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# MODAL LOGIC OF LATTICES

**Abstract**

We prove that the modal logic of lattices with the accessibility relation of being isomorphic to a sublattice is S4.2. The same is proven for modular and distributive lattices.

*Keywords*: modal logic of classes of structures, lattices, order, distributivity.

## 1. Introduction

Let $\mathcal{L}$ be a first-order language and $T$ a theory in said language. We consider the class $Mod(T)$ of all models of the theory $T$, along with the relation $\subsetneqq$ interpreted as *embeddability*; we write $M \subsetneqq N$ if there is an embedding $f\colon M \to N$. This gives rise to a Kripke frame $(Mod(T), \subsetneqq)$, whose modal logic we shall investigate.

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The paper is concerned with the modal logic of the lattice theory (by which we mean the modal logic of the frame $(Mod(T), \subsetneq)$, for $T =$ lattice theory), as well as some stronger theories: theory of modular lattices and theory of distributive lattices.

The study of modal logics of classes of structures began with the study of modal logics of set theory and arithmetic. In these and the following work the modal operators were interpreted a little bit differently than in our case — $\Box$ was interpreted as ”in all forcing extensions” [5], ”in all ground models” [7] etc. In these works it was demanded from the language $\mathcal{L}$ to be strong enough to be able to express the interpretation of the operator $\Box$ [9]. This has been generalised and more recent papers on the topic consider the case where $\mathcal{L}$ is a first-order language, which is not strong enough [6, 1]. We follow these authors in taking $\mathcal{L}$ to be first-order, as well as investigating the relation $\subsetneq$ on the class $Mod(T)$. Intuitively, as Saveliev and Shapirovsky describe ”robust” theories to be the ”true” modal logics of a given relation, we think of the modal logic of $(Mod(T), \subsetneq)$ to be the ”true” modal logic of a theory (especially since the said frame is bisimilar to a frame where the relation $\subsetneq$ is replaced with the direct extension relation $\subseteq$ [9]).

The cases where $T =$ graph theory and $T =$ theory of abelian groups are known thanks to the authors of [1] and [6]. Lattice theory seems to be a natural theory to be investigated in this manner, in order to see if the same is true for graphs, abelian groups and lattices. In this paper we prove that this is in fact true.

## 2. Preliminaries

Let $\mathcal{L}_\Box$ be a standard propositional modal language, that is a countable set of propositional variables $Var$ together with the set of logical symbols $\{\wedge, \neg, \Box\}$ — the symbols $\rightarrow, \leftrightarrow, \vee$ and $\Diamond$ are defined as usual. The set $Fm_\Box$ of modal formulas is defined in a standard way.

The set of modal formulas $\Lambda$ is called a *modal logic* if it is closed under modus ponens and substitution. If it is furthermore closed under necessitation $(\frac{\alpha}{\Box\alpha})$ and contains the K axiom (the formula $\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$) then it is called a *normal modal logic*.

The modal logic relevant to our research is well known logic S4.2, which is the smallest normal modal logic containing the following formulas:

$$\begin{aligned} \mathrm{T}: &\quad \Box p \to p, \\ 4: &\quad \Box p \to \Box\Box p, \\ .2: &\quad \Diamond\Box p \to \Box\Diamond p. \end{aligned}$$

Validity in a Kripke model $\mathbb{M} = (\mathcal{W}, R, v)$ — where $\mathcal{W}$ is a nonempty collection of *possible worlds*, $R$ is an accessibility relation on $\mathcal{W}$, and $v\colon Var \to \mathcal{P}(\mathcal{W})$ is a valuation — is defined in a standard way. If $M \in \mathcal{W}$, then:

$$\begin{aligned} M \Vdash p &\iff M \in v(p), \\ M \Vdash \alpha \wedge \beta &\iff M \Vdash \alpha \text{ and } M \Vdash \beta, \\ M \Vdash \neg\alpha &\iff M \nVdash \alpha, \\ M \Vdash \Box\alpha &\iff R(M,N) \Rightarrow N \Vdash \alpha, \text{ for all } N \in \mathcal{W}. \end{aligned}$$

A formula $\alpha$ is valid in a Kripke model $\mathbb{M}$ if for all $M \in \mathcal{W}$, $M \Vdash \alpha$. Similarly, a formula is valid in a Kripke frame $\mathbb{F} = (\mathcal{W}, R)$ if it is valid in every model built on that frame.

