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CONTINUA OF LOGICS RELATED TO INTUITIONISTIC AND MINIMAL LOGICS

Abstract

We analyze the relationship between logics around intuitionistic logic and minimal logic. We characterize the intersection of minimal logic and co-minimal logic introduced by Vakarelov, and reformulate logics given in the previous studies by Vakarelov, Bezhanishvili, Colacito, de Jongh, Vargas, and Niki in a uniform language. We also compare the new logic with other known logics in terms of the cardinalities of logics between them. Specifically, we apply Wronski's algebraic semantics, instead of neighborhood semantics used in the previous studies, to show the existence of continua of logics between known logics and the new logic. This result is an extension of the conventional results, and the proof is given in a simpler way.

Keywords: intuitionistic logic, minimal logic, subminimal logic, co-minimal logic, Yankov formula.

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1. Introduction

In this paper, we explore the relation between known and new logics with a focus on subminimal logics, which were introduced by [13, 12]. Since we have two backgrounds, we will first describe the backgrounds and then clarify the aims of the paper.

1.1. Vakarelov's logics and Minimal logic

Intuitionistic logic contains *ex contradictione quodlibet* (ECQ for short), which is the inference rule deriving any conclusion B from A and $\neg A$. Minimal logic, which is introduced by [6], is the logic excluding ECQ from Intuitionistic logic. When we treat classical, intuitionistic and minimal logics, negation is usually defined by making use of the absurdity constant and implication, i.e., $\neg A := A \to \bot$. We call this kind of negation *intuitionistic negation* by following the terminology used by [13, 12]. Using intuitionistic negation, minimal logic is the weakest logic with respect to the strength of negation, on the assumption that the implication is at least intuitionistic.

However, there is another way to treat negation in classical/intuitionistic/minimal logics. That is, to take negation as a *primitive* logical connective with one argument. We call this kind of negation *subminimal negation* by following the terminology used in [13] again. In [13], subminimal negation was introduced to analyze strong negation in a more general framework. By using subminimal negation, we can analyze properties of negation in a more detailed way than intuitionistic negation, and we can define logics weaker than minimal logic, which we shall call *subminimal logics*.

In this paper, we are interested in two systems introduced in [13]. One is the $\bot$-free fragment of **SUBMIN** (Definition 6.1), which is one of the subminimal logics. The other is co-minimal logic (**CO-MIN** for short) (Definition 2.7), which has ECQ. The positive fragment of **CO-MIN** coincides with that of intuitionistic logic, but **CO-MIN** is neither weaker nor stronger than minimal logic. Moreover, **CO-MIN** is stronger than **SUBMIN**. The relationship between the four systems discussed by Vakarelov can be summarized as follows.

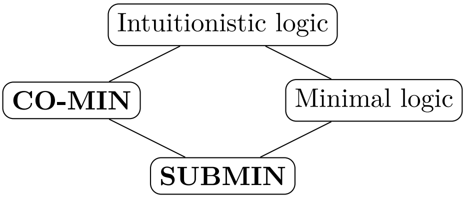


Figure 1: Main systems discussed by Vakarelov

Note here that the systems above the lines are strictly stronger than the lower systems.

1.2. Subminimal logics

Let us now move on to the second background. In [3, 4], several subminimal logics that are closely related to **SUBMIN** were introduced. By using subminimal negation instead of intuitionistic negation, we can treat negation separately from implication and absurdity. Then, the following properties of negation do not hold automatically.

(Co) $(A \to B) \to (\neg B \to \neg A)$ (Contraposition);

(NECQ) $(A \wedge \neg A) \to \neg B$ (Negative Ex Contradiction Quodibet);

(N) $(A \leftrightarrow B) \to (\neg A \leftrightarrow \neg B)$ (Congruence).

Hence we can obtain systems by adding these as axioms to the positive fragment of intuitionistic logic with subminimal negation and the following hierarchy.
Furthermore, [3] established the soundness and completeness theorems for the newly introduced subminimal logics using neighborhood semantics.

The relationship between logics has been analyzed by measuring of the cardinality of logics between them. In [17], the existence of a continuum of logics between classical and intuitionistic logics is proved, by using algebraic semantics. As a related result, [9] showed the existence of a continuum of logics between some intutitionistic modal logics. Another result that is

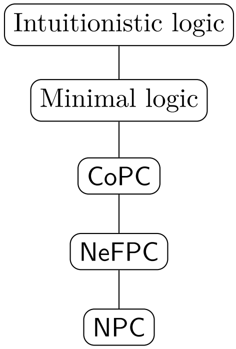


Figure 2: Systems discussed by Colacito, Bezhanishvili and de Jongh

directly related to the above systems can be found in [1] in which the existence of continua of logics between the systems included in the above figure is established.

Although **SUBMIN** was mentioned in [3] its relationship to the logics introduced in [13] was not clarified. It was Niki who revealed, in [8], the relations between **SUBMIN** and subminimal logics included in Figure 2. As a result, the systems are related as summarized in the following figure. Note that Niki refers to the $\perp$-free system **SUBMIN** (**SUBMIN**$^-$ for short) as $\mathsf{An^-PC}$.

1.3. The aims of the paper

Building on the two backgrounds, we have two aims for this paper. First, we investigate the systems introduced by Vakarelov in some further detail. More specifically, we observe that there is a simpler characterization of **CO-MIN** and introduce the intersection of **CO-MIN** and minimal logic. Second, we establish some new results concerning the existence of continua of logics between systems that are not discussed so far in the literature. The results are established by making use of the techniques introduced in [15], and this strategy has the advantage of simplifying the proofs for the previous results established in [1].

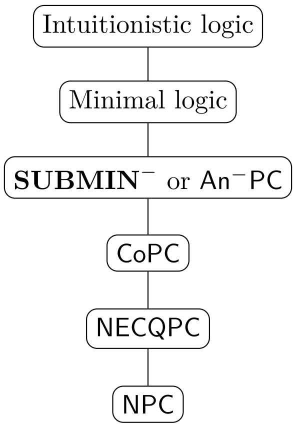


Figure 3: Systems discussed by Colacito, Bezhanishvili and de Jongh, after Niki's clarification

The remainder of this paper is structured as follows. In Section 2, we introduce proof systems for subsystems of intuitionistic logic and establish some results related to the first aim. In Section 3 we define algebraic semantics for the systems defined in Section 2, as a preparation for the main result. In Section 4, we show the existence of continua of logics between the logics discussed in this paper, and this will be related to the second aim. In Section 5, we conclude the paper by summarizing the main findings of the paper and pointing out some directions for further research. In the appendix, we prove the soundness and completeness theorems for the main logics of this paper using the Kripke semantics introduced by Vakarelov in [13].

2. Proof system

We shall use the language $\mathcal{L}_\neg$ consisting of denumerable propositional variables, $\wedge, \vee, \to$ and $\neg$. The set of propositional variables is denoted by $\mathcal{P}$. We define formulas of $\mathcal{L}_\neg$ as follows: $A \coloneqq p \mid \neg A \mid (A \wedge A) \mid (A \vee A) \mid (A \to A)$, and employ the abbreviation $A \leftrightarrow B := (A \to B) \wedge (B \to A)$. The formula $A[p_1/B_1, \ldots, p_n/B_n]$ is obtained by replacing all occurrences of p_i in A with B_i for each $i = 1, \ldots, n$ and leaving all other variables fixed. We call this operation *substitution*.

Definition 2.1 (NPC [3]). We define the Hilbert system NPC by adding a negative axioms to the positive axiom and rules of intuitionistic/minimal logic.

Axioms

- Ax1: $p \to (q \to p)$;
- Ax2: $(p \to (q \to r)) \to ((p \to q) \to (p \to r))$;
- Ax3: $p \to (p \vee q)$;
- Ax4: $q \to (p \vee q)$;
- Ax5: $(p \to r) \to [(q \to r) \to ((p \vee q) \to r)]$;
- Ax6: $(p \wedge q) \to p$; Ax7 : $(p \wedge q) \to q$;
- Ax8: $p \to (q \to (p \wedge q))$;
- (N): $(p \leftrightarrow q) \to (\neg p \leftrightarrow \neg q)$.

Inference rules

- (MP): If A and $A \to B$, then B;
- (Sub): If A, then $A[p_1/B_1, \ldots, p_n/B_n]$, where $p_1, \ldots, p_n$ are propositional variables in A and $B_1, \ldots, B_n$ are formulas.

For a formula A, a sequence $A_1, \ldots, A_n$ is a *proof* of A in NPC, if A_i satisfies one of the following for any $1 \leq i \leq n$:

1. A_i is an axiom;
2. A_i is the result of applying (MP) to formulas A_j and A_k for some $j, k < i$;
3. A_i is the result of applying (Sub) to a formula A_j for some $j < i$;
4. $A_n = A$.

$\mathsf{NPC} \vdash A$ denotes that there is a *proof* of A in NPC. Unless there is a risk of misunderstanding, the set $\{A \mid \mathsf{NPC} \vdash A\}$ is also denoted by NPC.

For a set of formulas Γ, a formula A, a sequence $A_1, \dots, A_n$ is a *deduction* of A from Γ in NPC, if A_i satisfies one of the following for any $1 \leq i \leq n$:

1. A_i is in NPC or in Γ;
2. A_i is the result of applying (MP) to formulas A_j and A_k for some $j, k < i$;
3. $A_n = A$.

$\Gamma \vdash_{\mathsf{NPC}} A$ denotes there is a deduction of A from Γ in NPC.

In considering deductions with hypothesis, (Sub) is not allowed. If (Sub) were allowed, $p \vdash q$ would be derived for any p and q.

We consider the following axioms related to negation.

Definition 2.2 (Additional Axioms).

- $(\mathsf{An}): \ (p \to \neg p) \to \neg p$;
- $(\mathsf{An}^-): \ (p \to \neg p) \to (\neg q \to \neg p)$;
- $(\mathsf{Co}): \ (p \to q) \to (\neg q \to \neg p)$;
- $(\mathsf{NECQ}): \ (p \wedge \neg p) \to \neg q$;
- $(\mathsf{ECQ}): \ (p \wedge \neg p) \to q$;
- $(\mathsf{AVQ}): \ \neg\neg(\neg(p \to p) \to q)$;
- $(\mathsf{An} \cap \mathsf{ECQ}): \ \neg\neg(p \to p) \vee (\neg(q \to q) \to r)$;
- $(\mathsf{CoECQ}): \ \neg p \to \neg(q \wedge \neg q)$.

By adding the above axioms, we obtain the following Hilbert systems.

- NECQPC is the Hilbert system adding (NECQ) to NPC. NECQ is an abbreviation of *negative ex contradiction quodibet*.
- CoPC is the Hilbert system adding (Co) to NPC (in this case, (N) is redundant [3]). Co is an abbreviation of *contraposition*.
- $\mathsf{An}^-\mathsf{PC}$ is the Hilbert system adding (An^-) to NPC.
- $\mathsf{CoECQPC}$ is the Hilbert system adding (CoECQ) to NPC. CoECQ is an abbreviation of *contraposition of ex contradiction quodibet*.
- $\mathsf{An} \cap \mathsf{ECQPC}$ is the Hilbert system adding ($\mathsf{An} \cap \mathsf{ECQ}$) to $\mathsf{An}^-\mathsf{PC}$.

- $\mathsf{MPC}_\neg$ is the Hilbert system adding (An) to NPC. An is an abbreviation for *absorption of negation*.
- ECQPC is the Hilbert system adding (ECQ) to NPC. ECQ is an abbreviation of *ex contradiction quodibet*.
- AVQPC is the Hilbert system adding (AVQ) and (An) to NPC. AVQ is an abbreviation of *avoidability of quodibet*.
- IPC is the Hilbert system adding (ECQ) and (An) to NPC. (In this case, (N) is redundant, see Lemma 2.5.)

Note that the system AVQPC was introduced in [14].[1] For each Hilbert system, the deducibility relation "$\Gamma \vdash A$" is defined as in the NPC case.

In [3, 8] the systems $\mathsf{NeF}, \mathsf{CoPC}, \mathsf{An}^-\mathsf{PC}$ and $\mathsf{MPC}_\neg$ are defined in the language with $\top$. The systems $\mathsf{NECQPC}, \mathsf{CoPC}, \mathsf{An}^-\mathsf{PC}$ and $\mathsf{MPC}_\neg$ we defined above are the $\top$-free fragments of them, respectively. This can be proved in the same way as **CO-MIN** and $\mathbf{CO\text{-}MIN}^{-\top,\bot}$ in Appendix.

Unless there is a risk of misinterpretation, for any Hilbert system H above, we denote the set $\{A \mid \mathsf{H} \vdash A\}$ by H in the same way as NPC.

For each Hilbert system H defined above, the set $\{A \mid \mathsf{H} \vdash A\}$ can be summed up in one concept. Because we use it later, we define it as follows.

Definition 2.3. A super-N-logic (sN-logic, for short) in the language $\mathcal{L}_\neg$ is any set L of $\mathcal{L}_\neg$-formulas satisfying the conditions:

- $\mathsf{NPC} \subseteq L$;
- L is closed under modus ponens;
- L is closed under substitution.

As mentioned, for each Hilbert system H defined above, the set $\{A \mid \mathsf{H} \vdash A\}$ is an sN-logic.

Lemma 2.4 (Deduction theorem). *Let $\Gamma \cup \{A, B\}$ be a set of formulas. The following holds:*

$$\Gamma \cup \{A\} \vdash_{\mathsf{NPC}} B \iff \Gamma \vdash_{\mathsf{NPC}} A \to B.$$

[1] This formulation was given by [7].

To prove Deduction theorem, we need only $\mathsf{Ax1}$ and $\mathsf{Ax2}$ and the fact that (MP) is the unique inference rule in deductions. So, it holds for all Hilbert systems defined above.

