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A STUDY OF A SPECIAL SEMI MAXIMAL FILTER IN BL-ALGEBRAS

Abstract

In this article, a specific type of semi maximal filters is introduced, which form a lattice structure. These filters are called J- semi maximal and NJ-semi maximal, and their key properties in BL-algebras are analyzed. Additionally, these special filters are compared with other defined filters, particularly semi maximal and maximal filters. The purpose of this article is to provide a new analysis of filters in BL-algebras.

Keywords: BL-algebra, semi maximal filter, J-semi maximal filter, NJ-semi maximal filter.

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1. Introduction

Hájek introduced BL-algebras as an algebraic approach to studying many-valued logic [5]. He gave an algebraic proof of the completeness theorem for Basic Logic (BL), which is based on continuous triangular norms commonly applied in fuzzy logic. Filter theory plays a central role in the study of these algebras, since different filters correspond to different sets of provable formulas. Hájek introduced the concepts of filters and prime filters in BL-algebras [5], and employed prime filters to prove the completeness of BL. Expanding upon Hájek's foundational contributions, Turunen systematically explored the structural characteristics of filters and prime filters within BL-algebras [11, 12], thereby advancing the theoretical understanding of their underlying algebraic framework. Building on these developments, M. Bedrood et al. introduced and investigated a specific subclass of prime filters, termed *J-prime filters*, offering further insight into the structural behavior of prime filters in BL-algebras [2]. BL-algebras offer a rigorous algebraic framework for supporting logical operations within fuzzy systems. This ensures that fuzzy reasoning is not only effective for managing imprecise information but also grounded in a mathematically sound structure, guaranteeing logical consistency and reliability.

Maximal filters in BL-algebras are particularly important in various fields such as non-classical logic, lattice theory, and mathematical and practical applications. These filters are key tools for analyzing complex algebraic structures and understanding the logical relationships between elements. With the development of new concepts like semi-maximal filters and their types, researchers have gained a more precise understanding of how these filters interact with the structures of BL-algebras.

S. Motamed et al. introduced and studied radical filters based on maximal filters [8]. A. Paad et al. defined and investigated semi maximal filters [9]. Based on these studies, A. Movahed et al. presented new results and an equivalent definition for semi maximal filters, comparing them with other types of filters [6].
In this paper, we investigate a distinguished subclass of generalized semi maximal filters in BL- algebras, with an emphasis on their structural and logical characteristics. Based on the concept of semi simple BL-algebras, we concluded that every semi maximal filter is also an NJ-semi maximal filter. Furthermore, every minimal prime filter is a J-semi maximal filter in BL-algebras. Additionally, we proved that for any filter in a Hyperarchimedean BL-algebra, every proper filter of it is a J-semi maximal filter. Finally, we introduced the concept of semi factors and provided a clear framework for understanding this idea.

2. Preliminaries

We recollect some definitions and results which will be used in the sequel:

Definition 2.1 ([5]). A BL-algebra is an algebra $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ equipped with an order $\leq$ satisfying the following:

- (BL_1) $(L, \wedge, \vee, 0, 1)$ is a bounded lattice,
- (BL_2) $(L, \odot, 1)$ is a commutative monoid,
- (BL_3) $\odot$ and $\rightarrow$ form an adjoint pair i.e., $z \leq x \rightarrow y$ if and only if $x \odot z \leq y$, for all $x, y, z \in L$,
- (BL_4) $x \wedge y = x \odot (x \rightarrow y)$, for all $x, y \in L$,
- (BL_5) $(x \rightarrow y) \vee (y \rightarrow x) = 1$, for all $x, y \in L$.

Throughout the paper, we denote L as a $BL-$algebra.

Definition 2.2 ([10]). An element, if $a \odot a = a$ and the collection of all idempotent elements is displayed with $B(L)$.

Also if every element of L is an idempotent element, then BL-algebra L is called a Boolean algebra.We say that a $BL-$algebra A is a $BL-$chain if the underlying order $\leq$ is total.

Definition 2.3 ([13]). A filter is a non-empty subset F of L satisfying the following conditions:

$(F1)$ If $a \in F,\ b \in L$ and $a \leq b$, then $b \in F$,

$(F2)$ If $a, b \in F$, then $a \odot b \in F$.

We denote by $F(L)$ the set of all filters of L. Also, it has also been proven that $(F(L), \wedge, \vee, \{1\}, L)$ is a complete Brouwerian lattice.

Lemma 2.4 ([10]).

(1) Let $X \subseteq L$. Denote by $(X]$ the filter generated by X. Then we have

$$(X] = \{a \in L : x_1 \odot x_2 \odot \ldots \odot x_n \leq a,\ n \in \mathbb{N}, x_1, x_2, \ldots, x_n \in X\}$$

In particular $(a] = \{x \in L : a^n < x, n \in \mathbb{N}\}$.

(2) For $F, G \in F(L)$ and $a, b \in L$

- $F \wedge G = F \cap G$;
- $F \vee G = (F \cup G] = \{x \in L | a \odot b \leq x \ for\ some\ a \in F\ and\ b \in G\}$;
- *If* $a \leq b$, *then* $(b] \subseteq (a]$;
- $(a] \vee (b] = (a \odot b] = (a \wedge b]$;
- $(a] \cap (b] = (a \vee b]$.

Definition 2.5 ([13, 3]). Let F be a filter of L.

- If $F \neq L$, then F is called a proper filter of L.
- A proper filter F of L is called prime filter if for all $a, b \in L, a \vee b \in F$, satisfies $a \in F$ or $b \in F$.

We denote by $Spec(L)$ the set of all prime filters of a BL-algebra L.

- A filter P of L is called a minimal prime filter of L when:

(1) $P \in Spec(L)$;
(2) If there exists $Q \in Spec(L)$ such that $Q \subseteq P$, then $P = Q$.

We denote by $Min(L)$ the set of all minimal prime filters of L.

- A proper filter F of L is called maximal if and only if for each filter $J \neq F$, if $F \subseteq J$, implies $J = L$.
  We denote by $Max(L)$ the set of all maximal filters of L.
  The intersection of all maximal filters of L is called the radical of L and it is denoted by $Rad(L)$. The intersection of all maximal filters of L which contain the filter F is called the radical of F and it is denoted by $Rad(F)$.

Note: Prime filter P of L is called minimal prime filter over filter F, if

(1) $F \subseteq P$;
(2) If there exists $Q \in Spec(L)$ such that $F \subseteq Q \subseteq P$, then $P = Q$.

