \check{n}} \nu L^1_\approx (\check{n} \in \lceil \Delta \rceil \subseteq \Delta)$$

Remark C.5. The active formula is right maximal by definition. The instance of $\nu L^1_\approx$ preserves right connectedness. Thus, the active formula is also left maximal.

Case $L^2_\approx$: We start with proof Π and apply the induction hypothesis to the subproofs Π_1 and Π_2 to get Π_1' and Π_2'.

$$\dfrac{\dfrac{\Pi_1}{\Gamma, A \approx B \vdash \Delta, B} \quad \dfrac{\Pi_2}{\Gamma, A \approx B, A \vdash \Delta}}{\Gamma, A \approx B \vdash \Delta} L^2_\approx$$

$$\rightsquigarrow \quad \dfrac{\Pi_1'}{\Gamma : \vec{n}, A \approx B : a_1 \vdash \Delta : \check{n}a_1, B : \check{n}a_1} \qquad \dfrac{\Pi_2'}{\Gamma : \vec{n}, A \approx B : a_1, A : a_2 \vdash \Delta : \check{n}a_1 a_2}$$

Then we use Lemma 3.2 to replace a_2 with $\check{n}a_1$ in Π'_2 and conclude with $L^2_{\approx}$ as follows:

$$\dfrac{\dfrac{\Pi'_1}{\Gamma : \vec{n}, A \approx B : a_1 \vdash \Delta : \check{n}a_1, B : \check{n}a_1} \quad \dfrac{\Pi'_2[\check{n}a_1/a_2]}{\Gamma : \vec{n}, A \approx B : a_1, A : \check{n}a_1 \vdash \Delta : \check{n}a_1}}{\Gamma : \vec{n}, A \approx B : a_1 \vdash \Delta : \check{n}a_1} \; L^2_{\approx}(\check{n}a_1 \in \Delta)$$

Remark C.6. The instance of $L^2_{\approx}$ preserves right connectedness in $\Pi'_2[\check{n}a_1/a_2]$. It also preserves right connectedness in Π'_1, even in the single-succedent restriction.

Case $L^3_{\approx}$: We start with the following proof:

$$\dfrac{\dfrac{\Pi_1}{\Gamma, A \approx B, C \approx D, (A \otimes C) \approx (B \otimes D) \vdash \Delta}}{\Gamma, A \approx B, C \approx D \vdash \Delta} \; L^3_{\approx}$$

We apply the induction hypothesis to the subproof Π_1 to get Π'_1.

$$\dfrac{\Pi'_1}{\Gamma : \vec{n}, A \approx B : a_1, C \approx D : a_2, (A \otimes C) \approx (B \otimes D) : a_3 \vdash \Delta : \check{n}a_1a_2a_3}$$

Then we use Lemma 3.2 to replace a_3 with a_1a_2.

$$\dfrac{\dfrac{\Pi'_1[a_1a_2/a_3]}{\Gamma : \vec{n}, A \approx B : a_1, C \approx D : a_2, (A \otimes C) \approx (B \otimes D) : a_1a_2 \vdash \Delta : \check{n}a_1a_2}}{\Gamma : \vec{n}, A \approx B : a_1, C \approx D : a_2 \vdash \Delta : \check{n}a_1a_2} \; L^3_{\approx}(a_1a_2 \in \Delta)$$

Remark C.7. The instance of $L^3_{\approx}$ preserves right connectedness.

Case $\nu L^3_{\approx}$: We start with the following proof:

$$\dfrac{\dfrac{\Pi_1}{\Gamma, A \approx B, C \approx D, (A \otimes C) \approx (B \otimes D) \vdash \Delta}}{\Gamma, A \approx B, C \approx D \vdash \Delta} \; L^3_{\approx}$$

We apply the induction hypothesis to the subproof Π_1 to get Π'_1.

$$\dfrac{\Pi'_1}{\Gamma : \vec{n}, A \approx B : a_1, C \approx D : a_2, (A \otimes C) \approx (B \otimes D) : a_3 \vdash \Delta : \check{n} a_1 a_2 a_3}$$

