Bulletin of the **S**ection of **L**ogic
Volume 54/3 (2025), pp. 383–406
https://doi.org/10.18778/0138-0680.2025.13

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ON INVOLUTIVE WEAK EXCHANGE ALGEBRAS

Abstract

In this paper, involutive weak exchange algebras (for short, involutive WE algebras) are introduced and studied. Their properties and characterizations are investigated. Some important results and examples are given. In particular, it is proven that in involutive WE algebras, the properties (BB), (B), (*), (**) and (Tr) are equivalent. Moreover, involutive BE, involutive GE, involutive pre-BCK and involutive pre-Hilbert algebras are considered, their connections are established. It is shown that involutive WE algebras (respectively, involutive GE algebras) satisfying the commutative property are Wajsberg algebras (respectively, Boolean algebras). Finally, the interrelationships between the classes of involutive algebras considered here are presented.

Keywords: (involutive) Hilbert algebra, pre-Hilbert algebra, BCK, BE, GE algebra, (positive) implicativity, commutativity.

2020 Mathematical Subject Classification: 03G25, 06F35.

Presented by: Janusz Ciuciura
Received: February 1, 2025, **Received in revised form**: July 11, 2025,
Accepted: July 18, 2025, **Published online**: November 28, 2025

1. Introduction

L. Henkin [6] introduced the notion of ”implicative model”, as a model of positive implicative propositional calculus. In 1960, A. Monteiro [16] has given the name ”Hilbert algebras” to the dual algebras of Henkin’s implicative models. In 1966, K. Iséki [11] introduced a new notion called a BCK algebra. It is an algebraic formulation of the BCK-propositional calculus system of C. A. Meredith (see [17], p. 316). In 1984, J. M. Font et al. [5] introduced Wajsberg algebras which are a model of $\aleph_0$-valued Łukasiewicz logic, studied by M. Wajsberg [18]. They were also considered earlier by Y. Komori in [14, 15] under the name of CN algebras. Hilbert and Wajsberg algebras are particular cases of BCK algebras.

In [13], as a generalization of BCK algebras, H. S. Kim and Y. H. Kim defined BE algebras. In 2008, A. Walendziak [19] proved that the commutative BE algebras coincide with the BCK algebras. Later on, in 2010, D. Buşneag and S. Rudeanu [3] introduced the notion of pre-BCK algebra. A BCK algebra is just a pre-BCK algebra satisfying also the antisymmetry. Recently, as a generalization of Hilbert algebras, R. Bandaru et al. [1] introduced GE algebras (generalized exchange algebras) and A. Walendziak [22] introduced pre-Hilbert algebras (the definition of a pre-Hilbert algebra is inspired by Henkin’s Positive Implicative Logic [6]). All of the algebras mentioned above are contained in the class of RML algebras (an RML algebra is an algebra $(A, \to, 1)$ of type $(2, 0)$ satisfying the identities: $x \to x = 1 = x \to 1$ and $1 \to x = x$), introduced by A. Iorgulescu in [8].

Many systems of propositional calculus have been developed, one of the axioms of which is the axiom of double negation: $\varphi \to \neg\neg\varphi$. This is one of the motivations for introducing algebras that satisfy the involution law $(x^-)^- = x$. Involutive BCK and involutive BE algebras were investigated in [7] and [2], respectively.

In this paper, we introduce and study involutive weak exchange algebras (involutive WE algebras, for short). We give some examples, properties and characterizations of them. In particular, we prove that in involutive WE algebras, the properties (BB), (B), (*), (**) and (Tr) are equivalent. Moreover, we consider involutive BE, involutive GE, involutive pre-BCK and involutive pre-Hilbert algebras. We show that they are particular cases of involutive weak exchange algebras and also establish other connections between them. We prove that involutive WE algebras (respectively, involutive GE algebras) satisfying the commutative property are Wajsberg algebras (respectively, Boolean algebras). Finally, we present the interrelationships between the classes of involutive algebras considered here.

2. Preliminaries

Let $\mathcal{A} = (A, \to, 1)$ be an algebra of type $(2, 0)$. We define the binary relation $\leq$ on A by: for all $x, y \in A$,

$$x \leq y \Longleftrightarrow x \to y = 1. \tag{\#}$$

We consider the following list **A** of properties ([8]) that can be satisfied by $\mathcal{A}$ (in fact, the properties in the list are the most important properties satisfied by a BCK algebra):

(An) (Antisymmetry) $(x \leq y$ and $y \leq x) \Longrightarrow x = y$,

(B) $y \to z \leq (x \to y) \to (x \to z)$,

(BB) $y \to z \leq (z \to x) \to (y \to x)$,

(C) (= (WE)) (Weak exchange) $x \to (y \to z) \leq y \to (x \to z)$,

(D) $y \leq (y \to x) \to x$,

(Ex) (Exchange) $x \to (y \to z) = y \to (x \to z)$,

(K) $x \leq y \to x$,

(L) (Last element) $x \leq 1$,

(M) $1 \to x = x$,

(Re) (Reflexivity) $x \leq x$,

(Tr) (Transitivity) $(x \leq y$ and $y \leq z) \Longrightarrow x \leq z$,

(*) $y \leq z \Longrightarrow x \to y \leq x \to z$,

(**) $y \leq z \Longrightarrow z \to x \leq y \to x$.