We denote by $Min(F)$ the set of all minimal prime filters over filter F.

COROLLARY 2.6 ([10]). Every prime filter of L is contained in a unique maximal filter of L.

THEOREM 2.7 ([8]). *A BL-algebra L is called semi simple if and only if $Rad(L) = \{1\}$.*

Remark 2.8 ([10]). For every $F \in F(L)$;

(1) $F = \cap \{P \in Spec(L) \mid F \subseteq P\}$.
(2) $\cap \{P \in Spec(L)\} = \{1\}$.

THEOREM 2.9 ([4]). *Let $P \in spec(L)$. Then $P \in Min(L)$ if and only if for each $a \in P$, there exists $r \in L \setminus P$ such that $r \vee a = 1$.*

DEFINITION 2.10 ([11]). Let X be a non-empty subset of L. $Co-Ann_L(X)$ is the Co- annihilator of X defined by:

$$Co - Ann_L(X) = \{a \in L | a \vee x = 1, \forall x \in X\}$$

THEOREM 2.11 ([4]). *Let $P \in Min(L)$ and F be finitely generated filter. Then $F \subseteq P$ if and only if $Co - Ann_L(F) \nsubseteq P$.*

Definition 2.12 ([10]). An element $a \in L$ is called archimedean if there is $n \in \mathbb{N}$, such that $a \vee (a^n)^* = 1$.

Theorem 2.13 ([10]). *L is Hyperarchimedean if and only if $Spec(L) = Max(L)$.*

Theorem 2.14 ([5]). *If $a \neq 1$, then there is a prime filter P of L such that $a \notin P$.*

Corollary 2.15. Let F be an filter of L and $a \in L \setminus F$. Then there exists $P \in spec(L)$ such that $F \subseteq P$ and $a \notin P$.

Theorem 2.16 ([10]). *For any L, the following statements are equivalent:*

(1) L is a BL-chain.

(2) Any proper filter of L is prime.

(3) $\{1\}$ is a prime filter.

(4) $Spec(L)$ is linearly ordered.

Proposition 2.17 ([6]). Let F be a proper filter of L and $P \in Spec(L)$ such that $F \subseteq P$. Then there exists $Q \in Min(F)$ such that $Q \subseteq P$.

Lemma 2.18 ([10]). *The proper filter P is a prime filter if and only if $F \cap G \subseteq P$, then $F \subseteq P$ or $G \subseteq P$, for all $F, G \in F(L)$.*

Definition 2.19 ([8]). Let F be a proper filter of L. If $Rad(F) = F$, then F is called a semi maximal filter of L.

Theorem 2.20 ([8]). *Every maximal filter of L is a semi maximal filter.*

Proposition 2.21 ([6]). Let F be a semi maximal filter of L and K a subset of L such that $K \nsubseteq F$. Then the set $(F : K) = \{x \in L \mid x \vee k \in F, \forall k \in K\}$ is also a semi maximal filter.

Note: Let $a \in L$ and F be a filter of L . We put

- $M_a = \cap \, \{M | \; M \in Max(L), \; a \in M\}$.
- $M(a) = \{M | \; M \in Max(L), \; a \in M\}$.
- $P_a = \cap \, \{P | \; P \in Min(L), \; a \in P\}$.

THEOREM 2.22 ([6]). *A proper filter F of L is a semi maximal filter if and only if for all $a \in F$, $M_a \subseteq F$.*

THEOREM 2.23 ([6]). *Let F be a proper filter of L. Then the following statements are equivalent:*

(1) F is a semi maximal filter in L.

(2) $M(a) \subseteq M(b)$ and $a \in F$, implies that $b \in F$.

(3) $M(a) = M(b)$ and $a \in F$ implies that $b \in F$.

THEOREM 2.24 ([6]). *If L is a semi simple $BL-algera$ and $a \in L$, then $M_a \subseteq P_a$.*

LEMMA 2.25 ([1]). *If $a \in L$, then $P_a = Co - Ann(Co - Ann(a))$.*

3. Semi maximal filters in BL-algebras

In this section, we introduce the notions of J-semi maximal and NJ-semi maximal filters, exploring their defining properties and the conditions under which various filters qualify as either J-semi maximal or NJ-semi maximal. The concept of J-semi maximal filters extends the idea of semi maximal filters, broadening their applicability. Additionally, we present the notion of semi factors and demonstrate that the set of semi factors of a filter forms a lattice.

DEFINITION 3.1. Let F and J be two filters of L. A filter F is called a J-semi maximal filter if $M_a \cap J \subseteq F$ for all $a \in F$. Also, if $J \nsubseteq F$ and F is a J-semi maximal filter, then F is called an NJ-semi maximal filter, and J is referred to as a semi factor of F.

Clearly, if $J \subseteq F$, then F is always a J-semi maximal filter. It follows that every NJ-semi maximal filter is, by definition, a J-semi maximal filter.

Example 3.2. (1) Let $L = \{0, a, b, c, d, 1\}$. where $0 < a, b < c < 1$ and $0 < b < d < 1$. Define $\odot$ and $\to$ as follows:

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	a	0	a
b	0	0	0	0	b	b
c	0	a	0	a	b	c
d	0	0	b	b	d	d
1	0	a	b	c	d	1

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	d	1	d	1
b	c	c	1	1	1	1
c	b	c	d	1	d	1
d	a	a	c	c	1	1
1	0	a	b	c	d	1

The relationships between the members are depicted in Figure 1.

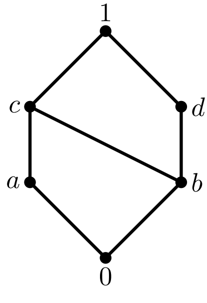


Figure 1: Relationships between the elements of $L = \{0, a, b, c, d, 1\}$.

Then $(L, \odot, \rightarrow, 0, 1)$ is a BL-algebra [10]. It has four filters:
$F_0 = \{1\}, F_1 = L, F_2 = \{1, a, c\}, F_3 = \{1, d\}$. Obviously, $Max(L) = \{F_2, F_3\}$ and $M_1 = F_0$, $M_a = F_2$, $M_b = L$, $M_c = F_2$, $M_d = F_3$. Then F_0, F_2 and F_3 are semi maximal filters.
It is easy to see that F_2 is a F_3-semi maximal filter and $F_3 \nsubseteq F_2$, so F_2 is

a NF_3-semi maximal filter and F_3 is a semi factor of the filter F_2. Also, F_0 is a F_2-semi maximal filter and F_0 is a NF_2-semi maximal filter.

