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SOME RESULTS CONCERNING AXIOMS FOR EQUIVALENTIAL CALCULUS

Abstract

One of the most important questions in the area of the equivalential calculus (EC) currently is the issue of the single shortest axiom. We show some new a single organic and inorganic axioms for EC which are either D-complete or R-complete. We also present a number of two-element sets of axioms which posses some special properties. Two matrix are also discussed, which exclude two formulas from the set of potential $2MP$-complete axioms.

Keywords: equivalential calculus, D-complete, R-complete, single axiom, condensed detachment.

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1. Introduction

The most significant point of the paper are not hitherto known single axioms for the equivalential calculus (EC). Paragraph 2 discusses the key concepts and results from the area of EC. The next paragraph (3) shows a hitherto unknown inorganic R-complete or D-complete axioms. Paragraph 4. discusses some organic axioms for EC. The next paragraph (5) comprises a number of two-element sets of axioms for EC. In Paragraph 6, we present two matrices which show that some formulas are not single $2MP$-complete axioms. At the end, there are presented two questions which remain open in the area of EC.

2. Equivalential calculus

The equivalential calculus (EC) is identical with all formulas that are tautologies of the standard matrix for the equivalence (E) from the classical propositional calculus. It has become widely accepted to use the Polish notation to talk about this calculus, so we use the symbol E instead of, e.g., $\equiv$. The results presented in this article can be directly interpreted into the language of algebra as well. We are speaking here mainly of algebras called E-groupoids (cf. [3, 4, 5]). The functor E is then understood as a certain definite algebraic action and is most often denoted by the symbol *cdot* in infix notation.

A well-formed formula (wff) of EC is a formula built from a binary connective E and some of denumerably many sentence letters $p, q, r, \ldots$. Each sentence letter is a wff. If α and β are wffs, so is $E\alpha\beta$.

The set of rules of the proof procedure, classically, is composed of the rule of Modus Ponens for E and the substitution rule. The rule of Modus Ponens allows one to detach α from the formula $E\alpha\beta$ and the result β. We name the set of these rules MPS. Since the mid-20th century, it has been noted that other rules could be introduced instead of these. Firstly, the set of rules of the proof procedure with the reversed rule of Modus Ponens for E and the substitution rule. The reversed rule allows one to detach β from the formula $E\alpha\beta$ and the result α. We name the set of these rules RMPS. Secondly, the condensed detachment rule (D) was introduced, which combines detachment with the best possible substitution. A detailed presentation of the rule D can be found in, for example, [6, 7, 10, 15].

Otherwise, instead of the rule D we may use the reversed condensed detachment rule (R). The difference between these rules is that the rule D allows one to detach α from the formula $E\alpha\beta$ and the result is the most general instant of β (from left to right). In contrast, the rule R allows one to detach β from the formula $E\alpha\beta$ and the result is the most general instant of α (from right to left). A detailed presentation of the rule R can be found in, e.g., [13, 8]. It has been shown that these rules are not inferentially equivalent.

We now give definitions of the concepts that will be used later in the text.

Definition 2.1 (C-complete). A set of axioms of EC is C-complete (classically) if and only if it forms a complete EC-theory by the rules MPS, as the only rules allowed in the proof.

Definition 2.2 (RC-complete). A set of axioms of EC is RC-complete (classically with reversed Modus Ponens) if and only if it forms a complete EC-theory by the rules RMPS, as the only rules allowed in the proof.

It is possible that we have in the set of rules of the proof procedure both the rule Modus Ponens and the reversed rule of Modus Ponens and the substitution rule. We name the set of these three rules (2MP).

Definition 2.3 ($2MP$-complete). A set of axioms of EC is $2MP$-complete if and only if it forms a complete EC-theory by both the rule Modus Ponens and the reversed rule of Modus Ponens and the substitution rule, as the only rules allowed in the proof.

Definition 2.4 (D-complete). A set of axioms of EC is D-complete if and only if it forms a complete EC-theory by the rule D, as the only rule allowed in the proof.

Definition 2.5 (R-complete). A set of axioms of EC is R-complete if and only if it forms a complete EC-theory by the rule R, as the only rule allowed in the proof.

It is possible that we have in the set of rules of the proof procedure both the rule D and the rule R. We name the set of these rules (DR).

Definition 2.6 (DR-complete). A set of axioms of EC is DR-complete if and only if it forms a complete EC-theory by both the rule D and the rule R, as the only rules allowed in the proof.