Then we use Lemma 3.2 to replace a_3 with $\check{n} a_1 a_2$.

$$\dfrac{\dfrac{\Pi'_1[\check{n} a_1 a_2 / a_3]}{\Gamma : \vec{n}, A \approx B : a_1, C \approx D : a_2, (A \otimes C) \approx (B \otimes D) : \check{n} a_1 a_2 \vdash \Delta : \check{n} a_1 a_2}}{\Gamma : \vec{n}, A \approx B : a_1, C \approx D : a_2 \vdash \Delta : \check{n} a_1 a_2} \; \nu L^3_{\approx} (\check{n} a_1 a_2 \in \lceil \Delta \rceil)$$

Remark C.8. The instance of $\nu L^3_{\approx}$ preserves right connectedness.

Case $L^{3*}_{\approx}$: The proof is similar to the one for $L^3_{\approx}$. We start with the proof:

$$\dfrac{\dfrac{\Pi_1}{\Gamma, A \approx B, (A \otimes A) \approx (B \otimes B) \vdash \Delta}}{\Gamma, A \approx B \vdash \Delta} \; L^{3*}_{\approx}$$

We apply the induction hypothesis to the subproof Π_1 to get Π'_1.

$$\dfrac{\Pi'_1}{\Gamma : \vec{n}, A \approx B : a_1, (A \otimes B) \approx (A \otimes B) : a_2 \vdash \Delta : \check{n} a_1 a_2}$$

Then we use Lemma 3.2 to replace a_2 with a_1 and conclude as follows:

$$\dfrac{\dfrac{\Pi'_1[a_1/a_2]}{\Gamma : \vec{n}, A \approx B : a_1, (A \otimes B) \approx (A \otimes B) : a_1 \vdash \Delta : \check{n} a_1}}{\Gamma : \vec{n}, A \approx B : a_1 \vdash \Delta : \check{n} a_1} \; L^{3*}_{\approx}$$

Case $aL^2_{\approx}$: We first apply the induction hypothesis to the subproof Π_1 to get Π'_1:

$$\dfrac{\dfrac{\Pi_1}{\Gamma, A \approx B, B \supset A \vdash \Delta}}{\Gamma, A \approx B \vdash \Delta} \; aL^2_{\approx} \quad \rightsquigarrow \quad \dfrac{\Pi'_1}{\Gamma : \vec{n}, A \approx B : a_1, B \supset A : a_2 \vdash \Delta : \check{n} a_1 a_2}$$

Then we use Lemma 3.2 to replace a_2 with a_1 and conclude as follows:

$$\dfrac{\dfrac{\Pi_1'[a_1/a_2]}{\Gamma : \vec{n}, A \approx B : a_1, B \supset A : a_1 \vdash \Delta : \check{n}a_1}}{\Gamma : \vec{n}, A \approx B : a_1 \vdash \Delta : \check{n}a_1}\, aL^2_{\approx}$$

Case $L^r_{\approx}$: We first apply the induction hypothesis to the subproof Π_1 to get Π_1':

$$\dfrac{\dfrac{\Pi_1}{\Gamma, A \approx B, D, D^A_B \vdash \Delta}}{\Gamma, A \approx B, D \vdash \Delta}\, L^r_{\approx} \quad \rightsquigarrow \quad \dfrac{\Pi_1'}{\Gamma : \vec{n}, A \approx B : a_1, D : a_2, D^A_B : a_3 \vdash C : \check{n}a_1a_2a_3}$$

Then we use Lemma 3.2 to replace a_3 with a_1a_2.

$$\dfrac{\dfrac{\Pi_1'[a_1a_2/a_3]}{\Gamma : \vec{n}, A \approx B : a_1, D : a_2, D^A_B : a_1a_2 \vdash \Delta : \check{n}a_1a_2}}{\Gamma : \vec{n}, A \approx B : a_1, D : a_2 \vdash \Delta : \check{n}a_1a_2}\, L^r_{\approx}(a_1a_2 \in \Delta)$$

Remark C.9. The instance of $L^r_{\approx}$ preserves right connectedness.