Our research concerns the modal logic of the frame $(Mod(T), \subsetneq)$. Since members of $Mod(T)$ are first-order structures, it is important to define valuations accordingly. First, any function $t\colon Var \to Fm_{\mathcal{L}}$, where $Fm_{\mathcal{L}}$ is the set of well formed formulas of the language $\mathcal{L}$, is called an *$\mathcal{L}$-translation*. Every $\mathcal{L}$-translation $t$ gives rise to a valuation $v_t$:

$$v_t(p) = \{M \in Mod(T) : M \models t(p)\}.$$

It is important to stress out, that our collection of possible worlds is a proper class, and so the ranges of valuations $v_t$ are power sets of proper classes. The functions are definable, since the relation of satisfiablity is definable in the language of set theory, so we use only valuations $v_t$ (for all $\mathcal{L}$-translations $t$) in the frame in order to avoid metamathematical issues.

In order to validate the soundness of a given modal logic in the frame one needs to check certain properties of the accessibility relation; if the relation satisfies properties that are characteristic for a certain logic, then this logic is valid in a frame.

Theorem 2.1. *If $\mathbb{F}$ is a frame such that its accessibility relation is transitive, reflexive and directed, then S4.2 is valid in $\mathbb{F}$.*

Proof: This is a simple proof one can find as an exercise in many handbooks concerning modal logic. See for example [2]. $\square$

The main tool for proving completeness results is the method of *control statements*, developed by Hamkins and Löwe [5]. Said technique is based on the Jankov-Fine formulas. It works in an environment where a given logic conjectured to be logic complete with respect to that frame (its upper bound) is a normal modal logic with the finite frame property. In this technique various kinds of control statements are used. In our case only two of those kinds will be useful: buttons and dials.

A sentence $\varphi \in Fm_{\mathcal{L}}$ is called a *button* in a first-order model $M$ iff for every $\mathcal{L}$-translation $t$ and every propositional variable $p$ such that $t(p) = \varphi$ we have $(Mod(T), \subsetneq, v_t), M \Vdash \Diamond\Box p$. A set of sentences $\{\varphi_0, \ldots, \varphi_n\} \subseteq Fm_{\mathcal{L}}$ is called a *dial* in a model $M$ iff for every $\mathcal{L}$-translation $t$ and every $p \in Var$ such that $t(p_i) = \varphi_i$ we have that $(Mod(T), \subsetneq, v_t), M \Vdash \Diamond p_i$, for every $i \in \{0, \ldots, n\}$, $(Mod(T), \subsetneq, v_t), M \Vdash \Box \bigvee_{j \in \{0,\ldots,n\}} p_j$ and $(Mod(T), \subsetneq, v_t), M \Vdash p_k$, for exactly one $k \in \{0, \ldots, n\}$. To put it in simpler words, the idea behind buttons, is that they are statements that are possibly necessary. They are true in some extension, and from that point onward they are true in all further extensions. A button $\beta$ in a model $M$ is said to be *pushed* if $M \Vdash \Box p$, for all propositional variables $p$, and all translations $t$ such that $t(p) = \beta$. Otherwise, the button is said to be *unpushed*. So one can push any buttons that are yet unpushed in a model by going to an extension in which they are already pushed — but once it is done, they will never become unpushed again. This idea works in this way only in the directed environment, so when the logic S4.2 is valid – in weaker logics, like S4 a stronger notion is needed for the same result (Hamkins, Leibman and Löwe introduce a notion of a weak button [4]). Dials on the other hand, can be set as desired always. For a given dial $\{\psi_0, \ldots, \psi_n\}$ model $M$ is said to have a dial value $j$ (for $j \leqslant n$) if $M \models \psi_j$. Dial is a set of statements in a model: exactly one of them is true, but one can switch to any other statement, changing the value of the dial — because all of the statements are possible. The value can go back and forth indefinitely.