(2) Let $L = \overline{\mathbb{Z}} \cup \{-\infty\} \cup \{0, a, b, 1\}$, where $\overline{\mathbb{Z}}$ is the set of negative integer numbers and $-\infty < \cdots < -2 < -1 < 0 < a, b < 1$. Operations $\odot$ and $\rightarrow$ are defined as follows:

$\odot$	$-\infty$	$\cdots$	-3	-2	-1	0	a	b	1
$-\infty$	$-\infty$	$\cdots$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
-3	$-\infty$	$\cdots$	-6	-5	-4	-3	-3	-3	-3
-2	$-\infty$	$\cdots$	-5	-4	-3	-2	-2	-2	-2
-1	$-\infty$	$\cdots$	-4	-3	-2	-1	-1	-1	-1
0	$-\infty$	$\cdots$	-3	-2	-1	0	0	0	0
a	$-\infty$	$\cdots$	-3	-2	-1	0	a	0	a
b	$-\infty$	$\cdots$	-3	-2	-1	0	0	b	b
1	$-\infty$	$\cdots$	-3	-2	-1	0	a	b	1

$\rightarrow$	$-\infty$	$\cdots$	-3	-2	-1	0	a	b	1
$-\infty$	1	$\cdots$	1	1	1	1	1	1	1
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
-3	$-\infty$	$\cdots$	1	1	1	1	1	1	1
-2	$-\infty$	$\cdots$	-1	1	1	1	1	1	1
-1	$-\infty$	$\cdots$	-1	1	1	1	1	1	1
0	$-\infty$	$\cdots$	-3	-2	-1	1	1	1	1
a	$-\infty$	$\cdots$	-3	-2	-1	b	1	b	1
b	$-\infty$	$\cdots$	-3	-2	-1	a	a	1	1
1	$-\infty$	$\cdots$	-3	-2	-1	0	a	b	1

The relationships between the members are depicted in Figure 2. Then $(L, \wedge, \vee, \odot, \rightarrow, -\infty, 1)$ is a BL-algebra [7]. It has filters:
$F_0 = \{1\}$, $F_1 = L$, $F_2 = \{1, b, a, 0\}$, $F_3 = \{1, a\}$, $F_4 = \{1, b\}$ and $F_5 = \{\ldots, -3, -2, -1, 0, a, b, 1\} \smallsetminus \{-\infty\}$ are filters. Obviously, $F_2 \in Max(L)$ and $M_a = F_2 \nsubseteq F_3$. Hence F_3 is not a semi maximal filter.

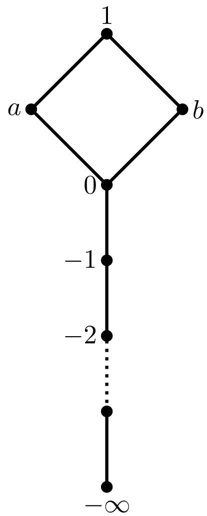


Figure 2: Relationships between the elements of $L = \overline{\mathbb{Z}} \cup \{-\infty\} \cup \{0, a, b, 1\}$.

It is easy to see that F_3 is not F_5-semi maximal filter(because $M_a \cap F_5 \nsubseteq F_3$). Also, consider $F_3 = \{1, a\}$ is not F_2-semi maximal filter.

Lemma 3.3. *If $a, b \in L$, then the following sentences hold:*

(1) $a \in M_b$ if and only if $M_a \subseteq M_b$;

(2) $M_a \cap M_b = M_{(a \vee b)}$;

(3) If $a \leq b$, then $M(a) \subseteq M(b)$ and $M_b \subseteq M_a$;

(4) $M_{(a \odot b)} = M_a \vee M_b$.

Proof: (1) It is trivial.

(2) Let $x \in M_a \cap M_b$, but $x \notin M_{(a \vee b)}$. Then there exists $M \in Max(L)$ such that $x \notin M$ and $a \vee b \in M$, since M is prime filter, Hence $a \in M$ or $b \in M$ and we get $x \notin M_a$ or $x \notin M_b$, which is a contradiction. Therefore

$M_a \cap M_b \subseteq M_{(a\vee b)}$. Now, suppose that $x \in M_{(a\vee b)}$ but $x \notin M_a \cap M_b$. Without loss of generality, suppose that $x \notin M_a$. So there exists $M \in Max(L)$ such that $x \notin M$ and $a \in M$. As $a \leq a \vee b$, hence we conclude that $a \vee b \in M$. Thus $x \notin M_{(a\vee b)}$, which is a contradiction. Hence $M_{(a\vee b)} \subseteq M_a \cap M_b$. Therefore $M_{a\vee b} = M_a \cap M_b$.

(3) Suppose that $M \in M(a)$, hence $M \in Max(L)$ and $a \in M$. As M is a filter and $a \leq b$, hence $b \in M$ and we conclude that $M \in M(b)$. It follows that

$$M(a) = \{M \in Max(L)|a \in M\} \subseteq \{M \in Max(L)|b \in M\} = M(b),$$

Also,

$$\bigcap\{M \in Max(L)|b \in M\} \subseteq \bigcap\{M \in Max(L)|a \in M\},$$

Thus $M_b \subseteq M_a$.

(4) Since $(a \odot b) \leq a, b$, by part (3) we get $M_a, M_b \subseteq M_{(a\odot b)}$. Thus we conclude that $M_a \vee M_b \subseteq M_{(a\odot b)}$. Since $a \in M_a$ and $b \in M_b$, it follows that $a, b \in M_a \vee M_b$. Thus $(a \odot b) \in M_a \vee M_b$. Consequently, we obtain $M_{(a\odot b)} \subseteq M_a \vee M_b$. Therefore $M_{(a\odot b)} = M_a \vee M_b$. □

Proposition 3.4. Let F, J and K be three filters of L.

(1) If F is a J-semi maximal filter of L and $P \in Min(F)$, then P is a J-semi maximal filter of L.

(2) Let F and J be K-semi maximal filters of L. Then $F \vee J$ is a K-semi maximal filter of L.

(3) Let $\{F_i\}_{i\in\Omega}$ be a non empty family of J-semi maximal filter. Then $\bigwedge_{i\in\Omega} F_i$ is a J-semi maximal filter.