We will use Deduction theorem without a mentioning in what follows.

LEMMA 2.5. *Let* $\mathsf{IPC}^{-(\mathsf{N})}$ *be the Hilbert system removing* (N) *from* IPC. $\mathsf{IPC}^{-(\mathsf{N})} \vdash (\mathsf{N})$.

PROOF: We can show $\mathsf{IPC}^{-(\mathsf{N})} \vdash (\neg p \wedge q \wedge (p \leftrightarrow q)) \to (p \wedge \neg p)$ as follows: We can prove $\neg p, q$ and $p \leftrightarrow q$ from $\neg p \wedge q \wedge (p \leftrightarrow q)$ by using $\mathsf{Ax6}, 7$. Since we can show p from q and $p \leftrightarrow q$ by (MP), we can show $p \wedge \neg p$ from $\neg p \wedge q \wedge (p \leftrightarrow q)$ by $\mathsf{Ax8}$. Then we have the following:

1. $\mathsf{IPC}^{-(\mathsf{N})} \vdash (\neg p \wedge q \wedge (p \leftrightarrow q)) \to (p \wedge \neg p)$
2. $\mathsf{IPC}^{-(\mathsf{N})} \vdash (\neg p \wedge q \wedge (p \leftrightarrow q)) \to \neg q$ ((ECQ) and 1)
3. $\mathsf{IPC}^{-(\mathsf{N})} \vdash (\neg p \wedge (p \leftrightarrow q)) \to (q \to \neg q)$ (From 2)
4. $\mathsf{IPC}^{-(\mathsf{N})} \vdash (\neg p \wedge (p \leftrightarrow q)) \to \neg q$ ((An) and 3)
5. $\mathsf{IPC}^{-(\mathsf{N})} \vdash (p \leftrightarrow q) \to (\neg p \to \neg q)$ (From 4)
6. $\mathsf{IPC}^{-(\mathsf{N})} \vdash (p \leftrightarrow q) \to (\neg q \to \neg p)$ (The same way as 1-5)
7. $\mathsf{IPC}^{-(\mathsf{N})} \vdash (p \leftrightarrow q) \to (\neg p \leftrightarrow \neg q)$ ($\mathsf{Ax8}, 5$ and 6). □

LEMMA 2.6. $\mathsf{NPC} \subseteq \mathsf{NECQPC} \subseteq \mathsf{CoPC} \subseteq \mathsf{An}^{-}\mathsf{PC} \subseteq \mathsf{MPC}_{\neg}$.

PROOF: By combining the results of [3, Page 12] and [8, Page 970], we have $\mathsf{NPC} \subseteq \mathsf{NeFPC} \subseteq \mathsf{CoPC} \subseteq \mathsf{An}^{-}\mathsf{PC} \subseteq \mathsf{MPC}_{\neg}$. In their proofs, the axiom for $\top$ is not used. Since $\mathsf{NPC}, \mathsf{NECQPC}, \mathsf{CoPC}, \mathsf{An}^{-}\mathsf{PC}, \mathsf{MPC}_{\neg}$ are $\top$-free fragment of them. □

Hereinafter Lemma 2.6 is used without reference.

We shall show that *co-minimal logic* (**CO-MIN** for short (cf. [13])) is a conservative extension of ECQPC.

To define **CO-MIN**, we shall use the language $\mathcal{L}_{\neg,\top,\bot}$ consisting of $\mathcal{L}_{\neg}$, $\top$, $\bot$. We define formulas of $\mathcal{L}_{\neg,\top,\bot}$ as follows: $A := p \mid \top \mid \bot \mid \neg A \mid (A \wedge A) \mid (A \vee A) \mid (A \to A)$.

Definition 2.7 (cf. [13]). **CO-MIN** is the Hilbert system obtained by adding the axioms $\bot \to p$, $p \to \top$, $\neg p \to \neg\neg\top$ and $\neg\top \to p$ to CoPC in $\mathcal{L}_{\neg,\top,\bot}$.

We define $\mathbf{CO\text{-}MIN}^{-\top,\bot}$ as the Hilbert system adding the axioms $\neg p \to \neg\neg(q \to q)$ and $\neg(p \to p) \to q$ to CoPC in $\mathcal{L}_{\neg}$.

The following lemma shows that **CO-MIN** is a conservative extension for $\mathbf{CO\text{-}MIN}^{-\top,\bot}$.

Lemma 2.8. *Given a formula A in the language of $\mathcal{L}_{\neg}$,*

$$\mathbf{CO\text{-}MIN} \vdash A \;\Leftrightarrow\; \mathbf{CO\text{-}MIN}^{-\top,\bot} \vdash A.$$

Proof: See Appendix. □

Next, we show the equivalence between $\mathbf{CO\text{-}MIN}^{-\top,\bot}$ and ECQPC.

Lemma 2.9. ECQPC *contains* $\mathsf{An}^{-}\mathsf{PC}$*.*

Proof: It suffices to show that $\mathsf{ECQPC} \vdash (p \to \neg p) \to (\neg q \to \neg p)$.

We can show $\mathsf{ECQPC} \vdash ((p \to \neg p) \wedge \neg q) \to (q \leftrightarrow p)$ as follows:

We can show $p \to \neg p$ and $\neg q$ from $(p \to \neg p) \wedge \neg q$ by $\mathsf{Ax6, 7}$. Assume q, then we can show p from $\neg q$ by (ECQ). Assume p, then we can show $\neg p$ from $p \to \neg p$, and so q by (ECQ) again.

Then we have the following:

1. $\mathsf{ECQPC} \vdash ((p \to \neg p) \wedge \neg q) \to (q \leftrightarrow p)$
2. $\mathsf{ECQPC} \vdash ((p \to \neg p) \wedge \neg q) \to (\neg q \to \neg p)$ $\quad$ ((N) and 1)
3. $\mathsf{ECQPC} \vdash (p \to \neg p) \to (\neg q \to \neg p)$ $\quad$ (From 2). □

Lemma 2.10. ECQPC *is equivalent to* $\mathbf{CO\text{-}MIN}^{-\top,\bot}$*.*

Proof: Since $p \to p$ is derivable, $\neg(p \to p) \to q$ is equivalent to the instance $((p \to p) \wedge \neg(p \to p)) \to q$ of ECQ. Then it suffices to show that $\mathsf{ECQPC} \vdash \neg p \to \neg\neg(q \to q)$.

1. $\mathsf{ECQPC} \vdash \neg(q \to q) \to \neg\neg(q \to q)$
2. $\mathsf{ECQPC} \vdash (\neg(q \to q) \to \neg\neg(q \to q)) \to (\neg p \to \neg\neg(q \to q))$ (An^-)
3. $\mathsf{ECQPC} \vdash \neg p \to \neg\neg(q \to q)$ (1 and 2).

For the other direction, it suffices to show $\mathbf{CO\text{-}MIN}^{-\top,\perp} \vdash (p \wedge \neg p) \to q$.

1. $\mathbf{CO\text{-}MIN}^{-\top,\perp} \vdash (p \wedge \neg p) \to (p \leftrightarrow (p \to p))$
2. $\mathbf{CO\text{-}MIN}^{-\top,\perp} \vdash (p \wedge \neg p) \to (\neg p \to \neg(p \to p))$ ((N) and 1)
3. $\mathbf{CO\text{-}MIN}^{-\top,\perp} \vdash (p \wedge \neg p) \to \neg(p \to p)$ (From 2)
4. $\mathbf{CO\text{-}MIN}^{-\top,\perp} \vdash \neg(p \to p) \to q$ (Ax of $\mathbf{CO\text{-}MIN}^{-\top,\perp}$)
5. $\mathbf{CO\text{-}MIN}^{-\top,\perp} \vdash (p \wedge \neg p) \to q$ (3 and 4). □

In order to show $\mathsf{An} \cap \mathsf{ECQPC}$ is the intersection of ECQPC and $\mathsf{MPC}_\neg$, we show the following lemma.

LEMMA 2.11. *The Hilbert system* $\mathsf{An}^-\mathsf{PC}^+$ *consisting of* $\mathsf{An}^-\mathsf{PC}$ *plus the axiom* $\neg\neg(p \to p)$ *is equivalent to* $\mathsf{MPC}_\neg$.

PROOF: $\mathsf{An}^-\mathsf{PC}^+ \vdash (p \to \neg p) \to \neg p$ can be shown as follows.

1. $\mathsf{An}^-\mathsf{PC}^+ \vdash (p \to \neg p) \to (\neg\neg(p \to p) \to \neg p)$ (An^-)
2. $\mathsf{An}^-\mathsf{PC}^+ \vdash \neg\neg(p \to p) \to ((p \to \neg p) \to \neg p)$ (From 1)
3.; $\mathsf{An}^-\mathsf{PC}^+ \vdash (p \to \neg p) \to \neg p$ (2 and Ax of $\mathsf{An}^-\mathsf{PC}^+$).

For the converse, it suffices to show $\mathsf{MPC}_\neg \vdash \neg\neg(p \to p)$, since $\mathsf{MPC}_\neg \vdash (\mathsf{An}^-)$.

1. $\mathsf{MPC}_\neg \vdash \neg(p \to p) \to \neg\neg(p \to p)$ (NECQ)
2. $\mathsf{MPC}_\neg \vdash (\neg(p \to p) \to \neg\neg(p \to p)) \to \neg\neg(p \to p)$ (An)
3. $\mathsf{MPC}_\neg \vdash \neg\neg(p \to p)$ (1 and 2). □

LEMMA 2.12. $\mathsf{An} \cap \mathsf{ECQPC}$ *is the intersection of* ECQPC *and* $\mathsf{MPC}_\neg$.

Proof: By Lemma 2.9, to prove $\mathsf{An} \cap \mathsf{ECQPC} \subseteq \mathsf{ECQPC}$ and $\mathsf{An} \cap \mathsf{ECQPC} \subseteq \mathsf{MPC}_{\neg}$, it suffices to show that $\mathsf{ECQPC} \vdash \neg\neg(p \to p) \vee (\neg(q \to q) \to r)$ and $\mathsf{MPC}_{\neg} \vdash \neg\neg(p \to p) \vee (\neg(q \to q) \to r)$, but it is immediate from Lemma 2.10 and Lemma 2.11.

For the other direction, note that $\mathsf{ECQPC} = \mathbf{CO\text{-}MIN}^{-\top,\bot}$ and $\mathsf{MPC}_{\neg} = \mathsf{An}^{-}\mathsf{PC}^{+}$, so $\mathsf{An}^{-}\mathsf{PC}$ is contained in the intersection of ECQPC and $\mathsf{MPC}_{\neg}$. Suppose that A is in the intersection of ECQPC and $\mathsf{MPC}_{\neg}$. Since $\mathsf{MPC}_{\neg} \vdash A$, $\mathsf{An}^{-}\mathsf{PC} \vdash B \to A$, where B is a conjunction of substitution instances of $\neg\neg(p \to p)$. Since $\mathsf{ECQPC} \vdash A$, $\mathsf{An}^{-}\mathsf{PC} \vdash C \to A$, where C is a conjunction of substitution instances of $\neg(q \to q) \to r$. By $\mathsf{Ax5}$, $\mathsf{An}^{-}\mathsf{PC} \vdash (B \vee C) \to A$. Since $B \vee C$ is equivalent to a conjunction of instances of $\mathsf{An} \cap\ \mathsf{ECQ}$, we obtain $\mathsf{An} \cap \mathsf{ECQPC} \vdash A$. $\square$

It will be seen below that **SUBMIN** can be further formalized differently.

Lemma 2.13. *$\mathsf{CoECQPC}$ is equivalent to $\mathsf{An}^{-}\mathsf{PC}$.*

Proof: We show that $\mathsf{CoECQPC} \vdash (p \to \neg p) \to (\neg q \to \neg p)$.

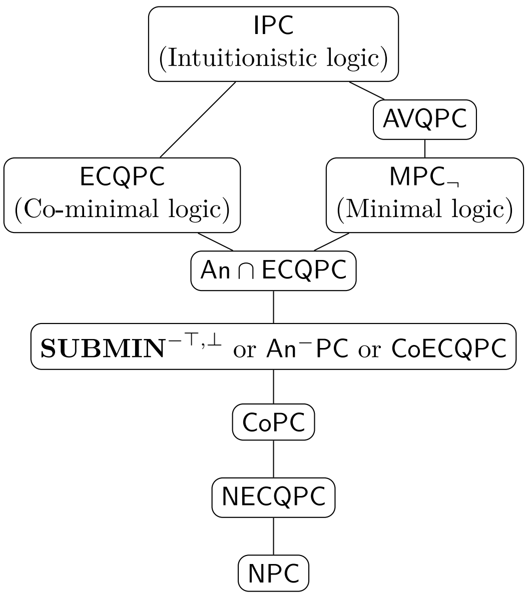
We can infer $\mathsf{CoECQPC} \vdash (p \to \neg p) \to ((p \wedge \neg p) \leftrightarrow p)$ by using $\mathsf{Ax6}, \mathsf{7}$.

1. $\mathsf{CoECQPC} \vdash (p \to \neg p) \to ((p \wedge \neg p) \leftrightarrow p)$
2. $\mathsf{CoECQPC} \vdash (p \to \neg p) \to (\neg(p \wedge \neg p) \leftrightarrow \neg p)$ ((N) and 1)
3. $\mathsf{CoECQPC} \vdash \neg(p \wedge \neg p) \to ((p \to \neg p) \to \neg p)$ (From 2)
4. $\mathsf{CoECQPC} \vdash \neg q \to \neg(p \wedge \neg p)$ ((CoECQ))
5. $\mathsf{CoECQPC} \vdash \neg q \to ((p \to \neg p) \to \neg p)$ (3 and 4)
6. $\mathsf{CoECQPC} \vdash (p \to \neg p) \to (\neg q \to \neg p)$ (From 5)

For the other direction, we show that $\mathsf{An}^{-}\mathsf{PC} \vdash \neg p \to \neg(q \wedge \neg q)$.