From Proposition 2.1 and Theorem 2.7 of [8] we have

Lemma 2.1. *Let $\mathcal{A} = (A, \to, 1)$ be an algebra of type $(2, 0)$.*
Then the following hold:

(i) *(M) + (B) imply (Re), (*), (**);*

(ii) *(M) + (*) imply (Tr);*

(iii) *(M) + (**) imply (Tr);*

(iv) *(Re) + (Ex) imply (WE);*

(v) *(M) + (BB) imply (B);*

(vi) *(An) + (WE) imply (Ex).*

Definition 2.2. An algebra $(A, \to, 1)$ is a/an:

1. *RML algebra* ([8]) if it satisfies (Re), (M), (L);
2. *BE algebra* ([13]) if it satisfies (Re), (M), (L), (Ex), that is, it is an RML algebra with the exchange property (Ex);
3. *pre-BCK algebra* ([3]) if it satisfies (Re), (M), (L), (Ex), (*) or, equivalently ([8], Theorem 2.3), (Re), (M), (L), (Ex), (B), that is, it is a BE algebra with (B);
4. *BCK algebra* ([11]) if it satisfies (An), (BB), (D), (Re), (L) or, equivalently ([8]), (An), (Re), (M), (L), (Ex), (B), that is, it is a pre-BCK algebra with (An).

Denote by **RML**, **BE**, **preBCK** and **BCK** the classes of RML, BE, pre-BCK and BCK algebras, respectively. By definition, **BE** = **RML** + (Ex), **preBCK** = **BE**+ (B), **BCK** = **preBCK** + (An).

Consider the following list **B** of particular properties that can be satisfied by $\mathcal{A} = (A, \rightarrow, 1)$:

(GE) (Generalized exchange) $x \rightarrow (y \rightarrow z) = x \rightarrow (y \rightarrow (x \rightarrow z))$,

(pi) (general positive implicativity) $x \rightarrow (x \rightarrow y) = x \rightarrow y$,

(pimpl) (positive implicativity) $x \rightarrow (y \rightarrow z) = (x \rightarrow y) \rightarrow (x \rightarrow z)$,

(p-1) $x \rightarrow (y \rightarrow z) \leq (x \rightarrow y) \rightarrow (x \rightarrow z)$,

(p-2) $(x \rightarrow y) \rightarrow (x \rightarrow z) \leq x \rightarrow (y \rightarrow z)$,

(impl) (implicativity) $(x \rightarrow y) \rightarrow x = x$,

(comm) (commutativity) $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$.

Remark 2.3. The properties (GE)–(p-2) in the list are the most important properties satisfied by a Hilbert algebra.

LEMMA 2.4. *Let* $(A, \rightarrow, 1)$ *be an algebra of type* $(2, 0)$*. Then the following hold:*

(i) *(Re) + (pimpl) imply (L);*

(ii) *(Re) + (L) + (pimpl) imply (K);*

(iii) *(Ex) + (pi) imply (GE);*

(iv) *(Re) + (M) + (pimpl) imply (pi), (GE);*

(v) *(M) + (comm) imply (An);*

(vi) *(M) + (L) + (comm) imply (Re).*

PROOF: (i) and (ii) follow from Propositions 6.4(i) and 6.9(1) of [8], respectively.

(iii) and (iv) follow from Propositions 2.7 and 3.7 of [21], respectively.

(v) and (vi) follow from Proposition 3.3(i) and (ii) of [20]. □

Definition 2.5. Let $\mathcal{A} = (A, \to, 1)$ be an algebra of type $(2, 0)$. We say that $\mathcal{A}$ is a:

5. *Hilbert algebra* [4] if it satisfies (An), (K), (p-1);
6. *pre-Hilbert algebra* [22] if it satisfies (M), (K), (p-1);
7. *GE algebra* [1] (*generalized exchange algebra*) if it satisfies (Re), (M), (GE).

Denote by **H**, **preH** and **GE** the classes of all Hilbert, pre-Hilbert and GE algebras, respectively.

Remark 2.6. In [4], A. Diego proved that Hilbert algebras satisfy (Re), (M), (L), (B), (Ex), (pi), (p-2), (pimpl). Moreover, he showed that the class of all Hilbert algebras is a variety.

Theorem 2.7 ([22, Theorem 3.9]). *Pre-Hilbert algebras satisfy (Re), (M), (L), (B), (WE), (K), (Tr), (p-1), (p-2).*

Pre-Hilbert algebras do not have to satisfy (An), (Ex), (GE), (pi), (pimpl); see the example below.

Example 2.8 ([23]). Consider the set $A = \{a, b, c, d, 1\}$ and the operation $\to$ given by the following table:

$\to$	a	b	c	d	1
a	1	c	b	d	1
b	a	1	1	d	1
c	a	1	1	d	1
d	a	c	c	1	1
1	a	b	c	d	1

The algebra $\mathcal{A} = (A, \to, 1)$ verifies properties (M), (K), (p-1). Then, $\mathcal{A}$ is a pre-Hilbert algebra. It does not verify (An) for $(x, y) = (b, c)$; (Ex) and (pimpl) for $(x, y, z) = (a, d, b)$; (pi) for $(x, y) = (a, b)$.

Remark 2.9. By definition and Theorem 2.7, **preH** = **RML** + (K) + (p-1). Since (An) + (K) + (p-1) imply (M) (see [4]), a Hilbert algebra is in fact a pre-Hilbert algebra verifying (An), that is, **H** =**preH** + (An).

THEOREM 2.10 ([21, Corollary 3.2]). *Any GE algebra satisfies (Re), (M), (L), (WE), (D), (K), (GE), (pi).*

Remark 2.11. Since GE algebras satisfy (Re), (M), (L), we get **GE** = **RML** + (GE).

GE algebras do not have to satisfy (An), (B), (Tr), (Ex), (p-1), (pimpl); see the example below.

Example 2.12. Let $A = \{a, b, c, d, e, 1\}$ and $\to$ be defined as follows:

$\to$	a	b	c	d	e	1
a	1	1	c	c	1	1
b	a	1	d	d	1	1
c	a	1	1	1	1	1
d	a	1	1	1	1	1
e	a	1	1	1	1	1
1	a	b	c	d	e	1

.