We say that a calculus is C-incomplete, if it is based on the rules MPS and there exists at least one formula satisfying the standard matrix for the equivalence functor that cannot be proved in the theory. Analogously we say that a calculus is RC-incomplete, if it is based on the rules RMPS, D-incomplete, if it is based on the rule D, and R-incomplete, if it is based on the rule R and there exists at least one formula satisfying the standard matrix for the equivalence functor that cannot be proved in the theory.

The first set of axioms of EC was proposed by Leśniewski [11]. The first single shortest axiom was found by Łukasiewicz [12]. To be precise, Łukasiewicz found three different such axioms. He proved that there exists no shorter single axiom. All the sets of axioms are C-complete, but they are either not RC-complete, or not D-complete or not R-complete. Currently, many different axiomatisations of EC are known, and research on EC is focused on the problem of finding a single shortest axiom depending on an established set of rules of the proof procedure.

We currently know fourteen single shortest (11-character long) axioms that are C-complete, the last one found in 2003 [19]. All these axioms are either not D-complete or not R-complete. Most of them are not RC-complete. The axiom

$$EEEpqrEqErp \tag{2.1}$$

which was found by Meredith and Prior [13] is both C-complete and RC-complete.

The converse formula is a formula in which every subformula of the form $E\alpha\beta$ is replaced with $E\beta\alpha$. It was proved that if a formula A is a C-complete (D-complete) axiom, then the converse of A is a RC-complete (R-complete) axiom. E.g., Meredith and Prior [13] proved that

$$EpEEqEprErq \tag{2.2}$$

is a single C-complete axiom. So, the formulas

$$EEEpqEEqrpr \tag{2.3}$$

$$EEEpqEqrEpr \tag{2.4}$$

are single RC-complete axioms, which was proved by Peterson [14].

Ulrich [15] has shown that Wajsberg's [17] set of axioms

$$EEpEqrErEqp \tag{2.5}$$

$$EEEpppp \tag{2.6}$$

is D-complete.

I proved [2] that the set of axioms

$$EEEpqrEErqp \tag{2.7}$$

$$EpEpEpp \tag{2.8}$$

is R-complete.

Ulrich [15] proved that any formula of the scheme

$$EsEsEsEsA, \tag{2.9}$$

where A is any single C-complete axiom and does not contain the variable s, is a D-complete axiom. So we have a number of single D-complete axioms, because we have a number of single C-complete axioms. I proved [2] that under the same conditions any formula of the scheme

$$EEEEAssss \tag{2.10}$$

is an R-complete axiom. A number of single R-complete axioms exist as well.

Leśniewski [11] noted that every theorem of EC is a formula in which each sentence letter occurs an even number of times. If each sentence letter occuring occurs two times in a formula, then we may say that this formula is two-property. It was proved that if we use either the rule D or the rule R to a two-property formula, then we can derive only a two-property formula.

So any two-property formula cannot be either a D-complete or R-complete single axiom. On the other hand, every theorem of EC can be derived from some two-property formula by the substitution rule.

3. Inorganic complete axioms

A hitherto unknown inorganic single axiom which is R-complete is shown below.

THEOREM 3.1. *The formula*

$$EsEsEsEsEEEpqrEqErp \tag{3.1}$$

is an R-complete axiom.

PROOF:

$$\begin{aligned} &1. EsEsEsEsEEEpqrEqErp \\ R1.1 = {} &2. EEEpqEEqrpr \end{aligned}$$

The first step of the proof is easy. We use the standard notation for the rule R. The description R1.1 means that the rule R was applied to line 1, which in this case was the premise for the rule R. By applying the rule R once, we obtain the formula $EEEpqEEqrpr$, i.e., the RC-complete axiom (2.3). From (2.3) the formula (2.7) and the formula

$$EEwExEyzEwExEyEzEEEpqrEqErp$$

can be derived, because both formulas are two-property, and it is known that every two-property formula can be derived from (2.3) by the rule R. Next we derive the formula (2.8), from this second formula by means of a one-way detachment by the R rule. Since (2.7) and (2.8) are R-complete, (3.1) is as well. □

Since the formula (3.1) is formed from the scheme $EsEsEsEsA$, (2.9) is a single D-complete axiom. The starting axiom A here is the formula $EEEpqrEqErp$ (2.1), which is both C-complete and RC-complete, but both D-incomplete and R-incomplete.

The same axiom (2.1) is the starting axiom for a hitherto unknown inorganic single axiom which is D-complete.

THEOREM 3.2. *The formula*

$$EEEEEEEpqrEqErpssss \tag{3.2}$$

is a D-complete axiom.