Case $\nu L^r_{\approx}$: We first apply the induction hypothesis to the subproof Π_1 to get Π_1':

$$\dfrac{\dfrac{\Pi_1}{\Gamma, A \approx B, D, D^A_B \vdash \Delta}}{\Gamma, A \approx B, D \vdash \Delta}\, L^r_{\approx} \quad \rightsquigarrow \quad \dfrac{\Pi_1'}{\Gamma : \vec{n}, A \approx B : a_1, D : a_2, D^A_B : a_3 \vdash C : \check{n}a_1a_2a_3}$$

Then we use Lemma 3.2 to replace a_3 with $\check{n}a_1a_2$.

$$\dfrac{\dfrac{\Pi_1'[\check{n}a_1a_2/a_3]}{\Gamma : \vec{n}, A \approx B : a_1, D : a_2, D^A_B : \check{n}a_1a_2 \vdash \Delta : \check{n}a_1a_2}}{\Gamma : \vec{n}, A \approx B : a_1, D : a_2 \vdash \Delta : \check{n}a_1a_2}\, \nu L^r_{\approx}(\check{n}a_1a_2 \in \lceil\Delta\rceil)$$

Remark C.10. The instance of $\nu L^r_{\approx}$ preserves right connectedness. □

C.1. Rule Variants that Fail and How to Fix Them

Let us consider the following increasing (Kripke-monotonic) and synchronizing (same-label) variants of $L^3_\approx$ and $L^r_\approx$:

$$\frac{\Gamma, A \approx B : x, C \approx D : y, A \otimes C \approx B \otimes D : y \vdash \Delta}{\Gamma, A \approx B : x, C \approx D : y \vdash \Delta}\ kL^3_\approx (x \subseteq y)$$

$$\frac{\Gamma, A \approx B : x, C \approx D : x, A \otimes C \approx B \otimes D : x \vdash \Delta}{\Gamma, A \approx B : x, C \approx D : x \vdash \Delta}\ sL^3_\approx$$

$$\frac{\Gamma, A \approx B : x, D : y, D^A_B : y \vdash \Delta}{\Gamma, A \approx B : x, D : y \vdash \Delta}\ kL^r_\approx (x \subseteq y) \qquad \frac{\Gamma, A \approx B : x, D : x, D^A_B : x \vdash \Delta}{\Gamma, A \approx B : x, D : x \vdash \Delta}\ sL^r_\approx$$

We show that Theorem 8.2 extends to the variants discussed above only in the presence of the following explicit left Kripke monotonicity rule:

$$\frac{\Gamma, A : x, A : y \vdash \Delta}{\Gamma, A : x \vdash \Delta}\ L_k (x \subseteq y)$$

An immediate consequence is that using the variants in place of the original rules does not allow fully structural-free complete calculi. Let us also note that L_k subsumes contraction, which corresponds to the special case when $x = y$.

Case $kL^3_\approx$ **(fails):** We start with the following proof:

$$\frac{\dfrac{\Pi_1}{\Gamma, A \approx B, C \approx D, (A \otimes C) \approx (B \otimes D) \vdash \Delta}}{\Gamma, A \approx B, C \approx D \vdash \Delta}\ L^3_\approx$$

We apply the induction hypothesis to the subproof Π_1 to get Π_1'.

$$\frac{\Pi_1'}{\Gamma : \vec{n}, A \approx B : a_1, C \approx D : a_2, (A \otimes C) \approx (B \otimes D) : a_3 \vdash \Delta : \check{n} a_1 a_2 a_3}$$

Then we use Lemma 3.2 to replace a_2 and a_3 with $a_1 a_2$ and since $a_1 \subseteq a_1 a_2$ we can apply an instance of $kL^3_\approx$.

$$\dfrac{\begin{array}{c}\Pi'_1[a_1a_2/a_2; a_1a_2/a_3] \\ \hdashline \Gamma : \vec{n}, A \approx B : a_1, C \approx D : a_1a_2, (A \otimes C) \approx (B \otimes D) : a_1a_2 \vdash \Delta : \check{n}a_1a_2\end{array}}{\Gamma : \vec{n}, A \approx B : a_1, C \approx D : a_1a_2 \vdash \Delta : \check{n}a_1a_2}\, kL^3_{\approx}(a_1 \subseteq a_1a_2)$$

However, the conclusion in the labeled proof is not a translation of the conclusion in the label-free proof because $C \approx D$ should be labeled with a_2.