We can observe that the properties (Re), (M), (L), (GE) (hence (pi)) are satisfied. Therefore, $(A, \to, 1)$ is a GE algebra. It does not satisfy (An) for $(x, y) = (c, d)$; (Ex) for $(x, y, z) = (a, b, c)$; (Tr), (B), (p-1) and (pimpl) for $(x, y, z) = (a, e, c)$.

It is known that $\leq$ is an order relation in BCK and Hilbert algebras. By definition, in RML, BE and GE algebras, $\leq$ is a reflexive relation. Since (M) + (B) imply (Tr), see Lemma 2.1(i) and (ii), in pre-BCK and pre-Hilbert algebras, $\leq$ is reflexive and transitive (i.e., it is a pre-order relation).

DEFINITION 2.13. Let $\mathcal{A} = (A, \to, 1)$ be an RML algebra. We say that $\mathcal{A}$ is:

8. *implicative* if property (impl) is satisfied;
9. *positive implicative* if property (pimpl) is satisfied;
10. *commutative* if property (comm) is satisfied.

Note that positive implicative RML algebras are also called, for short, pimpl-RML algebras. Denote by **impl-RML**, **pimpl-RML** and **comm-RML** the classes of implicative, positive implicative and commutative

RML algebras, respectively; similarly for the subclasses of the class of all RML algebras. For example, **pimpl-preH** denotes the class of all positive implicative pre-Hilbert algebras.

Remark 2.14. Note that from [8] it follows that in RML algebras, (pimpl) implies (pi). For BCK algebras, (pimpl) and (pi) are equivalent (cf. Theorem 8 of [12]). From Remark 6.7 of [8] we see that $\mathbf{H} = \mathbf{pimpl\text{-}BCK}$.

THEOREM 2.15 ([12, Theorem 9]). *In commutative BCK algebras, we have*

$$\text{(pi)} \Longleftrightarrow \text{(pimpl)} \Longleftrightarrow \text{(impl)}.$$

THEOREM 2.16 ([12, Theorem 10]). *Every implicative BCK algebra is commutative and positive implicative.*

THEOREM 2.17 ([19]). *The commutative BE algebras coincide with the commutative BCK algebras.*

THEOREM 2.18 ([1, Theorem 3.9]). *Any commutative GE algebra is a Hilbert algebra.*

Let $\mathcal{A} = (A, \to, 0, 1)$ be an algebra of type $(2, 0, 0)$ and let $\leq$ be the binary relation on A defined by (#). We say that $\mathcal{A}$ is a *bounded RML algebra* if it satisfies

(F) (First element) $0 \to x = 1$

(that is, $0 \leq x$) for all $x \in A$ and its reduct $(A, \to, 1)$ is an RML algebra.

Denote by $\mathbf{RML}^b$ (where "b" means "bounded") the class of all bounded RML algebras; similarly for the subclasses of the class **RML**. For example, $\mathbf{comm\text{-}BCK}^b$ denotes the class of bounded commutative BCK algebras.

3. Involutive WE algebras

Let $\mathcal{A} = (A, \to, 0, 1)$ be a bounded RML algebra. Define a *negation* $^-$, by [12], for all $x \in A$,

$$x^- = x \to 0.$$

If the negation $^-$ satisfies:

(DN) (Double Negation) $(x^-)^- = x$,

we say that it is *involutive* and that the algebra is *involutive*, that is, $\mathcal{A} = (A, \to, 0, 1)$ is called an *involutive RML* algebra if it is a bounded RML algebra satisfying (DN).

PROPOSITION 3.1. In an involutive RML algebra $(A, \to, 0, 1)$, the following properties hold:

(Neg1) $0^- = 1$ and $1^- = 0$,

(Neg2) $((x^-)^-)^- = x^-$,

(Neg3) $x^- = y^- \Longrightarrow x = y$.

PROOF: (Neg1) By (Re), $0^- = 0 \to 0 = 1$. By (M), $1^- = 1 \to 0 = 0$.
(Neg2) and (Neg3) follow from (DN). □

PROPOSITION 3.2. Let $\mathcal{A} = (A, \to, 0, 1)$ be a bounded RML algebra. Then the following statements are equivalent:

(i) $\mathcal{A}$ is involutive;

(ii) $((x^-)^-)^- = x^-$, and $x^- = y^- \Longrightarrow x = y$, for all $x, y \in A$;

(iii) $(x \to 0) \to 0 = (0 \to x) \to x$ for every $x \in A$.

PROOF: (i) $\Longrightarrow$ (ii) follows from Proposition 3.1.
(ii) $\Longrightarrow$ (i) is obvious.
(i) $\Longrightarrow$ (iii) Let $x \in A$. We have $(x \to 0) \to 0 = (x^-)^- = x = 1 \to x = (0 \to x) \to x$.
(iii) $\Longrightarrow$ (i) Let $x \in A$. We obtain $(x^-)^- = (x \to 0) \to 0 = (0 \to x) \to x = 1 \to x = x$. □

DEFINITION 3.3. An algebra $\mathcal{A} = (A, \to, 0, 1)$ is called an *involutive weak exchange algebra*, or *involutive WE algebra* for short, if it is an involutive RML algebra satisfying the weak exchange property (WE).

Denote by $\mathbf{RML}_{(DN)}$, $\mathbf{WE}_{(DN)}$ the classes of involutive RML and involutive WE algebras, respectively. Similarly, if $\mathbf{X}$ is a subclass of $\mathbf{RML}^b$, then let us denote by $\mathbf{X}_{(DN)}$ the subclass of involutive algebras of the class $\mathbf{X}$.

Example 3.4. Consider the set $A = \{0, a, b, c, 1\}$ and the operation $\to$ given by the following table:

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	1	1	1
b	a	1	1	b	1
c	c	a	1	1	1
1	0	a	b	c	1

The algebra $\mathcal{A} = (A, \to, 0, 1)$ satisfies properties (Re), (M), (L), (F), (DN), (WE). Then, $\mathcal{A}$ is an involutive WE algebra. It does not satisfy (An) for $(x, y) = (a, b)$; (Ex) for $(x, y, z) = (b, c, 0)$; (GE) for $(x, y, z) = (c, b, 0)$; (B), (BB), (Tr), (p-1), (pimpl) for $(x, y, z) = (c, b, a)$; (pi) for $(x, y) = (b, c)$.

