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PERFECT AND LOCAL BOUNDED EQUALITY ALGEBRAS

Abstract

In this paper, by considering the notion of bounded equality algebra, we introduce the concepts of perfect, integral and local bounded equality algebras and we state and prove some related results. Specially, we define the notions of some filters such as primary and perfect filters and we get the relation between perfect filters with perfect bounded equality algebras and primary filters in the local bounded equality algebras. Finally, by defining the concepts of radical, locally finite, and peculiar bounded equality algebra, we give a representation for any bounded p-equality and local bounded equality algebras.

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1. Introduction

Novák developed implication-based fuzzy type theory as a higher-order fuzzy logic, which corresponds to the fuzzy version of classical type theory in classical higher-order logic. While classical-type theory models rely on equality, with identity as the principal connective, this approach differs from the equality-based models. To address this diversity, Novák and Baets introduced a specific algebra of truth values for fuzzy type theory, known as EQ-algebras. Jenei [8] later introduced equality algebras motivated by EQ-algebras. This structure contains an equivalence, a meet operation, and a constant 1. A non-commutative version of this structure was also studied in [9]. Jenei [8] found a close connection between equality algebras and BCK-algebras. Ciungu [4] and then Dvurečenskij et.al [6] improved this result not only for equality algebras but also for pseudo equality algebras. Internal states, and states on equality algebras as well as special types of deductive systems on equality algebras were studied in [5], [21], and [22], respectively.

In recent times, various aspects of equality algebras have been studied, such as commutative equality and &-equality in [10], radical filter on equality algebras in [11], implicative equality, annihilators, right and left maps on equality in [1, 13], different representation of equality algebras in [12], the concept of involution in [3], Stone filters and MBJ-neutrosophic filters on equality algebras in [7, 19], prime filter, co-annihilators and ordinal sum on equality algebras in [14, 15, 16, 17, 20], the concept of derivation and state on equality algebras in [18, 21] and the relation between equality algebra and other logical algebra in [22].

In this paper, by considering the notion of bounded equality algebra, we introduce the concepts of perfect, integral and local bounded equality algebras and we state and prove some related results. Specially, we define the notions of some filters such as primary and perfect filters and we get the relation between perfect filters with perfect bounded equality algebras and primary filters in the local bounded equality algebras. Finally, by defining the concepts of radical, locally finite, and peculiar bounded equality algebra, we give a representation for any bounded p-equality and local bounded equality algebras.

2. Preliminaries

In this section, we have some basic concepts about equality algebra that will be used in later sections. We only remind you of some definitions and results.

Definition 2.1 ([8]). An algebraic structure $(E;\wedge,\sim,1)$ of type $(2,2,0)$ is called an *equality algebra* if it satisfies the following conditions, for all $x,y,z\in E$,

$(E1)$ $(E,\wedge,1)$ is a $\wedge$-semilattice with the greatest element 1,

$(E2)$ $x\sim y=y\sim x$,

$(E3)$ $x\sim x=1$,

$(E4)$ $x\sim 1=x$,

$(E5)$ $x\leq y\leq z$ implies $x\sim z\leq y\sim z$ and $x\sim z\leq x\sim y$,

$(E6)$ $x\sim y\leq (x\wedge z)\sim(y\wedge z)$,

$(E7)$ $x\sim y\leq (x\sim z)\sim(y\sim z)$.

An equality algebra $(E,\leq)$ with a binary relation defined by "$x\leq y$ if and only if $x\wedge y=x$" is a poset. Also, we define the operation "$\rightarrow$" on E as: $x\rightarrow y=x\sim(x\wedge y)$.

The set $\emptyset\neq X\subseteq E$ is called a *sub-equality algebra* if it is closed under all operations on E. It means that for any $x,y\in X$, $x\wedge y, x\sim y\in X$.

An equality algebra $(E,\sim,\wedge,0,1)$ is called a *bounded equality algebra* if for every $x\in E$, $0\leq x$. In a bounded equality algebra E, we define the negation operation "$'$" by: $x' = x \to 0 = x \sim 0$, for all $x \in E$. If for any $x \in E$, $x'' = x$, then E is called an *involutive equality algebra*. A lattice equality algebra is an equality algebra that is a lattice. An equality algebra E is called a positive equality algebra or p-equality if for any $x, y, z \in E$, $x \to (y \to z) = (x \to y) \to (x \to z)$ (see [22]).

Note. From now on, let $(E; \wedge, \sim, 1)$ or E be an equality algebra, unless otherwise stated.

Proposition 2.2 ([8, 22]). The following conditions hold, for any $x, y, z \in E$:

(i) $x \leq (x \to y) \to y$,

(ii) $x \to (y \to z) = y \to (x \to z)$,

(iii) $x \leq y$ implies $y \to z \leq x \to z$ and $z \to x \leq z \to y$,

(iv) $((x \to y) \to y) \to y = x \to y$,

(v) $x \to y \leq (x \wedge z) \to (y \wedge z)$,

(vi) $x \to y \leq (z \to x) \to (z \to y)$ and $x \to y \leq (y \to z) \to (x \to z)$.

Definition 2.3 ([9]). Let F be a non-empty subset of E. Then F is called a *filter* of E if for all $x, y \in E$, we have

(i) $x \in F$ and $x \leq y$ imply $y \in F$,

(ii) $x \in F$ and $x \sim y \in F$ imply $y \in F$.

The set of all filters of E is denoted by $\mathcal{F}(E)$.

Proposition 2.4 ([5, 9]). Let $\emptyset \neq F \subseteq E$. Then $F \in \mathcal{F}(E)$ if and only if, for all $x, y \in E$, $1 \in F$, and if $x \in F$ and $x \to y \in F$, then $y \in F$.

Clearly, for any $F \in \mathcal{F}(E)$, $1 \in F$, and F is called a *proper filter* of E if $F \neq E$. If E is bounded, then $F \in \mathcal{F}(E)$ is proper if and only if it does not contain 0.

Let $F \in \mathcal{F}(E)$. Define the relation θ_F on E by

$$(x, y) \in \theta_F \quad \text{if and only if} \quad \{x \to y, y \to x\} \subseteq F.$$

The set of all congruence relations on E is denoted by $\mathcal{C}on(E)$.

Let $\dfrac{E}{F} = \{[x] \mid x \in E\}$, where $[x] = \{y \in E \mid \ (x, y) \in \theta_F\}$. Then define the binary relation $\leq_F$ on $\dfrac{E}{F}$ by:

$$[x] \leq_F [y] \quad \text{if and only if} \quad x \to y \in F,$$

which is an order relation on $\dfrac{E}{F}$. For any $x, y \in E$, define

$$[x] \sim_F [y] = [x \sim y] \quad \text{and} \quad [x] \wedge_F [y] = [x \wedge y].$$

Then $\left(\dfrac{E}{F}, \sim_F, \wedge_F, 1_{\frac{E}{F}}\right)$ is called a *quotient equality algebra* and denoted by $\dfrac{E}{F}$, where $1_{\frac{E}{F}} = [1]_{\theta_F} = F$.

Definition 2.5 ([17]). Let $\emptyset \neq A \subseteq E$. The smallest filter of E containing A is called *the generated filter by A in E*, which is denoted by $\langle A \rangle$. Indeed, $\langle A \rangle = \bigcap\limits_{A \subseteq F \in \mathcal{F}(E)} F$.

Proposition 2.6 ([17]). Let $\emptyset \neq A \subseteq E$. Then

$$\langle A \rangle = \{x \in E \mid a_1 \to (a_2 \to (... \to (a_n \to x)...)) = 1, \text{ for some } n \in \mathbb{N} \\ \text{and } a_1, ..., a_n \in A\}.$$

In particular, for any element $a \in E$, we have $\langle a \rangle = \{x \in E \mid a \overset{n}{\to} x = 1, \text{ for some } n \in \mathbb{N}\}$, where $x \overset{0}{\to} y = y$ and $x \overset{n}{\to} y = x \to (x \overset{n-1}{\to} y)$.

Definition 2.7 ([2]). Let F be a non-empty subset of E such that $1 \in F$ and $x, y, z \in E$. Then

(i) F is called a *positive implicative filter* of E if $x \to (y \to z) \in F$ and $x \to y \in F$ imply $x \to z \in F$,

(ii) F is called an *implicative filter* of E if $z \to ((x \to y) \to x) \in F$ and $z \in F$, then $x \in F$.

(iii) F is called a *fantastic filter* of E, if $z \in F$ and $z \to (x \to y) \in F$ imply $((y \to x) \to x) \to y \in F$.