With an explicit rule for left Kripke monotonicity, we can fix the problem as follows:

$$\dfrac{\dfrac{\begin{array}{c}\Pi'_1[a_1a_2/a_2; a_1a_2/a_3] \\ \hdashline \Gamma : \vec{n}, A \approx B : a_1, C \approx D : a_1a_2, (A \otimes C) \approx (B \otimes D) : a_1a_2 \vdash \Delta : \check{n}a_1a_2\end{array}}{\Gamma : \vec{n}, A \approx B : a_1, C \approx D : a_1a_2 \vdash \Delta : \check{n}a_1a_2}\, kL^3_{\approx}(a_1 \subseteq a_1a_2)}{\Gamma : \vec{n}, A \approx B : a_1, C \approx D : a_2 \vdash \Delta : \check{n}a_1a_2}\, L_k(a_2 \subseteq a_1a_2)$$

Remark C.11. The instances of L_k and $kL^3_{\approx}$ preserve right connectedness.

Case $sL^3_{\approx}$ **(fails):** We start with the following proof:

$$\dfrac{\begin{array}{c}\Pi_1 \\ \hdashline \Gamma, A \approx B, C \approx D, (A \otimes C) \approx (B \otimes D) \vdash \Delta\end{array}}{\Gamma, A \approx B, C \approx D \vdash \Delta}\, L^3_{\approx}$$

We apply the induction hypothesis to the subproof Π_1 to get Π'_1.

$$\begin{array}{c}\Pi'_1 \\ \hdashline \Gamma : \vec{n}, A \approx B : a_1, C \approx D : a_2, (A \otimes C) \approx (B \otimes D) : a_3 \vdash \Delta : \check{n}a_1a_2a_3\end{array}$$

Then we use Lemma 3.2 to replace a_2 and a_3 with a_1.

$$\dfrac{\begin{array}{c}\Pi'_1[a_1/a_2; a_1/a_3] \\ \hdashline \Gamma : \vec{n}, A \approx B : a_1, C \approx D : a_1, (A \otimes C) \approx (B \otimes D) : a_1 \vdash \Delta : \check{n}a_1a_2\end{array}}{\Gamma : \vec{n}, A \approx B : a_1, C \approx D : a_1 \vdash \Delta : \check{n}a_1}\, sL^3_{\approx}$$

However, the conclusion in the labeled proof is not a translation of the conclusion in the label-free proof because $C \approx D$ should be labeled with a_2.

With an explicit rule for left Kripke monotonicity, we can fix the problem as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\Pi_1'[a_1a_2/a_2; a_1a_2/a_3]}{\Gamma : \vec{n}, A \approx B : a_1a_2, C \approx D : a_1a_2, (A \otimes C) \approx (B \otimes D) : a_1a_2 \vdash \Delta : \check{n}a_1a_2}}{\Gamma : \vec{n}, A \approx B : a_1a_2, C \approx D : a_1a_2 \vdash \Delta : \check{n}a_1a_2}\, sL^3_{\approx}}{\Gamma : \vec{n}, A \approx B : a_1a_2, C \approx D : a_2 \vdash \Delta : \check{n}a_1a_2}\, L_k(a_2 \subseteq a_1a_2)}{\Gamma : \vec{n}, A \approx B : a_1, C \approx D : a_2 \vdash \Delta : \check{n}a_1a_2}\, L_k(a_1 \subseteq a_1a_2)$$

Remark C.12. The instances of L_k and $sL^3_{\approx}$ preserve right connectedness.

Case $kL^r_{\approx}$ **(fails):** Similar to $kL^3_{\approx}$.