1. $\mathsf{An}^-\mathsf{PC} \vdash \neg q \to \neg(q \wedge \neg q)$ ($\mathsf{Ax7}$ and (Co))
2. $\mathsf{An}^-\mathsf{PC} \vdash (q \wedge \neg q) \to \neg q$ ($\mathsf{Ax7}$)
3. $\mathsf{An}^-\mathsf{PC} \vdash (q \wedge \neg q) \to \neg(q \wedge \neg q)$ (1 and 2)
4. $\mathsf{An}^-\mathsf{PC} \vdash ((q \wedge \neg q) \to \neg(q \wedge \neg q)) \to (\neg p \to \neg(q \wedge \neg q))$ ((An^-))
5. $\mathsf{An}^-\mathsf{PC} \vdash \neg p \to \neg(q \wedge \neg q)$ (3 and 4) $\square$

The results concerning the relations between the systems so far can be summarized as follows.



See the Appendix for **SUBMIN**$^{-\top,\bot}$. The systems other than $\mathsf{An} \cap \mathsf{ECQPC}$ were defined using different languages and the relations were already known. The relation of $\mathsf{An} \cap \mathsf{ECQPC}$ to other systems is a new result of this paper, but at this point, it is not yet clear whether it can be separated from other systems. In Section 4, using algebraic semantics, we separate $\mathsf{An} \cap \mathsf{ECQPC}$ from the others. Before presenting the main results, we will first prepare some tools that we will use in the proofs of the main results.

3. Algebraic semantics

We now turn to introduce algebraic semantics for logics in the previous section.

DEFINITION 3.1 (N-algebra (cf. [3])). Let $\langle |\mathfrak{A}|, \wedge_{\mathfrak{A}}, \vee_{\mathfrak{A}} \rangle$ be a lattice with the greatest element $1_{\mathfrak{A}}$. The order $\leq_{\mathfrak{A}}$ in the lattice is defined by

$$a \leq_{\mathfrak{A}} b :\Leftrightarrow a \wedge_{\mathfrak{A}} b = a$$

for any $a, b \in |\mathfrak{A}|$. The N-algebra $\langle |\mathfrak{A}|, 1_{\mathfrak{A}}, \wedge_{\mathfrak{A}}, \vee_{\mathfrak{A}}, \to_{\mathfrak{A}}, \neg_{\mathfrak{A}} \rangle$ is given by defining a binary operator $\to_{\mathfrak{A}}$ and an unary operator $\neg_{\mathfrak{A}}$ over $|\mathfrak{A}|$ as follows:

$$a \to_{\mathfrak{A}} b := \max\{c \in |\mathfrak{A}| \mid a \wedge_{\mathfrak{A}} c \leq_{\mathfrak{A}} b\};$$

$$(a \leftrightarrow_{\mathfrak{A}} b) \to_{\mathfrak{A}} (\neg_{\mathfrak{A}} a \leftrightarrow_{\mathfrak{A}} \neg_{\mathfrak{A}} b) = 1_{\mathfrak{A}},$$

where $a \leftrightarrow_{\mathfrak{A}} b$ is an abbreviation of $(a \to_{\mathfrak{A}} b) \wedge_{\mathfrak{A}} (b \to_{\mathfrak{A}} a)$.

For any N-algebra $\mathfrak{A}$, the following holds in the same manner as Heyting algebras:

$$a \leq_{\mathfrak{A}} b \Longleftrightarrow a \to_{\mathfrak{A}} b = 1_{\mathfrak{A}}.$$

This relation will be used without notice in what follows.

We now define the following algebraic conditions that correspond to the axioms.

DEFINITION 3.2 (Additional conditions for negation). The following conditions corresponding to the additional inference rules in Definition 2.2 are defined as follows:

1. $(\mathsf{An})^E$: $(x \to \neg x) \to \neg x = 1$;

2. $(\mathsf{An}^-)^E$: $(x \to \neg x) \to (\neg y \to \neg x) = 1$;
3. $(\mathsf{Co})^E$: $(x \to y) \to (\neg y \to \neg x) = 1$;
4. $(\mathsf{NECQ})^E$: $(x \wedge \neg x) \to \neg y = 1$;
5. $(\mathsf{ECQ})^E$: $(x \wedge \neg x) \to y = 1$;
6. $(\mathsf{AVQ})^E$: $\neg\neg(\neg(x \to x) \to y) = 1$;
7. $(\mathsf{An} \cap \mathsf{ECQ})^E$: $\neg\neg(x \to x) \vee (\neg(y \to y) \to z) = 1$;
8. $(\mathsf{CoECQ})^E$: $\neg x \to \neg(y \wedge \neg y) = 1$.

These conditions are also denoted without E unless there is a risk of misunderstanding.

In what follows, we fix $\Psi = \{\wedge, \vee, \to\}$. Recall that $\mathcal{P}$ is the set of propositional variables.

Definition 3.3 (Valuation of N-algebra). For any N-algebra and any map v from $\mathcal{P}$ to $|\mathfrak{A}|$, the *valuation* $\overline{v}$ in $\mathfrak{A}$ is defined as follows:

1. If A is a propositional variable, then $\overline{v}(A) := v(A)$;
2. If $A = \neg B$, then $\overline{v}(\neg B) := \neg_{\mathfrak{A}}\overline{v}(B)$;
3. If $A = B \otimes C$, then $\overline{v}(B \otimes C) := \overline{v}(B) \otimes_{\mathfrak{A}} \overline{v}(C)$, for $\otimes \in \Psi$.

Since $\overline{v}$ is uniquely determined for any v, by this definition the valuation $\overline{v}$ is also written as v unless there is a risk of misunderstanding.

For given a valuation v and any formula A, if $v(A) = 1_{\mathfrak{A}}$, then we say that A is *true in* v. If A is true in v for any valuation v in $\mathfrak{A}$, then we say that A is *true in* $\mathfrak{A}$, which is denoted by $\mathfrak{A} \models A$. For a class $\mathcal{C}$ of N-algebras, if $\mathfrak{A} \models A$ for any $\mathfrak{A}$ in $\mathcal{C}$, then we say that A is *true in* $\mathcal{C}$. For any set Δ of formulas, any formula A and any class $\mathcal{C}$ of N-algebras, if, for any N-algebra $\mathfrak{A} \in \mathcal{C}$ and valuation v in $\mathfrak{A}$, A is true in v whenever B is true in v for any $B \in \Delta$, then we say A is true under Δ in $\mathcal{C}$ which is denoted by $\Delta \models_{\mathcal{C}} A$. If Δ is empty, it is denoted by $\models_{\mathcal{C}} A$.

We can prove the completeness theorem for the logics defined above and these N-algebras. In particular, here we show the completeness theorem regarding ECQPC.

THEOREM 3.4. *Let $\mathcal{C}_{\mathsf{ECQ}}$ be the class of N-algebras that validate ECQ. For any formula A, the following holds:*

$$\mathsf{ECQPC} \vdash A \;\Leftrightarrow\; \models_{\mathcal{C}_{\mathsf{ECQ}}} A.$$

PROOF: With regard to the forward implication, it can be proved by induction on the length of the deduction. For the other direction, it can be proved in the same manner as Heyting algebras (cf. [2, page 195]). □

Thus, completeness theorems can be proved in the same manner for the remaining logics. In more general, we can show the following.

THEOREM 3.5 (Completeness theorem). *For sN-logic L, let $\mathcal{C}_L$ be the class of N-algebras in which all theorems of L are true. For any formula A, the following holds:*

$$L \vdash A \;\Leftrightarrow\; \models_{\mathcal{C}_L} A.$$

PROOF: We can show this theorem in the same way as [3, page 51] by replacing $\top$ with $p \to p$. □

4. The existence of continua of logics between pairs of logics below intuitionistic logic

In this section, we show the main theorem: there are continua of logics between logics introduced in Section 2. In order to show it, we introduce the following definitions and lemmas. Most of them are introduced by [15]. The Heyting algebras are defined as usual.

4.1. Preliminaries

DEFINITION 4.1 (Second greatest element (cf. [15])). Given a N-algebra $\mathfrak{A}$, we say that $\mathfrak{A}$ has *the second greatest element* if the greatest element exists in $|\mathfrak{A}| \setminus \{1_{\mathfrak{A}}\}$. We write the second greatest element in $\mathfrak{A}$ as $\star_{\mathfrak{A}}$ if it exists.

The role of the second greatest element will be explained when we introduce Yankov formulas.

Recall that $\Psi = \{\wedge, \vee, \to\}$.

Definition 4.2 (Ψ-reduct, Ψ-embedding (cf. [15])). For any Heyting algebra $\mathfrak{A}$, we define the following:

1. The algebra $\langle |\mathfrak{A}|, 1_{\mathfrak{A}}, 0_{\mathfrak{A}}, \langle \otimes_{\mathfrak{A}} \mid \otimes \in \Psi \rangle \rangle$ is the Ψ-*reduct*.
2. For a Heyting algebra $\mathfrak{B}$, a Ψ-*embedding* from $\mathfrak{A}$ to $\mathfrak{B}$ is an embedding from the Ψ-reduct of $\mathfrak{A}$ to the Ψ-reduct of $\mathfrak{B}$.

The following is a generalization of Yankov formulas (cf. [16]).

Definition 4.3 (Yankov formula of N-algebra). In what follows, for any function g and $a \in dom(g)$, g_a denotes the value of g at a. For any finite N-algebra $\mathfrak{A}$ with the second greatest element, any injection g from $|\mathfrak{A}|$ to $\mathcal{P}$, and any formula A, the *Yankov formula* $\chi^g_A(\mathfrak{A})$ with $\mathfrak{A}$, g and A is defined as follows:

$\chi^g_A(\mathfrak{A}) := \bigwedge\{(g_a \otimes g_b) \to g_{a \otimes_{\mathfrak{A}} b} \mid \otimes \in \Psi, a, b \in |\mathfrak{A}|\} \cup \{g_{a \otimes_{\mathfrak{A}} b} \to (g_a \otimes g_b) \mid \otimes \in \Psi, a, b \in |\mathfrak{A}|\} \to (g_{\star_{\mathfrak{A}}} \vee A)$.

The antecedent of $\chi^g_A(\mathfrak{A})$ is denoted by $\theta(\chi^g_A(\mathfrak{A}))$.

Yankov formula will be used when we separate logics and make continua of logics. For any finite N-algebra $\mathfrak{A}$ with the second greatest element, the antecedent of Yankov formula asserts that algebraic operators except for negation are rewritten by corresponding logical connectives. The succedent of Yankov formula is defined by using the second greatest element. In a valuation v making the antecedent of Yankov formula true, $v(g_{\star_{\mathfrak{A}}})$ is interpreted as the second greatest element in the model. If we choose N-algebra appropriately, the succedent of Yankov formula is interpreted as the second greatest element, then the N-algebra refutes the Yankov formula.

The difference from the conventional definition of the Yankov formula is in the succedent of the formula: We added the formula A. In order to separate logics, we take an instance of the axioms concerning negation as A in the proof of the main theorem.

In the following, we will prove some lemmata to prove the main theorem.

An N-algebra with the minimum element can be reformed into a Heyting algebra by modifying the negation.

Definition 4.4 (Heyting algebraization of N-algebra). For any N-algebra $\mathfrak{A} = \langle |\mathfrak{A}|, 1_{\mathfrak{A}}, \wedge_{\mathfrak{A}}, \vee_{\mathfrak{A}}, \rightarrow_{\mathfrak{A}}, \neg_{\mathfrak{A}} \rangle$ with the minimum element $0_{\mathfrak{A}}$, the Heyting algebra $\mathfrak{A}^H = \langle |\mathfrak{A}|, 1_{\mathfrak{A}^H}, 0_{\mathfrak{A}^H}, \wedge_{\mathfrak{A}^H}, \vee_{\mathfrak{A}^H}, \rightarrow_{\mathfrak{A}^H}, \neg_{\mathfrak{A}^H} \rangle$ is defined as follows:

$|\mathfrak{A}^H| := |\mathfrak{A}|, 1_{\mathfrak{A}^H} := 1_{\mathfrak{A}}, 0_{\mathfrak{A}^H} := 0_{\mathfrak{A}}, \wedge_{\mathfrak{A}^H} := \wedge_{\mathfrak{A}}, \vee_{\mathfrak{A}^H} := \vee_{\mathfrak{A}}, \rightarrow_{\mathfrak{A}^H} := \rightarrow_{\mathfrak{A}}$, and

$$\neg_{\mathfrak{A}^H} a := a \rightarrow_{\mathfrak{A}} 0_{\mathfrak{A}}.$$

Definition 4.5 (Filter). For any lattice H, a *filter* F of H is a nonempty subset of H satisfying the following 1 and 2:

1. If $a, b \in F$, then $a \wedge_H b \in F$;
2. If $a \in F$ and $a \leq_H b$, then $b \in F$.

Given $a \in H$, the filter $\{b \in H \mid a \leq b\}$ is the smallest filter containing a. We refer to this as the *filter generated by* a.

Proposition 4.6. For any Heyting algebra $\mathfrak{A}$ and any subset F of $|\mathfrak{A}|$, F is a filter of $\mathfrak{A}$ if and only if F satisfies both of the following 1 and 2:

1. $1_{\mathfrak{A}} \in F$;
2. If $a \in F$ and $a \rightarrow_{\mathfrak{A}} b \in F$, then $b \in F$.