(4) Let F be a J-semi maximal filter and $K \subseteq J$. Then F is a K-semi maximal filter.

(5) If $J = L$, then L-semi maximal filters and semi maximal filters are the same.

(6) Every maximal filter is a J-semi maximal filter.

(7) Every semi maximal filter is a J-semi maximal filter.

(8) If $F = \{1\}$ is a J-semi maximal filter, then $J \cap Rad(L) = \{1\}$.

(9) Let $J \neq \{1\}$ and $J \cap Rad(L) = \{1\}$. Then $\{1\}$ is a NJ-semi maximal filter.

Proof: (1) Suppose that $a \in P$. By Theorem 2.9 , there exists $r \in L \backslash P$ such that $r \vee a = 1 \in F$. Since F is a J-semi maximal filter, so $M_{a \vee r} \cap J \subseteq F$. It follows from Lemma 3.3 part(2) that $M_a \cap M_r \cap J \subseteq F \subseteq P$. It is clear that $M_r \nsubseteq P$ and by Lemma 2.18 we conclude that $M_a \cap J \subseteq P$. Therefore, P is a J-semi maximal filter.

(2) Let $x \in F \vee J$, hence $a \odot b \leq x$, for some $a \in F$ and $b \in J$. Since F and J are K-semi maximal filters, so we get $M_a \cap K \subseteq F$ and $M_b \cap K \subseteq J$. Hence we have $M_a \cap K \subseteq F \vee J$ and $M_b \cap K \subseteq F \vee J$. we get $K \cap (M_a \vee M_b) = (M_a \cap K) \vee (M_b \cap K) \subseteq F \vee J$. By Lemma 3.3 part (4) We get $K \cap M_{(a \odot b)} \subseteq F \vee J$. We have $a \odot b \leq x$, it follows from Lemma 3.3 part (3) that $M_x \subseteq M_{(a \odot b)}$. Therefore $M_x \cap K \subseteq M_{(a \odot b)} \cap K \subseteq F \vee J$, hence we get $M_x \cap K \subseteq F \vee J$. Thus $F \vee J$ is a K-semi maximal filter.

(3) Suppose that $a \in \bigwedge_{i \in \Omega} F_i$. Hence $a \in F_i$, for all $i \in \Omega$. Since F_i is a J-semi maximal filter, so $M_a \cap J \subseteq F_i$, for all $i \in \Omega$. Obviously, $M_a \cap J \subseteq \bigwedge_{i \in \Omega} F_i$, for $a \in \bigwedge_{i \in \Omega} F_i$. Therefore, $\bigwedge_{i \in \Omega} F_i$ is a J-semi maximal filter.

(4) Assume that F is a J-semi maximal filter. This means that for all $a \in F$, $M_a \cap J \subseteq F$. By the given hypothesis, we have $K \subseteq J$. This implies that $M_a \cap K \subseteq M_a \cap J$. Therefore, F is a K-semi maximal filter.

(5) Let F be a L-semi maximal filter. Then $M_a \cap L \subseteq F$, for all $a \in F$. Always, $M_a \cap L = M_a$, hence F is a semi maximal filter.

(6) Suppose that M is a maximal filter of L. For all $a \in M$,we have $M_a \subseteq M$. On the other hand, for all $J \in F(L)$, we always have $M_a \cap J \subseteq M_a \subseteq M$, as the intersection of M_a with any filter J is always contained within M. Therefore, we can conclude that M is a J-semi maximal filter.

(7) Let F be a semi maximal filter. Then $M_a \subseteq F$, for all $a \in F$. It is clear that $M_a \cap J \subseteq M_a$, for all $J \in F(L)$. Hence F is a J-semi maximal filter.

(8) Suppose that $F = \{1\}$ is a J-semi maximal filter. This means that $M_1 \cap J \subseteq \{1\}$. Since $M_1 = Rad(L)$ so $Rad(L) \cap J = \{1\}$.

(9) By hypothesis, $J \cap Rad(L) = \{1\}$. Hence the filter $\{1\}$ is a J-semi maximal filter. Since $J \neq \{1\}$, so the $\{1\}$ is a NJ-semi maximal filter. □

Corollary 3.5. Let J be a filter of L and $\Sigma = \{F \in F(L) |\ F$ is a J-semi maximal filter$\}$. Then

(1) $(\Sigma, \subseteq)$ is a poset.

(2) $(\Sigma, \vee, \wedge)$ is a lattice.

Proof: By Proposition 3.4, parts (2) and (3), it is clear. □

In Example 3.2 part (1) for the filter $J = F_3$, the set of all J-semi maximal filter is $\Sigma = \{F_0, F_1, F_2, F_3\}$ that forms a lattice.

Proposition 3.6. Let P be a prime filter of L and J be a proper filter of L. Then the following conditions are equivalent:

(1) P is a J-semi maximal filter.

(2) P is either a semi maximal filter or $J \subseteq P$.

Proof: (1) $\Rightarrow$ (2) Suppose that $J \nsubseteq P$, we prove that P is a semi maximal filter. Since P is a J-semi maximal filter, so $M_a \cap J \subseteq P$, for all $a \in P$. By Lemma 2.18, we deduce that $M_a \subseteq P$, for all $a \in P$. Therefore P is a J-semi maximal filter.

(2) $\Rightarrow$ (1) It is clear. □

Corollary 3.7. Let F and J be two filters of L.

(1) If F is a NJ-semi maximal filter of L, then there exists $P \in Min(F)$ such that P is a semi maximal filter of L.

(2) Every prime NJ-semi maximal filter is a semi maximal filter.

(3) Let L be a BL-chain. Then every proper NJ-semi maximal filter is a semi maximal filter.

PROOF: (1) F is a NJ-semi maximal filter, hence $J \nsubseteq F$ and F is a J-semi maximal filter. Since $J \nsubseteq F$, so there exists $x \in J \setminus F$. Obviously, if we take $K := (x]$ thus F is a K-semi maximal filter. Also, $x \notin F$ hence by Corollary 2.15, we deduce that there exists $Q \in Spec(L)$ containing F such that $x \notin Q$. It follows from Theorem 2.17 that there exists $P \in Min(F)$ such that $P \subseteq Q$. Clearly, $x \notin P$, so $K \nsubseteq P$. By Proposition 3.4 part(1) and Proposition 3.6, therefore P is a semi maximal filter.