PROOF: The proof is quite similar to the proof of Theorem 3.1.

$$1. EEEEEEEpqrEqErpssss$$
$$D1.1 = 2. EpEEqEprErq$$

By applying the rule D once, we obtain the formula $EpEEqEprErq$, i.e., the C-complete axiom (2.2). From (2.2) the formula (2.5) and the formula $EEEEEEEpqrEqErpwxyzEEEwxyz$ can be derived, because both formulas are two-property, and it is known that every two-property formula can be derived from (2.2) by the rule D. Next we derive the formula (2.6), from this second formula, by means of a one-way detachment by the D rule. Since (2.5) and (2.6) are D-complete, (3.1) is as well. □

It is clear that the axiom (3.2) is R-complete as well. It is built according to the scheme (2.10).

The fact that (3.1) is an R-complete axiom has not been announced so far. The same is true of the fact that (3.2) is a D-complete axiom. All R-complete or D-complete inorganic single axioms known so far are 19 characters long.

4. Organic axioms

So far, we have only shown inorganic axioms. There are organic single axioms that are R-complete. In [2] I mentioned without a proof that the formula $EEEpqrEsEsEsEsEqErp$ is a single DR-complete axiom. It turns out that this is true. Moreover, this axiom is R-complete.

Theorem 4.1. *The formula*

$$EEEpqrEsEsEsEsEqErp \tag{4.1}$$

is an R-complete axiom.

Proof:

$$1.EEEpqrEsEsEsEsEqErp$$
$$R1.1 = 2.EEEpqEEqrpr$$

The formula $EEEpqEEqrpr$ is the axiom (2.3), which is RC-complete. Two formulas which are two-property can be derived from it by the rule R:

$$3.EEEpqrEErqp$$

It is the axiom (2.7).

$$4.EExEyEzwEEEpqrExEyEzEwEqErp$$

From the formula we can derive:

$$RD4.1 = 5.EpEpEpp,$$

which is the axiom (2.8). Since the set of axioms (2.7) and (2.8) is R-complete, (4.1) is as well. □

In [2], I proved that the formula $EEEEEEqEprssssErEqp$ is a DR-complete axiom. The proof makes significant use of both the rule D and the rule R. I will now present a simpler proof of this fact. It is a proof of the fact that the formula is an R-complete axiom. The proof only uses the rule R. Obviously, at the same time it is a proof that the formula is a DR-complete axiom.

Theorem 4.2. *The formula*

$$EEEEEEqEprssssErEqp \tag{4.2}$$

is an R-complete axiom.

PROOF:

$$
\begin{aligned}
&1. EEEEEEEpqssssErpErq \\
R1.1 = \; &2. EEEEEEtEqpuuuuEEEEEEErpssssEqrt \\
R2.1 = \; &3. EEEEEEEEqpEqptttt \\
RRRR3.1.1.1.1 = \; &4. EEpqEpq \\
R1.4 = \; &5. EEEEEEpErqssssEErqp \\
R2.5 = \; &6. EEEEEEEErpEpqErqssss
\end{aligned}
$$

The last formula is an R-complete axiom, because it falls under the scheme 2.10. In it, A is the axiom (2.4) which is a single RC-complete axiom. So, the formula (4.2) is a R-complete axiom. □

Since the axiom (4.2) is a single R-complete axiom, its converse formula

$$EEEEEEEpqrssssEqErp \tag{4.3}$$

is a single D-complete axiom. The axiom 4.3 is a DR-complete axiom as well (cf. [2]).

The fact that the axiom (4.1) is a single R-complete axiom has not been hitherto known. We can say the same about the axiom (4.2). Furthermore, here we have a shorter proof of the fact that it is a single DR-complete axiom. The same is true for the single organic D-complete axiom (4.3).

5. Two-element sets of axioms for EC

In this paragraph, we present several sets of axioms that prove to be helpful in the study of EC. All of these sets of axioms are either C-complete or RC-complete. So in this paragraph we no longer talk about the rule D or the rule R.

Each of sets of axioms under this paragraph contains two formulas. One of these is the following axiom:

$$EEpqEqp \tag{5.1}$$

which indicates that the equivalence is symmetrical. The most obvious second axiom is

$$EEpqEEqrEpr \tag{5.2}$$

which indicates that the equivalence is transitive. Axioms (5.1) and (5.2) constitute a a set of axioms which is D-complete. By the way, the reflexivity of the equivalence (Epp) can be derived from the set [18]. The proof is easy:

1.	$EEpqEqp$	5.1
2.	$EEEpqEqrEpr$	5.2
3.	$EEppEpp$	$1 : p/q$
4.	$EEEppEppEpp$	$2 : p/q, p/r$
5.	Epp	$MP : 4, 3$

The formula

$$EEEpqEprErq, \tag{5.3}$$

is the converse of (5.2). The set of axioms (5.1) and is RC-complete. It was proved that the formula is not a single axiom for EC.