PROPOSITION 2.8 ([2]). Let E be bounded and $F \in \mathcal{F}(E)$. Then F is an implicative filter of E if and only if for any $x \in E$, $(x' \to x) \to x \in F$.

DEFINITION 2.9 ([2, 14, 17]). Suppose $F \in \mathcal{F}(E)$ is proper and $x, y \in E$. Then F is called

(i) a *maximal filter* of E if it is not included in any other proper filter of E. The set of all maximal filters of E is denoted by $\mathcal{M}ax(E)$.

(ii) a *prime filter* of E if for any $x, y \in E$, $x \to y \in F$ or $y \to x \in F$. The set of all prime filters of E is denoted by $\mathcal{P}rime(E)$.

(iii) a $\vee$-*irreducible* if $x \vee y \in F$ implies $x \in F$ or $y \in F$, where E is a lattice. The set of all $\vee$-irreducible filters of E is denoted by $\mathcal{S}pec(E)$.

PROPOSITION 2.10 ([14]). Suppose E is a lattice. Then

(i) $F \in \mathcal{S}pec(E)$ if and only if $\dfrac{E}{F}$ is a chain.

(ii) $\mathcal{P}rime(E) \subseteq \mathcal{S}pec(E)$.

PROPOSITION 2.11 ([14]).

(i) Let E be bounded. $M \in \mathcal{M}ax(E)$ if and only if ($x \notin M$ if and only if there exists $n \in \mathbb{N}$ such that $x \xrightarrow{n} 0 \in M$).

(ii) Let E be bounded. If $F \in \mathcal{F}(E)$ is proper, then there exists $M \in \mathcal{M}ax(E)$ such that $F \subseteq M$.

(iii) Each maximal filter of E is prime. Indeed, $\mathcal{M}ax(E) \subseteq \mathcal{P}rime(E)$.

Let F be a proper filter of E. The intersection of all maximal filters of E that contain F is called *the radical of* F and is denoted by $Rad(F)$. If

$F = \{1\}$, then $Rad(E)$ is the intersection of all maximal filters of E (see [11]). Also, we have $Rad'(E) = \{x \in E \mid x' \in Rad(E)\}$. (For more details, see [11]).

THEOREM 2.12 ([11]). *Consider $F \in \mathcal{F}(E)$. Then*

$$\begin{aligned} Rad(F) &= \{x \in E \mid (x \xrightarrow{n} 0) \to x \in F, \textit{for any } n \in \mathbb{N}\} \\ &\subseteq \{x \in E \mid \ x' \to x \in F\}. \end{aligned}$$

Note. From now on, consider that $(E, \wedge, \sim, 0, 1)$ is a bounded equality algebra, unless otherwise stated.

3. Perfect and integral equality algebras

In this section, we introduce the notion of order on bounded equality algebra and by using this concept we define different kinds of bounded equality algebras such as perfect and integral bounded equality algebras and investigate the relation between them.

DEFINITION 3.1. Let $x \in E$. Then the least natural number n such that $x \xrightarrow{n} 0 = 1$ is called the *order* of x and is denoted by $ord(x) = n$. If there is no natural number such that $x \xrightarrow{n} 0 = 1$, then $ord(x) = \infty$.

Remark 3.2. It is clear that $ord(0) = 1$ and $ord(1) = \infty$.

Example 3.3. (i) Let $(E = \{0, a, b, 1\}, \leq)$ be a chain where $0 \leq a \leq b \leq 1$. Define the operation "$\sim$" on E as follows:

$\sim$	0	a	b	1
0	1	a	0	0
a	a	1	a	a
b	0	a	1	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

Then $(E, \wedge, \vee, \sim, 0, 1)$ is a bounded equality algebra where $ord(a) = 2$ and $ord(b) = ord(1) = \infty$.

(ii) Let $E = \{0, a, b, c, d, 1\}$ be a poset where $0 \leq a, b \leq c \leq d \leq 1$. Define "$\sim$" on E as follows:

$\sim$	0	a	b	c	d	1
0	1	d	d	d	c	0
a	d	1	c	d	c	a
b	d	c	1	d	c	b
c	d	d	d	1	d	c
d	c	c	c	d	1	d
1	0	a	b	c	d	1

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	d	1	1	1
b	d	d	1	1	1	1
c	d	d	d	1	1	1
d	c	c	c	d	1	1
1	0	a	b	c	d	1

Then $(E, \wedge, \vee, \sim, 0, 1)$ is a bounded equality algebra. Clearly, $ord(a) = ord(b) = ord(c) = 2$, $ord(d) = 3$ and $ord(1) = \infty$.

(iii) Let $E = [0, 1]$ and $x, y \in E$. Define

$$x \rightarrow y = \begin{cases} 1 & x \leq y \\ (1-x) \vee y & x > y \end{cases} \qquad \text{and} \quad x \sim y = (x \rightarrow y) \wedge (y \rightarrow x).$$

Then $(E, \wedge, \sim, 0, 1)$ is a bounded equality algebra where for any $x \leq \frac{1}{2}$, $ord(x) = 2$ and for any $x > \frac{1}{2}$, $ord(x) = \infty$.

Proposition 3.4. Let $x, y \in E$.

(i) $\langle x \rangle$ is proper if and only if $ord(x) = \infty$.

(ii) If $x \leq y$ and $ord(y) < \infty$, then $ord(x) < \infty$.

(iii) If $x \leq y$ and $ord(x) = \infty$, then $ord(y) = \infty$.

(iv) Let $M \in \mathcal{M}ax(E)$. Then $M \subseteq \{x \in E \mid ord(x) = \infty\}$.

Proof: (i) Suppose $\langle x \rangle$ is proper and $ord(x) < \infty$, by the contrary. Then there is $n \in \mathbb{N}$ such that $x \overset{n}{\rightarrow} 0 = 1$. According to the definition of $\langle x \rangle$, $0 \in \langle x \rangle$ and so $\langle x \rangle = E$, which is a contradiction. Conversely, suppose $ord(x) = \infty$. Then for any $n \in \mathbb{N}$, $x \overset{n}{\rightarrow} 0 \neq 1$. Thus $0 \notin \langle x \rangle$. Hence, $\langle x \rangle$ is a proper filter of E.

(ii) Suppose $x \leq y$ and $ord(y) < \infty$. Then there is $n \in \mathbb{N}$ such that $y \overset{n}{\rightarrow} 0 = 1$. By Proposition 2.2(iii), $y \rightarrow 0 \leq x \rightarrow 0$. Also, by Proposition 2.2(iii), $y \rightarrow (y \rightarrow 0) \leq y \rightarrow (x \rightarrow 0)$. Again, by Proposition 2.2(iii), $y \rightarrow (x \rightarrow 0) \leq x \rightarrow (x \rightarrow 0)$. So we have $y \rightarrow (y \rightarrow 0) \leq x \rightarrow (x \rightarrow 0)$.

By repeating this method, we get $1 = y \overset{n}{\to} 0 \leq x \overset{n}{\to} 0$. Hence, $x \overset{n}{\to} 0 = 1$, and so $ord(x) = n$. Therefore, $ord(x) < \infty$.

(iii) Let $x \leq y$. Similar to the proof of (ii), $y \overset{n}{\to} 0 \leq x \overset{n}{\to} 0$. Since for any $n \in \mathbb{N}$, $x \overset{n}{\to} 0 \neq 1$, we get that for any $n \in \mathbb{N}$, $y \overset{n}{\to} 0 \neq 1$. Hence, $ord(y) = \infty$.

(iv) Let $M \in \mathcal{M}ax(E)$ and $\mathcal{C} = \{x \in E \mid ord(x) = \infty\}$. We prove that $M \subseteq \mathcal{C}$. For this, suppose $x \in M$. If $ord(x) < \infty$, then there exists $n \in \mathbb{N}$ such that $x \overset{n}{\to} 0 = 1$ and so $x \overset{n}{\to} 0 \in M$. Since $x \in M$ and $M \in \mathcal{F}(E)$, we get $0 \in M$, a contradiction. Therefore, $ord(x) = \infty$ and so $x \in \mathcal{C}$. □

Proposition 3.5. Assume E is a chain and $F \in \mathcal{F}(E)$. Then

(i) If $x \notin Rad(F)$, then $ord(x) < \infty$.

(ii) $Rad(F) = \{x \in E \mid ord(x) = \infty\}$.