Definition 4.7 (Quotient algebra of Heyting algebra). For any Heyting algebra $\mathfrak{A}$ and any filter F of $\mathfrak{A}$, the binary relation $\sim_F$ over $\mathfrak{A}$ is defined as follows:

$$a \sim_F b :\Longleftrightarrow a \rightarrow_{\mathfrak{A}} b \in F \text{ and } b \rightarrow_{\mathfrak{A}} a \in F \text{ for any } a, b \in |\mathfrak{A}|.$$

It is easy to show that this binary relation $\sim_F$ is a congruence relation over $\mathfrak{A}$. We define the congruence class of $a \in |\mathfrak{A}|$ with respect to $\sim_F$ by $[a]_F := \{b \in |\mathfrak{A}| \mid a \sim_F b\}$. The set of congruence classes $\{[a]_F \mid a \in |\mathfrak{A}|\}$ is denoted by $|\mathfrak{A}|/F$.

The *quotient algebra*
$\mathfrak{A}/F = \langle |\mathfrak{A}|/F, 1_{\mathfrak{A}/F}, 0_{\mathfrak{A}/F}, \wedge_{\mathfrak{A}/F}, \vee_{\mathfrak{A}/F}, \rightarrow_{\mathfrak{A}/F}, \neg_{\mathfrak{A}/F} \rangle$ is defined as follows: for any $[a]_F, [b]_F \in |\mathfrak{A}|/F$

1. $1_{\mathfrak{A}/F} := [1_{\mathfrak{A}}]_F$;
2. $0_{\mathfrak{A}/F} := [0_{\mathfrak{A}}]_F$;

3. $[a]_F \wedge_{\mathfrak{A}/F} [b]_F \coloneqq [a \wedge_{\mathfrak{A}} b]_F$;
4. $[a]_F \vee_{\mathfrak{A}/F} [b]_F \coloneqq [a \vee_{\mathfrak{A}} b]_F$;
5. $[a]_F \to_{\mathfrak{A}/F} [b]_F \coloneqq [a \to_{\mathfrak{A}} b]_F$;
6. $\neg_{\mathfrak{A}/F}[a]_F \coloneqq [\neg_{\mathfrak{A}} a]_F$.

It is easy to show that $\mathfrak{A}/F$ is a Heyting algebra.

The following is a generalization of Lemma 2 in [15] to Yankov formula of N-algebra.

LEMMA 4.8. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be finite N-algebras with the second greatest elements such that, for each filter F of $\mathfrak{B}^H$, there is no Ψ-embedding from $\mathfrak{A}^H$ to $\mathfrak{B}^H/F$. Then, for any formula A and any injection $g : |\mathfrak{A}| \to \mathcal{P}$, $\mathfrak{B} \models \chi^g_A(\mathfrak{A})$ holds.*

PROOF: Recall that any Heyting algebra is an N-algebra. We first show the statement for finite Heyting algebras $\mathfrak{C}$ and $\mathfrak{D}$ with the second greatest elements.

To show the contraposition, assume $\mathfrak{D} \not\models \chi^g_A(\mathfrak{C})$ for some $g : |\mathfrak{C}| \to \mathcal{P}$ and a formula A. Then there is a valuation v such that $v(\theta(\chi^g_A(\mathfrak{C}))) \not\leq_{\mathfrak{D}} v(g_{\star_{\mathfrak{C}}} \vee A)$, where $\theta(\chi^g_A(\mathfrak{C}))$ is the antecedent of $\chi^g_A(\mathfrak{C})$. Consider another Yankov formula $\chi^g_{g_{\star_{\mathfrak{C}}}}(\mathfrak{C})$ for a propositional variable $g_{\star_{\mathfrak{C}}}$. Since $v(\theta(\chi^g_{g_{\star_{\mathfrak{C}}}}(\mathfrak{C}))) = v(\theta(\chi^g_A(\mathfrak{C})))$ and $v(g_{\star_{\mathfrak{C}}}) \leq_{\mathfrak{D}} v(g_{\star_{\mathfrak{C}}} \vee A)$, $v(\theta(\chi^g_{g_{\star_{\mathfrak{C}}}}(\mathfrak{C}))) \leq_{\mathfrak{D}} v(g_{\star_{\mathfrak{C}}})$ implies a contradiction, namely $v(\theta(\chi^g_A(\mathfrak{C}))) \leq_{\mathfrak{D}} v(g_{\star_{\mathfrak{C}}} \vee A)$. Therefore $v(\theta(\chi^g_{g_{\star_{\mathfrak{C}}}}(\mathfrak{C}))) \not\leq_{\mathfrak{D}} v(g_{\star_{\mathfrak{C}}})$ and so $\mathfrak{D} \not\models \chi^g_{g_{\star_{\mathfrak{C}}}}(\mathfrak{C})$ holds.

Take the filter G of $\mathfrak{D}$ generated by $v(\theta(\chi^g_{g_{\star_{\mathfrak{C}}}}(\mathfrak{C})))$. From the construction of G, we have the following:

$$v(g_{\star_{\mathfrak{C}}}) \notin G; \tag{9}$$

$$[v(g_{c \otimes d})]_G = [v(g_c \otimes g_d)]_G \text{ for any } c, d \in |\mathfrak{C}| \text{ and } \otimes \in \Psi. \tag{10}$$

Let $v' : \mathfrak{C} \to \mathfrak{D}/G$ be the map defined by $v'(c) \coloneqq [v(g_c)]_G$ for any $c \in |\mathfrak{C}|$. We prove v' is a Ψ-embedding.

We see that each operators in Ψ is preserved by v'. For $\wedge$, we have the following equation for each $c, d \in |\mathfrak{C}|$.

$$\begin{aligned}
v'(c \wedge_{\mathfrak{C}} d) &= [v(g_{c \wedge_{\mathfrak{C}} d})]_G \\
&= [v(g_c \wedge g_d)]_G \\
&= [v(g_c) \wedge_{\mathfrak{D}} v(g_d)]_G \\
&= [v(g_c)]_G \wedge_{\mathfrak{D}/G} [v(g_b)]_G \\
&= v'(c) \wedge_{\mathfrak{D}/G} v'(d).
\end{aligned}$$

The cases $\vee$ and $\rightarrow$ can be shown in the same manner.

We can see that v' is an injection as follows.

Assume $v'(c) = v'(d)$. Then $[v(g_c)]_G = [v(g_d)]_G$ holds, and so for any $c, d \in |\mathfrak{C}|$,

$$\begin{aligned}
v(g_c) \leftrightarrow_{\mathfrak{D}} v(g_d) &= (v(g_c) \rightarrow_{\mathfrak{D}} v(g_d)) \wedge_{\mathfrak{D}} (v(g_d) \rightarrow_{\mathfrak{D}} v(g_c)) \\
&= v(g_c \rightarrow g_d) \wedge_{\mathfrak{D}} v(g_d \rightarrow g_c)
\end{aligned}$$

By $[v(g_c)]_G = [v(g_d)]_G$, we have $v(g_c \rightarrow g_d) \wedge_{\mathfrak{D}} v(g_d \rightarrow g_c) \in G$, and so $v(g_{c \rightarrow_{\mathfrak{C}} d}), v(g_{d \rightarrow_{\mathfrak{C}} c}) \in G$ from (10).

Furthermore, the equation $1_{\mathfrak{D}} \leftrightarrow_{\mathfrak{D}} v(g_{1_{\mathfrak{C}}}) = v(g_c \rightarrow g_c) \leftrightarrow_{\mathfrak{D}} v(g_{c \rightarrow_{\mathfrak{C}} c})$ holds. Since we have $v(g_c \rightarrow g_c) \leftrightarrow_{\mathfrak{D}} v(g_{c \rightarrow_{\mathfrak{C}} c}) \in G$, we obtain that $v(g_{1_{\mathfrak{C}}}) \in G$.

If we assume $c \neq d$, then $c \rightarrow_{\mathfrak{C}} d \neq 1_{\mathfrak{C}}$ or $d \rightarrow_{\mathfrak{C}} c \neq 1_{\mathfrak{C}}$ hold. Since $\mathfrak{C}$ has the second greatest element $\star_{\mathfrak{C}}$, we have $c \rightarrow_{\mathfrak{C}} d \leq_{\mathfrak{C}} \star_{\mathfrak{C}}$ or $d \rightarrow_{\mathfrak{C}} c \leq_{\mathfrak{C}} \star_{\mathfrak{C}}$. We assume $c \rightarrow_{\mathfrak{C}} d \leq_{\mathfrak{C}} \star_{\mathfrak{C}}$. Then the equation $v(g_{c \rightarrow_{\mathfrak{C}} d} \rightarrow g_{\star_{\mathfrak{C}}}) \leftrightarrow_{\mathfrak{D}} v(g_{(c \rightarrow_{\mathfrak{C}} d) \rightarrow_{\mathfrak{C}} \star_{\mathfrak{C}}}) = (v(g_{c \rightarrow_{\mathfrak{C}} d}) \rightarrow_{\mathfrak{D}} v(g_{\star_{\mathfrak{C}}})) \leftrightarrow_{\mathfrak{D}} v(g_{1_{\mathfrak{C}}})$ holds.

Since we have $v(g_{c \rightarrow_{\mathfrak{C}} d} \rightarrow g_{\star_{\mathfrak{C}}}) \leftrightarrow_{\mathfrak{D}} v(g_{(c \rightarrow_{\mathfrak{C}} d) \rightarrow_{\mathfrak{C}} \star_{\mathfrak{C}}}), v(g_{1_{\mathfrak{C}}}), v(g_{c \rightarrow_{\mathfrak{C}} d}) \in G$, we obtain $v(g_{\star_{\mathfrak{C}}}) \in G$. But this contradicts (9). If $d \rightarrow_{\mathfrak{C}} c \leq_{\mathfrak{C}} \star_{\mathfrak{C}}$ we can derive a contradiction in the same manner. Therefore we obtain $c = d$, and v' is an embedding. This completes a proof of the statements for Heyting algebras $\mathfrak{C}$ and $\mathfrak{D}$.

Next we show the statement for any finite N-algebra $\mathfrak{A}$ and $\mathfrak{B}$ with the second greatest elements. Assume that, for each filter F of $\mathfrak{B}^H$, there is no Ψ-embedding from $\mathfrak{A}^H$ to $\mathfrak{B}^H/F$. Then, from the above argument, $\mathfrak{B}^H \models \chi^g_{\star_{\mathfrak{A}}}(\mathfrak{A})$ holds for any injection $g : |\mathfrak{A}| \rightarrow \mathcal{P}$. Since $\chi^g_{\star_{\mathfrak{A}}}(\mathfrak{A})$ does not contain the negation $\neg$, it implies $\mathfrak{B} \models \chi^g_{\star_{\mathfrak{A}}}(\mathfrak{A})$. Then, for any valuation v, any formula A and any $g : |\mathfrak{A}| \to \mathcal{P}$, we have $v(\theta(\chi^g_A(\mathfrak{A}))) = v(\theta(\chi^g_{\star_{\mathfrak{A}}}(\mathfrak{A}))) \leq_{\mathfrak{B}} v(g_{\star_{\mathfrak{A}}}) \leq_{\mathfrak{B}} v(g_{\star_{\mathfrak{A}}} \vee A)$, and so $\mathfrak{B} \models \chi^g_A(\mathfrak{A})$. □

Definition 4.9. For any natural number n, subsets of natural numbers a_n, r_n and s_n are defined as follows:

$$a_n := \{i \mid i < n\}, r_n := a_n \cup \{n+1\}, s_n := r_n \cup \{n+3\}$$

Lemma 4.10. *For any Heyting algebras $\mathfrak{A}$, let $\underline{\mathfrak{A}}$ be the algebra obtained by adding a new minimum element $0_{\underline{\mathfrak{A}}}$ to $\mathfrak{A}$ and changing the definition of negation $\neg$ into $\neg_{\underline{\mathfrak{A}}} a := a \to_{\underline{\mathfrak{A}}} 0_{\underline{\mathfrak{A}}}$. Then $\underline{\mathfrak{A}}$ is a Heyting algebra.*

Proof: It is immediate that $\underline{\mathfrak{A}}$ is a lattice by the definition. For any $a \in |\underline{\mathfrak{A}}|$, $a \to_{\underline{\mathfrak{A}}} 0_{\underline{\mathfrak{A}}}$ and $0_{\underline{\mathfrak{A}}} \to_{\underline{\mathfrak{A}}} a$ exist as follows. If $a = 0_{\underline{\mathfrak{A}}}$, then both of them are equal to $1_{\underline{\mathfrak{A}}}$. If $a \neq 0_{\underline{\mathfrak{A}}}$, then $c \neq 0_{\underline{\mathfrak{A}}}$ implies $0_{\mathfrak{A}} \leq_{\underline{\mathfrak{A}}} a \wedge_{\underline{\mathfrak{A}}} c$ and so $a \wedge_{\underline{\mathfrak{A}}} c \not\leq 0_{\underline{\mathfrak{A}}}$. Therefore, $a \to_{\underline{\mathfrak{A}}} 0_{\underline{\mathfrak{A}}} = 0_{\underline{\mathfrak{A}}}$. The equation $0_{\underline{\mathfrak{A}}} \to_{\underline{\mathfrak{A}}} a = 1_{\underline{\mathfrak{A}}}$ is immediate from the definition of $\to$. □

In order to distinct logics, we construct countably many algebras in the following.

For any natural number n, let us put $|\mathfrak{A}_n| = \{a_0, \ldots, a_{n+8}\} \cup \{r_0, \ldots, r_{n+6}\} \cup \{s_0, s_{n+4}, \mathbb{N}\}$ and for any $x, y \in |\mathfrak{A}_n|, x \to_{\mathfrak{A}_n} y = \bigcup\{z \in |\mathfrak{A}_n| \mid x \cap z \subseteq y\}, \neg_{\mathfrak{A}_n} x = x \to_{\mathfrak{A}_n} a_0$.

