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Maciej A. Hałapacz

ON CERTAIN COMPLETENESS RESULTS FOR S4.2 MODAL LOGIC AND ITS APPLICATIONS IN MODAL LOGICS OF CLASSES OF STRUCTURES

Abstract

In this paper, we obtain new classes of Kripke frames complete for the modal logic S4.2, through the means of the so-called *control statements*. We provide applications of the theorem in the field of modal logics of classes of structures.

Keywords: control statements, modal logic, S4.2, modal logics of classes of structures.

1. Introduction

The modal logic S4.2 is a well-known logic. It is a normal modal logic axiomatised by the following formulas:

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- K: $\Box(p \to q) \to (\Box p \to \Box q)$,
- T: $\Box p \to p$,
- 4: $\Box p \to \Box\Box p$,
- .2: $\Diamond\Box p \to \Box\Diamond p$.

It is widely known to be charaterised by the class of frames that are directed partial preorders (quasiorders), that is, preorders satisfying the condition

$$wRu \;\&\; wRu' \Rightarrow \exists z(uRz \;\&\; u'Rz). \tag{1.1}$$

In [2] Hamkins and Löwe provided another characterisation. First, we need the following definition:

Definition 1.1. A *pre-Boolean algebra* is a preorder $\langle P, \leqslant\rangle$ such that its quotient $\langle P/_{\sim}, \leqslant /_{\sim}\rangle$ by a relation $\sim$ defined

$$w \sim u \Leftrightarrow wRu \;\&\; uRw,$$

is a Boolean lattice.

We can similarly understand the concept of a pre-lattice. Now we can understand the following theorem:

Theorem 1.2 ([2, Theorem 10]). *Modal logic S4.2 is sound and complete with respect to the class of finite pre-Boolean algebras.*

The completeness proof utilises the notion of a *control statements*, namely *buttons* and *switches*. The authors used this result in order to obtain a method for proving that S4.2 is an upper bound on modal validities in a certain class of structures, connected by a model-theoretic relation to construct a certain Kripke model, where possible worlds are first-order structures.

This paper aims to provide other completeness results (countably many) of the logic S4.2 by means of another type of control statements introduced by Hamkins and Löwe, which are *ratchets*. This leads also to another method for proving the completeness of Kripke models of first-order structures.

2. Modal logic

The language we use is that of standard modal logic: let Var be a set of denumerably many propositional variables and $\mathcal{L} = Var \cup \{\wedge, \neg, \Box\}$. The symbols $\vee, \to, \leftrightarrow, \Diamond$ are defined as usual. The set of modal propositional formulas $Fm_{\mathcal{L}}$ is defined in the standard way.

In this paper only the modal logic S4.2 concerns us – we provided its axioms in the introduction. As a normal modal logic, it is closed under the rules of necessitation ($\frac{\alpha}{\Box\alpha}$), modus ponens and substitution.

Let us quickly review the semantics we use here. It is that of standard Kripke models: a Kripke model is a triple $\mathcal{M} = \langle W, R, v\rangle$, where W is a nonempty set, a collection of so called *possible worlds*, $R \subseteq W \times W$ is an *accessibility relation*, and $v\colon Var \to \mathcal{P}(W)$ is a valuation. A Kripke frame is a tuple $\mathcal{F} = \langle W, R\rangle$ consisting only of the first two elements of the Kripke model. Satisfaction in a Kripke model is defined in a standard way:

If $w \in W$, then:

$$\begin{aligned} w \Vdash p &\iff w \in v(p),\\ w \Vdash \alpha \wedge \beta &\iff w \Vdash \alpha \text{ and } w \Vdash \beta,\\ w \Vdash \neg\alpha &\iff w \nVdash \alpha,\\ w \Vdash \Box\alpha &\iff wRu \Rightarrow u \Vdash \alpha, \text{ for all } u \in W. \end{aligned}$$

We say that a formula $\alpha \in Fm_{\mathcal{L}}$ is valid in the model $\mathcal{M}$ iff it is true in every $w \in W$. Similarly, we say that α is valid in a frame $\mathcal{F}$ iff it is valid in all models built on that frame.