Case $sL^r_{\approx}$ **(fails):** Similar to $sL^3_{\approx}$.

C.2. Technical Height-Preserving Lemmas

Let us write $\vdash^n_{\mathsf{S}}$ instead of $\vdash_{\mathsf{S}}$ to indicate provability in a proof-system S but only for proofs with height less than some natural number n.

LEMMA C.13. *Let* s *be a labeled sequent* $\Gamma \vdash \Delta$, $[u/c]$ *be a label substitution such that* $c \in \mathbf{L}^1$ *or* $c = \emptyset$, *and* $s[u/c]$ *be the sequent obtained from* s *by simultaneously applying* $[u/c]$ *to all of the labeled formulas occurring in* s. *If* $\vdash^n_{\mathsf{L3_{ISCI}}}$ s *then* $\vdash^n_{\mathsf{L3_{ISCI}}}$ $s[u/c]$.

PROOF: By induction on the height h of the proof of $\Gamma \vdash \Delta$. The base case $h = 0$ is when s is the conclusion of an axiom.

Case id**:** This case subsumes both id_p and $id_{\approx}$. Suppose that s is of the form $\Gamma, A : x \vdash \Delta, A : y$ with $x \subseteq y$. If $c \not\subseteq y$ then $s[u/c] = s$ and the result is immediate. Otherwise, $c \subseteq y$ and $y = (y - c) \cup c$. Since $x \subseteq y$, $y = (y - x) \cup x$ implies $y = (y - (x \cup c)) \cup (x - c) \cup c$. Hence, $y[u/c] = (y - (x \cup c)) \cup (x - c) \cup u$. We then show that $s[u/c]$ remains an axiom for

A by showing that $x[u/c] \subseteq y[u/c]$. If $c \not\subseteq x$ then $x[u/c] = x = x - c$ and $x - c \subseteq y[u/c]$. If $c \subseteq x$ then $x[u/c] = ((x-c) \cup c)[u/c] = (x-c) \cup u$ and $(x - c) \cup u \subseteq y[u/c]$.

Cases $L_{\bot}^{K}$ **and** $L_{\bot}^{B}$**:** Similar to Case id.

For the inductive case $h = n + 1$, let r be the last rule applied (which has s as a conclusion). If r requires the introduction of eigenvariables we proceed as follows.

Case $L_{\vee}^{B}$**:** Suppose that s is of the form $\Gamma, A \vee B : x \vdash \Delta, C : y$ and is obtained by the rule $L_{\vee}^{B}$ from the premises $s_1 = \Gamma, A : xa \vdash \Delta, C : ya$ and $s_2 = \Gamma, B : xb \vdash \Delta, C : yb$, where $a, b \not\subseteq \Gamma \cup \Delta$, which have proofs Π_1, Π_2 such that $h(\Pi_1), h(\Pi_2) \leqslant n$. We choose two labels $a' \neq b'$ such that $a', b' \not\subseteq \Gamma \cup \Delta$ and $a', b' \not\subseteq xyuabc$. By induction hypothesis on Π_1 and Π_2 with substitutions $[a'/a]$ and $[b'/b]$ we get proofs Π_1' and Π_2' of $\Gamma, A : xa' \vdash \Delta, C : ya'$ and $\Gamma, B : xb' \vdash \Delta, C : yb'$. Then, by induction hypothesis on Π_1' and Π_2' with substitution $[u/c]$, we get proofs Π_1'' and Π_2'' of $\Gamma[u/c], A \vee B : x[u/c], A : x[u/c]a' \vdash \Delta[u/c], C : y[u/c]a'$ and $\Gamma[u/c], A \vee B : x[u/c], B : x[u/c]b' \vdash \Delta[u/c], C : y[u/c]b'$ from which we infer the conclusion $\Gamma[u/c], A \vee B : x[u/c] \vdash \Delta[u/c]$ by the rule $L_{\vee}^{B}$.

Case $R_{\supset}$**:** Similar to Case $L_{\vee}^{B}$.