Remark 3.5. Since (Re) and (Ex) imply (WE), see Lemma 2.1(iv), any involutive BE algebra (hence also any involutive pre-BCK and involutive BCK algebra) is an involutive WE algebra. Examples of involutive BE and involutive BCK algebras are presented in [2] and [7], respectively; below we give an example of an involutive pre-BCK algebra.

Example 3.6. Let $A = \{0, a, b, 1\}$ and $\to$ be defined as follows:

$\to$	0	a	b	1
0	1	1	1	1
a	b	1	1	1
b	a	1	1	1
1	0	a	b	1

We can observe that algebra $\mathcal{A} = (A, \to, 0, 1)$ satisfies properties (Re), (M), (L), (F), (DN), (Ex), (B). Therefore, $\mathcal{A}$ is an involutive pre-BCK algebra. It does not satisfy (An) for $(x, y) = (a, b)$; (pi) for $(x, y) = (a, 0)$; (p-1) and (pimpl) for $(x, y, z) = (a, a, 0)$; (GE) for $(x, y, z) = (a, 1, 0)$.

Remark 3.7. Since GE algebras and pre-Hilbert algebras satisfy the property (WE), see Theorems 2.7 and 2.10, we conclude that $\mathbf{GE}_{(DN)}$ and $\mathbf{preH}_{(DN)}$ are subclasses of $\mathbf{WE}_{(DN)}$.

Now we give two examples: Example 3.8 of an involutive GE algebra and Example 3.9 of an involutive pre-Hilbert algebra.

Example 3.8. Consider the set $A = \{0, a, b, c, d, 1\}$ and the operation $\rightarrow$ given by the following table:

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	b	1	b	1	1	1
b	a	a	1	a	1	1
c	d	1	1	1	d	1
d	c	c	1	c	1	1
1	0	a	b	c	d	1

The algebra $\mathcal{A} = (A, \rightarrow, 0, 1)$ satisfies properties (Re), (M), (L), (F), (DN), (GE) (hence (WE), (pi)). Then $\mathcal{A}$ is an involutive GE algebra. It does not satisfy (An) for $x = a$, $y = c$; (Ex) for $x = b$, $y = d$, $z = 0$; (B), (Tr), (p-1), (pimpl) for $x = c$, $y = b$, $z = d$.

Example 3.9. Let $A = \{0, a, b, c, d, 1\}$ and $\rightarrow$ be defined as follows:

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	b	1	b	b	1	1
b	a	a	1	1	a	1
c	d	a	1	1	a	1
d	c	1	c	c	1	1
1	0	a	b	c	d	1

The algebra $\mathcal{A} = (A, \rightarrow, 0, 1)$ satisfies (Re), (M), (L), (F), (DN), (K), (p-1). Then $\mathcal{A}$ is an involutive pre-Hilbert algebra. It does not satisfy (An) for $x = b$, $y = c$; (Ex) for $x = a$, $y = d$, $z = 0$; (pimpl) for $x = y = c$, $z = 0$; (pi) for $x = c$, $y = 0$; (GE) for $x = c$, $y = 1$, $z = 0$.

Proposition 3.10. Let $\mathcal{A} = (A, \to, 0, 1)$ be an involutive weak exchange algebra. Then, for all $x, y \in A$, we have:

(Neg4) $x \to y^- \leq y \to x^-$,

(Neg5) $y^- \to x^- \leq x \to y$,

(Neg6) $x^- \to y \leq y^- \to x$,

(Neg7) $x \to y \leq y^- \to x^-$,

(Neg8) $x \leq y \Longleftrightarrow y^- \leq x^-$.

Proof: (Neg4) follows from (WE).

(Neg5) Using (Neg4) and (DN), we have $y^- \to x^- \leq x \to (y^-)^- = x \to y$.

(Neg6) We get $x^- \to y \overset{(DN)}{=} x^- \to (y^-)^- \overset{(Neg4)}{\leq} y^- \to (x^-)^- \overset{(DN)}{=} y^- \to x$.

(Neg7) By (DN) and (Neg4), $x \to y = x \to (y^-)^- \leq y^- \to x^-$, that is, (Neg7) holds.

(Neg8) Let $x \leq y$. Then, by (Neg7), $1 = x \to y \leq y^- \to x^-$. Hence $y^- \to x^- = 1$, that is, $y^- \leq x^-$. Conversely, if $y^- \leq x^-$, then $(x^-)^- \leq (y^-)^-$. Therefore $x \leq y$ by (DN). □

Corollary 3.11. Involutive weak exchange algebras satisfy (Neg1)–(Neg8).

Proposition 3.12. Let $\mathcal{A} = (A, \to, 0, 1)$ be an involutive BE algebra. Then, for all $x, y \in A$, we have:

(Neg4') $x \to y^- = y \to x^-$,

(Neg5') $x \to y = y^- \to x^-$,

(Neg6') $x^- \to y = y^- \to x$.

Proof: (Neg4') follows from (Ex). Using (Neg4') and (DN), we get (Neg5') and (Neg6'). □

Proposition 3.13. Let $\mathcal{A} = (A, \to, 0, 1)$ be an involutive RML algebra. Then the following are equivalent:

(Neg9) $x \to x^- = x^-$,

(Neg10) $x^- \to x = x$.