The control statements in standard Kripke models are an easy concept: they are special kinds of formulas. In a Kripke model $\mathcal{M}$, in a world $w \in W$ a formula β is a button iff $w \Vdash \Diamond\Box\beta$. Similarly, a formula σ is a switch in w iff $w \Vdash \Box(\Diamond\sigma \wedge \Diamond\neg\sigma)$. Furthermore, we say that a button β in w is *pushed*, if $w \Vdash \beta$, otherwise it is *unpushed*. Similarly, we say that a switch σ is *on* in w, if $w \Vdash \sigma$, otherwise it is *off*. It is useful to take such formulas into sets, which then can be called sets of independent control statements. A set X of control statements is *independent* in w iff:

- for any $B \subseteq X$ of unpushed buttons there exists $u \in W$ such that

wRu and $u \Vdash \alpha$, for $\alpha \in B$ and $u \nVdash \beta$, for any button $\beta \in X \setminus B$ that is unpushed in w,

- for any $S \subseteq X$ of switches there exists $u \in W$ such that wRu and $u \Vdash \sigma \wedge \neg\psi$, for $\sigma \in S$ and switches $\psi \in X \setminus S$.

This framework allowed Hamkins and Löwe to obtain the following result:

THEOREM 2.1 ([2, Lemma 9]). *A class of frames $\mathbb{F}$ is sound and complete for S4.2 if and only if every frame in $\mathbb{F}$ is a S4.2 frame and there are Kripke models with frames in $\mathbb{F}$ for arbitrarily large finite independent families of buttons and switches.*

3. Completeness

In order to provide our result we need to present one special set of control statements introduced by Hamkins and Löwe, called ratchets (volume control in [2], ratchets in [1] and later works). A *ratchet* $\{\beta_1, \ldots, \beta_n\}$ in a world w is a finite sequence of buttons, such that:

- $w \Vdash \Diamond(\beta_i \wedge \neg\beta_k)$, for $k > i$,
- $w \Vdash \Box\beta_m \to \beta_j$, for $j \leqslant m$.

We say that a ratchet's volume at w is i iff $w \Vdash \Box\beta_i$.

A ratchet is not an independent set of control statements, but it can be independent of another set of independent control statements S in w, if:

- pushing buttons in S and turning switches on or off does not imply increasing ratchet's volume,
- increasing ratchet's volume does not imply that any buttons in S that were unpushed are pushed, nor that the pattern of switches is changed.

One more definition is in order:

DEFINITION 3.1. A modal logic Λ is consistent with a set S of control statements, iff there is a Kripke model M for Λ and there is $w \in M$ such that S is an independent set of control statements in w.

Let us establish the following connection between ratchets and buttons:

THEOREM 3.2. *Suppose Λ is a modal logic and $n \geqslant 2$. The following conditions are equivalent:*

1. *Λ is consistent with arbitrarily large finite families of independent buttons,*
2. *Λ is consistent with n arbitrarily large, finite, independent ratchets,*
3. *Λ is consistent with 2 arbitrarily large, finite, independent ratchets.*

PROOF: (1) $\Rightarrow$ (2): Suppose Λ is consistent with arbitrarily large families of independent buttons. That means that there are models of Λ with worlds admitting any finite-sized families of independent buttons. For n ratchets of length m, we take a model M, a world $w \in M$ and B, a family of $n \cdot m$ buttons in w. Divide it in any way into disjoint subsets of cardinality m. For each such subset B_0, the set $\{\beta_1, \beta_1 \wedge \beta_2, \ldots, \bigwedge_{1 \leqslant j \leqslant m}(\beta_j)\}$ forms a ratchet in w of length m (for $\beta_j \in B_0$), independent of other ratchets.

(2) $\Rightarrow$ (3): Let $B_i = \{\beta_{i_k} : 0 \leqslant k \leqslant m\}$ be the ith independent ratchet of length $m+1$. The sequences $\{\bigwedge_{1 \leqslant i \leqslant n}(\beta_{i_0}), \bigwedge_{1 \leqslant i \leqslant n}(\beta_{i_0} \wedge \beta_{i_1}), \ldots, \bigwedge_{1 \leqslant i \leqslant n}(\bigwedge_{1 \leqslant j \leqslant m}(\beta_{i_j}))\}$ for even $i \leqslant n$ form one ratchet, while for odd $i \leqslant n$ form the other.