The set $\mathfrak{A}_n$ forms a Heyting algebra by taking algebraic operations $\vee$ and $\wedge$ as set operations $\cup$ and $\cap$.

Lemma 4.11 (cf. Lemma 5.7 in [15]). *For any natural number n, $\mathfrak{A}_n = \langle |\mathfrak{A}_n|, \mathbb{N}, a_0, \cap, \cup, \to_{\mathfrak{A}_n}, \neg_{\mathfrak{A}_n} \rangle$ is a Heyting algebra.*

Hence $\underline{\mathfrak{A}_n}$ is a Heyting algebra with the second greatest element a_{n+8} for any natural number n by Lemma 4.11.

$\underline{\mathfrak{A}_0}, \underline{\mathfrak{A}_1}$ and $\underline{\mathfrak{A}_2}$ are represented with Hasse diagrams as in Figure 4.

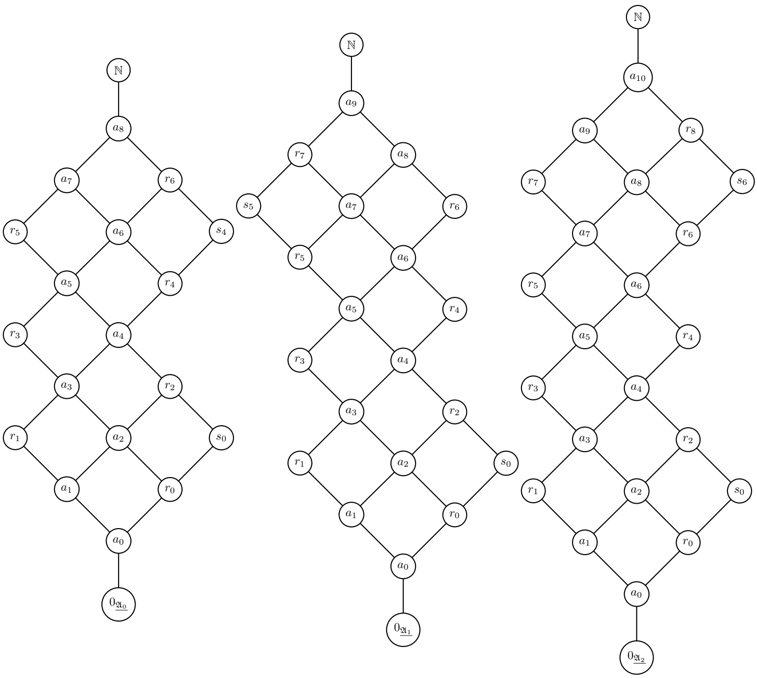


Figure 4: Heyting algebras $\underline{\mathfrak{A}_0}, \underline{\mathfrak{A}_1}$ and $\underline{\mathfrak{A}_2}$

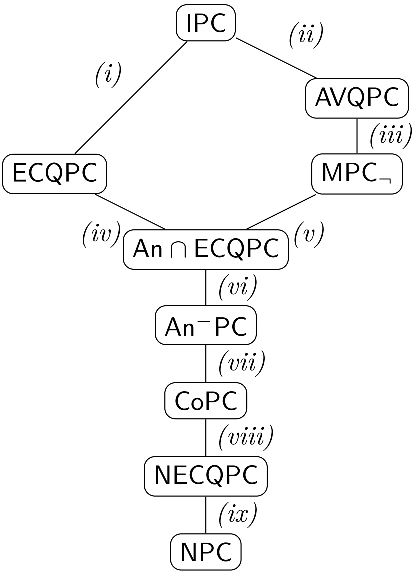
Consider an $\{\rightarrow\}$ embedding from $\mathfrak{A}_n$ to $\mathfrak{A}_m$. Since it preserves the order $\leq_{\mathfrak{A}_n}$, r_1, a_2, s_0 must be mapped into themselves respectively, and $r_{n+5}, a_{n+6}, s_{n+4}$ must be mapped into $r_{m+5}, a_{m+6}, s_{m+4}$, via $\{\rightarrow\}$-embedding. This is the key point of the lemma below.

Lemma 4.12 (cf. [15]). *For any natural numbers n and m, the following statement holds: If $n \neq m$, then $\underline{\mathfrak{A}_n}$ cannot be $\{\rightarrow\}$-embedded into a quotient algebra $\underline{\mathfrak{A}_m}/F$ of $\underline{\mathfrak{A}_m}$ for any filter F of $\underline{\mathfrak{A}_m}$.*

4.2. Main Results

The main results of this paper are summarized as the following theorem.

THEOREM 4.13. *There exist continua of logics in each line in the figure below.*



Hereafter, when the number in the above figure is used as anything other than a subscript, it is used to indicate a proof of the existence of continua of logics between the two logics.

LEMMA 4.14. *There is a continuum of logics between* IPC *and* ECQPC*. The same holds between* $\mathsf{MPC}_\neg$ *and* $\mathsf{An} \cap \mathsf{ECQPC}$*.*

PROOF: We prove this lemma as follows.

Step 1. We construct countably many models $N^1(\underline{\mathfrak{A}_n})$ of ECQPC based on $\underline{\mathfrak{A}_n}$ in Lemma 4.11.

Step 2. For each natural number n and each set $I \subset \mathbb{N}$ of natural numbers, we define Yankov formula $\mathcal{A}_n$ and logic $L^1(I)$.

Step 3. We show $N^1(\underline{\mathfrak{A}_k}) \models \mathcal{A}_l$ if $k \neq l$.

Step 4. We show $\mathcal{A}_n \in L^1(I)$ iff $n \in I$ by proving $N^1(\underline{\mathfrak{A}_n}) \not\models \mathcal{A}_n$.

Step 5. We show that there is continuum of logics between IPC and ECQPC.

Step 1. For any natural number n, the algebra

$$N^1(\underline{\mathfrak{A}_n}) = \langle |N^1(\underline{\mathfrak{A}_n})|, 1_{N^1(\underline{\mathfrak{A}_n})}, \wedge_{N^1(\underline{\mathfrak{A}_n})}, \vee_{N^1(\underline{\mathfrak{A}_n})}, \rightarrow_{N^1(\underline{\mathfrak{A}_n})}, \neg_{N^1(\underline{\mathfrak{A}_n})} \rangle$$

is defined as follows:

$$\begin{aligned} |N^1(\underline{\mathfrak{A}_n})| &:= |\underline{\mathfrak{A}_n}|, \\ 1_{N^1(\underline{\mathfrak{A}_n})} &:= \mathbb{N}, \\ \wedge_{N^1(\underline{\mathfrak{A}_n})} &:= \wedge_{\underline{\mathfrak{A}_n}}, \\ \vee_{N^1(\underline{\mathfrak{A}_n})} &:= \vee_{\underline{\mathfrak{A}_n}}, \\ \rightarrow_{N^1(\underline{\mathfrak{A}_n})} &:= \rightarrow_{\underline{\mathfrak{A}_n}}, \end{aligned}$$

and

$$\neg_{N^1(\underline{\mathfrak{A}_n})} a := \begin{cases} a_{n+8} & \text{if } a = 0_{\underline{\mathfrak{A}_n}}; \\ 0_{\underline{\mathfrak{A}_n}} & \text{otherwise,} \end{cases} \text{ for any } a \in |N^1(\underline{\mathfrak{A}_n})|.$$

We show that $N^1(\underline{\mathfrak{A}_n})$ validates the conditions (N), (ECQ) and $(\mathsf{An} \cap \mathsf{ECQ})$.

For (N), let $a, b \in |N^1(\underline{\mathfrak{A}_n})|$. If $a \neq 0_{\underline{\mathfrak{A}_n}}$ and $b \neq 0_{\underline{\mathfrak{A}_n}}$, then

$$\begin{aligned} (a \leftrightarrow_{N^1(\underline{\mathfrak{A}_n})} b) &\rightarrow_{N^1(\underline{\mathfrak{A}_n})} (\neg_{N^1(\underline{\mathfrak{A}_n})} a \leftrightarrow_{N^1(\underline{\mathfrak{A}_n})} \neg_{N^1(\underline{\mathfrak{A}_n})} b) \\ &= (a \leftrightarrow_{N^1(\underline{\mathfrak{A}_n})} b) \rightarrow_{N^1(\underline{\mathfrak{A}_n})} (0_{\underline{\mathfrak{A}_n}} \leftrightarrow_{N^1(\underline{\mathfrak{A}_n})} 0_{\underline{\mathfrak{A}_n}}) \\ &= (a \leftrightarrow_{N^1(\underline{\mathfrak{A}_n})} b) \rightarrow_{N^1(\underline{\mathfrak{A}_n})} \mathbb{N} \\ &= \mathbb{N}. \end{aligned}$$

Hence (N) holds for $a \neq 0_{\underline{\mathfrak{A}_n}}$ and $b \neq 0_{\underline{\mathfrak{A}_n}}$. If $a = 0_{\underline{\mathfrak{A}_n}}$ and $b \neq 0_{\underline{\mathfrak{A}_n}}$, then

$$\begin{aligned}(0_{\underline{\mathfrak{A}_n}} \leftrightarrow_{N^1(\underline{\mathfrak{A}_n})} b) \rightarrow_{N^1(\underline{\mathfrak{A}_n})} (\neg_{N^1(\underline{\mathfrak{A}_n})} 0_{\underline{\mathfrak{A}_n}} \leftrightarrow_{N^1(\underline{\mathfrak{A}_n})} \neg_{N^1(\underline{\mathfrak{A}_n})} b) \\ = (0_{\underline{\mathfrak{A}_n}} \leftrightarrow_{N^1(\underline{\mathfrak{A}_n})} b) \rightarrow_{N^1(\underline{\mathfrak{A}_n})} (a_{n+8} \leftrightarrow_{N^1(\underline{\mathfrak{A}_n})} 0_{\underline{\mathfrak{A}_n}}) \\ = 0_{\underline{\mathfrak{A}_n}} \rightarrow_{N^1(\underline{\mathfrak{A}_n})} 0_{\underline{\mathfrak{A}_n}} \\ = \mathbb{N}.\end{aligned}$$

The equality $0_{\underline{\mathfrak{A}_n}} \leftrightarrow_{N^1(\underline{\mathfrak{A}_n})} b = 0_{\underline{\mathfrak{A}_n}}$ follows from $0_{\underline{\mathfrak{A}_n}} \leftrightarrow_{N^1(\underline{\mathfrak{A}_n})} b = (0_{\underline{\mathfrak{A}_n}} \rightarrow_{N^1(\underline{\mathfrak{A}_n})} b) \wedge_{N^1(\underline{\mathfrak{A}_n})} (b \rightarrow_{N^1(\underline{\mathfrak{A}_n})} 0_{\underline{\mathfrak{A}_n}})$ and $b \rightarrow_{N^1(\underline{\mathfrak{A}_n})} 0_{\underline{\mathfrak{A}_n}} = 0_{\underline{\mathfrak{A}_n}}$. Hence (N) holds for $a = 0_{\underline{\mathfrak{A}_n}}$ and $b \neq 0_{\underline{\mathfrak{A}_n}}$. The remaining case can be proven in the same manner. So (N) holds in $N^1(\underline{\mathfrak{A}_n})$.

For (ECQ), let $a, b \in |N^1(\underline{\mathfrak{A}_n})|$. Then

$$\begin{aligned}(a \wedge_{N^1(\underline{\mathfrak{A}_n})} \neg_{N^1(\underline{\mathfrak{A}_n})} a) \rightarrow_{N^1(\underline{\mathfrak{A}_n})} b = 0_{\underline{\mathfrak{A}_n}} \rightarrow_{N^1(\underline{\mathfrak{A}_n})} b \\ = \mathbb{N}.\end{aligned}$$

So $N^1(\underline{\mathfrak{A}_n})$ validates (ECQ).

It is immediate that (An $\cap$ ECQ) follows from (ECQ).

Step 2. Take a nonempty proper subset I of natural numbers and an injection $g : |N^1(\underline{\mathfrak{A}_n})| \rightarrow \mathcal{P}$, and let $\mathcal{A}_n$ be $\chi^g_{(g_{0_{\underline{\mathfrak{A}_n}}} \rightarrow \neg g_{0_{\underline{\mathfrak{A}_n}}}) \rightarrow \neg g_{0_{\underline{\mathfrak{A}_n}}}}(N^1(\underline{\mathfrak{A}_n}))$. Let $L^1(I)$ be the logic adding axioms $\mathcal{A}_n$ into ECQPC for $n \in I$. Note that $L^1(I)$ is a sN-logic.

Step 3. We show that $N^1(\underline{\mathfrak{A}_k}) \models \mathcal{A}_l$ for natural numbers k, l with $k \neq l$. Note that $N^1(\underline{\mathfrak{A}_l})^H = \underline{\mathfrak{A}_l}$. There is no $\{\rightarrow\}$-embedding from $\underline{\mathfrak{A}_l}$ to $\underline{\mathfrak{A}_k}/F$ for any natural number k with $k \neq l$ and any filter F of $\underline{\mathfrak{A}_k}$ by Lemma 4.12, and so there is no Ψ-embedding $\underline{\mathfrak{A}_l}$ to $\underline{\mathfrak{A}_k}/F$ for any natural number k and l with $k \neq l$ and any filter F of $\underline{\mathfrak{A}_k}$. Therefore for any natural number k and l with $k \neq l$, $N^1(\underline{\mathfrak{A}_k}) \models \mathcal{A}_l$ holds from Lemma 4.8.