A finite set of buttons is said to be *independent* of a finitely long dial in a model $M$ iff one can push only the desired buttons and set the dial as desired as well without interfering with one another. To put it more formally, if $\{\varphi_0, \ldots, \varphi_n\}$ is a set of buttons and $\{\psi_0, \ldots, \psi_k\}$ is a dial, then for every $I_0 \subseteq I_1 \subseteq \{0, \ldots, n\}$ and for every $m \in \{0, \ldots, k\} = J$ and every $\mathcal{L}$-translation $t$ such that $t(p_i) = \varphi_i$ and $t(q_j) = \psi_j$ we have that a formula:

$$(\bigwedge_{i\in I_0} \Box p_i \wedge \bigwedge_{i\notin I_0} \neg\Box p_i \wedge q_j \wedge \bigwedge_{l\in J\setminus\{j\}} \neg q_l) \to \Diamond(\bigwedge_{i\in I_1} \Box p_i \wedge \bigwedge_{i\notin I_1} \neg\Box p_i \wedge q_m \wedge \bigwedge_{l\in J\setminus\{m\}} \neg q_l)$$

is valid in $(Mod(T), \subsetneq, v_t), M$, for some $j \in J$.

This terminology is used by Hamkins and Löwe in the proof of the following theorem:

THEOREM 2.2 ([5, 4]). *If a world $M$ in a frame validates arbitrarily large finite sets of buttons independent of an arbitrarily large finite dial, then its modal logic is contained within S4.2, as long as the buttons are unpushed — if they all are pushed, the validities are contained within S5.*

## 3. The modal logic of lattice theory

In this section we shall consider the case, where $T$ = lattice theory, that is the first-order theory in a first-order language augmented by two binary functional symbols $\wedge, \vee$ and characterised by the universal closure of the following axioms:

$$\begin{aligned} x \vee (y \vee z) &= (x \vee y) \vee z, & x \wedge (y \wedge z) &= (x \wedge y) \wedge z; \\ x \vee y &= y \vee x, & x \wedge y &= y \wedge x; \\ x \vee (x \wedge y) &= x, & x \wedge (x \vee y) &= x. \end{aligned}$$

The theorem we aim to prove is the following:

THEOREM 3.1. *The modal logic of the lattice theory (the modal logic of the frame $(Mod(T), \subsetneq)$, for $T$ = lattice theory) is exactly S4.2.*

Proof: To establish S4.2 as a lower bound we simply need to check that the relation $\subsetneqq$ is reflexive, transitive and directed. The first two are trivial, and lattices exhibit even a stronger quality with respect to embeddings than directedness — any two lattices $L_1, L_2$ have a common extension (take for example a direct product of lattices $L_1$ and $L_2$).

In order to establish an upper bound, we are going to use the beforementioned technique of control statements. For an arbitrary lattice $L$, we are going to find arbitrarily long sets of buttons, as well as an arbitrarily long dial.

**Dials:** For dials of any finite length (larger than 1) we take for each $n \in \mathbb{N}$ the sequences $\{\varphi_0, \ldots, \varphi_{n-1}, \varphi_n\}$, where $\varphi_0$ states that there are no atoms or no least element, $\varphi_i$, for $1 \leqslant i < n$ states that "there are exactly $i$ atoms", and $\varphi_n$ states that "there are at least $n$ atoms" (atoms are understood in the standard way, as elements immediately above the least element of the lattice, a statement "there are exactly $i$ atoms" implies existence of the least element). In any lattice $L$ one of the sentences of any of such dials is always true, as the lattice always has some amount of atoms: let $i$ be the cardinality of the set of atoms of the lattice $L$ ($i$ may be infinite). If $i < n$, then $L \models \varphi_i$, otherwise $L \models \varphi_n$. So any lattice satisfies exactly one of the sentences $\varphi_i$ from a given dial.

Any of the sentences $\varphi_i$ is possible, necessarily so: one can always take any lattice $L$, no matter the dial volume it satisfies, and add a descending infinite chain below all the elements of $L$ to bring the value down to $\varphi_0$ in such a lattice $L'$. From there one can add any lattice $L_k$, with $k$ atoms below all of $L'$'s elements. Let $L_k$ be a lattice $M_k$ (a $k$-wide diamond). This new extension satisfies any desired dial value.