(2) By Proposition 3.6, it is clear.

(3) Suppose that F is proper J-semi maximal filter such that $J \nsubseteq F$. Hence $M_a \cap J \subseteq F$, for all $a \in F$. Since L is a BL-chain, by Theorem 2.16, F is a prime filter. Also, by part(2) of this corollary, we conclude that $M_a \subseteq F$, for all $a \in F$. Therefore F is a semi maximal filter. □

Note: Let F be a filter of L. We define:

$$F_s = \{b \in L \mid \ b \in M_a, \ for \ some \ a \in F\}.$$

LEMMA 3.8. *Let F and J be filters of L. Then the following sentences hold:*

(1) F_s is a semi maximal filter of L.

(2) $F \subseteq F_s$.

(3) $F_s = \cap\{Q \mid F \subseteq Q$ and Q is a semi maximal filter$\}$.

(4) $(F \cap J)_s = F_s \cap J_s$.

(5) If $F \subseteq J$, then $F_s \subseteq J_s$.

(6) If F is a semi maximal filter, then $F = F_s$.

PROOF: (1) First, we prove that F_s is a filter. It is clear that $1 \in F_s$. Suppose that $b, c \in F_s$, hence $b \in M_a$, for some $a \in F$ and $c \in M_t$, for some $t \in F$. By Lemma 3.3 part (1), we have $M_b \subseteq M_a$ and $M_c \subseteq M_t$. Thus $M_b \vee M_c = M_{b \odot c} \subseteq M_{a \odot t}$. Since F is a filter, so $a \odot t \in F$ and we deduce that $b \odot c \in F_s$. Also, if $b \leq c$ and $b \in F_s$, then $b \in M_a$, for some $a \in F$. It follows from Lemma 3.3 that $c \in M_a$ for some $a \in F$, that is, $c \in F_s$. Hence F_s is a filter. Now, we show that F_s is a semi maximal filter. This is proven by Theorem 2.23. Let $M(a) \subseteq M(b)$ and $a \in F_s$. Then $M_b \subseteq M_a$ and there exists $t \in F$ such that $a \in M_t$. Hence by lemma 3.3 part (1), $M_a \subseteq M_t$ and we get $b \in M_t$. Therefore $b \in F_s$.

(2) Suppose that $a \in F$. Since $a \in M_a$, so we obtain $a \in F_s$.

(3) Let $K = \cap\{ Q \mid F \subseteq Q$ and Q is a semi maximal filter $\}$. Then by parts (1) and (2) of this lemma, it is clear that $K \subseteq F_s$. By contrary, if $F_s \nsubseteq K$. Suppose that $b \in F_s$ but $b \notin K$. Hence there exists a semi maximal filter Q such that $F \subseteq Q$ and $b \notin Q$. According to Theorem 2.23 and using the equivalent definition so, there is not $a \in Q$ such that $M_b \subseteq M_a$. We deduce that $b \notin M_a$, thus $b \notin F_s$, which is a contradiction.

(4) Assume that $b \in (F \cap J)_s$, hence there exists $a \in F \cap J$ such that $b \in M_a$. Since $a \in F$ and $b \in M_a$, so $b \in F_s$. Similarly, $b \in J_s$. Therefore $(F \cap J)_s \subseteq F_s \cap J_s$. Now, suppose that $b \in F_s \cap J_s$. Then there exist $a \in F$ and $c \in J$ such that $b \in M_a$ and $b \in M_c$. Obviously, $a \vee c \in F \cap J$ and $b \in M_a \cap M_c = M_{a \vee c}$. We conclude that $b \in (F \cap J)_s$ and $F_s \cap J_s \subseteq (F \cap J)_s$.

(5) It is clear.

(6) By part (3) of this lemma, the proof is evident. □

Example 3.9. Let $F_1 = \{1, a, c\}$ be the filter from Example 3.2 (1), it is easy to clarify $(F_1)_s = \{1, a, c\}$. (Because F_1 is a semi maximal filter.) Also, for the filter $F_2 = \{1, b, a, 0\}$ in Example 3.2 (2), $(F_2)_s = \{1, b, a, 0\}$. On the other hand, $F \subseteq F_s$.

PROPOSITION 3.10. Suppose that F and J are filters of L. The following sentences are equivalent:

(1) F is a J-semi maximal filter.

(2) $F_s \cap J \subseteq F$.

(3) There exists a semi maximal filter K containing F such that $K \cap J \subseteq F$.

(4) For each $a \in F$ and $b \in J$ if $M_b \subseteq M_a$, then $b \in F$.

Proof: (1) $\Rightarrow$ (2) Let $b \in F_s \cap J$. Then there exists $a \in F$ such that $b \in M_a$, hence $M_b \subseteq M_a$. Since F is a J-semi maximal filter, so $M_a \cap J \subseteq F$. We have $b \in M_b \cap J \subseteq M_a \cap J \subseteq F$, thus $b \in F$.

(2) $\Rightarrow$ (3) Consider $K := F_s$.

(3) $\Rightarrow$ (4) By hypothesis $a \in F$ and $F \subseteq K$, thus $a \in K$. Since K is a semi maximal filter, so $M_a \subseteq K$. We have $b \in J$ and $b \in M_b$ hence $b \in M_b \cap J$. Therefore $b \in M_b \cap J \subseteq M_a \cap J \subseteq K \cap J \subseteq F$. As a result $b \in F$.

(4) $\Rightarrow$ (1) We show that $M_a \cap J \subseteq F$, for all $a \in F$. Assume that $b \in M_a \cap J$, then $M_b \subseteq M_a$ and $b \in J$. By part (4), we conclude that $b \in F$. Therefore F is a J-semi maximal filter. □

Proposition 3.11. Let $F, J \in F(L)$. Then F is a J-semi maximal filter (NJ-semi maximal filter) if and only if $F \cap J$ is a J-semi maximal filter (NJ-semi maximal filter).