Step 4. We can also prove the following.

$$\mathcal{A}_n \in L^1(I) \Leftrightarrow n \in I.$$

Since $\Leftarrow$ is obvious, we show $\Rightarrow$. We assume $n \notin I$ and show that $\mathcal{A}_n \notin L^1(I)$. Since $N^1(\underline{\mathfrak{A}_n})$ validates (N), (ECQ) and $\mathcal{A}_m$ for each $m \neq n$, it is enough to show $N^1(\underline{\mathfrak{A}_n}) \not\models \mathcal{A}_n$. Let g^{-1} be the inverse map of g, i.e., $g^{-1} : \{g_a \in \mathcal{P} \mid a \in |N^1(\underline{\mathfrak{A}_n})|\} \to |N^1(\underline{\mathfrak{A}_n})|$ such that $g^{-1}(g(a)) = a$. Take a valuation v into $|N^1(\underline{\mathfrak{A}_n})|$ which is an extension of g^{-1}. Then $v(\theta(\chi^g_{(g_{0_{\underline{\mathfrak{A}_n}}} \to \neg g_{0_{\underline{\mathfrak{A}_n}}}) \to \neg g_{0_{\underline{\mathfrak{A}_n}}}}(N^1(\underline{\mathfrak{A}_n})))) = \mathbb{N}$ since $v(g_a \otimes g_b) = a \otimes_{N^1(\underline{\mathfrak{A}_n})} b = v(g_{a \otimes_{N^1(\underline{\mathfrak{A}_n})} b})$ for any $a, b \in |N^1(\underline{\mathfrak{A}_n})|$ and any $\otimes \in \Psi$. On the other hand, $v((g_{0_{\underline{\mathfrak{A}_n}}} \to \neg g_{0_{\underline{\mathfrak{A}_n}}}) \to \neg g_{0_{\underline{\mathfrak{A}_n}}}) \neq \mathbb{N}$ and so $v(((g_{0_{\underline{\mathfrak{A}_n}}} \to \neg g_{0_{\underline{\mathfrak{A}_n}}}) \to \neg g_{0_{\underline{\mathfrak{A}_n}}}) \vee g_{\star_{\mathfrak{A}}}) = \star_{\mathfrak{A}} \neq \mathbb{N}$. Therefore, $N^1(\underline{\mathfrak{A}_n}) \not\models \chi^g_{(g_{0_{\underline{\mathfrak{A}_n}}} \to \neg g_{0_{\underline{\mathfrak{A}_n}}}) \to \neg g_{0_{\underline{\mathfrak{A}_n}}}}(N^1(\underline{\mathfrak{A}_n}))$. Then

$$
\begin{aligned}
&v((g_{0_{\underline{\mathfrak{A}_n}}} \to \neg g_{0_{\underline{\mathfrak{A}_n}}}) \to \neg g_{0_{\underline{\mathfrak{A}_n}}}) \\
&\qquad = (v(g_{0_{\underline{\mathfrak{A}_n}}}) \to_{N^1(\underline{\mathfrak{A}_n})} \neg_{N^1(\underline{\mathfrak{A}_n})} v(g_{0_{\underline{\mathfrak{A}_n}}})) \to_{N^1(\underline{\mathfrak{A}_n})} \neg_{N^1(\underline{\mathfrak{A}_n})} v(g_{0_{\underline{\mathfrak{A}_n}}})) \\
&\qquad = (0_{\underline{\mathfrak{A}_n}} \to_{N^1(\underline{\mathfrak{A}_n})} a_{n+8}) \to_{N^1(\underline{\mathfrak{A}_n})} a_{n+8} \\
&\qquad = \mathbb{N} \to_{N^1(\underline{\mathfrak{A}_n})} a_{n+8},
\end{aligned}
$$

and so $v((g_{0_{\underline{\mathfrak{A}_n}}} \to \neg g_{0_{\underline{\mathfrak{A}_n}}}) \to \neg g_{0_{\underline{\mathfrak{A}_n}}}) \neq \mathbb{N}$. Therefore $N^1(\underline{\mathfrak{A}_n}) \not\models \mathcal{A}_n$ from the construction of $\mathcal{A}_n$. Since $L^1(I)$ is the minimum logic containing each axioms of ECQPC and $\mathcal{A}_m$ for $m \in I$, we have $L^1(I) \subseteq \{A \mid N^1(\underline{\mathfrak{A}_n}) \models A\}$. Thus we obtain $\mathcal{A}_n \notin L^1(I)$. This is the end of Step 4.

Step 5. By the result of Step 4, it follows that $L^1(I) \neq L^1(J)$ for any nonempty proper subsets I and J of natural numbers with $I \neq J$.

Since $\mathsf{IPC} \vdash (p \to \neg p) \to \neg p$, IPC proves succedent of $\mathcal{A}_n$ for any natural number n, and so $\mathsf{IPC} \vdash \mathcal{A}_n$.

Then, for any nonempty proper subset I of natural numbers, $\mathsf{ECQPC} \subsetneq L^1(I) \subsetneq \mathsf{IPC}$ hold. Since the choice of I is continuum, there is continuum of logics between IPC and ECQPC. We can show (v) by letting $L^5(I)$ be the logic adding axioms $\mathcal{A}_n$ into $\mathsf{An} \cap \mathsf{ECQPC}$ for $n \in I$ in the same manner as (i). □

LEMMA 4.15. *There is a continuum of logics between* IPC *and* AVQPC. *The same holds between* ECQPC *and* $\mathsf{An} \cap \mathsf{ECQPC}$.

PROOF: For any natural number n, the algebra

$$N^2(\underline{\mathfrak{A}_n}) = \langle |N^2(\underline{\mathfrak{A}_n})|, 1_{N^2(\underline{\mathfrak{A}_n})}, \wedge_{N^2(\underline{\mathfrak{A}_n})}, \vee_{N^2(\underline{\mathfrak{A}_n})}, \to_{N^2(\underline{\mathfrak{A}_n})}, \neg_{N^2(\underline{\mathfrak{A}_n})} \rangle$$

is defined in the same way as Lemma 4.14 except for negation. $\neg_{N^2(\underline{\mathfrak{A}_n})}$ is defined as follows:

$$\neg_{N^2(\underline{\mathfrak{A}_n})} a := \mathbb{N} \text{ for any } a \in |N^2(\underline{\mathfrak{A}_n})|.$$

First, we see that $N^2(\underline{\mathfrak{A}_n})$ validates the conditions $(\mathsf{N}), (\mathsf{An}), (\mathsf{AVQ})$ and $(\mathsf{An} \cap \mathsf{ECQ})$.

For (N), take any $a, b \in |N^2(\underline{\mathfrak{A}_n})|$. Then

$$\begin{aligned}(a \leftrightarrow_{N^2(\underline{\mathfrak{A}_n})} b) \to_{N^2(\underline{\mathfrak{A}_n})} (\neg_{N^2(\underline{\mathfrak{A}_n})} a \leftrightarrow_{N^2(\underline{\mathfrak{A}_n})} \neg_{N^2(\underline{\mathfrak{A}_n})} b) \\ &= (a \leftrightarrow_{N^2(\underline{\mathfrak{A}_n})} b) \to_{N^2(\underline{\mathfrak{A}_n})} (\mathbb{N} \leftrightarrow_{N^2(\underline{\mathfrak{A}_n})} \mathbb{N}) \\ &= (a \leftrightarrow_{N^2(\underline{\mathfrak{A}_n})} b) \to_{N^2(\underline{\mathfrak{A}_n})} \mathbb{N} \\ &= \mathbb{N}.\end{aligned}$$

So $N^2(\underline{\mathfrak{A}_n})$ validates (N).

For (An), let $a \in |N^2(\underline{\mathfrak{A}_n})|$. Then

$$\begin{aligned}(a \to_{N^2(\underline{\mathfrak{A}_n})} \neg_{N^2(\underline{\mathfrak{A}_n})} a) \to_{N^2(\underline{\mathfrak{A}_n})} \neg_{N^2(\underline{\mathfrak{A}_n})} a &= (a \to_{N^2(\underline{\mathfrak{A}_n})} \mathbb{N}) \to_{N^2(\underline{\mathfrak{A}_n})} \mathbb{N} \\ &= \mathbb{N}.\end{aligned}$$

So $N^2(\underline{\mathfrak{A}_n})$ validates (An). The case (AVQ) can be shown in the same way.

It is immediate that $(\mathsf{An} \cap \mathsf{ECQ})$ holds from (An).

For any natural number n, we define $\mathcal{B}_n := \chi^g_{(g_{\mathbb{N}} \wedge \neg g_{\mathbb{N}}) \to g_{0_{\underline{\mathfrak{A}_n}}}}(N^2(\underline{\mathfrak{A}_n}))$. Using $N^2(\underline{\mathfrak{A}_n})$, $\mathcal{B}_n$ and (ECQ) instead of $N^1(\underline{\mathfrak{A}_n})$, $\mathcal{A}_n$ and (An), respectively, we can show (ii) and (iv) in the same manner as (i). □

We can show the remining cases in the same way as Lemma 4.14.

LEMMA 4.16. *There is a continuum of logics between* AVQPC *and* $\mathsf{MPC}_\neg$.

PROOF: For any natural number n, the algebra

$$N^3(\underline{\mathfrak{A}_n}) = \langle |N^3(\underline{\mathfrak{A}_n})|, 1_{N^3(\underline{\mathfrak{A}_n})}, \wedge_{N^3(\underline{\mathfrak{A}_n})}, \vee_{N^3(\underline{\mathfrak{A}_n})}, \to_{N^3(\underline{\mathfrak{A}_n})}, \neg_{N^3(\underline{\mathfrak{A}_n})} \rangle$$

is defined in the same way as Lemma 4.14 except for negation. $\neg_{N^3(\underline{\mathfrak{A}_n})}$ is defined as follows:

$$\neg_{N^3(\underline{\mathfrak{A}_n})}a := a \rightarrow_{N^3(\underline{\mathfrak{A}_n})} a_0 \text{ for any } a \in |N^3(\underline{\mathfrak{A}_n})|.$$

First, we see that $N^3(\underline{\mathfrak{A}_n})$ validates conditions (N) and (An).

For (N), take any $a, b \in |N^3(\underline{\mathfrak{A}_n})|$. It is immediate that (N) holds in $N^3(\underline{\mathfrak{A}_n})$ in view of the following.
$(a \rightarrow_{N^3(\underline{\mathfrak{A}_n})} b) \wedge_{N^3(\underline{\mathfrak{A}_n})} (b \rightarrow_{N^3(\underline{\mathfrak{A}_n})} a_0) \leq_{N^3(\underline{\mathfrak{A}_n})} a \rightarrow_{N^3(\underline{\mathfrak{A}_n})} a_0$,
$(b \rightarrow_{N^3(\underline{\mathfrak{A}_n})} a) \wedge_{N^3(\underline{\mathfrak{A}_n})} (a \rightarrow_{N^3(\underline{\mathfrak{A}_n})} a_0) \leq_{N^3(\underline{\mathfrak{A}_n})} b \rightarrow_{N^3(\underline{\mathfrak{A}_n})} a_0$.

For (An), take any $a \in |N^3(\underline{\mathfrak{A}_n})|$. Then

$$\begin{aligned}(a \rightarrow_{N^3(\underline{\mathfrak{A}_n})} &\neg_{N^3(\underline{\mathfrak{A}_n})} a) \rightarrow_{N^3(\underline{\mathfrak{A}_n})} \neg_{N^3(\underline{\mathfrak{A}_n})} a\\ &= (a \rightarrow_{N^3(\underline{\mathfrak{A}_n})} (a \rightarrow_{N^3(\underline{\mathfrak{A}_n})} a_0)) \rightarrow_{N^3(\underline{\mathfrak{A}_n})} (a \rightarrow_{N^3(\underline{\mathfrak{A}_n})} a_0)\\ &= (a \rightarrow_{N^3(\underline{\mathfrak{A}_n})} a_0) \rightarrow_{N^3(\underline{\mathfrak{A}_n})} (a \rightarrow_{N^3(\underline{\mathfrak{A}_n})} a_0)\\ &= \mathbb{N}.\end{aligned}$$

Hence (An) hold in $N^3(\underline{\mathfrak{A}_n})$.

For any natural number n, we define
$\mathcal{D}_n := \chi^g_{\neg\neg(\neg(g_{\mathbb{N}} \rightarrow g_{\mathbb{N}}) \rightarrow g_{0_{\underline{\mathfrak{A}_n}}})}(N^3(\underline{\mathfrak{A}_n}))$. Using $N^3(\underline{\mathfrak{A}_n})$, $\mathcal{D}_n$ and (AVQ) instead of $N^1(\underline{\mathfrak{A}_n})$, $\mathcal{A}_n$ and (An), respectively, we can show (iii) in the same way as (i). □

Lemma 4.17. *There is a continuum of logics between* $\mathsf{An} \cap \mathsf{ECQPC}$ *and* $\mathsf{An}^-\mathsf{PC}$.

Proof: For any natural number n, the algebra

$$N^6(\underline{\mathfrak{A}_n}) = \langle |N^6(\underline{\mathfrak{A}_n})|, 1_{N^6(\underline{\mathfrak{A}_n})}, \wedge_{N^6(\underline{\mathfrak{A}_n})}, \vee_{N^6(\underline{\mathfrak{A}_n})}, \rightarrow_{N^6(\underline{\mathfrak{A}_n})}, \neg_{N^6(\underline{\mathfrak{A}_n})}\rangle$$

is defined in the same way as Lemma 4.14 except for negation. $\neg_{N^4(\underline{\mathfrak{A}_n})}$ is defined as follows:

$$\neg_{N^6(\underline{\mathfrak{A}_n})}a := a_{n+8} \text{ for any } a \in |N^6(\underline{\mathfrak{A}_n})|.$$

We can show $N^6(\underline{\mathfrak{A}_n})$ validates (N) and (An^-) in the same way as (ii).