**Buttons:** The idea used for buttons is that of a cycle from graph theory. To mimic a notion of a cycle of a length $n$ (for $n \geqslant 3$) in a lattice we construct a lattice $W_n$ in a following way: take sets $\{1\}, \ldots, \{n\}$. Then take each pair of numbers of the form $\{m, m+2\}$ (for $1 \leqslant m \leqslant n-2$), as well as the pairs $\{1,2\}$ and $\{n-1, n\}$ additionally. Those sets together with the sets $\{1, \ldots, n\}, \varnothing$ form a lattice $W_n$ together with two operations: $W_n = (L_n, \cup^*, \cap)$, where $L_n = \{\varnothing, \{1, \ldots, n\}, \bigcup\{\{m\} : m \in \{1, \ldots, n\}\}, \bigcup\{\{k, k+2\} : k \in \{1, \ldots, n-2\}\}, \{1,2\}, \{n-1, n\}\}$ and $a \cup^* b =$

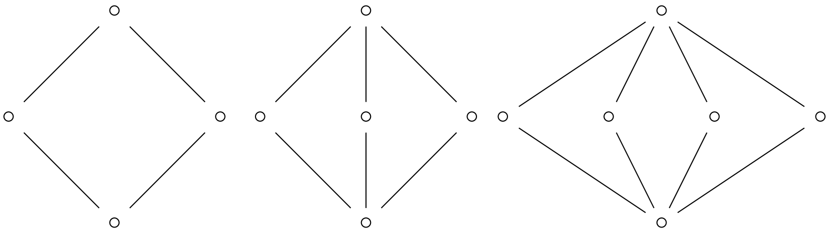

Figure 1: Lattices $M_2, M_3$ and $M_4$ respectively.

$min\{x \in L : a \cup b \subseteq x\}$. Each of these lattices is finite, so any lattice $W_n$ can be characterised as a sublattice by a first-order sentence $\psi_n$. For buttons we take those sentences $\psi_n$, stating that a lattice $W_n$ is a sublattice of a lattice under consideration.

We need to prove that they are independent of dials: this is easy, since the dial is concerned about what happens at the lowest parts of the lattice, and one can easily extend the lattice downwards, so that other parts of the lattice are not interfered with. The fact that none of the lattices $M_n$ contain any of $W_m$ as sublattice is trivial. What remains to be proven is that none of the lattices $W_m$ contain any lattice $W_l$ as a sublattice, for $m \neq l$.

Let $m, l$ be any two natural numbers such that $l < m$. We will show $W_m$ does not contain $W_l$ as a subblattice.

Suppose that $f \colon W_l \to W_m$ is a injection. We will show that $f$ is not a homomorphism. First, observe that if there is $i \leqslant l$ such that $f(\{i\}) \neq \{n\}$, for some $n \leqslant m$, then $f$ is not a homomorphism. Of course, if $f(\{i\}) = \varnothing$ or $\{1, \ldots, m\}$ then $f$ is not a homomorphism. Let us assume, that $f(\{i\}) = \{n, k\}$, for some $n, k \leqslant m$. By the definition of $W_l$, there exists $j \leqslant l$ such that $\{i\} \vee_{W_l} \{j\} = \{i, j\}$. In $W_l$ we have $\{i\} \leqslant_{W_l} \{i, j\} \leqslant_{W_l} \{1, \ldots, l\}$, but since $f$ is an injection and $f(\{i\}) = \{n, k\}$ this chain cannot be replicated by $f(\{i\}), f(\{i, j\}), f(\{1, \ldots, l\})$.