Proof: (i) Assume $x \notin Rad(F)$. Then by Proposition 2.12, there exists $n \in \mathbb{N}$ such that $(x \overset{n}{\to} 0) \to x \notin F$. So, $x \overset{n}{\to} 0 \nleq x$. Since if $x \overset{n}{\to} 0 \leq x$, then $(x \overset{n}{\to} 0) \to x = 1 \in F$, a contradiction. By assumption, E is a chain, thus $x \leq x \overset{n}{\to} 0$, and so $x \overset{n+1}{\to} 0 = 1$. Hence, $ord(x) = n + 1 < \infty$.

(ii) Let $x \in Rad(F)$ and $ord(x) < \infty$. Since $x \in Rad(F)$, for any $M \in \mathcal{M}ax(E)$ such that $F \subseteq M$, $x \in M$. In addition, $ord(x) < \infty$, so there exists $n \in \mathbb{N}$ such that $x \overset{n}{\to} 0 = 1 \in M$. Since $x \in M$ and $M \in \mathcal{F}(E)$, we get $0 \in M$, which is a contradiction. So $ord(x) = \infty$. Hence, by (i), the proof is clear. □

In the following example we show that the condition "chain" in Proposition 3.5 is necessary.

Example 3.6. Let $E = \{0, a, b, 1\}$ be a poset where $0 \leq a, b \leq 1$. Define the operation "$\sim$" on E as follows:

$\sim$	0	a	b	1
0	1	b	a	0
a	b	1	0	a
b	a	0	1	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	b	1	b	1
b	a	a	1	1
1	0	a	b	1

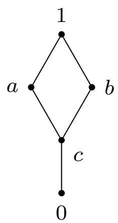
Then $(E, \wedge, \sim, 0, 1)$ is a bounded equality algebra. Clearly, $\mathcal{M}ax(E) = \{\{a, 1\}, \{b, 1\}\}$ and so $Rad(E) = \{1\}$. In addition, $a, b \notin Rad(E)$ and $ord(a) = ord(b) = \infty$.

Definition 3.7. Assume $\emptyset \neq X \subseteq E$, is a sub-equality algebra of E. Then X is called *perfect* if for any $x \in X$,

$$ord(x) < \infty \text{ if and only if } ord(x') = \infty.$$

Example 3.8. (i) In Example 3.3(i), E is not perfect, since $ord(a) = ord(a') = 2 < \infty$.

(ii) Consider $(E = \{0, a, b, c, 1\}, \leq)$ to be a poset where $0 \leq c \leq a, b \leq 1$. The operation "$\sim$" and the implication "$\rightarrow$" on E are given by the following tables:



$\sim$	0	a	b	c	1
0	1	0	0	0	0
a	0	1	c	b	a
b	0	c	1	a	b
c	0	b	a	1	c
1	0	a	b	c	1

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	b	b	1
b	0	a	1	a	1
c	0	1	1	1	1
1	0	a	b	c	1

Then $E = (E, \wedge, \vee, \sim, 0, 1)$ is a bounded equality algebra. Clearly, E is perfect.

(iii) According to Example 3.3(iii), E is perfect.

Definition 3.9. Let E be a lattice. An element $a \in E$ is called the *complemented* if there exists an element $b \in E$ such that $a \vee b = 1$ and $a \wedge b = 0$. Then the element b is called a *complement of* a. The set of all complemented elements is denoted by $\mathcal{C}(E)$, i.e., $\mathcal{C}(E) = \{a \in E \mid \exists b \in E;\ a \vee b = 1 \ \&\ a \wedge b = 0\}$.

Obviously, in any chain equality algebra, $\mathcal{C}(E) = \{0, 1\}$.

Theorem 3.10. *Suppose E has just one maximal filter such as M such that $M = E \setminus \{0\}$. Then the following statements hold:*

(i) *E is perfect and for any $x \in M$, $ord(x) = \infty$.*

(ii) *If E is Boolean, then $E = \{0, 1\}$.*

Proof: (i) By Proposition 3.4(iv), $M \subseteq \{x \in E \mid ord(x) = \infty\}$. Since $M = E \setminus \{0\}$, we get that for any $x \in E \setminus \{0\}$, $ord(x) = \infty$. Also, for $x \in E$, if $x = 0$, then $x' = 1$ and so $ord(x) = 1 < \infty$ and $ord(x') = \infty$. If $x \neq 0$, then $x \in M$ and so $ord(x) = \infty$. Clearly, $x' \notin M$, so $x' = 0$ and $ord(x') < \infty$. Therefore, E is perfect.

(ii) Let $x \in E \setminus \{0, 1\}$. Then $x \in M$ and so $x' \notin M$. Thus, $x' = 0$ for any $x \in E \setminus \{0\}$. Then $x \vee x' = x \vee 0 = x \neq 1$, so $x \notin \mathcal{C}(E)$. Hence, $\mathcal{C}(E) = \{0, 1\}$. Now, if E is Boolean, then $E = \mathcal{C}(E) = \{0, 1\}$. □

Example 3.11. In Example 3.8(ii), $E \setminus \{0\} = M = \{a, b, c, 1\} \in \mathcal{M}ax(E)$.

Proposition 3.12. If $Rad(E) \cup Rad'(E)$ is a subalgebra of E, then it is perfect.

Proof: Let $x \in Rad(E) \cup Rad'(E)$, then we have the following two cases:

Case 1: $x \in Rad(E)$. Then for any $M \in \mathcal{M}ax(E)$, $x \in M$ and by Proposition 3.4(iv), $ord(x) = \infty$. In addition, since $x \in Rad(E)$, by Proposition 2.12, $x' \to x = 1$ and so $x' \leq x$. By Proposition 2.2(i),

$$1 = x \to (x' \to 0) \leq x' \to (x' \to 0) = x' \xrightarrow{2} 0,$$

and so $x' \xrightarrow{2} 0 = 1$. Hence, $ord(x') < \infty$.

Case 2: $x \in Rad'(E)$. Then $x' \in Rad(E)$, and so $ord(x') = \infty$. By Case 1, $ord(x'') < \infty$ and since $x \leq x''$, by Proposition 3.4(ii), we get $ord(x) < \infty$. Therefore, according to the above cases, $Rad(E) \cup Rad'(E)$ is perfect. □

Definition 3.13. Let $F \in \mathcal{F}(E)$ be proper and $x \in E$. Then the least $n \in \mathbb{N}$ such that $x \xrightarrow{n} 0 \in F$ is called *order of x respect to F* and denoted by $ord_F(x) = n$. If there is no $n \in \mathbb{N}$ such that $x \xrightarrow{n} 0 \in F$, then $ord_F(x) = \infty$.

Example 3.14. According to Example 3.3(i), let $F = \{b, 1\}$. Then $ord_F(a) = 2$ and $ord_F(b) = ord_F(1) = \infty$.

Definition 3.15. A proper filter P of E is called a *perfect filter* if for any $x \in E$,

$$ord_F(x) < \infty \text{ if and only if } ord_F(x') = \infty.$$

Example 3.16. (i) According to Example 3.8(ii), the filter $F = \{a, 1\}$ is a perfect filter of E.
(ii) According to Example 3.3(i), $F = \{b, 1\}$ and $G = \{1\}$ are not perfect filters since $ord_F(a) = ord_F(a') < \infty$.

Remark 3.17. If $\{1\}$ is a perfect filter of E, then every filter of E is perfect. Since if $F \in \mathcal{F}(E)$ such that for $x \in E$, $ord_F(x) = \infty$, then for any $n \in \mathbb{N}$, $x \stackrel{n}{\to} 0 \notin F$, we get $x \stackrel{n}{\to} 0 \neq 1 \notin \{1\}$. Thus, $ord_{\{1\}}(x) = \infty$. Moreover, $\{1\}$ is perfect, thus by definition we get $ord_{\{1\}}(x') < \infty$, and so there exists $m \in \mathbb{N}$ such that $x' \stackrel{m}{\to} 0 \in \{1\} \subseteq F$. Hence, $ord_F(x') < \infty$, and so F is perfect.

Proposition 3.18. Let F be a perfect filter of E and $F \subseteq G$, where G is a proper filter of E. Then G is a perfect filter of E.