For any natural number n, we define

$$\mathcal{F}_n := \chi^g_{\neg\neg(g_{0_{\underline{\mathfrak{A}_n}}} \to g_{0_{\underline{\mathfrak{A}_n}}}) \vee (\neg(g_{\mathbb{N}} \to g_{\mathbb{N}}) \to g_{0_{\underline{\mathfrak{A}_n}}})}(N^6(\underline{\mathfrak{A}_n})).$$

Using $N^6(\underline{\mathfrak{A}_n})$, $\mathcal{F}_n$ and $(\mathsf{An} \cap \mathsf{ECQ})$ instead of $N^1(\underline{\mathfrak{A}_n})$, $\mathcal{A}_n$ and (An) respectively, we can show (vi) in the same manner as (i). $\square$

LEMMA 4.18. *There is a continuum of logics between* $\mathsf{An}^-\mathsf{PC}$ *and* CoPC.

PROOF: For any natural number n, the algebra

$$N^7(\underline{\mathfrak{A}_n}) = \langle |N^7(\underline{\mathfrak{A}_n})|, 1_{N^7(\underline{\mathfrak{A}_n})}, \wedge_{N^7(\underline{\mathfrak{A}_n})}, \vee_{N^7(\underline{\mathfrak{A}_n})}, \to_{N^7(\underline{\mathfrak{A}_n})}, \neg_{N^7(\underline{\mathfrak{A}_n})} \rangle$$

is defined in the same way as Lemma 4.14 except for negation. $\neg_{N^7(\underline{\mathfrak{A}_n})}$ is defined as follows:

$$\neg_{N^7(\underline{\mathfrak{A}_n})} a := \begin{cases} \mathbb{N} & \text{if } a = 0_{\underline{\mathfrak{A}_n}}; \\ a_0 & \text{otherwise ,} \end{cases} \text{ for any } a \in |N^7(\underline{\mathfrak{A}_n})|.$$

First, we see that $N^7(\underline{\mathfrak{A}_n})$ validates conditions (Co) and (N).
We see that (Co). Let $a, b \in N^7(\underline{\mathfrak{A}_n})$. If $a \neq 0_{\underline{\mathfrak{A}_n}}$ and $b \neq 0_{\underline{\mathfrak{A}_n}}$, then

$$\begin{aligned}(a \to_{N^7(\underline{\mathfrak{A}_n})} b) \to_n (\neg_{N^7(\underline{\mathfrak{A}_n})} b \to_{N^7(\underline{\mathfrak{A}_n})} \neg_{N^7(\underline{\mathfrak{A}_n})} a) &\\ = (a \to_{N^7(\underline{\mathfrak{A}_n})} b) \to_{N^7(\underline{\mathfrak{A}_n})} (a_0 \to_{N^7(\underline{\mathfrak{A}_n})} a_0) &\\ = (a \to_{N^7(\underline{\mathfrak{A}_n})} b) \to_{N^7(\underline{\mathfrak{A}_n})} \mathbb{N} &\\ = \mathbb{N}. &\end{aligned}$$

Hence (Co) holds for $a \neq 0_{\underline{\mathfrak{A}_n}}$ and $b \neq 0_{\underline{\mathfrak{A}_n}}$. If $a \neq 0_{\underline{\mathfrak{A}_n}}$ and $b = 0_{\underline{\mathfrak{A}_n}}$, then

$$\begin{aligned}(0_{\underline{\mathfrak{A}_n}} \to_{N^7(\underline{\mathfrak{A}_n})} b) \to_{N^7(\underline{\mathfrak{A}_n})} (\neg_{N^7(\underline{\mathfrak{A}_n})} b \to_{N^7(\underline{\mathfrak{A}_n})} \neg_{N^7(\underline{\mathfrak{A}_n})} 0_{\underline{\mathfrak{A}_n}}) &\\ = \mathbb{N} \to_{N^7(\underline{\mathfrak{A}_n})} (a_0 \to_{N^7(\underline{\mathfrak{A}_n})} \mathbb{N}) &\\ = \mathbb{N}. &\end{aligned}$$

Hence (Co) holds for $a \neq 0_{\underline{\mathfrak{A}_n}}$ and $b = 0_{\underline{\mathfrak{A}_n}}$. If $a \neq 0_{\underline{\mathfrak{A}_n}}$ and $b \neq 0_{\underline{\mathfrak{A}_n}}$, then

$$\begin{aligned}(a \to_{N^7(\underline{\mathfrak{A}_n})} 0_{\underline{\mathfrak{A}_n}}) \to_{N^7(\underline{\mathfrak{A}_n})} (\neg_{N^7(\underline{\mathfrak{A}_n})} 0_{\underline{\mathfrak{A}_n}} \to_{N^7(\underline{\mathfrak{A}_n})} \neg_{N^7(\underline{\mathfrak{A}_n})} a) \\ = (a \to_{N^7(\underline{\mathfrak{A}_n})} 0_{\underline{\mathfrak{A}_n}}) \to_{N^7(\underline{\mathfrak{A}_n})} (\mathbb{N} \to_{N^7(\underline{\mathfrak{A}_n})} a_0) \\ = 0_{\underline{\mathfrak{A}_n}} \to_{N^7(\underline{\mathfrak{A}_n})} a_0 \\ = \mathbb{N}.\end{aligned}$$

Hence (Co) holds for $a \neq 0_{\underline{\mathfrak{A}_n}}$ and $b \neq 0_{\underline{\mathfrak{A}_n}}$. The remaining case can be proven in the same manner. So (Co) holds in $N^7(\underline{\mathfrak{A}_n})$. The cases (N) can be shown in the same manner.

For any natural number n, we define $\mathcal{G}_n := \chi^{g}_{(g_{a_0} \to \neg g_{a_0}) \to (\neg g_{0_{\underline{\mathfrak{A}_n}}} \to \neg g_{a_0})}(N^7(\underline{\mathfrak{A}_n}))$. Using $N^7(\underline{\mathfrak{A}_n})$, $\mathcal{G}_n$ and (An^-) instead of $N^1(\underline{\mathfrak{A}_n})$, $\mathcal{A}_n$ and (An) respectively, we can show (vii) in the same manner as (i). □

Lemma 4.19. *There is a continuum of logics between* CoPC *and* NECQPC.

Proof: For any natural number n, the algebra

$$N^8(\underline{\mathfrak{A}_n}) = \langle |N^8(\underline{\mathfrak{A}_n})|, 1_{N^8(\underline{\mathfrak{A}_n})}, \wedge_{N^8(\underline{\mathfrak{A}_n})}, \vee_{N^8(\underline{\mathfrak{A}_n})}, \to_{N^8(\underline{\mathfrak{A}_n})}, \neg_{N^8(\underline{\mathfrak{A}_n})} \rangle$$

is defined in the same way as Lemma 4.14 except for negation. $\neg_{N^8(\underline{\mathfrak{A}_n})}$ is defined as follows:

$$\neg_{N^8(\underline{\mathfrak{A}_n})} a := \begin{cases} \mathbb{N} & \text{if } a = a_{n+8}; \\ a_{n+8} & \text{otherwise }, \end{cases} \quad \text{for any } a \in |N^8(\underline{\mathfrak{A}_n})|.$$

First, we see that $N^8(\underline{\mathfrak{A}_n})$ validates the conditions (N) and (NECQ). For (N), let $a, b \in N^8(\underline{\mathfrak{A}_n})$. If $a \neq a_{n+8}$ and $b \neq a_{n+8}$, then

$$\begin{aligned}(a \leftrightarrow_{N^8(\underline{\mathfrak{A}_n})} b) \to_{N^8(\underline{\mathfrak{A}_n})} (\neg_{N^8(\underline{\mathfrak{A}_n})} a \leftrightarrow_{N^8(\underline{\mathfrak{A}_n})} \neg_{N^8(\underline{\mathfrak{A}_n})} b) \\ = (a \leftrightarrow_{N^8(\underline{\mathfrak{A}_n})} b) \to_{N^8(\underline{\mathfrak{A}_n})} (a_{n+8} \leftrightarrow_{N^8(\underline{\mathfrak{A}_n})} a_{n+8}) \\ = (a \leftrightarrow_{N^8(\underline{\mathfrak{A}_n})} b) \to_{N^8(\underline{\mathfrak{A}_n})} \mathbb{N} \\ = \mathbb{N}.\end{aligned}$$

sHence (N) holds for $a \neq a_{n+8}$ and $b \neq a_{n+8}$. If $a = a_{n+8}$ and $b \neq a_{n+8}$, then

$$\begin{aligned}(a_{n+8} \leftrightarrow_{N^8(\underline{\mathfrak{A}_n})} b) \rightarrow_{N^8(\underline{\mathfrak{A}_n})} (\neg_{N^8(\underline{\mathfrak{A}_n})} a_{n+8} \leftrightarrow_{N^8(\underline{\mathfrak{A}_n})} \neg_{N^8(\underline{\mathfrak{A}_n})} b)&\\ = (a_{n+8} \leftrightarrow_{N^8(\underline{\mathfrak{A}_n})} b) \rightarrow_{N^8(\underline{\mathfrak{A}_n})} (\mathbb{N} \leftrightarrow_{N^8(\underline{\mathfrak{A}_n})} a_{n+8})&\\ = (a_{n+8} \leftrightarrow_{N^8(\underline{\mathfrak{A}_n})} b) \rightarrow_{N^8(\underline{\mathfrak{A}_n})} a_{n+8}&\\ = \mathbb{N}.&\end{aligned}$$

The equality $\mathbb{N} \leftrightarrow_{N^8(\underline{\mathfrak{A}_n})} a_{n+8} = a_{n+8}$ follows from $\mathbb{N} \leftrightarrow_{N^8(\underline{\mathfrak{A}_n})} a_{n+8} = (\mathbb{N} \rightarrow_{N^8(\underline{\mathfrak{A}_n})} a_{n+8}) \wedge_{N^8(\underline{\mathfrak{A}_n})} (a_{n+8} \rightarrow_{N^8(\underline{\mathfrak{A}_n})} \mathbb{N})$ and $\mathbb{N} \rightarrow_{N^8(\underline{\mathfrak{A}_n})} a_{n+8} = a_{n+8}$. Hence (N) holds for $a = a_{n+8}$ and $b \neq a_{n+8}$. The remaining case can be proven in the same manner.

For (NECQ), let $a, b \in |N^8(\underline{\mathfrak{A}_n})|$, then

$$\begin{aligned}(a \wedge_{N^8(\underline{\mathfrak{A}_n})} \neg_{N^8(\underline{\mathfrak{A}_n})} a) \rightarrow_{N^8(\underline{\mathfrak{A}_n})} \neg_{N^8(\underline{\mathfrak{A}_n})} b &= a_{n+8} \rightarrow_{N^8(\underline{\mathfrak{A}_n})} \neg_{N^8(\underline{\mathfrak{A}_n})} b\\ &= \mathbb{N}.\end{aligned}$$

Hence (NECQ) holds in $N^8(\underline{\mathfrak{A}_n})$.

For any natural number n, we define $\mathcal{H}_n := \chi^g_{(g_{a_0} \rightarrow g_{a_{n+8}}) \rightarrow (\neg g_{a_{n+8}} \rightarrow \neg g_{a_0})}(N^8(\underline{\mathfrak{A}_n}))$. Using $N^8(\underline{\mathfrak{A}_n})$, $\mathcal{H}_n$ and (Co) instead of $N^1(\underline{\mathfrak{A}_n})$, $\mathcal{A}_n$ and (An) respectively, we can show (viii) in the same manner as (i). □

Lemma 4.20. *There is a continuum of logics between* NECQPC *and* NPC.

Proof: For any natural number n, the algebra

$$N^9(\underline{\mathfrak{A}_n}) = \langle |N^9(\underline{\mathfrak{A}_n})|, 1_{N^9(\underline{\mathfrak{A}_n})}, \wedge_{N^9(\underline{\mathfrak{A}_n})}, \vee_{N^9(\underline{\mathfrak{A}_n})}, \rightarrow_{N^9(\underline{\mathfrak{A}_n})}, \neg_{N^9(\underline{\mathfrak{A}_n})} \rangle$$

is defined in the same way as Lemma 4.14 except for negation. $\neg_{N^9(\underline{\mathfrak{A}_n})}$ is defined as follows:

$$\neg_{N^9(\underline{\mathfrak{A}_n})} a := \begin{cases} \mathbb{N} & \text{if } a = \mathbb{N};\\ a_{n+8} & \text{otherwise}\ , \end{cases} \quad \text{for any } a \in |N^9(\underline{\mathfrak{A}_n})|.$$

As in the case of (viii), we can show that $N^9(\underline{\mathfrak{A}_n})$ validates (N).

For any natural number n, we define $\mathcal{H}_n := \chi^g_{(g_{\mathbb{N}} \wedge \neg g_{\mathbb{N}}) \rightarrow \neg g_{a_{n+8}}}(N^9(\underline{\mathfrak{A}_n}))$. Using $N^9(\underline{\mathfrak{A}_n})$, $\mathcal{H}_n$ and (NECQ) instead of $N^1(\underline{\mathfrak{A}_n})$, $\mathcal{A}_n$ and (An) respectively, we can show (ix) in the same manner as (i). □

Lemma 4.19 and Lemma 4.20 are already proved in [1, Proposition 3.5] using neighborhood semantics.