If r does not require eigenvariables, we apply the induction hypothesis on the premises of r since they have proofs of height strictly less than $n + 1$ and we conclude $s[u/c]$ by reapplying r. □

Lemma C.14 shows that weakening is height-preserving admissible for all calculi in the $\mathsf{L3_{ISCI}}$ family.

LEMMA C.14. *Weakening is height-preserving eliminable in* $\mathsf{L3_{ISCI}}$, *that is, if* $\vDash^{n}_{\mathsf{L3_{ISCI}}} \Gamma \vdash \Delta$, *then* $\vDash^{n}_{\mathsf{L3_{ISCI}}} \Gamma, \Gamma' \vdash \Delta$ *and* $\vDash^{n}_{\mathsf{L3_{ISCI}}} \Gamma \vdash \Delta, \Delta'$.

PROOF: The proof is by induction on the height h of a proof Π of $\Gamma \vdash \Delta$. For $h = 0$, it is clear that if $\Gamma \vdash \Delta$ is an axiom, then so are $\Gamma, \Gamma' \vdash \Delta$ and $\Gamma \vdash \Delta, \Delta'$.

For $h = n + 1$, let r be the last rule applied in Π. If r is not $R_{\supset}$ or $L^{B}_{\vee}$, we apply the induction hypothesis on the premises of r and conclude by reapplying r. Otherwise, we first use Lemma C.13 to replace the eigenvariables in all of the premises of r with variables not occurring in $\Gamma \cup \Gamma' \cup \Delta \cup \Delta'$ and then apply the induction hypothesis to the modified premises before concluding with a new instance of r. □

Corollary C.15. The proof translation given in Theorem 8.2 is height-preserving. Hence, if $\vdash^{n}_{\mathsf{G3_{ISCI}}} A$, then $\vdash^{n}_{\mathsf{L3_{ISCI}}} A$.

Proof: Let us first observe that label substitution and weakening admissibility are height-preserving. It then follows that the translation described in Theorem 8.2 is also height-preserving since it is a one-to-one mapping of each rule in $\mathsf{G3_{ISCI}}$ to the corresponding rule in $\mathsf{L3_{ISCI}}$. □

D. Appendix: Kripke Soundness of $\mathsf{G3^{K}_{ISCI}}$

Theorem D.1. $\mathsf{G3^{K}_{ISCI}}$ *is sound w.r.t. the Kripke semantics of* ISCI*: if* $\vdash_{\mathsf{G3^{K}_{ISCI}}} A$ *then* $\models_{\mathsf{K}} A$.

Proof: Let S be a finite set of formulas $\{F_1, \ldots, F_n\}$. We define $\bigwedge S$ and $\bigvee S$ as the formulas $F_1 \wedge F_2 \wedge \ldots \wedge F_n$ and $F_1 \vee F_2 \ldots \vee F_n$ with the special cases $\bigwedge \emptyset = \top$ and $\bigvee \emptyset = \bot$.

We define the realizability of a sequent $\Gamma \vdash \Delta$ as the following property: forall Kripke models $\mathcal{M}$ and all words m in $\mathcal{M}$, if $m \Vdash \bigwedge \Gamma$ then $m \Vdash \bigvee \Delta$. For all rules in $\mathsf{G3^{K}_{ISCI}}$, we show that if all premises are realizable, then so is the conclusion.

Case id: This case subsumes both id_p and $id_{\approx}$.

Suppose we have $\dfrac{}{\Gamma, A \vdash \Delta, A}\,\mathrm{id}$ and let m be a world in a Kripke model. If $m \Vdash \bigwedge \Gamma \wedge A$, then $m \Vdash A$. Hence $m \Vdash \bigvee \Delta \vee A$.

Case $L_{\bot}$: Suppose we have $\dfrac{}{\Gamma, \bot \vdash \Delta}\,L_{\bot}$ and let m be a world in a Kripke model. Since $m \nVdash \bot$, we immediately have that $m \Vdash \bigwedge \Gamma \wedge \bot$ implies $m \Vdash \bigvee \Delta$.