Proof: Consider F is a perfect filter of E. If $G \in \mathcal{F}(E)$ such that for $x \in E$, $ord_G(x) = \infty$, then for any $n \in \mathbb{N}$, $x \stackrel{n}{\to} 0 \notin G$, we get $x \stackrel{n}{\to} 0 \notin F$. Thus, $ord_F(x) = \infty$. Moreover, by assumption F is perfect, thus by definition we get $ord_F(x') < \infty$, and so there exists $m \in \mathbb{N}$ such that $x' \stackrel{m}{\to} 0 \in F \subseteq G$. Hence, $ord_G(x') < \infty$, and so G is perfect. □

Theorem 3.19. *Let $P \in \mathcal{F}(E)$ be proper. Then $\dfrac{E}{P}$ is perfect if and only if P is a perfect filter of E.*

Proof: Let P be a perfect filter of E, and $[x] \in \dfrac{E}{P}$. If $ord([x]) < \infty$, then there exists $n \in \mathbb{N}$ such that $[x] \stackrel{n}{\to} [0] = [1]$, and so $[x \stackrel{n}{\to} 0] = [1]$. Then for $n \in \mathbb{N}$, $x \stackrel{n}{\to} 0 \in P$. Thus, $ord_P(x) < \infty$ and by assumption, $ord_P(x') = \infty$. Then for any $m \in \mathbb{N}$, $x' \stackrel{m}{\to} 0 \notin P$ and so $[x'] \stackrel{m}{\to} [0] \neq [1]$. Hence, $ord([x']) = \infty$. Now, suppose $ord([x]) = \infty$. Then for any $n \in \mathbb{N}$, $[x] \stackrel{n}{\to} [0] \neq [1]$ and so $[x \stackrel{n}{\to} 0] \neq [1]$. Thus, for any $n \in \mathbb{N}$, $x \stackrel{n}{\to} 0 \notin P$, and so $ord_P(x) = \infty$. Since P is perfect, we get $ord_P(x') < \infty$. Then there exists $m \in \mathbb{N}$ such that $x' \stackrel{m}{\to} 0 \in P$ and so this implies that $[x'] \stackrel{m}{\to} [0] = [1]$. Hence $ord([x']) < \infty$. Therefore, $\dfrac{E}{P}$ is perfect.

Conversely, suppose $\dfrac{E}{P}$ is perfect and for $x \in E$, $ord_P(x) = \infty$. Thus, for any $n \in \mathbb{N}$, $x \overset{n}{\to} 0 \notin P$. Hence, for any $n \in \mathbb{N}$, $[x] \overset{n}{\to} [0] \neq [1]$ and so $ord([x]) = \infty$. Since $\dfrac{E}{P}$ is perfect, $ord([x']) < \infty$. So there exists $m \in \mathbb{N}$ such that $[x'] \overset{m}{\to} [0] = [1]$. Hence, $x' \overset{m}{\to} 0 \in P$, and so $ord_P(x') < \infty$. The proof of the other side is similar. Therefore, P is a perfect filter. $\square$

PROPOSITION 3.20. *E is perfect if and only if for any proper filter $F \in \mathcal{F}(E)$, F is a perfect filter of E.*

PROOF: If for any $F \in \mathcal{F}(E)$, F is a perfect filter of E, then $\{1\}$ is a perfect filter of E and by Theorem 3.19, $\dfrac{E}{\{1\}}$ is perfect, and so E is perfect.

Conversely, suppose E is perfect, $F \in \mathcal{F}(E)$ and for $x \in E$, $ord_F(x) = \infty$. Then, for any $n \in \mathbb{N}$, $x \overset{n}{\to} 0 \notin F$. Hence, $x \overset{n}{\to} 0 \neq 1$, and so $x \overset{n}{\to} 0 \notin \{1\}$. It means that $ord_{\{1\}}(x) = \infty$. Moreover, since E is perfect, by Theorem 3.19, we get $\{1\}$ is a perfect filter of E and so $ord_{\{1\}}(x') < \infty$. Thus, there exists $m \in \mathbb{N}$ such that $x' \overset{m}{\to} 0 \in \{1\}$. Hence, there exists $m \in \mathbb{N}$ such that $x' \overset{m}{\to} 0 \in F$, and so $ord_F(x') < \infty$. Therefore, F is a perfect filter of E. $\square$

DEFINITION 3.21. A proper filter P of E is called a *primary filter of $E*, if for any $x, y \in E$, $x \to y' \in P$ implies that there is $n \in \mathbb{N}$ such that $x \overset{n}{\to} 0 \in P$ or $y \overset{n}{\to} 0 \in P$.

Example 3.22. (i) According to Example 3.3(i), $P = \{b, 1\}$ is primary.
(ii) According to Example 3.6, $F = \{1\}$ is not primary.

Open Problem: Let F be a primary filter of E and $F \subseteq G$, where G is a proper filter of E. Is G a primary filter of E?

THEOREM 3.23. *Every perfect filter is a primary filter.*

PROOF: Suppose a proper filter P is perfect and $y \to x' \in P$. Let for any $n \in \mathbb{N}$, $x \overset{n}{\to} 0 \notin P$. Then $ord_P(x) = \infty$. Since P is perfect, we get $ord_P(x') < \infty$, and so there exists $m \in \mathbb{N}$ such that $x' \overset{m}{\to} 0 \in P$. Hence, $[x'] \overset{m}{\to} [0] = [1]$. Since $y \to x' \in P$, we get $[y] \leq [x']$. By the proof of

Proposition 3.4(ii), $[x'] \stackrel{m}{\to} [0] \leq [y] \stackrel{m}{\to} [0]$ and so $[y] \stackrel{m}{\to} [0] = [1]$. Hence, $y \stackrel{m}{\to} 0 \in P$. Therefore, P is a primary filter of E. $\square$

In the following example we show that the converse of Theorem 3.23 does not hold, in general.

Example 3.24. According to Example 3.3(i), $\{b, 1\}$ is a primary filter but is not perfect, since $ord(a) = ord(a') < \infty$.

Definition 3.25. E is called *integral*, if for any $x, y \in E$, $x \leq y'$ implies $x' = 1$ or $y' = 1$.

Example 3.26. (i) In Example 3.3(i), E is not integral. Since $a \leq a' = a$ but $a' = a \neq 1$.

(ii) In Example 3.8(ii), E is integral.

Theorem 3.27. *Consider $P \in \mathcal{F}(E)$ to be proper. If $\dfrac{E}{P}$ is integral, then P is primary.*

Proof: Suppose $\dfrac{E}{P}$ is integral and for $x, y \in E$, $x \to y' \in P$. Then $[x] \to [y'] = [1]$, and so $[x] \leq [y']$. Since $\dfrac{E}{P}$ is integral, we get $[x'] = [1]$ or $[y'] = [1]$. Thus, $[x \to 0] = [1]$ or $[y \to 0] = [1]$. Hence, $x' \in P$ or $y' \in P$ (choose $n = 1$ in definition of primary). Therefore, P is primary. $\square$

The following example shows that the converse of Proposition 3.27 does not hold, in general.

Example 3.28. According to Example 3.3(i), the filter $\{b, 1\}$ is primary but since $a \leq a'$, we can not consequence that $a' = 1$.

Theorem 3.29. *Let E be p-equality. Then $\dfrac{E}{P}$ is integral if and only if P is primary.*

Proof: $(\Rightarrow)$ By Theorem 3.27, the proof is clear.

$(\Leftarrow)$ Assume $[x] \leq [y']$, then $x \to y' \in P$. Since P is primary, there exists $n \in \mathbb{N}$ such that $x \stackrel{n}{\to} 0 \in P$ or $y \stackrel{n}{\to} 0 \in P$. Since E is a p-equality, we have $x \to 0 = x \stackrel{n}{\to} 0 \in P$ or $y \to 0 = y \stackrel{n}{\to} 0 \in P$, and so $x' \in P$ or $y' \in P$.

Hence, $[x'] = [1]$ or $[y'] = [1]$. Therefore, $\dfrac{E}{P}$ is integral. □

PROPOSITION 3.30. Let E be a chain and $F \in \mathcal{F}(E)$. If $F = Rad(F)$, then F is primary.

PROOF: Suppose $x \to y' \in F$, for $x, y \in E$. Then we have the following cases:

Case 1: If $x \in F$, then $y' \in F$, and so F is primary.

Case 2: If $y \in F$, then by Proposition 2.2(ii), $y \to x' = x \to y' \in F$, and so $x' \in F$. Hence, F is primary.

Case 3: If $x, y \notin F$, then $x, y \notin Rad(F)$. By Proposition 3.5(i), there exist $n, m \in \mathbb{N}$ such that $x \overset{n}{\to} 0 = 1 \in F$ and $y \overset{m}{\to} 0 = 1 \in F$. Hence, F is primary.
Therefore, in all the above cases we prove that F is primary. □

THEOREM 3.31. *If P is a positive implicative and maximal filter of E, then $\dfrac{E}{P}$ is integral and so P is primary.*

PROOF: Suppose $[x] \leq [y']$. Then $x \to y' \in P$, and so we have the following cases:

Case 1: If $x \in P$ and $y \notin P$, since $x \to y' \in P$ and $P \in \mathcal{F}(E)$, then $y' \in P$, and so $[y'] = [1]$.