Proof: Let $\mathcal{A}$ satisfy (Neg9) and $x \in A$. Applying (DN) and (Neg9), we get $x^- \to x = x^- \to (x^-)^- = (x^-)^- = x$, that is, (Neg10) holds. Now let $\mathcal{A}$ satisfy (Neg10). We have $x \to x^- = (x^-)^- \to x^- = x^-$. Thus we get (Neg9). □

Proposition 3.14. Let $\mathcal{A} = (A, \to, 0, 1)$ be an involutive WE algebra. Then the following are equivalent:

(Neg11) $(x \to y)^- \leq x$,

(Neg12) $x^- \leq x \to y$.

Proof: Let $\mathcal{A}$ satisfy (Neg11). By (Neg8) and (DN), $x^- \leq ((x \to y)^-)^- = x \to y$. Thus (Neg12) holds in $\mathcal{A}$. Conversely, if $x^- \leq x \to y$, then $(x \to y)^- \leq (x^-)^- = x$. Therefore, (Neg11) holds. □

Theorem 3.15. *Let* $\mathcal{A} = (A, \to, 0, 1)$ *be an algebra of type* $(2, 0, 0)$. *Then the following statements are equivalent:*

(i) $\mathcal{A}$ *is an involutive weak exchange algebra;*

(ii) $\mathcal{A}$ *satisfies (M), (WE), (F), (DN);*

(iii) $\mathcal{A}$ *satisfies (M), (WE), (L), (DN).*

Proof: (i) $\Longrightarrow$ (ii) is obvious.

(ii) $\Longrightarrow$ (iii) Let $x \in A$. We have $1 \overset{(F)}{=} 0 \to (x \to 0) \overset{(WE)}{\leq} x \to (0 \to 0) \overset{(F)}{=} x \to 1$. Hence, by (M), we get $x \to 1 = 1$, that is, $x \leq 1$. Thus $\mathcal{A}$ satisfies (L).

(iii) $\Longrightarrow$ (i) First we observe that (F) holds in $\mathcal{A}$. Let $x \in A$. We obtain

$$1 \overset{(L)}{=} x^- \to 1 \overset{(DN)}{=} x^- \to (1^-)^- = x^- \to (1^- \to 0)$$

$$\overset{\text{(WE)}}{\leq} 1^- \to (x^- \to 0) = 0 \to (x^-)^- \overset{\text{(DN)}}{=} 0 \to x.$$

Hence, by (M), $0 \leq x$. Thus (F) holds. Now we prove that $\mathcal{A}$ satisfies (Re). For $x \in A$, we get $x \overset{\text{(DN)}}{=} (x^-)^- \overset{\text{(M)}}{=} 1 \to (x^- \to 0) \overset{\text{(WE)}}{\leq} x^- \to (1 \to 0) \overset{\text{(M)}}{=} x^- \to 0 = (x^-)^- \overset{\text{(DN)}}{=} x$, i.e., $x \leq x$. □

Proposition 3.16. Involutive WE algebras satisfy properties (D) and (K).

Proof: Let $\mathcal{A} = (A, \to, 0, 1)$ be an involutive WE algebra and $x, y \in A$. By (Re) and (WE), $1 = (y \to x) \to (y \to x) \leq y \to ((y \to x) \to x)$. From (M) we see that $y \to ((y \to x) \to x) = 1$, that is, $y \leq (y \to x) \to x$. Thus (D) holds in $\mathcal{A}$. Now we have

$$1 \overset{\text{(L)}}{=} y \to 1 \overset{\text{(Re)}}{=} y \to (x \to x) \overset{\text{(WE)}}{\leq} x \to (y \to x).$$

Then $x \to (y \to x) = 1$ by (M). Therefore $x \leq y \to x$. Thus (K) also holds. □

Theorem 3.17. *If $\mathcal{A} = (A, \to, 0, 1)$ is an involutive WE algebra, then*

$$\text{(BB)} \iff \text{(B)} \iff \text{(**)} \iff \text{(Tr)} \iff \text{(*)}.$$

Proof: Let $x, y, z \in A$. By Lemma 2.1(v), (i), (BB) $\Longrightarrow$ (B) $\Longrightarrow$ (**). Observe that (Tr) $\Longleftarrow$ (**). Let $\mathcal{A}$ satisfy (Tr) and suppose that $y \leq z$. By Proposition 3.16, $\mathcal{A}$ satisfies (D). Then $z \leq (z \to x) \to x$. Using (Tr), we have $y \leq (z \to x) \to x$. By (WE), $z \to x \leq y \to x$. Thus (**) holds in $\mathcal{A}$. The converse follows from Lemma 2.1(iii).

Now we prove that (**) implies (*). Let $y \leq z$. By (Neg8), $z^- \leq y^-$. Applying (**), we obtain $y^- \to x^- \leq z^- \to x^-$. From (Neg7) we conclude that $x \to y \leq y^- \to x^-$ and from (Neg5) we see that $z^- \to x^- \leq x \to z$. Consequently, $x \to y \leq x \to z$, that is (*) holds.

Finnaly, we show that (*) implies (BB). By (D), $z \leq (z \to x) \to x$. Using (*) and (WE), we get $y \to z \leq y \to ((z \to x) \to x)$ and $y \to ((z \to x) \to x) \leq (z \to x) \to (y \to x)$. Since $\mathcal{A}$ satisfy (Tr), see Lemma 2.1 (ii), we deduce that $y \to z \leq (z \to x) \to (y \to x)$. Thus (BB) holds. □

Remark 3.18. Since (Re) + (Ex) imply (WE), Theorem 3.17 is a generalization of [10, Theorem 3.9].

4. Connections

By definition, involutive pre-Hilbert algebras are characterized by the axioms: (M), (K), (p-1), (F) and (DN). Every pre-Hilbert algebra satisfies (L) and (WE), see Theorem 2.7, and by the proof of Theorem 3.15, (L) + (WE) + (DN) imply (F). Then, we have

PROPOSITION 4.1. Let $\mathcal{A} = (A, \to, 0, 1)$ be an algebra of type $(2, 0, 0)$. Then $\mathcal{A}$ is an involutive pre-Hilbert algebra if and only if it satisfies (M), (K), (p-1) and (DN).