Case $R_\vee$**:** We consider the rule $\dfrac{\Gamma \vdash \Delta, A_1, A_2}{\Gamma \vdash \Delta, A_1 \vee A_2} R_\vee$ and m a world in a Kripke model such that $m \Vdash \bigwedge \Gamma$.
By assumption from the premiss we get $m \Vdash \bigvee \Delta \vee A_1 \vee A_2$.

Case $L^K_\vee$**:** We consider the rule $\dfrac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta} L^K_\vee$ and m a world in a Kripke model such that $m \Vdash \bigwedge \Gamma \wedge (A \vee B)$.
If $m \Vdash A$ then $m \Vdash \bigwedge \Gamma \wedge A$ and by assumption from the first premiss $m \Vdash \bigwedge \Gamma \wedge A$ implies $m \Vdash \bigwedge \Delta$. Otherwise, since $m \Vdash A \vee B$, we necessarily have $m \Vdash B$ and by assumption from the second premiss $m \Vdash \bigwedge \Gamma \wedge B$ implies $m \Vdash \bigvee \Delta$. Hence, $m \Vdash \bigvee \Delta$.

Case $R_\supset$**:** We consider the rule $\dfrac{\Gamma, A \vdash B}{\Gamma \vdash \Delta, A \supset B} R_\supset$ and m a world in a Kripke model such that $m \Vdash \bigwedge \Gamma$.
If $m \Vdash \bigvee \Delta$ then $m \Vdash \bigvee \Delta \vee (A \supset B)$. Otherwise, we need to show that $m \Vdash A \supset B$. Suppose some arbitrary n such that $m \leqslant n$. If $n \Vdash A$ then, since by Kripke monotonicity we also have $n \Vdash \bigwedge \Gamma$, it follows that $n \Vdash \bigwedge \Gamma \wedge A$. Hence, by assumption from the premiss, we have $n \Vdash B$. Hence, $m \Vdash A \supset B$, which implies $m \Vdash \bigvee \Delta \vee (A \supset B)$.

Case $L^1_\approx$**:** We consider the rule $\dfrac{\Gamma, A \approx A \vdash \Delta}{\Gamma \vdash \Delta} L^1_\approx$ and m a world in a Kripke model such that $m \Vdash \bigwedge \Gamma$.
By condition $\mathcal{M}_{\approx_1}$ of Kripke models, we have $m \Vdash A \approx A$. Hence, by assumption from the premiss, we get $m \Vdash \bigvee \Delta$.

Case $L^2_\approx$**:** Similar to Case $L^3_\approx$.

Case $aL^2_\approx$**:** Similar to Case $L^3_\approx$.

Case $L^3_\approx$**:** We consider the rule $\dfrac{\Gamma, A \approx B, C \approx D, A \otimes C \approx B \otimes D \vdash \Delta}{\Gamma, A \approx B, C \approx D \vdash \Delta} L^3_\approx$ and m a world in a Kripke model such that $m \Vdash \bigwedge \Gamma \wedge A \approx B \wedge C \approx D$.
Since $m \Vdash A \approx B$ and $m \Vdash C \approx D$, we have $m \Vdash A \otimes C \approx B \otimes D$ by

condition $\mathcal{M}_{\approx_4}$ of Kripke models. Therefore, by assumption from the premiss, we get $\mathrm{m} \Vdash \bigvee \Delta$.

Case $\mathrm{L}^{3*}_{\approx}$: Similar to Case $\mathrm{L}^{3}_{\approx}$.

Case $\mathrm{L}^{r}_{\approx}$: We consider the rule $\dfrac{\Gamma, A \approx B, D, D^A_B \vdash \Delta}{\Gamma, A \approx B, D \vdash \Delta}\,\mathrm{L}^{r}_{\approx}$ and m a world in a Kripke model such that $\mathrm{m} \Vdash \bigwedge \Gamma \wedge A \approx B \wedge D$.
By the replacement law, we have $\mathrm{m} \Vdash D^A_B$. Hence, by assumption from the premiss, we get $\mathrm{m} \Vdash \Delta$.

The other cases are similar. □

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