(3) $\Rightarrow$ (1): Let n be any positive natural number. We will construct a set of n independent buttons. Let R_{n^2} and L_{n^2} be two ratchets of length n^2. Let r_k be a sentence that says that the volume of the ratchet R is k. We analogously define l_m. Let $A = \{r_k \wedge l_m : k + m = n^2\}$. Let us define sets of independent buttons of cardinality $2 \leqslant i \leqslant n$ inductively:

- $(r_k \wedge l_m), (r_{k'} \wedge l_{m'}) \in A$ are two independent buttons if $k \neq k'$,
- if $\varphi_1, \ldots, \varphi_j$ are independent buttons, then take sentences $\psi_1, \ldots, \psi_{2j} \in A$ such that $\neg(\Box \bigwedge_{s \leqslant 2j}(\psi_s) \to \varphi_i)$, for all $1 \leqslant i \leqslant 2j$, and if $\psi_i = r_k \wedge l_m$, $\psi_{i'} = r_{k'} \wedge l_{m'}$ and $i \leqslant i'$, then $k \leqslant k'$. Since the ratchets R and L can be arbitrarily large, we will find such sentences. The sentences $\varphi_i \vee \psi_{2i}$ together

with the sentence $\bigvee_{2n+1=k\leqslant 2j}(\psi_k)$ form a set of independent buttons, and its cardinality is $n+1$. It is easy to check that these sentences are buttons and are independent of each other. $\square$

Corollary 3.3. Let $m \geqslant 2$ be a natural number greater than 1. The logic S4.2 is sound and complete with respect to the following classes of frames:

$$\langle \prod_{1\leqslant i\leqslant m}(C_i) \times \mathcal{P}(S_k), \leqslant^*\rangle,$$

for all m-sized sets of finite chains C_i and all k, where S_k is a set of cardinality k, and the order is defined as $\langle a_1, \ldots, a_m, S\rangle \leqslant^* \langle b_1, \ldots, b_m, Z\rangle \iff a_1 \leqslant b_1 \ \& \ \ldots \ \& \ a_m \leqslant b_m$.

The intuition here is that these frames are direct products of linearly ordered lattices, where each node is replaced with a cluster of mutually accessible worlds (the last parameter, S, is responsible for that).

Proof: By the theorem by Hamkins and Löwe [2] it suffices to show that each such frame is an S4.2 frame and that there are frames for arbitrarily large families of independent buttons and switches. For any $m \geqslant 2$ it is easy to see that these frames are directed (in the sense of condition (1.1)) preorders, hence S4.2 is valid in them.

To prove the second part, by the Theorem 3.2, it suffices to find two arbitrarily long ratchets independent of each other and of arbitrarily large sets of independent switches in models built on those frames. Let us show it for $m = 2$, but this works for all $m \geqslant 2$. Let C_i be a linear order of length i. For a given frame $\langle C_i \times C_j \times \mathcal{P}(S_k), \leqslant^*\rangle$, let us assume without loss of generality that $C_i = \langle\{0, \ldots, i-1\}, \leqslant\rangle, C_j = \langle\{0, \ldots, j-1\} \leqslant\rangle, S_k = \{0, \ldots, k\}$. Let $P = \{p_0, \ldots, p_i\}, Q = \{q_0, \ldots, q_j\}$ and $R = \{r_0, \ldots, r_k\}$ be sets of distinct variables. Any valuation v satisfying the following condition:

$$(a, b, S) \in v(p) \Leftrightarrow p \leqslant p_a \text{ or } p \leqslant q_b \text{ or } p \in S$$

is a valuation that allows us to find two independent ratchets of length i (the set P) and j (the set Q), as well as a set (R) of k independent switches. Since in a class there are arbitrarily large finite frames of this sort, we are able to produce arbitrarily large ratchets and sets of switches, which concludes the proof. □

This result provides us with infinitely many completeness theorems for S4.2, for all $m \in \mathbb{N}$. The frames are pre-lattices, and the lattices obtained by making the quotient algebra by the relation $\sim$ (defined as in Definition 1.1) are distributive.

4. Applications

The authors of [2] used their result (Theorem 1.2) in order to prove the following:

THEOREM 4.1 ([2, Theorem 14]). *If a world w in a Kripke model $\mathcal{M}$ has arbitrarily large families of independent buttons and switches, then its modal logic is contained within S4.2.*

This comes in handy when working in a setting of classes of first-order structures. This setting originates in works on arithmetic and set theory, and considers an ordered tuple $\langle \mathcal{C}, \mathcal{R} \rangle$, something similar to a Kripke frame – $\mathcal{C}$ is a certain (typically proper) class of first-order structures, and $\mathcal{R}$ is a model-theoretic relation connecting members of $\mathcal{C}$. This intends to work as a frame, whose modal logic we investigate. To achieve that goal, one needs to define valuations accordingly: let L be a standard first-order language. Any function $t\colon Var \to Fm_L$ we call a L-translation. The valuations are then defined in the following way: $v_t(p) = \{M \in \mathcal{C} : M \models t(p)\}$. Only such valuations are considered in order to avoid metamathematical issues.