Proof: Let $F \cap J$ be a J-semi maximal filter. By Proposition 3.10, we have $(F \cap J)_s \cap J \subseteq F \cap J$. Now, by Lemma 3.8, we get

$$(F \cap J)_s \cap J = F_s \cap J_s \cap J = F_s \cap J \text{ (since } J \subseteq J_s).$$

On the other hand, $F_s \cap J = (F \cap J)_s \cap J \subseteq F \cap J \subseteq F$. Therefore by Proposition 3.10, F is a J-semi maximal filter. Additionally, by the hypothesis, $F \cap J \subsetneqq J$. This implies that $J \nsubseteq F$ and we can conclude that F is a NJ-semi maximal filter. The other side is clear (by Prposition 3.4). Now, let F be $NJ-$semi maximal filter. For a filter $F \cap J$ there exists a filter $J \nsubseteq F \cap J$ such that $F \cap J$ is a J-semi maximal filter. Thus $F \cap J$ is a NJ-semi maximal filter. Now, suppose that $F \cap J$ is NJ-semi maximal filter. Then for a filter F there exists a filter $J \nsubseteq F$, and also we have F is J-semi maximal filter, hence we conclude that F is a NJ-semi maximal filter. □

Proposition 3.12. A filter F of L is a NJ-semi maximal filter if and only if there exists $b \in L \setminus F$ such that $(b] \cap M_a \subseteq F$, for all $a \in F$.

Proof: Let F be a NJ-semi maximal filter. Then F is a J-semi maximal filter, for some filter J of L such that $J \nsubseteq F$. It is enough to take $b \in J \setminus F$.

Conversely, by hypothesis there exists $b \in L \setminus F$, we take $J = (b]$, it is clear that F is a J-semi maximal filter and $J \nsubseteq F$. Therefore F is a NJ-semi maximal filter. □

PROPOSITION 3.13. Let F, J, K and H be filters in L. Then the following sentences hold:

(1) F is a J-semi maximal filter if and only if F is a $(F \vee J)-$semi maximal filter.

(2) Let J be a semi maximal filter and $F \subseteq J$. Then F is a J-semi maximal filter if and only if F is a semi maximal filter.

(3) If F is a semi maximal filter, then J is a $F-$semi maximal filter if and only if $F \cap J$ is a semi maximal filter.

(4) $F \cap J$ is a J-semi maximal filter and is a $F-$semi maximal filter if and only if F is a J-semi maximal filter and J is a $F-$semi maximal filter.

(5) Let M be a maximal filter of L. Then $F \cap M$ is a semi maximal filter if and only if F is a semi maximal filter.

(6) If $F \subseteq J$ and F is a J-semi maximal filter and J is a K-semi maximal filter, then F is a K-semi maximal filter.

(7) If $F \subseteq J$ and $H \subseteq K$ and F is a J-semi maximal filter and H is a K-semi maximal filter, then $F \cap H$ is a $(J \cap K)-$semi maximal filter.

(8) $F_s \cap J$ is the smallest J-semi maximal filter containing $F \cap J$.

(9) $J \cap Rad(L)$ is the smallest J-semi maximal filter.

(10) If F is a J-semi maximal filter and a K-semi maximal filter, then F is a $(J \vee K)-$semi maximal filter.

PROOF: (1) Let F be a J-semi maximal filter. By Lemma 3.8, we have $F \subseteq F_s$ and implies that $F_s \cap (F \vee J) = (F_s \cap F) \vee (F_s \cap J)$. Since F is a J-semi maximal filter, so $F_s \cap J \subseteq F$. Thus, we obtain $F_s \cap (F \vee J) = (F_s \cap F) \vee (F_s \cap J) \subseteq F \vee F = F$. It follows from Proposition 3.10, that F is a $(F \vee J)-$semi maximal filter. Conversely, it is obvious.

(2) Assume that F is a J-semi maximal filter, hence for all $a \in F$, $M_a \cap J \subseteq F$. Since J is a semi maximal filter and $a \in F \subseteq J$, so we have $M_a \subseteq J$. We deduce that $M_a \cap J = M_a \subseteq F$. So F is a semi maximal filter. As every semi maximal filter is a J-semi maximal filter, conversely is clear.

(3) First, we show that $F \cap J$ is a semi maximal filter. Since J is a $F-$semi maximal filter, so by Proposition 3.11, we deduce that $F \cap J$ is a $F-$semi maximal filter. By hypothesis, F is a semi maximal filter and $F \cap J \subseteq F$, hence by part (2) of this proposition, we conclude that $F \cap J$ is a semi maximal filter.

Conversely, it follows from Proposition 3.11 and part (2) of Proposition 3.13.

(4) By Proposition 3.11, it is clear.

(5) M is maximal filter and F is a semi maximal filter, hence M is a semi maximal filter so $F \cap M$ is a semi maximal filter. Now suppose that $F \cap M$ is a semi maximal filter. If $F \subseteq M$, then $F = F \cap M$ is a semi maximal filter. Assume that $F \nsubseteq M$, since $F \cap M$ is a semi maximal filter and M is a semi maximal filter, so by this proposition part (3), we conclude that F is a $M-$semi maximal filter. By part (1), we get F is a $(F \vee M)-$semi maximal filter. Hence F is a $M-$semi maximal filter. Therefore F is a semi maximal filter.

(6) By hypothesis and Proposition 3.10, we have $F_s \cap J \subseteq F$ and $J_s \cap K \subseteq J$. Also, $F_s \subseteq J_s$. Hence we get $F_s \cap K = F_s \cap J_s \cap K \subseteq F_s \cap J \subseteq F$. It means that F is a K-semi maximal filter.

(7) It follows from Proposition 3.10, that $F_s \cap J \subseteq F$ and $H_s \cap K \subseteq H$. So, we have

$$(F \cap H)_s \cap (J \cap K) = F_s \cap H_s \cap J \cap K = (F_s \cap J) \cap (H_s \cap K) \subseteq F \cap H.$$

(8) We know that, $F \subseteq F_s$, so $F \cap J \subseteq F_s \cap J$. It is clear that $(F_s)_s = F_s$, hence $(F_s \cap J)_s \cap J \subseteq F_s \cap J$ and we conclude that $F_s \cap J$ is a J-semi maximal filter. Now, suppose that there exists a filter K of L such that it

is a J-semi maximal filter contains $F \cap J$. Hence $F_s \cap J = F_s \cap J_s \cap J = (F \cap J)_s \cap J \subseteq K_s \cap J \subseteq K$.

(9) Suppose that F is a J-semi maximal filter, thus $J \cap Rad(L) \subseteq J \cap F_s \subseteq F$.

(10) We have $F_s \cap (J \vee K) = (F_s \cap J) \vee (F_s \cap K) \subseteq F$. Hence F is a $(J \vee K)-$semi maximal filter. □

Corollary 3.14. Let J be a semi factor of a filter F. Then $F \vee J$ is a semi factor of F containing F.