Wajsberg [16] has shown that the formula

$$EEEpqrEpEqr \tag{5.4}$$

together with the axiom (5.1) constitute a set of axioms which is C-complete. A proof of this fact can be also found in [1]. On the other hand, Wajsberg [17] introduced the axiom

$$EEpEqrEEpqr \tag{5.5}$$

The set of axioms (5.1) and 5.5 is C-complete. It is easy to see that the formula (5.4) is its own converse, as is the formula (5.1). The same is true of the axiom (5.5), which is also its own converse. It is known in the area of the equivalential calculus that if a set of axioms is C-complete then the set of converse formulas is RC-complete. Thus the pairs of axioms $\{5.1, 5.4\}, \{5.1, 5.5\}$ are RC-complete.

THEOREM 5.1. *The set of formulas composed of*

$$EEpEqrEErpq \tag{5.6}$$

and the axiom (5.1) *is* C*-complete and* RC*-complete.*

PROOF: We first prove that the set is C-complete.

1.	$EEpqEqp$	5.1
2.	$EEpEqrEErpq$	5.6
3.	$EEEpEqrEErpqEEErpqEpEqr$	$1 : EpEqr/p, EErpq/q$
4.	$EEErpqEpEqr$	$MP : 3, 2$

The formula $EEErpqEpEqr$ is the axiom (2.1). It is a single C-complete axiom. So the set of axioms (5.1) and (5.6) is C-complete. It is known that if a set is C-complete, then a set of converse formulas is RC-complete. Because axioms (5.1) and (5.6) are their own converses, then the set of axioms is RC-complete. □

Further pairs of axioms are formed by adding to the axiom (5.1) following formulas:

$$EEpEqrEqErp, \tag{5.7}$$

$$EEpEEqprErq, \tag{5.8}$$

$$EEpEEpqrErq, \tag{5.9}$$

$$EEpqEEqEprr, \tag{5.10}$$

$$EEpqEEqErpr, \tag{5.11}$$

$$EEEEpqErqrp. \tag{5.12}$$

THEOREM 5.2. *Pairs of axioms (each separately):* $\{5.1, 5.7\}, \{5.1, 5.8\}, \{5.1, 5.9\}, \{5.1, 5.10\}, \{5.1, 5.11\}, \{5.1, 5.12\}$ *constitute sets of axioms which are* C*-complete.*

PROOF (Sketch): The proof for each set of axioms always follows the same pattern, we make a detachment by the rule of Modus Ponens for E from the axiom (5.1) with the given axiom, resulting in a single C-complete axiom. Accordingly, from (5.7) to $EEpEqrErEpq$ [13], from (5.8) to $EEpqErEEqrp$ [13], from (5.9) to $EEpqErEErqp$ [13], from (5.10) to $EEEpEqrrEqp$ [13], from (5.11) to $EEEpEqrqErp$ [13] and with (5.12) infers $EpEEEpqErqr$ [19]. □

Let's consider two more formulas:

$$EEEpqrEEqrp, \tag{5.13}$$

$$EpEqEErqErp. \tag{5.14}$$

THEOREM 5.3. *Pairs of axioms (each separately):* $\{5.1, 5.13\}, \{5.1, 5.14\}$ *constitute sets of axioms which are R-complete.*

PROOF (Sketch): The proof runs similarly, it is sufficient to take a detachment in one step using the reverse Modus Ponens rule. Accordingly, from (5.13) we obtain $EEEpqrEErpq$, while from (5.14) we derive $EEpEEqpEqrr$. The formulas which are derived are single R-complete axioms [14]. □

It is known that none of the formulas from (5.6) to (5.14) is a single C-complete axiom or RC-complete. On the other hand, it has been proved that the expressions (5.6), (5.9), (5.10), (5.12) and (5.14), each separately, are single $2MP$-complete axioms [8].