Case 2: If $y \in P$ and $x \notin P$, since by Proposition 2.2(ii), $x \to y' = y \to x' \in P$ and $P \in \mathcal{F}(E)$, then $x' \in P$, and so $[x'] = [1]$.

Case 3: If $x, y \in P$, then since by Proposition 2.2(ii), $x \to y' = y \to x' \in P$ and $P \in \mathcal{F}(E)$, then $[y'] = [1]$ and $[x'] = [1]$, and so $x', y' \in P$. From $x, y \in P$, we get $0 \in P$, which is a contradiction, since $P \in \mathcal{M}ax(E)$.

Case 4: If $x, y \notin P$, then since $P \in \mathcal{M}ax(E)$, by Proposition 2.11(iii), we get $P \in \mathcal{P}rime(E)$, so for any $x, y \in E$, we get $x \to y \in P$ or $y \to x \in P$. If $x \to y \in P$, since $x \to y' \in P$ and P is positive implicative, then $x \to 0 \in P$ and so $x' \in P$. Hence, $[x'] = [1]$. If $y \to x \in P$, then since by Proposition 2.2(ii), $x \to y' = y \to x' \in P$ and P positive implicative, we get $y' \in P$ and so $[y'] = [1]$.

Therefore, in all above cases, $\frac{E}{P}$ is integral and by Theorem 3.27, P is primary. $\square$

Definition 3.32. The set $\mathcal{D}_s(E)$ is called a *dense set* of E and is defined as follows:

$$\mathcal{D}_s(E) = \{x \in E \mid x' = 0\}.$$

Example 3.33. According to Example 3.3(i), $\mathcal{D}_s(E) = \{b, 1\}$ and in Example 3.8(ii), $\mathcal{D}_s(E) = \{a, b, c, 1\}$.

Proposition 3.34. $\mathcal{D}_s(E) \in \mathcal{F}(E)$.

Proof: Clearly, $1 \in \mathcal{D}_s(E)$, and so $\mathcal{D}_s(E) \neq \emptyset$. Consider $x, x \to y \in \mathcal{D}_s(E)$. Then $x' = 0 = (x \to y)'$. By Proposition 2.2(ii) and (vi), we have

$$0 = x' = x \to 0 = x \to (x \to y)' = (x \to y) \to x' \geq y \to 0 = y'.$$

Thus, $y' = 0$ and so $y \in \mathcal{D}_s(E)$. Therefore, $\mathcal{D}_s(E) \in \mathcal{F}(E)$. $\square$

Proposition 3.35. Let $F \in \mathcal{F}(E)$. If $\frac{E}{F}$ is involutive, then $\mathcal{D}_s(E) \subseteq F$.

Proof: Since $\frac{E}{F}$ is involutive, for any $[x] \in \frac{E}{F}$, we have $[x] = [x'']$. Suppose $x \in \mathcal{D}_s(E)$. Then $x' = 0$ and so $x'' = 1$. Since, $[x] = [x''] = [1]$, we have $x \in F$. Hence, $\mathcal{D}_s(E) \subseteq F$. $\square$

In the following example, we show that the converse of Proposition 3.35, may not be true in general.

Example 3.36. According to Example 3.3(ii), let $F = \{1\}$. Clearly, $\mathcal{D}_s(E) \subseteq F$, but $E \cong \frac{E}{F}$ is not involutive.

Note. We call E has *Gödel negation property* if $\mathcal{D}_s(E) = E \setminus \{0\}$.

Example 3.37. According to Example 3.8(ii), $\mathcal{D}_s(E) = \{a, b, c, 1\} = E \setminus \{0\}$. Hence, E has Gödel negation property.

Proposition 3.38. Let $F \in \mathcal{F}(E)$ and E has Gödel negation property. Then $\frac{E}{F}$ is involutive if and only if $\mathcal{D}_s(E) \subseteq F$.

PROOF: Suppose $\mathcal{D}_s(E) \subseteq F$ and there exists $[x] \in \dfrac{E}{F}$ such that $[x] \neq [x'']$. By Proposition 2.2(i), we know $[x] \leq [x'']$, thus, $[x''] \nleq [x]$. It means that $x'' \to x \notin F$. By assumption, $x'' \to x \notin \mathcal{D}_s(E)$, and so $(x'' \to x)' \neq 0$, which is a contradiction with Gödel negation property. The proof of the converse holds by Proposition 3.35. $\square$

PROPOSITION 3.39. Assume E has Gödel negation property. If F is a proper fantastic (implicative) filter, then $F = E \setminus \{0\}$.

PROOF: Since E has Gödel negation property, we get $\mathcal{D}_s(E) = E \setminus \{0\}$. By assumption, F is a proper filter, so $0 \notin F$, and so $F \subseteq E \setminus \{0\}$. Now, suppose $x \in E \setminus \{0\}$. From E has Gödel negation property, $x' = 0$.

If F is fantastic, since $1 \to (0 \to x) = 1 \in F$ and $1 \in F$, by assumption, we have $x'' \to x = ((x \to 0) \to 0) \to x \in F$. Since $x'' = 1 \in F$, we get $x \in F$. Hence, $F = E \setminus \{0\}$.

Now, if F is implicative, then by Proposition 2.8, for any $x \in E$, $(x' \to x) \to x \in F$ and so $x \in F$. Hence, $F = E \setminus \{0\}$. $\square$

COROLLARY 3.40. Consider E has Gödel negation property. Then every proper fantastic (implicative) filter of E is maximal.

DEFINITION 3.41. A proper filter F of E is called an *integral filter of* E if $x \to y' \in F$ implies $x' \in F$ or $y' \in F$.

Example 3.42. (i) According to Example 3.8(ii), the filter $F = \{b, 1\}$ is an integral filter of E.

(ii) In Example 3.22(ii), $F = \{1\}$ is not integral.

Remark 3.43. Every integral filter of E is primary. According to Example 3.3(i), $F = \{b, 1\}$ is a primary filter but is not integral, since $a \to a' = a \to a = 1$ but $a' = a'' = a \notin \{b, 1\}$. So the converse may not be true in general.

THEOREM 3.44. *Assume E is involutive, F is an integral filter of E and $F \subseteq G$, where G is a proper filter of E. Then G is an integral filter of E.*

PROOF: Let $x, y \in E$ such that $x \to y' \in G$. Since $(x \to y')' \to (x \to y')' = 1 \in F$ and F is an integral filter of E, we have $(x \to y')'' \in F$ or $(x \to y')' \in F$. If $(x \to y')'' \in F$, then since E is involutive, we get $x \to y' \in F$, and by assumption, $x' \in F$ or $y' \in F$. From $F \subseteq G$, we get

$x' \in G$ or $y' \in G$. Hence, G is integral. If $(x \to y') \to 0 = (x \to y')' \in F$, since $F \subseteq G$ and $x \to y' \in G$, then $0 \in G$, a contradiction, since G is proper. Therefore, G is an integral filter of E. $\square$

PROPOSITION 3.45. Assume P is a proper filter of E. Then P is integral if and only if $\dfrac{E}{P}$ is integral.

PROOF: Suppose $\dfrac{E}{P}$ is integral and $x \to y' \in P$. Then $[x] \leq [y']$ and by assumption, $[x'] = [1]$ or $[y'] = [1]$. Hence, $x' \in P$ or $y' \in P$. Therefore, P is an integral filter of E. The proof of the converse is similar. $\square$

PROPOSITION 3.46. E is integral if and only if E has Gödel negation property.

PROOF: Let E be integral. Since $0' = 1$, clearly, $0 \notin \mathcal{D}_s(E)$. So $\mathcal{D}_s(E) \subseteq E\backslash\{0\}$. Suppose $x \in E\backslash\{0\}$. Since E is integral, we get $\dfrac{E}{\{1\}}$ is integral and by Proposition 3.45, $\{1\}$ is an integral filter of E. By Proposition 2.2(i), $x \to x'' = 1 \in \{1\}$. Since $\{1\}$ is integral, we have $x' = 1$ or $x'' = 1$. If $x' = 1$, by Proposition 2.2(i), $x \leq x'' = 0$, and so $x = 0$, a contradiction. If $x'' = 1$, by Proposition 2.2(iv), $x' = x''' = 0$, and so $x \in \mathcal{D}_s(E)$. Hence, $E \setminus \{0\} \subseteq \mathcal{D}_s(E)$. Therefore, $\mathcal{D}_s(E) = E \setminus \{0\}$. Hence, E has Gödel property.