Proof: Since F is a J-semi maximal filter, so by Proposition 3.13 part(1), F is a $(F \vee J)-$semi maximal filter. Thus, $F \vee J$ is a semi factor of F. Obviously, $F \subseteq F \vee J$. □

Corollary 3.15. Let F be a filter of L. Consider,

$$\Omega = \{J \in F(L) \mid J \ is \ a \ semi \ factor \ of \ F\}.$$

(1) $(\Omega, \subseteq)$ is a poset.
(2) $(\Omega, \vee, \wedge)$ is a lattice.

Proposition 3.16. Let F, J be filters and P, Q be prime filters of L. Then

(1) If $F \cap P$ is a J-semi maximal filter (NJ-semi maximal filter), then either F is a J-semi maximal filter (NJ-semi maximal filter) or P.
(2) If $P \cap Q$ is a NJ-semi maximal filter, then either P is a semi maximal filter or Q.

Proof: (1) Obviously, if $F \subseteq P$, then F is a J-semi maximal filter (NJ-semi maximal filter.) Assume that $F \nsubseteq P$, hence there exists $a \in F \setminus P$ and for all $b \in P$, we have $a \vee b \in F \cap P$. Since $F \cap P$ is a J-semi maximal filter, so $M_{(a \vee b)} \cap J \subseteq F \cap P$. On the other hand, $M_a \cap M_b \cap J \subseteq P$ and $M_a \nsubseteq P$. By Lemma 2.18, we conclude that $M_b \cap J \subseteq P$, for all $b \in P$. Thus, P is a J-semi maximal filter. Since $F \cap P$ is a NJ-semi maximal filter, so there is a filter J such that $J \nsubseteq F \cap P$ and $(F \cap P)_s \cap J \subseteq F \cap P$.

Hence $(F_s \cap J) \cap P_s \subseteq P$, by Lemma 2.18, we have $F_s \cap J \subseteq P$ or $P_s \subseteq P$. We consider two cases:

Case 1: If $F_s \cap J \subseteq P$, then $J \subseteq P$. (Since $F \nsubseteq P$, so $F_s \nsubseteq P$.) On the other hand, $J \subseteq P \subseteq P_s$, hence $J \cap P_s = J$. Since $F \cap P$ is J-semi maximal filter, so $F_s \cap J = F_s \cap J \cap P_s = (F \cap P)_s \cap J \subseteq F \cap P \subseteq F$ implies that F is a J-semi maximal filter. Now, we show that $J \nsubseteq F$. We have $J \subseteq P$, if $J \subseteq F$, then $J \subseteq F \cap P$, which is a contradiction. Therefore F is a NJ-semi maximal filter.

Case 2: If $P_s \subseteq P$, then $P_s = P$ and P is a semi maximal filter. It follows from Proposition 3.6, that P is a NJ-semi maximal filter.

(2) It follows from Proposition 3.6 and part (1). □

Proposition 3.17. Let F and J be filters of L such that F is a J-semi maximal filter. Then $(F : x)$ is a J-semi maximal filter of L, for all $x \in L \backslash F$.

Proof: Following Proposition 3.10, assume that $M_b \subseteq M_a$ and $a \in (F : x)$ and $b \in J$. Since F is a J-semi maximal filter and $a \vee x \in F$, so we get $M_{(a \vee x)} \cap J \subseteq F$. Also, $b \leq b \vee x$ and J is a filter, hence $b \vee x \in J$. It is clear that by Lemma 3.3 part (2), we have $M_{(b \vee x)} \cap J = M_b \cap M_x \cap J \subseteq M_a \cap M_x \cap J = M_{(a \vee x)} \cap J \subseteq F$ and $b \vee x \in M_{(b \vee x)} \cap J$. Thus, we conclude that $b \vee x \in F$ and $b \in (F : x)$. □

Corollary 3.18. Let F, J be filters of L such that F be a J-semi maximal filter and X be a subset of L such that $X \nsubseteq F$. Then $(F : X) = \{a \in L | a \vee x \in F, for\ all\ x \in X\}$ is a J-semi maximal filter.

In the following theorem, we explain that if J is not a semi maximal filter and $Rad(L)$ does not contain J, then there is always a non-trivial J-semi maximal filter.

Theorem 3.19. *Let J be a filter of L such that $J \nsubseteq Rad(L)$ and it is not a semi maximal filter. Then there exists a filter F such that $F \subsetneqq J$ and is a J-semi maximal filter but not a semi maximal filter.*

Proof: Since $J \nsubseteq Rad(L)$, so there exists maximal filter M in L such that $J \nsubseteq M$. Consider $F := J \cap M$, it is clear that $F \subsetneqq J$. (If $F = J$, then $J \cap M = J$ and $J \subseteq M$, which is a contradiction). M is a maximal filter, so it is a semi maximal filter. Also, it is a J-semi maximal filter and by Proposition 3.11, $J \cap M$ is a J-semi maximal filter.Therefore, F is a J-semi maximal filter. We know that M is maximal and J is not a semi maximal filter, so by Proposition 3.13, part (5) we conclude that F is not a semi maximal filter. □

To summarize, if $J \nsubseteq Rad(L)$, then there are many J-semi maximal filters in L. Consider a maximal filter K such that $J \nsubseteq K$. It follows that K is a semi maximal filter. Thus, for each $k \in K$, $M_k \cap J$ is a J-semi maximal filter and, in fact, an NJ-semi maximal filter.

However, if $J \subseteq Rad(L)$, then only trivial J-semi maximal filters exist. Specifically, if K is a non-trivial J-semi maximal filter, then $J \nsubseteq K$, and for all $a \in K$, $M_a \cap J \subseteq K$. On the other hand, $J = Rad(L) \cap J \subseteq M_a \cap J \subseteq K$, which contradicts $J \nsubseteq K$.

As a result, $J \nsubseteq Rad(L)$ if and only if there exists a non-trivial J-semi maximal filter. Now, we are going to study J-semi maximal filters in Hyperarchimedean and semi-simple $BL-$algebras and get some results.
Note: In Example 3.2(2), the F_0 is not a J-semi maximal filter where $J = \{1, b\}$ (Since $M_1 \cap J = F_5 \cap J = \{1, b\} \nsubseteq F_5$). Next, we characterize $BL-$algebras in which the filter $\{1\}$ is a J-semi maximal filter.