Now we give a characterization of involutive positive implicative pre-Hilbert algebras.

PROPOSITION 4.2. Let $\mathcal{A} = (A, \to, 0, 1)$ be an algebra of type $(2, 0, 0)$. Then $\mathcal{A}$ is an involutive pimpl-pre-Hilbert algebra if and only if it satisfies (M), (Re), (pimpl) and (DN).

PROOF: Let $\mathcal{A}$ satisfy (M), (Re), (pimpl) and (DN). By Lemma 2.4(i), (Re) + (pimpl) imply (L). Moreover, by Lemma 2.4 (ii), (Re) + (L) + (pimpl) imply (K). Obviously, (Re) + (pimpl) imply (p-1). Then $\mathcal{A}$ is an involutive pimpl-pre-Hilbert algebra. The converse is obvious. $\square$

Remark 4.3. By Proposition 4.2, $\mathbf{pimpl\text{-}RML}_{(DN)} = \mathbf{pimpl\text{-}preH}_{(DN)}$.

Example 4.4. Consider the set $A = \{0, a, b, c, d, 1\}$ and the operation $\to$ given by the following table:

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	b	1	b	1	b	1
b	a	a	1	a	1	1
c	d	1	d	1	d	1
d	c	c	1	c	1	1
1	0	a	b	c	d	1

We can observe that algebra $\mathcal{A} = (A, \to, 0, 1)$ satisfies properties (Re), (M), (L), (F), (DN), (pimpl). Therefore, $\mathcal{A}$ is an involutive positive implicative pre-Hilbert algebra. It does not satisfy (An) for $(x, y) = (a, c)$; (Ex) for $(x, y, z) = (a, c, 0)$.

Remark 4.5. Since (Re) + (M) + (pimpl) imply (GE), see Lemma 2.4 (iv), we deduce that involutive pimpl-RML algebras are involutive GE algebras, that is, $\mathbf{pimpl\text{-}preH}_{(DN)} \subset \mathbf{GE}_{(DN)}$. Moreover, $\mathbf{GE}_{(DN)}$ + (pimpl) = $\mathbf{RML}_{(DN)}$ + (GE) + (pimpl) = $\mathbf{RML}_{(DN)}$ + (pimpl) = $\mathbf{pimpl\text{-}RML}_{(DN)}$ = $\mathbf{pimpl\text{-}preH}_{(DN)}$.

Recall from [10] the definition of Wajsberg algebras:

DEFINITION 4.6. A *Wajsberg algebra* is an algebra $(A, \to, 0, 1)$ of type $(2, 0, 0)$ satisfying: (M), (Ex), (L), (F), (comm).

THEOREM 4.7. *Let* $\mathcal{A} = (A, \to, 0, 1)$ *be an algebra of type* $(2, 0, 0)$*. Then the following statements are equivalent:*

(i) $\mathcal{A}$ *is an involutive weak exchange algebra with the property (comm);*

(ii) $\mathcal{A}$ *satisfies (M), (WE), (F), (comm);*

(iii) A *is a Wajsberg algebra;*

(iv) A *is a bounded commutative BE algebra;*

(v) $\mathcal{A}$ *is a bounded commutative BCK algebra.*

PROOF: (i) $\Longrightarrow$ (ii) is obvious.

(ii) $\Longrightarrow$ (iii) By the proof of Theorem 3.15, (M) + (WE) + (F) imply (L). Hence, if $\mathcal{A}$ satisfies (M), (WE), (F), (comm), then $\mathcal{A}$ also satisfies (L). From Lemma 2.4(v) we see that (An) holds in $\mathcal{A}$. By Lemma 2.1(vi), (Ex) also holds. Thus $\mathcal{A}$ is a Wajsberg algebra.

(iii) $\Longrightarrow$ (iv) From Lemma 2.4(vi) we conclude that Wajsberg algebras satisfy (Re). Then, if $\mathcal{A}$ is a Wajsberg algebra, then it is a bounded commutative BE algebra.

(iv) $\Longleftrightarrow$ (v) follows from Theorem 2.17.

(v) $\Longrightarrow$ (i) Observe that (M) + (F) + (comm) imply (DN). Indeed, let $x \in A$. We have $(x \to 0) \to 0 = (0 \to x) \to x = 1 \to x = x$. Then $\mathcal{A}$ is an involutive WE algebra satisfying (comm). $\square$

Remark 4.8. From Theorem 4.7 we see that $\mathbf{comm\text{-}WE}_{(DN)} = \mathbf{W} = \mathbf{comm\text{-}BE}^b = \mathbf{comm\text{-}BCK}^b$, where $\mathbf{W}$ denotes the class of Wajsberg algebras.

PROPOSITION 4.9. Let $\mathcal{A} = (A, \rightarrow, 0, 1)$ be an involutive BE algebra. Then

$$\text{(Neg9)} \Longleftrightarrow \text{(Neg10)} \Longleftrightarrow \text{(pi)}.$$

PROOF: By Proposition 3.13, (Neg9) $\Longleftrightarrow$ (Neg10). Let $x \in A$ and $x \rightarrow x^- = x^-$. Then $y^- \rightarrow (x \rightarrow x^-) = y^- \rightarrow x^-$. Applying (Ex), we obtain $x \rightarrow (y^- \rightarrow x^-) = y^- \rightarrow x^-$. In involutive BE algebras, we have $y^- \rightarrow x^- = x \rightarrow y$ (see Proposition 3.12). Hence $x \rightarrow (x \rightarrow y) = x \rightarrow y$, that is, (pi) holds. Conversely, $x \rightarrow x^- = x \rightarrow (x \rightarrow 0) \overset{\text{(pi)}}{=} x \rightarrow 0 = x^-$. Thus (Neg9) is satisfied. □