Conversely, let E has Gödel negation property. For proving that E is integral it is enough to prove that $\{1\}$ is integral. For this, suppose $x \to y' \in \{1\}$, where $x, y \in E$.

If $x = y = 0$, then $x' = y' = 1$, and so $x \to y' = 1 \in \{1\}$ and clearly, $x', y' \in \{1\}$.

If $x \neq 0$ and $y = 0$, then $y' = 1$, and so $y' \in \{1\}$.

If $x = 0$ and $y \neq 0$, then $x' = 1 \in \{1\}$.

If $x, y \neq 0$, then by assumption $x' = y' = 0$, so $0 = x' = x \to 0 = x \to y' \in F$, a contradiction, since F is proper.

Hence, in all the above cases, $\{1\}$ is integral, and by Proposition 3.45, E is integral. $\square$

PROPOSITION 3.47. If E is a lattice and integral, then for any $x, y \in E$,

$x \to y' = x' \vee y'$.

Proof: By Proposition 3.46, since E is integral, we have $E \backslash \{0\} = \mathcal{D}_s(E)$. Now, let $x, y \in E$.

If $x = y = 0$, then $x \to y' = 1$ and $x' = y' = 1$, and so $x \to y' = x' \vee y'$.
If $x = 0$ and $y \neq 0$, then $x \to y' = 1$, and so $x \to y' = x' \vee y'$.
If $x \neq 0$ and $y = 0$, then $x \to y' = 1 = x' \vee y'$.
If $x, y \neq 0$, then $x' = y' = 0$ and so $x \to y' = 0 = x' \vee y'$.
Hence, in all above cases we have $x \to y' = x' \vee y'$. □

Proposition 3.48. If E is a chain, then E is integral if and only if for any $x, y \in E$, $x \to y' = x' \vee y'$.

Proof: Let for any $x, y \in E$, $x \to y' = x' \vee y'$. It is enough to prove that E has Gödel negation property. Clearly, $\mathcal{D}_s(E) \subseteq E \setminus \{0\}$. Now, let $x \in E \setminus \{0\}$. Since $x \to x'' = 1$ and by assumption $x \to y' = x' \vee y'$, we get $x' \vee x'' = 1$. Since E is a chain, we get $x' = 1$ or $x'' = 1$. If $x' = 1$, then $x'' = 0$ and by Proposition 2.2(i), $x = 0$, a contradiction. If $x'' = 1$, then by Proposition 2.2(iv), $x' = x''' = 0$, and so $x \in \mathcal{D}_s(E)$. Hence, $\mathcal{D}_s(E) = E \setminus \{0\}$. Now, by Proposition 3.46, E is integral. The proof of the converse, by Proposition 3.47 is clear. □

Proposition 3.49. Consider E to be an integral involutive lattice. If F is an integral filter of E, then $F \in \mathcal{S}pec(E)$.

Proof: Let F be an integral filter of E and $x, y \in E$ such that $x \vee y \in F$. Then by Proposition 3.47 and assumption, $x' \to y'' = x'' \vee y'' = x \vee y \in F$. Since F is an integral filter of E, we have $x = x'' \in F$ or $y = y'' \in F$. Hence, $F \in \mathcal{S}pec(E)$. □

Clearly, converse of Proposition 3.49 does not hold in general. Since according to Example 3.3(i), $F = \{b, 1\} \in \mathcal{S}pec(E)$ but is not integral, since $a \to a' = a \to a = 1$ but $a' = a'' = a \notin F$.

Theorem 3.50. *E is an integral equality algebra if and only if $Rad(E) = E \setminus \{0\}$.*

Proof: Consider E to be an integral equality algebra. Then by Proposition 3.46, $\mathcal{D}_s(E) = E \backslash \{0\}$. Let $x \in Rad(E)$. Then by Proposition 2.12, for any $n \in \mathbb{N}$, $(x \overset{n}{\to} 0) \to x = 1$, and so for any $n \in \mathbb{N}$, $x \overset{n}{\to} 0 \leq x$. If $x = 0$, then $1 = x \overset{n}{\to} 0 \leq x = 0$, which is a contradiction, so $Rad(E) \subseteq E \setminus \{0\}$. Now, suppose $x \in E \setminus \{0\}$. Then $x \in \mathcal{D}_s(E)$, and so $x' = 0$. Thus,

$$x \overset{n}{\to} 0 = x \overset{n-1}{\to} (x \to 0) = x \overset{n-1}{\to} 0 = \cdots = x \to 0 = 0.$$

Hence, for any $x \in E \setminus \{0\}$, $x \overset{n}{\to} 0 \leq x$ and so $(x \overset{n}{\to} 0) \to x = 1 \in F$. Hence, $x \in Rad(E)$. Therefore, $Rad(E) = E \setminus \{0\}$.
Conversely, by Proposition 3.46, it is enough to prove that $\mathcal{D}_s(E) = E \setminus \{0\}$. Let $x \in E \setminus \{0\} = Rad(E)$. If $x' \neq 0$, then $x' \in Rad(E)$ and so $x, x' \in Rad(E)$ which is a contradiction. Thus, $x' = 0$, and so $x \in \mathcal{D}_s(E)$. Hence, $E \setminus \{0\} \subseteq \mathcal{D}_s(E) \subseteq E \setminus \{0\}$. Therefore, $\mathcal{D}_s(E) = E \setminus \{0\}$. □

Corollary 3.51. Let E be integral and $F \in \mathcal{F}(E)$. Then

(i) $Rad(F) = E \setminus \{0\}$.

(ii) $Rad(F) \in \mathcal{M}ax(E)$.

Proof: (i) Let $F \in \mathcal{F}(E)$. Since $\{1\} \subseteq F \subseteq E$. By [11, Proposition 3.14(ii)], $Rad(\{1\}) \subseteq Rad(F) \subseteq Rad(E)$. Then by Theorem 3.50, we have $E \setminus \{0\} \subseteq Rad(F) \subseteq E \setminus \{0\}$. Hence, $Rad(F) = E \setminus \{0\}$.

(ii) By (i) the proof is clear. □

Example 3.52. According to Example 3.3(i), $F = \{b, 1\} \in \mathcal{M}ax(E)$. Clearly, $F \neq E \setminus \{0\}$. In addition, E is not integral.

Proposition 3.53. Let E be integral, $F, G \in \mathcal{F}(E)$ and $F \subseteq G$ such that G be proper. If $Rad(F) = F$, then $Rad(G) = G$.

Proof: By Corollary 3.51(i), $Rad(F) = E \setminus \{0\}$. Since $\{1\} \subseteq F$, we have $Rad(\{1\}) \subseteq Rad(F)$ and so $E \setminus \{0\} \subseteq Rad(F) = F \subseteq G \subset E$. Thus, $E \setminus \{0\} = F = Rad(F) = G$. Since $F \subseteq G$, by [11, Proposition 3.14(ii)], $Rad(F) \subseteq Rad(G)$. Hence, $Rad(G) = G$. □

4. Local equality algebras

In this section, we define the notion of local bounded equality algebra and investigate the relation between local and other kinds of bounded equality algebras. We prove that if E is a bounded lattice local equality algebra, then $\mathcal{C}(E) = \{0, 1\}$. In addition, we show that every local bounded equality algebra is perfect or locally finite or peculiar.

DEFINITION 4.1. E is called *local*, if for any $x \in E$, $ord(x) < \infty$ or $ord(x') < \infty$.

Example 4.2. (i) According to Example 3.8(ii), clearly, $ord(1) = ord(c) = ord(a) = ord(b) = \infty$ and $ord(c') = ord(a') = ord(b') = ord(0) = 1$. Hence, E is local.

(ii) According to Example 3.3(iii), E is local.

(iii) According to Example 3.3(i), E is local.

(iv) In Example 3.22(ii), E is a bounded equality algebra which is not local since $ord(a) = ord(a') = \infty$.