Suppose then, that for all $i \leqslant l$ there exists $n \leqslant m$ such that $f(\{i\}) = \{n\}$. Let us call $\{a\}, \{b\} \in W_s$ neighbours in $W_s$ iff $\{a, b\} \in W_s$. Because $l < m$, there exists $n \leqslant m$ such that there is no $i \leqslant j$ such that $f(\{i\}) = \{n\}$

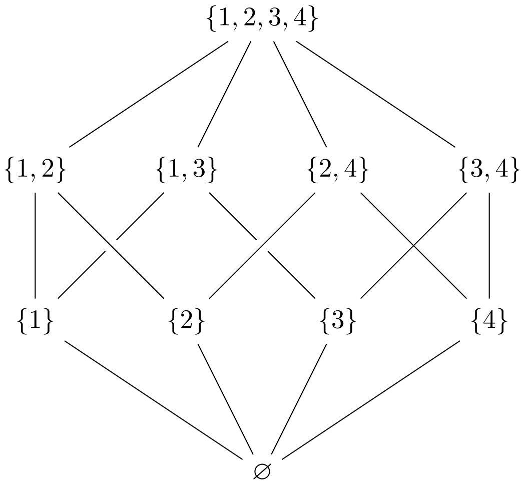

Figure 2: Lattice $W_4$.

and $\{n\}$ has a neighbour $\{k\}$ in $W_m$ such that there is some $a \leqslant l$ such that $f(\{a\}) = \{k\}$. It follows that $\{a\}$ has a neighbour $\{b\}$ in $W_l$, and $f(\{b\})$ is not a neighbour of $f(\{a\})$ in $W_m$, and hence $f(\{a\}) \vee_{W_m} f(\{b\}) \neq f(\{a, b\})$ or $f(\{a, b\}) > f(\{1, \dots, l\})$. Hence, $f$ is not a homomorphism. Obviously, it is also not the other way around, so the sets of buttons are independent. It is also fairly obvious that not all lattices have these buttons pushed.

Using Theorem 2.2 we establish a completeness result and conclude the proof of the theorem. □

## 4. Modal logic of modular and distributive lattices

Since the buttons (of index $\geqslant 4$) used to prove the above result are implying that there is a $N_5$ sublattice in any lattice satisfying them (so those lattices are not modular), one may wonder what the modal logic of stronger theories is, mainly the cases where $T$ = theory of modular lattices and $T$ = theory of distributive lattices, so the lattice theory augmented with the universal closures of the following axioms, respectively:

$$x \leqslant y \Rightarrow x \vee (z \wedge y) = (x \vee z) \wedge y, \tag{M}$$

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z). \tag{D}$$

Let us start with investigating modular lattices. It turns out that the modal logic of their theory is the same:

THEOREM 4.1. *The modal logic of modular lattice theory is exactly S4.2.*

PROOF: Lower bound is as easy as before: $\subsetneq$ on modular lattices is directed, as every product of modular lattices is a modular lattice. Obviously it is also transitive and reflexive.

For dials we do not need to change anything. We can use the same idea as before.

**Buttons:** We shall formulate different buttons, as the ones used before do not work in a modular environment. Instead of working as before and defining an infinite sequence of statements, we will define different sets of buttons for every natural number $n$:

In the paper [10] Wroński introduces the operation $\oplus$ on lattices. The finite lattice $L$ is said to be a *sum* of $A$ and $B$, $L = A \oplus B$ in symbols, if $A$ and $B$ are proper sublattices of $L$, such that $A \cup B = L$, and moreover, $A \cap B$ is a filter in $A$, and $A \cap B$ is an ideal in $B$ (see [8], and [3, Chapter 4] for details). We will write $L_1 \oplus_i L_2$ for Wroński's sum where $L_1 \cap L_2 = S_i$, for some $i \in I$.

Consider a lattice $C_n$ that is a chain of length $n$ (for $n \geqslant 3$). Let $D_n = C_{n+2} \times C_2$ be a product lattice. Consider two copies of $D_n$ ($D_n$ and $D_n^*$) and all Wroński's sums $L_{n_{i-2}} = D_n \oplus_i D_n^*$ where $D_n \cap D_n^*$ is a chain of cardinality $i \geqslant 3$. For each $n \in \mathbb{N}$ there are $n$ such sums.

For any $n \in \mathbb{N}$ all $L_{n_k}$ (for $1 \leqslant k \leqslant n$) are finite, hence characterizable by a first order formula $\varphi_k$. For a set of buttons of length $n$ we take sentences $\varphi_1, \ldots, \varphi_n$ charaterising respectively $L_{n_1}, \ldots, L_{n_n}$ as sublattices.