COROLLARY 4.10. Let $\mathcal{A} = (A, \rightarrow, 0, 1)$ be a bounded commutative BCK algebra. Then

$$\text{(Neg9)} \Longleftrightarrow \text{(Neg10)} \Longleftrightarrow \text{(pi)} \Longleftrightarrow \text{(pimpl)} \Longleftrightarrow \text{(impl)}.$$

PROOF: If $\mathcal{A}$ is a bounded commutative BCK algebra, then it is involutive. Hence, by Proposition 4.9, (Neg9) $\Longleftrightarrow$ (Neg10) $\Longleftrightarrow$ (pi). From Theorem 2.15 we see that (pi) $\Longleftrightarrow$ (pimpl) $\Longleftrightarrow$ (impl). □

PROPOSITION 4.11. Let $\mathcal{A} = (A, \rightarrow, 0, 1)$ be a bounded BE algebra. Then

$$\text{(DN)} \Longleftrightarrow \text{(Neg5')} \Longleftrightarrow \text{(Neg6')}.$$

PROOF: (DN) $\Longrightarrow$ (Neg5') Let $\mathcal{A}$ satisfy (DN) and $x, y \in A$. We have $x \rightarrow y = x \rightarrow (y^- \rightarrow 0) = y^- \rightarrow (x \rightarrow 0) = y^- \rightarrow x^-$, that is, (Neg5') holds.

(Neg5') $\Longrightarrow$ (DN) Let $\mathcal{A}$ satisfy (Neg5') and $x \in A$. By (Neg5'), $x = 1 \rightarrow x = x^- \rightarrow 1^- = x^- \rightarrow 0 = (x^-)^-$. Hence $\mathcal{A}$ satisfies (DN).

(Neg5') $\Longrightarrow$ (Neg6') Let $\mathcal{A}$ satisfy (Neg5') and $x, y \in A$. We get $y^- \rightarrow x = x^- \rightarrow (y^-)^- = x^- \rightarrow y$, that is, (Neg6') holds.

(Neg6') $\Longrightarrow$ (DN) Let $\mathcal{A}$ satisfy (Neg6') and $x \in A$. We have $(x^-)^- = x^- \rightarrow 0 = 0^- \rightarrow x = 1 \rightarrow x = x$. □

Lemma 4.12. *Let $\mathcal{A} = (A, \to, 0, 1)$ be an involutive pimpl-BE algebra. Then A satisfies (comm).*

Proof: By Lemma 2.4 (iv), (Re) + (M) + (pimpl) imply (pi). Then $\mathcal{A}$ satisfies (Neg10) by Proposition 4.9. From Proposition 4.11 we see that $\mathcal{A}$ also satisfies (Neg5') and (Neg6'). Let $x, y \in A$. We obtain $x \to y \overset{\text{(Neg10)}}{=} x \to (y^- \to y) \overset{\text{(Ex)}}{=} y^- \to (x \to y) \overset{\text{(pimpl)}}{=} (y^- \to x) \to (y^- \to y) \overset{\text{(Neg10)}}{=} (y^- \to x) \to y$.

Therefore, we have

(1) $x \to y = (y^- \to x) \to y$.

Observe that

(2) $((x \to y) \to y) \to y = x \to y$.

Indeed, $((x \to y) \to y) \to y \overset{\text{(Neg5')}}{=} ((y^- \to x^-) \to y) \to y \overset{(1)}{=} (x^- \to y) \to y \overset{\text{(Neg6')}}{=} (y^- \to x) \to y \overset{(1)}{=} x \to y$. Thus (2) holds. Now we prove that $(x \to y) \to y = (y \to x) \to x$. We get $(x \to y) \to y \overset{(1)}{=} ((y^- \to x) \to y) \to y \overset{\text{(Neg6')}}{=} ((x^- \to y) \to y) \to y \overset{(2)}{=} x^- \to y$. Similarly, $(y \to x) \to x = y^- \to x \overset{\text{(Neg6')}}{=} x^- \to y$. Consequently, (comm) holds. □

Proposition 4.13. Let $\mathcal{A} = (A, \to, 0, 1)$ be a bounded pimpl-BCK algebra (= a bounded Hilbert algebra). Then

$$\text{(DN)} \iff \text{(comm)} \iff \text{(impl)}.$$

Proof: From Lemma 4.12 we conclude that (DN) implies (comm). By Theorem 2.15, (comm) implies (impl). Now, suppose that $\mathcal{A}$ is implicative. Let $x \in A$. By (D), $x \leq (x \to 0) \to 0$. Hence $x \leq (x^-)^-$. On the other hand, since $0 \leq x$, by (*), $(x \to 0) \to 0 \leq (x \to 0) \to x$. Then $(x^-)^- \leq x$. Applying (An), we get (DN). □

Theorem 4.14. *Let $\mathcal{A} = (A, \to, 0, 1)$ be a bounded RML algebra. Then the following statements are equivalent:*

(i) $\mathcal{A}$ *satisfies (K), (p-1), (comm);*

(ii) $\mathcal{A}$ *is a bounded commutative Hilbert algebra;*

(iii) $\mathcal{A}$ *is an involutive Hilbert algebra;*

(iv) $\mathcal{A}$ *satisfies (GE), (comm);*

(v) $\mathcal{A}$ *is a bounded commutative GE algebra;*

(vi) $\mathcal{A}$ *is a bounded commutative BCK algebra satisfying (pi);*

(vii) $\mathcal{A}$ *is a Wajsberg algebra with (pi);*

(viii) $\mathcal{A}$ *is a bounded implicative BCK algebra.*

PROOF: (i) $\Longrightarrow$ (ii) By Lemma 2.4(v), (M) + (comm) imply (An). Hence $\mathcal{A}$ satisfies the antisymmetry property. Thus $\mathcal{A}$ is a bounded commutative Hilbert algebra.

(ii) $\Longrightarrow$ (iii) follows from Proposition 4.13.