THEOREM 4.3. *If E has a unique maximal filter, then E is local.*

PROOF: Suppose $x \in E$ such that $ord(x) = ord(x') = \infty$. By Proposition 3.4(i), $\langle x \rangle$ and $\langle x' \rangle$ are two proper filters of E. By Proposition 2.11(ii) and assumption, $\langle x \rangle, \langle x' \rangle \subseteq M$ and so $x, x' = x \to 0 \in M$. Thus, $0 \in M$, which is a contradiction. So, $ord(x) < \infty$ or $ord(x') < \infty$. Hence, E is local. $\square$

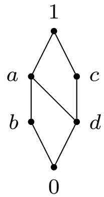
THEOREM 4.4. *Consider E to be a p-equality algebra. Then E is local if and only if E has a unique maximal filter.*

PROOF: Suppose E is local and $M \in \mathcal{M}ax(E)$. By Proposition 3.4(iv), $M \subseteq \{x \in E \mid ord(x) = \infty\}$. Now, assume $x \notin M$ and $ord(x) = \infty$. By Proposition 2.11(i), there exists $n \in \mathbb{N}$ such that $x \overset{n}{\to} 0 \in M$. Since E is p-equality and $x \to 0 = x \overset{n}{\to} 0$, we get $x \to 0 \in M$. Thus, by Proposition 3.4(iv), $ord(x \to 0) = \infty$. Since E is local, $ord((x \to 0) \to 0) < \infty$ and by Propositions 3.4(ii) and 2.2(i), $ord(x) < \infty$, which is a contradiction. Thus, for any $M \in \mathcal{M}ax(E)$, $M = \{x \in E \mid ord(x) = \infty\}$. Hence, E has a unique maximal filter. The proof of the converse by Theorem 4.3 is hold.

□

In the following example we show that the condition p-equality in Theorem 4.4 is necessary.

Example 4.5. Consider $E = \{0, a, b, c, d, 1\}$ is a lattice with the next Hasse diagram. Define the operation "$\sim$" on E as follows:



$\sim$	0	a	b	c	d	1
0	1	d	c	b	a	0
a	d	1	a	d	c	a
b	c	a	1	0	d	b
c	b	d	0	1	a	c
d	a	c	d	a	1	d
1	0	a	b	c	d	1

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	a	c	c	1
b	c	1	1	c	c	1
c	b	a	b	1	a	1
d	a	1	a	1	1	1
1	0	a	b	c	d	1

Then $(E, \sim, \wedge, 1)$ is an equality algebra which is not a p-equality algebra, since

$$c = a \rightarrow d = a \rightarrow (a \rightarrow 0) \neq (a \rightarrow a) \rightarrow (a \rightarrow 0) = 1 \rightarrow d = d.$$

Also, E is not local, since $ord(b) = ord(b') = \infty$.

THEOREM 4.6. *Let $P \in \mathcal{F}(E)$ be proper. Then $\dfrac{E}{P}$ is local if and only if P is primary.*

PROOF: Suppose $\dfrac{E}{P}$ is local, for any $x, y \in E$, $x \rightarrow y' \in P$ and for any $n \in \mathbb{N}$, $x \overset{n}{\rightarrow} 0 \notin P$. Then $[x] \overset{n}{\rightarrow} [0] \neq [1]$. Since $x \rightarrow y' \in P$, we have $[x] \leq [y']$ and similar to the proof of Proposition 3.4(ii), we have $[y'] \overset{n}{\rightarrow} [0] \leq [x] \overset{n}{\rightarrow} [0] \neq [1]$, for any $n \in \mathbb{N}$. Thus, $[y'] \overset{n}{\rightarrow} [0] \neq [1]$, and so $[y' \xrightarrow{n} 0] \neq [1]$. Hence, $ord([y']) = \infty$. Since $\frac{E}{P}$ is local, we have $ord([y']') < \infty$. By Proposition 2.2(i), $ord([y]) < \infty$. So there exists $k \in \mathbb{N}$ such that $[y] \xrightarrow{k} [0] = [1]$ and so $y \xrightarrow{k} 0 \in P$.

Conversely, by Proposition 2.2(i), $x \to x'' = 1 \in P$. Since P is primary, there exists $n \in \mathbb{N}$ such that $x \xrightarrow{n} 0 \in P$ or $x' \xrightarrow{n} 0 \in P$. If $ord([x]) = \infty$, then for any $n \in \mathbb{N}$, $[x] \xrightarrow{n} [0] \neq [1]$ and so $[x \xrightarrow{n} 0] \neq [1]$. Hence, $x \xrightarrow{n} 0 \notin P$, thus, $x' \xrightarrow{n} 0 \in P$, and so $[x'] \xrightarrow{n} [0] = [1]$. Hence, $ord([x']) < \infty$. The proof of the other case is similar. Therefore, $\frac{E}{P}$ is local □

Proposition 4.7. If E is integral, then E is local.

Proof: Since E is integral, we get $\frac{E}{\{1\}}$ is integral. Then by Theorem 3.27, $\{1\}$ is primary. By Theorem 4.6, $\frac{E}{\{1\}}$ is local. Hence, E is local. □

According to Example 3.3(i), E is local but E is not integral, since $a \leq a' = a$ and so $a = a' \neq 1$.

Corollary 4.8. Let $F \in \mathcal{F}(E)$. If $F = Rad(F)$, then $\frac{E}{F}$ is local.

Proof: By Proposition 3.30 and Theorem 4.6, the proof is clear. □

Corollary 4.9. $\{1\}$ is a primary filter of E if and only if E is local.

Proposition 4.10. If E is perfect and p-equality, then $E = Rad(E) \cup Rad'(E)$.

Proof: Since E is perfect, we get $\frac{E}{\{1\}}$ is perfect, and by Theorem 3.19, $\{1\}$ is perfect. By Theorem 3.23, $\{1\}$ is primary and by Corollary 4.9, E is local. Hence, E has a unique maximal filter M that contains all $x \in E$ such that $ord(x) = \infty$. Suppose $x \in E$. If $ord(x) = \infty$, then $x \in Rad(E)$. If $ord(x) < \infty$, since E is perfect, then $ord(x') = \infty$ and so $x' \in Rad(E)$. Thus, $x \in Rad'(E)$. Hence, $E = Rad(E) \cup Rad'(E)$. □

Corollary 4.11. Consider E to be a p-equality algebra. A proper filter $P \subseteq E$ is primary if and only if there is a unique $M \in \mathcal{M}ax(E)$ such that

$P \subseteq M$.

Proof: By Proposition 4.6, P is primary if and only if $\frac{E}{P}$ is local if and only if by Theorem 4.4, $\frac{E}{P}$ has only one maximal filter, such as $\frac{M}{P}$. Since there is a one-to-one correspondence between the filters of E and $\frac{E}{P}$, we get there is a unique $M \in \mathcal{M}ax(E)$ such that $P \subseteq M$. $\square$

Corollary 4.12. If E is local, then for any $G \in \mathcal{F}(E)$, $\frac{E}{G}$ is local, too.

Proposition 4.13. Consider E to be a local p-equality. Then E is perfect if and only if $\frac{E}{Rad(E)} = \{[0], [1]\}$.

Proof: Since E is perfect, we get $\frac{E}{\{1\}}$ is perfect, and by Theorem 3.19, $\{1\}$ is perfect. By Theorem 3.23, $\{1\}$ is primary and by Corollary 4.9, E is local. Thus, it has just one maximal filter M such that $Rad(E) = M$. In addition, since E is perfect, by Proposition 4.10, $E = Rad(E) \cup Rad'(E)$. Let $x \in E$. If $x \in Rad(E)$, then $[x] = [1]$. If $x \in Rad'(E)$, then $x' \in Rad(E)$, and so $[x'] = [1]$. Thus, $[x''] = [0]$. By Proposition 2.2(i), $[x] \leq [x'']$, and so $[x] = [0]$. Therefore, $\frac{E}{Rad(E)} = \{[0], [1]\}$.

Conversely, let $x \in E$. Then $[x] \in \frac{E}{Rad(E)} = \{[0], [1]\}$. If $[x] = [1]$, then $x \in Rad(E)$. If $[x] = [0]$, then $[x'] = [1]$, and so $x' \in Rad(E)$. Hence, $x \in Rad'(E)$. Thus, $ord(x) = \infty$ or $ord(x') = \infty$. If $ord(x) = \infty$ and $ord(x') = \infty$, then by Proposition 3.4(i), $\langle x \rangle$ and $\langle x' \rangle$ are two proper filters of E, and by Proposition 2.11(i) and assumption, $x, x' = x \to 0 \in M$, where $M \in \mathcal{M}ax(E)$, so $0 \in M$, which is a contradiction. Hence, E is perfect. $\square$

Corollary 4.14. Every chain equality algebra is local.