The cardinality of the lattice $L_{n_k}$ is bigger than the cardinality of the lattice $L_{n_j}$, for $j > k$ and all $n \in \mathbb{N}$, so in order to prove independence of the buttons, we need to check that $L_{n_k} \not\subseteq L_{n_j}$. This is also the case, because in each $L_{n_i}$ there are exactly $i$ different triples of elements such that they are incomparable and together with their suprema and infima they form a covering sublattice $L_{1_1}$ — one cannot embed $k$ such triples into a lattice where only $j$ of them exist without identifying some of them.

The dial is independent of the buttons as well, since it concerns what happens at the lowest parts of the lattice, and one can always extend a lattice adding more elements at the bottom to accommodate a given dial value, without interfering in the upper parts. Furthermore, lattices $M_n$ do not contain any of the lattices $L_{n_k}$, and Theorem 2.2 concludes the proof. □

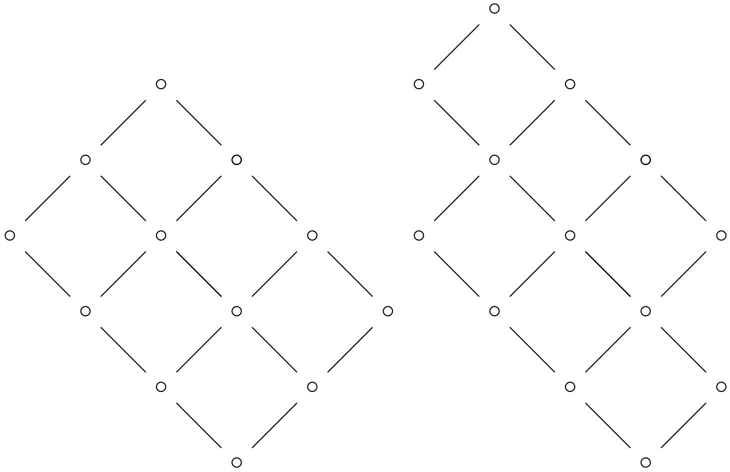

Figure 3: Lattices $L_{2_2}$ and $L_{2_1}$.

Corollary 4.2. The modal logic of the theory of distributive lattices is exactly S4.2.

Proof: Just as before, the product of distributive lattices is a distributive lattice. Thus, $S4.2 \subseteq$ modal logic of distributive lattices.

The lattices implied by the buttons used to prove theorem 4.1 are distributive, so the same idea works here as well.

The dials need to be changed, as lattices $M_n$ are not distributive. We cannot use the idea of atoms, because we might by mistake embed a lattice that satisfies one of our buttons. The idea here is that $\{\psi_0, \ldots, \psi_{n-1}, \psi_n\}$ is a dial, where $\psi_0 =$ "there is no least element", $\psi_m =$ "there are exactly $m - 1$ $\wedge$-irreducible elements between the least element and the least $\wedge$-reducible element" and $\psi_n =$ "there are at least $n-1$ $\wedge$-irreducible elements between the least element and the least $\wedge$-reducible element". This works similarly as before. The theorem 2.2 concludes the proof. ☐

## 5. Summary

Our results in fact are a little bit more general and state that the robust modal logic of the above frames is S4.2 (see [9] section 5). Furthermore, our results can be easily extended to some stronger theories i.e. Stone algebras, or more narrow classes of lattices (they not need to be first-order theories, so in this case the theorem concerns the modal logic of the frame $(\mathcal{S}, \subsetneq)$, where $\mathcal{S} =$ class of all such lattices) using a similar idea. It cannot be done indefinitely, since complete theories have Triv as their modal logic [6]. This leads us to ask a following question:

*Question 5.1.* Is there a theory extending lattice theory that its modal logic is S4.2, and all strictly stronger theories have a different modal logic?

We can ask as well not only about theories, but classes of lattices:

*Question 5.2.* Is there a class $\mathcal{C}$ of lattices such whose modal logic (the modal logic of the frame $(\mathcal{C}, \subsetneq)$) is still S4.2, yet all proper subclasses of $\mathcal{C}$ have a different modal logic?

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