(iii) $\Longrightarrow$ (iv) Hilbert algebras satisfy (Ex) and (pi) (see Remark 2.6). Therefore, by Lemma 2.4(iii), $\mathcal{A}$ satisfies (GE). Moreover, from Proposition 4.13 we see that $\mathcal{A}$ also satisfies (comm).

(iv) $\Longrightarrow$ (v) by definition.

(v) $\Longrightarrow$ (vi) Let $\mathcal{A}$ be a bounded commutative GE algebra. From Theorem 2.18 we deduce that $\mathcal{A}$ is a bounded Hilbert algebra. Hence it is a bounded BCK algebra and (pi) holds in $\mathcal{A}$. Thus $\mathcal{A}$ is a bounded commutative BCK algebra satisfying (pi).

(vi) $\Longleftrightarrow$ (vii) By Theorem 4.7, the bounded commutative BCK algebras coincide with the Wajsberg algebras. Hence (vi) and (vii) are equivalent.

(vi) $\Longrightarrow$ (viii) In BCK algebras, (pi) and (pimpl) are equivalent and **H** = **pimpl-BCK** (see Remark 2.14). Then, if $\mathcal{A}$ is a bounded commutative BCK algebra satisfying (pi), then it is implicative (see Corollary 4.10).

(viii) $\Longrightarrow$ (i) Obviously, BCK algebras satisfy (K). By Theorem 2.16, any implicative BCK algebra is commutative and positive implicative. Therefore (viii) implies (i). □

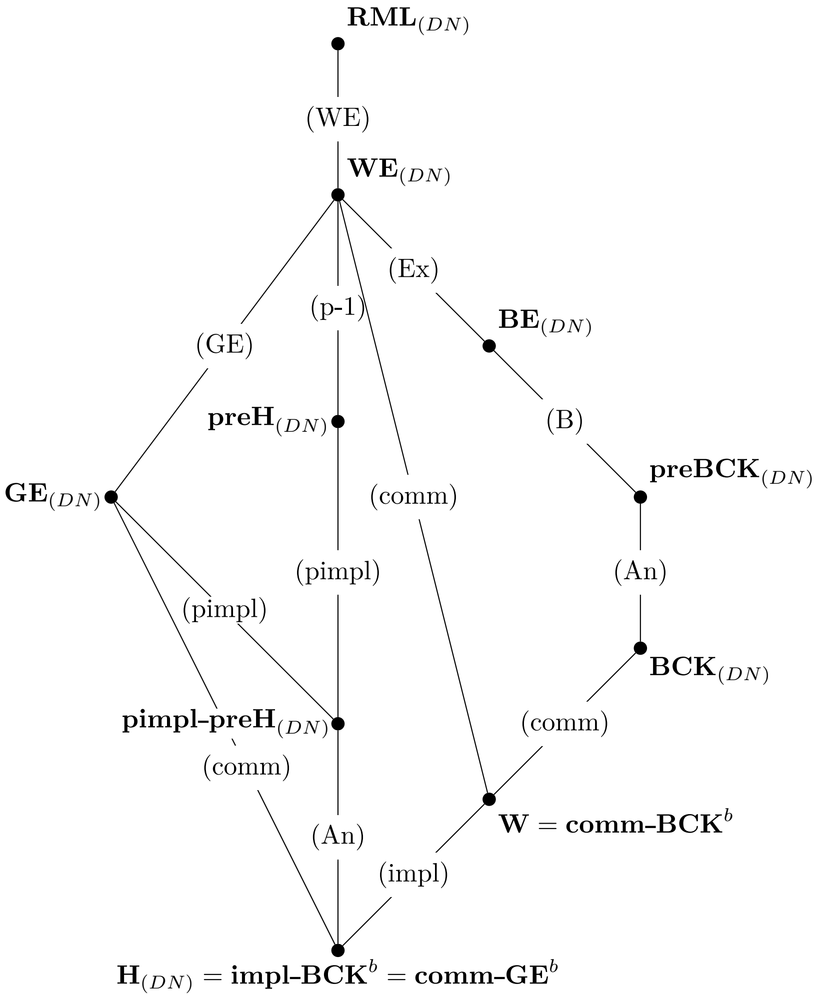

Figure 1: Connections between some subclasses of $\mathbf{RML}_{(DN)}$

Remark 4.15. By Theorem 4.14, we have **comm-H**^{b} = $\mathbf{H}_{(DN)}$ = **comm-GE**^{b} = **comm-BCK**^{b}+ (pi) = **W** + (pi) = **impl-BCK**^{b}.

By Definitions and Remarks 4.5, 4.8 and 4.15 we can draw Figure 1.

Note that from [12, Theorem 12] it follows that the bounded implicative BCK algebras are categorically equivalent to the Boolean algebras.

5. Summary and future work

In this article, we introduced and studied involutive WE algebras. We obtained their properties and characterizations. Moreover, we considered involutive BE, involutive GE, involutive pre-BCK and involutive pre-Hilbert algebras. We showed that they are particular cases of involutive WE algebras and also established other connections between them. We proved that involutive WE algebras (respectively, involutive GE algebras) satisfying the commutative property are Wajsberg algebras (respectively, Boolean algebras). Finally, we presented the interrelationships between the classes of algebras considered here.

The results obtained in the paper can be a starting point for future research. We suggest the following topics:

(1) Studying more deeply the involutive pre-BCK, involutive GE and involutive pre-Hilbert algebras.

(2) Describing connections of implicative-Boolean algebras (introduced in 2009 by A. Iorgulescu [9]) with other involutive algebras.

Acknowledgements. The author would like to thank the referees for their valuable suggestions and comments.

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Funding information: Not applicable.

Conflict of interests: None.

Ethical considerations: The Author assures of no violations of publication ethics and takes full responsibility for the content of the publication.

Declaration regarding the use of GAI tools: Not used.