Other single axioms for which it has been proved that they are complete together with these three rules are:

$$EEEEpqrEqrp \tag{5.15}$$

$$EpEEqrEqErp \tag{5.16}$$

$$EEEEpqrpEqr \tag{5.17}$$

$$EEpqErEpEqr \tag{5.18}$$

If we have those three rules in the set of rules of the proof procedure, then one can also find the set of axioms built from the axiom 5.1 and one additional axiom:

$$EpEEEpqrEqr, \tag{5.19}$$
$$EEEpqEpEqrr, \tag{5.20}$$
$$EEpqEEErpqr, \tag{5.21}$$
$$EEpEqErpEqr. \tag{5.22}$$

THEOREM 5.4. *Pairs of axioms (each separately):* {5.1, 5.19}, {5.1, 5.20}, {5.1, 5.21}, {5.1, 5.22} *are sets of axioms which are* $2MP$*-complete.*

PROOF (Sketch): The proof is also based on a single detachment, except that now with two kinds of Modus Ponens rules available, but in each case we obtain the same results. With (5.19) we derive $EEEEpqrCqrp$, from (5.20) we get $EpEEqrEqErp$, from (5.21) we infer $EEEEpqrpEqr$ and from (5.22) we get $EEpqErEpEqr$. It is proved that four resulting formulas are (each separately) a single $2MP$-complete axiom [14]. □

6. Exclusion of potential $2MP$-complete axiom

About the formulas (5.21) and (5.22) it is proved that they are neither a single shortest axiom which are C-complete, RC-complete nor $2MP$-complete. On the other hand, about the formulas (5.19) and (5.20) it has not been yet decided whether they are $2MP$-complete, but we know that they are neither C-complete nor RC-complete.

From the 630 length-11 theorems of EC that were potential single axiom we know all that are C-complete or R-complete. There now remain six such formulas about which it has not been decided whether they can be single $2MP$-complete axioms. The axioms (5.19) and (5.20), and additionally:

$$EEEEpEqrrqp, \tag{6.1}$$
$$EpEqErEErqp, \tag{6.2}$$
$$EpEEEpEqrrq, \tag{6.3}$$
$$EEpEqEEqprr. \tag{6.4}$$

Until recently, two other formulas were not known to be $2MP$-complete axioms, but D. Ulrich has now ruled this out no earlier than 2004:

$$EEEEEpqrqpr \tag{6.5}$$

$$EpEqErEpErq \tag{6.6}$$

I am not aware of any scientific publication that can be quoted here that includes Ulrich's results. The results were published only on Ulrich's website. He passed away in 2020. I found another six-valued matrix that is valid for 6.5 and both the rule of Modus Ponens and the reversed rule of Modus Ponens, but some theorems are rejected by the matrix.

	1*	2*	3	4	5	6
1	1	1	4	4	4	4
2	2	2	3	6	5	6
3	3	3	2	5	6	5
4	4	4	1	1	1	1
5	5	5	6	3	2	3
6	6	6	5	2	3	2

(6.7)

In the set of designeted values are two elements: $\{1, 2\}$. The matrix validates the theorem (6.5), but rejects, e.g., the theorem $EEpEqpq$. It is easy to see for $p = 3$ and $q = 1$, we get $EE3E131$, so next $EE341$, from that we get $E51$ and finally the value of the theorem is 5. Thus from the formula (6.5) the theorem $EEpEqpq$ cannot be derived, so (6.5) cannot be a single $2MP$-complete axiom.

I found another six-valued matrix that is valid for 6.6 and both the rule of Modus Ponens and the reversed rule of Modus Ponens, but the theorem $EEEpqpq$ is rejected by the matrix.

	1*	2*	3	4	5	6
1	2	2	3	4	5	6
2	2	2	3	4	5	6
3	3	4	1	2	6	5
4	5	4	6	2	1	3
5	5	4	6	2	1	3
6	6	4	5	2	3	1

(6.8)

Here is also the set of designated values that consists of two elements: $\{1, 2\}$. The matrix validates the theorem (6.6), but rejects the theorem $EEEpqpq$. It is easy to see for $p = 2$ and $q = 3$, we get $EEE2323$, so next $EE323$, from that we get $E43$ and finally the value of the theorem is 5. Thus, from the formula (6.6) the theorem $EEEpqpq$ can not be derived so (6.6) cannot be a single $2MP$-complete axiom.

7. Open questions for EC

We still have some open questions in EC. Some of the most important ones related to the subject of this paper are:

1. Is there a single axiom shorter than 19 characters long which is either D-complete, R-complete or DR-complete?

   We do not now any single such axiom, neither organic nor inorganic. We know that the length of a potential formula has to be 15 characters and it is highly probable that in that formula some sentence letter will be occur four times. All formulas shorter than 15 characters were excluded by the proof of Łukasiewicz [12] and further research [13, 14, 9, 20, 19].

2. Is any of the formulas (5.19, 5.20, 6.1, 6.2, 6.3, 6.4) a single $2MP$-complete axiom?

   We know that neither of the formulas is C-complete, RC-complete, D-complete, R-complete or DR-complete.

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