Proposition 3.20. Let L be Hyperarchimedean and $J \in F(L)$. Then every proper filter is a J-semi maximal filter.

Proof: Suppose that F is a proper filter. for all $a \in F$;

$$\{P \in Spec(L) \mid F \subseteq P\} \subseteq \{P \in Spec(L) \mid a \in P\}$$
$$\bigcap\{P \in Spec(L) \mid a \in P\} \subseteq \bigcap\{P \in Spec(L) \mid F \subseteq P\}.$$

By Theorem 2.13 and by Corollary 2.8,

$$M_a = \bigcap\{\ P \in Spec(L) \mid a \in P\} \subseteq \bigcap\{\ P \in Spec(L) \mid F \subseteq P\} = F.$$

We always have $M_a \cap J \subseteq M_a$, for all $a \in F$. On the other hand, $M_a \cap J \subseteq M_a \subseteq F$ for all $a \in F$. Therefore, F is a J-semi maximal filter. □

Theorem 3.21. *Let L be a semi-simple $BL-$algebra, X be a subset of L and J be a filter of L.Then the following sentences hold:*

(1) The filter $\{1\}$ is a J-semi maximal filter. In particular, if $J \neq \{1\}$ is a filter, then the filter $\{1\}$ is a NJ-semi maximal filter.

(2) $Co - Ann(X)$ is a J-semi maximal filter.

(3) Every minimal prime filter of L is a J-semi maximal filter.

Proof: (1) Since L is semi-simple, so $M_1 = Rad(L) = \{1\}$. For each filter J of L, we have $M_1 \cap J \subseteq \{1\}$, hence the filter $\{1\}$ is a J-semi maximal filter.

(2) By part (1) of this proposition, filter $\{1\}$ is a J-semi maximal filter. Put $F := \{1\}$. Now by Corollary 3.18, $(\{1\} : X)$ is a J-semi maximal filter. Obviously, $(\{1\} : X) = Co - Ann(X)$. Thus $Co - Ann(X)$ is a J-semi maximal filter.

(3) In Proposition 3.4 (1), take $F = \{1\}$ and by part (1) of this proposition, we conclude that every minimal prime filter is a J-semi maximal filter. □

Theorem 3.22. *Let L be a semi-simple $BL-$algebra and F be a filter of L such that $Co - Ann(F) \neq \{1\}$. Then there is a filter J of L such that F is a NJ-semi maximal filter of L.*

Proof: Since L is semi-simple, so $Rad(L) = \{1\}$. We first prove that $M_a \cap Co - Ann(F) \subseteq Rad(L)$, for all $a \in F$. Let $b \in M_a \cap Co - Ann(F)$. Then $b \in M_a$ and $b \vee a = 1$. Hence $M_b \subseteq M_a$. We get $b \in M_b = M_b \cap M_a = M_{(a \vee b)} = M_1 = Rad(L)$. Now, take $J = Co - Ann(F)$. Moreover, $M_a \cap Co - Ann(F) = Rad(L) = \{1\} \subseteq F$, for all $a \in F$. Thus F is a J-semi maximal filter.

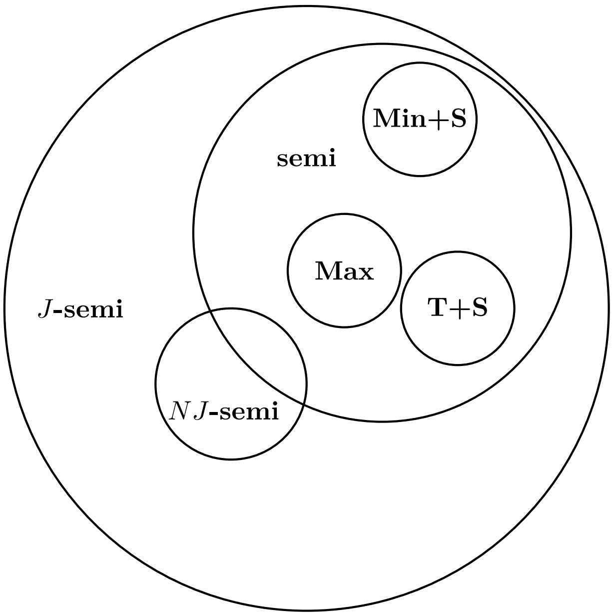
Now, we show that $J \nsubseteq F$. By contrary, suppose that $Co - Ann(F) \subseteq F$ and let $1 \neq a \in Co - Ann(F)$. Hence $a \in F$ and $a \vee t = 1$ for all $t \in F$. Hence $a \vee a = 1$, as a result $a = 1$, which is a contradiction. Therefore F is a NJ-semi maximal filter. □

Corollary 3.23. Let $F, J \in F(L)$. Then every filter F of a semi-simple $BL-$algebra L is a J-semi maximal filter or $Co - Ann(F) = \{1\}$.

Proof: First, assume that $Co - Ann(F) \neq \{1\}$. By Proposition 3.22, we conclude that there exists a filter J such that F is a NJ-semi maximal filter. Now, suppose that F is not a NJ-semi maximal filter in L. Again by Proposition 3.22, we have L is not semi-simple or $Co - Ann(F) = \{1\}$. By hypothesis L is semi-simple, hence $Co - Ann(F) = \{1\}$. □

4. Conclusions

The defined J-semi maximal filters exhibit a higher level of generality compared to semi maximal filters. We have shown that every semi maximal filter is a J-semi filter, and that every NJ-semi filter is also a J-semi filter. Furthermore, it has been proven that every prime NJ-semi filter is a semi maximal filter. We have established that the set of all J-semi filters forms a lattice. A detailed investigation of these filters has been carried out within various classes of BL-algebras. It has been proven that in any BL-chain, each proper NJ-semi filter is a semi maximal filter. Moreover, it has been demonstrated that if the intersection of two prime filters is an NJ-semi filter, then at least one of them must be a semi maximal filter. Additionally, we have shown that a prime filter is a J-semi filter if and only if it is either a semi maximal filter or contains the filter J. Finally, we concluded that there exists a minimal prime filter above every NJ-semi filter.
Next, we show a summary of the relationship between semi maximal filters in a diagram.



S:= Semi-Simple, Max:= Maximal filter, Min:= Minimal prime filter, T:=Trivial filter, J-semi:=J-semi maximal, NJ-semi:=J-semi maximal, semi:=semi maximal filter

Figure 3: The relationships between J-semi maximal filters and the other filters in BL-algebras.

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