Note. Set

$$\mathcal{D}(E) = \{x \in E \mid ord(x) = \infty\}$$

and
$$\mathcal{D}^*(E) = \{x \in E \mid \exists\, y \in \mathcal{D}(E) \text{ s.t } x \leq y'\}.$$

THEOREM 4.15. *Assume E is local. Then E is perfect if and only if $E = \mathcal{D}(E) \cup \mathcal{D}^*(E)$.*

PROOF: Suppose $x \in E$. If $ord(x) = \infty$, then $x \in \mathcal{D}(E)$. If $ord(x) < \infty$, then $x \notin \mathcal{D}(E)$. Since E is perfect, we get $ord(x') = \infty$, and so $x' \in \mathcal{D}(E)$. By Proposition 2.2(i), we know $x \leq x''$ and by definition of $\mathcal{D}^*(E)$, we get $x \in \mathcal{D}^*(E)$. Hence, $E \subseteq \mathcal{D}(E) \cup \mathcal{D}^*(E)$. The proof of the other side is clear. Therefore, $E = \mathcal{D}(E) \cup \mathcal{D}^*(E)$.
Conversely, let $E = \mathcal{D}(E) \cup \mathcal{D}^*(E)$ and $x \in E$. If $ord(x) < \infty$, then $x \notin \mathcal{D}(E)$. By assumption, $x \in \mathcal{D}^*(E)$. So, there is $y \in \mathcal{D}(E)$ such that $x \leq y'$. By Proposition 2.2(i) and (iii), $y \leq y'' \leq x'$. Since $y \in \mathcal{D}(E)$, we get $ord(y) = \infty$ and by Proposition 3.4(iii), $ord(x') = \infty$. Thus, in this case E is perfect. Now, if $ord(x) = \infty$, then $x \in \mathcal{D}(E)$. If $ord(x') = \infty$, the E is not local, which is a contradiction. Thus, $ord(x') < \infty$. Therefore, in both cases, E is perfect. □

PROPOSITION 4.16. Consider E to be a p-equality algebra and $\mathcal{N}(E) = \{x \in E \mid \exists\, n \in \mathbb{N}\ s.t\ x \overset{n}{\to} 0 = 1\}$. Then

(i) $E = \mathcal{N}(E) \cup Rad(E)$ if and only if E is local.

(ii) If E is a lattice and local, then $\mathcal{C}(E) = \{0, 1\}$.

PROOF: (i) Suppose E is local. Then by Theorem 4.4, E has a unique maximal filter such as M and so $Rad(E) = M$. Let $x \in E$. Then by assumption $ord(x) < \infty$ or $ord(x') < \infty$. If $ord(x) < \infty$, then clearly $x \in \mathcal{N}(E)$. If $ord(x) = \infty$, then by Theorem 4.4, $x \in M = Rad(E)$. Hence, $E = \mathcal{N}(E) \cup Rad(E)$.

Conversely, suppose $M_1, M_2 \in \mathcal{M}ax(E)$ such that $M_1 \neq M_2$. Then there are $x \in M_1 \setminus M_2$. Since $x \in E$, by assumption, $x \in \mathcal{N}(E)$ or $x \in Rad(E)$. If $x \in \mathcal{N}(E)$, then there exists $n \in \mathbb{N}$ such that $x \overset{n}{\to} 0 = 1 \in M_1$, and so $0 \in M_1$, which is a contradiction. If $x \in Rad(E)$, then according to definition of $Rad(E)$, $x \in M_2$, which is a contradiction. The proof of the other case is similar. Hence, E is local.

(ii) Suppose E is local. Then by Theorem 4.4, E has a unique maximal filter such as M. Let $x \in \mathcal{C}(E)$. Then $x \vee x' = 1 \in M$. Since by Propositions 2.10(ii) and 2.11(iii) $M \in \mathcal{S}pec(E)$, we get $x \in M$ or $x' \in M$. Thus, $ord(x) = \infty$ or $ord(x') = \infty$. By Proposition 3.4(i), if $ord(x) = \infty$ and $ord(x') = \infty$, then by Proposition 3.19(ii), $x, x' \in M$, and so $0 \in M$, a contradiction. If $ord(x) = \infty$, then $ord(x') < \infty$, so there exists $n \in \mathbb{N}$ such that $x' \overset{n}{\to} 0 = 1$. Since E is p-equality algebra, we get $x' \to 0 = 1$ and so $x'' = 1$. Thus, by Proposition 2.2(v), $x' = x''' = 0$. Since $x \vee x' = 1$, we get $x = 1$. If $ord(x) < \infty$, then there exists $n \in \mathbb{N}$ such that $x \overset{n}{\to} 0 = 1$, and so $x \to 0 = 1$. Hence, $x' = 1$. Thus, by Proposition 2.2(iv), $x \leq x'' = 0$. Thus, $x = 0$. Therefore, $\mathcal{C}(E) = \{0, 1\}$. □

Definition 4.17. An equality algebra E is called *locally finite* if for any $x \in E \setminus \{1\}$, $ord(x) < \infty$.

Example 4.18. Let $E = \{0, a, b, c, 1\}$ be a chain where $0 \leq a \leq b \leq c \leq 1$. Define the operation "$\sim$" on E as follows:

$\sim$	0	a	b	c	0
0	1	c	b	b	1
a	c	1	c	c	a
b	b	c	1	c	b
c	b	c	c	1	c
1	0	a	b	c	1

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	c	1	1	1	1
b	b	c	1	1	1
c	b	c	c	1	1
1	0	a	b	c	1

Then $(E, \wedge, \sim, 0, 1)$ is a locally finite equality algebra.

Definition 4.19. E is called a *peculiar equality algebra* if it is local and there are $x, y \in E \setminus \{0, 1\}$ such that $ord(x) = \infty$, $ord(y) < \infty$ and $ord(y') < \infty$.

Example 4.20. (i) In Example 3.3(i), E is a peculiar equality algebra.

(ii) In Example 3.3(ii), E is a peculiar equality algebra.

(iii) In Example 3.8(ii), E is not a peculiar equality algebra.

Theorem 4.21. *Let E be local. Then E is perfect or locally finite or peculiar.*

Proof: We have the following cases:

Case 1: Suppose that E is not locally finite and peculiar. We prove that E is perfect. Since E is not locally finite, there exists $x \in E \setminus \{1\}$ such that $ord(x) = \infty$. Since E is local, we have $ord(x') < \infty$. Moreover, since $ord(x) = \infty$, if there exists $y \in E$ such that $ord(y), ord(y') < \infty$, then E is peculiar, which is a contradiction. Hence, for all $y \in E$, $ord(y) = \infty$ or $ord(y') = \infty$. If there exists $y \in E$ such that $ord(y) = ord(y') = \infty$, then E is not local, which is a contradiction. Therefore, E is perfect.

Case 2: Suppose that E is not perfect and locally finite. We prove that E is peculiar. By assumption, E is local. Since E is not locally finite, there exists $x \in E \setminus \{1\}$ such that $ord(x) = \infty$. Also, E is not perfect. Then there exists $y \in E$ such that $ord(y) = ord(y') = \infty$ or $ord(y), ord(y') < \infty$. If $ord(y) = ord(y') = \infty$, then E is not local, which is a contradiction. Hence, there exists $y \in E$ such that $ord(y), ord(y') < \infty$. Therefore, E is peculiar.

Case 3: Suppose that E is not perfect and peculiar. We prove that E is locally finite. Since E is not perfect, there exists $y \in E$ such that $ord(y) = ord(y') = \infty$ or $ord(y), ord(y') < \infty$. If $ord(y) = ord(y') = \infty$, then E is not local, which is a contradiction. Thus, there exists $y \in E$ such that $ord(y), ord(y') < \infty$. Now, let x be an arbitrary element of $E \setminus \{1\}$. If $x = y$ or $x = y'$, then $ord(x) < \infty$. Suppose that $x \neq y$. If $ord(x) = \infty$, then E is peculiar, which is a contradiction. Thus, $ord(x) < \infty$. Hence, for all $x \in E \setminus \{1\}$, $ord(x) < \infty$. Therefore, E is locally finite. $\square$

5. Conclusion

By considering the notion of bounded equality algebra, the concepts of local and perfect bounded equality algebras are defined and some related results are proved. Specially, the notions of order of elements in bounded equality algebras and some filters such as primary and perfect filters are introduced and the relation between perfect filters with perfect bounded equality algebras and primary filter with the local bounded equality algebras are investigated. In addition, by defining the concept of locally finite bounded equality algebra, a relation between local and perfect bounded equality algebras are studied. In addition, it was shown that every integral bounded equality algebra is local and every local bounded equality algebra is perfect or locally finite or peculiar.

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