It might be seen as arbitrary to pick the first-order language for such translations, especially strange since many interesting model-theoretic relations would be inexpressible in it. Thanks to the results by Saveliev and Shapirovsky [3] the modal logic obtained via these valuations will be the stronger than in any stronger language. Furthermore, if we intend to use the Theorem 4.1 or similar in order to prove that the lower bound is the same as the upper bound on modal validities in a given class, we obtain the result that this logic is the robust logic of this class, i.e. the logic of this class does not depend on the choice of language.

In order to understand this method in the mentioned framework we need to understand what does it mean to be a button or a switch in a first-order structure. A first-order sentence $\varphi \in Fm_L$ is called a button in M iff for all translations t and all $p \in Var$ such that $t(p) = \varphi$ $M \Vdash \Diamond\Box p$. Switch is understood similarly through the means of translation.

Due to Theorem 3.2 we can state the theorem in terms of ratchets:

Corollary 4.2. If a world w in a Kripke model $\mathcal{M}$ has two arbitrarily large independent ratchets independent of arbitrarily large families of switches, then its modal logic is contained within S4.2.

As an example we will provide a tuple $\langle \mathcal{C}, \mathcal{R} \rangle$ such that its modal logic in the above sense is exactly S4.2.

Let $L_<$ be a first-order language together with the set $\{<, c_0, c_1\}$, where $<$ is a binary relational symbol and c_0, c_1 are constant symbols. Let $T =$ theory of linear orders together with the following two axioms:

1. $\forall x \exists y (y > x)$,
2. $c_0 < c_1$.

Theorem 4.3. *The robust modal logic of the frame $\langle \mathcal{C}, \mathcal{R} \rangle$, for $\mathcal{C} = Mod(T)$ (class of models of T) and $\mathcal{R} = \precsim$ (embeddability relation) is S4.2.*

Proof: Soundness is easy, as it suffices to show that condition (1.1) is met by the class $\mathcal{C}$ under $\mathcal{R}$. In $\mathcal{C}$ even a stronger property is exhibited, as any two linear orders have a common extension (take, for example, a direct lexicographical product); hence S4.2 is valid.

For completeness, we will find two independent ratchets independent of an arbitrary length, as well as arbitrarily large sets of independent switches. Let $M \in \mathcal{C}$ be any model of T. Let $S = \{\sigma_n : n \in \mathbb{N}\}$, where σ_n is a sentence that says "there are two objects greater than c_1 such that they do not have an immediate predecessor, between them are no such objects, and are n objects without an immediate successor". These are our switches: one can switch them on in any extension of M where this is satisfied, then one can make them off in any discrete extension they embed into. For ratchets, take sequences φ_m, ψ_k for $m, k \in \mathbb{N}$, where φ_m is a sentence that says "there are at least m elements smaller (with respect to $<$) than c_0"

and ψ_k is a sentence that says "there are at least k elements between c_0 and c_1. It is straightforward that these are buttons in M and those with bigger indexes imply those with smaller, as well as the fact that one can find $N \supsetneq M$ in which any desired button is true, and the one with greater index is not. Since sentences σ_n concern what happens in the top part of linear order, sentences ψ_k what is true in its middle, and sentences φ_m what happens at the bottom, they don't interfere with one another, and are independent. So there are two independent ratchets of arbitrary length and arbitrary large families of independent switches, so from Corollary 4.3 follows completeness. □

Note that proving completeness by the means of Theorem 4.1 would be significantly more complicated.

This result brings the following corollary:

Corollary 4.4. Let T_n, for $n \geqslant 3$ be theory in language with $<$ binary relational symbol and $c_0, c_1, \ldots, c_{n-1}$ constant symbols, extending T and having as additional axioms $c_i < c_j$, for $i < j$. Robust modal logic of $\langle Mod(T_n), \subsetneq \rangle$ is S4.2.

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Maciej A. Hałapacz
University of Wrocław
Department of Logic and Methodology of Sciences
Koszarowa 3
51-168 Wrocław, Poland
e-mail: maciej.halapacz@uwr.edu.pl

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