5. Concluding remarks

5.1. Summary of the main results

In [13], it was clear that **SUBMIN** is a subsystem of both co-minimal logic and minimal logic, but it was not clear if **SUBMIN** is the strongest logic among the systems that are contained in both co-minimal logic and minimal logic. We identified that **SUBMIN** is not the strongest logic, but $\mathsf{An} \cap \mathsf{ECQPC}$ is the strongest logic. Moreover, we presented new formalizations of co-minimal logic and **SUBMIN** that clarify their relationships with ECQ. Furthermore, we applied the method used in [15] and simplified the proofs of the results offered in [1] and obtained new results that are not included in [1]. Figure 5 summarizes the main results of this paper.
Note here that there exist continua of logics in each line in the figure above, as we proved in Theorem 4.13.

5.2. Future Directions

There is still much work to be done in this area of research. Some directions that seem to be worth exploring are described here.

ECQ and substructural logics For proof systems, this paper focused on Hilbert systems. But we can also define sequent calcului for these logics (cf. [8, 3, 11]). Then, given the deep connections between substructural logics and algebraic semantics, it will be interesting to explore the substructural versions of LN, which is a sequent calculus version of NPC, and its extensions. Note that the existence of continua of logics between pairs of substructural logics is explored in [10]. Therefore, it would be interesting to examine whether the method used in this paper can be applied to substructural versions of LN and related systems.

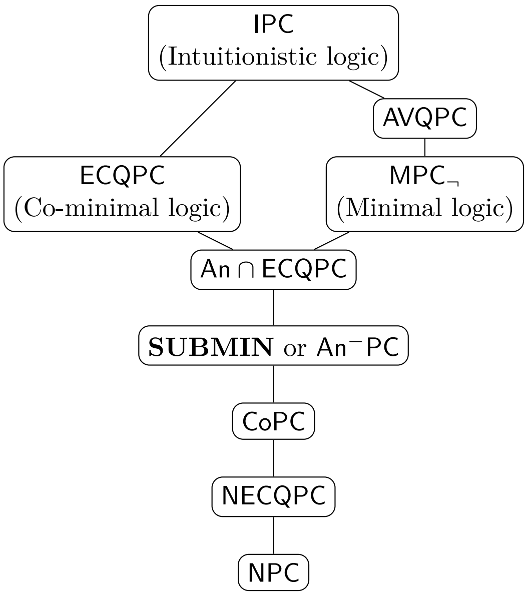


Figure 5: Summary of the results presented in the paper

Another method We used algebraic semantics for showing the existence of continua of logics between pairs of logics related to intuitionistic logic and minimal logic. It is known that there is a duality ("Priestley duality") between the class of Priestley spaces and the class of bounded distributive lattices. Since N-algebras are distributive lattices, we might be able to use topological semantics to obtain yet another proof for the existence of continua of logics.[2]

[2]This idea was pointed out by Prof. Hanamantagouda P. Sankappanavar after the presentation based on an earlier draft of this paper at *Non-Classical Logics: Theory and Applications 2024*. I would like to thank Prof. Sankappanavar for this interesting comment.

No gap between logics This paper focused on the existence of continua of logics between logics related to intuitionistic logic and minimal logic. However, there are pairs of intermediate logics where no logic exists between the pair, as shown in [5]. More specifically, if we consider the family of extensions of intuitionistic logic, known as the n-valued Gödel logic G_n, then for $n \geq 3$, no logic exists between G_n and G_{n+1}. In view of this result, it will be interesting to see if extensions of the logics we considered in this paper will have similar results.

First-order logics We discussed only propositional logics in this paper, but of course it will be interesting to consider first-order logics. If we add universal and existential quantifiers to these logics with the usual axioms and rules, *without* the additional axioms such as the constant domain axiom, then we can show immediately the existence of continua of logics between first-order expansions of logics discussed in this paper by considering propositional logics to be predicate logics with only zero argument predicate symbols. However, other cases remain to be explored in further detail.

Appendix: Kripke semantics for logics above $\mathsf{An^-PC}$

DEFINITION 6.1. We define the following Hilbert systems:

- **SUBMIN** is the Hilbert system obtained by adding axioms $\bot \to p$, $p \to \top$ and $\neg p \to \neg\neg\top$ to CoPC in $\mathcal{L}_{\neg,\top,\bot}$.
- **SUBMIN**$^{-\top,\bot}$ as the Hilbert system adding axioms $\neg p \to \neg\neg(q \to q)$ to CoPC in $\mathcal{L}_{\neg}$.
- **MIN** is the Hilbert system obtained by adding axioms $\bot \to p$, $p \to \top$, $\neg p \to \neg\neg\top$ and $\neg\neg\top$ to CoPC in $\mathcal{L}_{\neg,\top,\bot}$.
- **MIN**$^{-\top,\bot}$ as the Hilbert system adding axioms $\neg p \to \neg\neg(q \to q)$ and $\neg\neg(p \to p)$ to CoPC in $\mathcal{L}_{\neg}$.

Kripke semantics for the logic **SUBMIN** ((F, G)-semantics for short) was introduced in [13].

Definition 6.2 (cf. [13]). An (F, G)-frame $\mathcal{F}$ for $\mathsf{An^{-}PC}$ is a quadruple $(W, \leq, F, G)$ satisfying the following:

- $(W, \leq)$ is a quasi-order, i.e., $\leq$ is a reflexive and transitive relation on W (we call an element of W a world);
- F, G are upward closed subsets of W such that F is a subset of G.

An (F, G)-model $\mathcal{M}$ is a pair $(\mathcal{F}, \mathcal{V})$ satisfying the following:

- $\mathcal{F}$ is an (F, G)-frame;
- $\mathcal{V}$ is a mapping assigning an upward closed set of worlds to each propositional variable.

A valuation of formulas in $\mathcal{L}_{\neg,\top,\bot}$ is inductively defined as follows:

- $\mathcal{M}, w \Vdash_{(F,G)} \top$ for all $w \in W$;
- $\mathcal{M}, w \nVdash_{(F,G)} \bot$ for all $w \in W$;
- $\mathcal{M}, w \Vdash_{(F,G)} p \iff w \in \mathcal{V}(p)$;
- $\mathcal{M}, w \Vdash_{(F,G)} A \wedge B \iff \mathcal{M}, w \Vdash_{(F,G)} A$ and $\mathcal{M}, w \Vdash_{(F,G)} B$;
- $\mathcal{M}, w \Vdash_{(F,G)} A \vee B \iff \mathcal{M}, w \Vdash_{(F,G)} A$ or $\mathcal{M}, w \Vdash_{(F,G)} B$;
- $\mathcal{M}, w \Vdash_{(F,G)} A \to B \iff \forall v \geq w[\mathcal{M}, v \Vdash_{(F,G)} A \Rightarrow \mathcal{M}, v \Vdash_{(F,G)} B]$;
- $\mathcal{M}, w \Vdash_{(F,G)} \neg A \iff \forall v \geq w[\mathcal{M}, v \Vdash_{(F,G)} A \Rightarrow v \in F]$ and $w \in G$.

$\mathcal{F} \models_{(F,G)} A$ denotes that $(\mathcal{F}, \mathcal{V}), w \Vdash_{(F,G)} A$ for all $\mathcal{V}$ and $w \in W$. $\mathcal{M} \models_{(F,G)} A$ denotes that $\mathcal{M}, w \Vdash_{(F,G)} A$ for all $w \in W$. For any set of formulas Γ, $\Gamma \models_{(F,G)} A$ denotes $\mathcal{M}, w \Vdash_{(F,G)} B$ for any $B \in \Gamma$ implies $\mathcal{M}, w \Vdash_{(F,G)} A$, for any $\mathcal{M}$ and $w \in W$, where W is the set of worlds of $\mathcal{M}$.

By imposing conditions on (F, G)-frame, the soundness and completeness theorems holds with some already defined logics.

Fact 6.3 (cf. [13]).

1. $\Gamma \vdash_{\mathbf{SUBMIN}} A \iff \Gamma \models_{(F,G)} A$.
2. $\Gamma \vdash_{\mathbf{MIN}} A \iff \Gamma \models_{(F,G)} A$ for the class of (F, G)-semantics with $G = W$.
3. $\Gamma \vdash_{\mathbf{CO\text{-}MIN}} A \iff \Gamma \models_{(F,G)} A$ for the class of (F, G)-semantics with $F = \varnothing$.
4. $\Gamma \vdash_{\mathbf{IPC}} A \iff \Gamma \models_{(F,G)} A$ for the class of (F, G)-semantics with $G = W$ and $F = \varnothing$, where **IPC** is intuitionistic logic.

By excluding the conditions on $\top$ and $\bot$, (F, G)-semantics for formulas in $\mathcal{L}_\neg$ can be given. Then we can show the soundness and completeness in the same way.

FACT 6.4.

1. $\Gamma \vdash_{\mathbf{SUBMIN}^{-\top,\bot}} A \Longleftrightarrow \Gamma \models_{(F,G)} A$.
2. $\Gamma \vdash_{\mathbf{MIN}^{-\top,\bot}} A \Longleftrightarrow \Gamma \models_{(F,G)} A$ for the class of (F, G)-semantics with $G = W$.
3. $\Gamma \vdash_{\mathbf{CO\text{-}MIN}^{-\top,\bot}} A \Longleftrightarrow \Gamma \models_{(F,G)} A$ for the class of (F, G)-semantics with $F = \varnothing$.
4. $\Gamma \vdash_{\mathbf{IPC}^{-\top,\bot}} A \Longleftrightarrow \Gamma \models_{(F,G)} A$ for the class of (F, G)-semantics with $G = W$ and $F = \varnothing$.

Hence, for $\top$ and $\bot$ free formula A, $\mathbf{CO\text{-}MIN} \vdash A \Leftrightarrow \mathbf{CO\text{-}MIN}^{-\top,\bot} \vdash A$ is immediate from the above facts. The same argument holds for the remaining cases. Furthermore, $\mathbf{SUBMIN}^{-\top,\bot}$ is equivalent to $\mathsf{An^-PC}$, and $\mathbf{MIN}^{-\top,\bot}$ is equivalent to $\mathsf{MPC}_\neg$ (cf. [8]).

The soundness and completeness theorems can be shown for $\mathsf{An} \cap \mathsf{ECQPC}$ and AVQPC by adding similar conditions for F and G.

DEFINITION 6.5 ($\mathsf{An} \cap \mathsf{ECQ}$-frame and AVQ-frame). For an (F, G)-frame without $\top$ and $\bot$, F' denotes $:= \{w \in W \mid \forall v \geq w(v \notin F)\}$. We define the following conditions:

- $\mathsf{An} \cap \mathsf{ECQ}$-frames are (F, G)-frames with $F' \cup G = W$.
- AVQ-frames are (F, G)-frames such that $\forall u \geq w[u \notin F']$ implies $w \in F$ for all $w \in W$ and $G = W$ hold.

We write $\Vdash_{\mathsf{An \cap ECQ}}, \models_{\mathsf{An \cap ECQ}}, \Vdash_{\mathsf{AVQ}}$ and $\models_{\mathsf{AVQ}}$ for validity with respect to the classes of $\mathsf{An} \cap \mathsf{ECQ}$-frames and AVQ-frames.

DEFINITION 6.6 (cf. [14]). A sub-normal Kripke frame is a tuple $(W, \leq, F)$ satisfying the following:

- $(W, \leq)$ is a quasi-order, i.e., $\leq$ is a reflexive and transitive relation on W (we call an element of W a world);
- F is an upward closed subset of W such that for every $w \notin F$, there is $v \in W$ such that $w \leq v$ and for every $u \in W$, if $v \leq u$, then $u \notin F$.

For any AVQ-frame $(W, \leq, F, G)$, G is uniquely determined only by W and F, and so the AVQ-frame can be rewritten as $(W, \leq, F)$. In [14], *sub-normal Kripke frame* $(W, \leq, F)$ was introduced. Because the condition for F in AVQ-frane is the contraposition of the condition for F in sub-normal Kripke frame, AVQ-frame is sub-normal frame, and vice versa. Because our definition of AVQ-frame is more convenient for the proof for soundness, we use the definition of AVQ-frame.

We show the soundness and completeness of $\mathsf{An} \cap \mathsf{ECQPC}$ for the class of $\mathsf{An} \cap \mathsf{ECQPC}$-frames.

THEOREM 6.7 (Soundness of $\mathsf{An} \cap \mathsf{ECQPC}$). *If* $\Gamma \vdash_{\mathsf{An}\cap\mathsf{ECQPC}} A$, *then* $\Gamma \models_{\mathsf{An}\cap\mathsf{ECQ}} A$.

PROOF: We show this statement by induction on the length of the deduction. Here we show only the cases for the negative axiom ($\mathsf{An} \cap \mathsf{ECQ}$), since the remaining case can be proven in the same manner as [8]. Let $(\mathcal{F}, \mathcal{V})$ and $w \in W$ be arbitrary.

Suppose $(\mathcal{F}, \mathcal{V}), w \nVdash_{\mathsf{An}\cap\mathsf{ECQ}} \neg\neg(p \to p)$. First, we observe that $(\mathcal{F}, \mathcal{V}), v \Vdash_{\mathsf{An}\cap\mathsf{ECQ}} \neg\neg(p \to p)$ iff $v \in G$ for any $v \in W$. We show $(\mathcal{F}, \mathcal{V}), u \nVdash_{\mathsf{An}\cap\mathsf{ECQ}} \neg(q \to q)$ for every $u \geq w$, which implies $w \Vdash_{\mathsf{An}\cap\mathsf{ECQ}} \neg(q \to q) \to r$. Since $w \notin G$, $w \in F'$ and so $u \notin F$ for every $u \geq w$. By the definition of the valuation, $(\mathcal{F}, \mathcal{V}), u \nVdash_{\mathsf{An}\cap\mathsf{ECQ}} \neg(q \to q)$ for every $u \geq w$. $\Box$

In what follows, we call a set of formulas Δ saturated if the following conditions hold,

- There is a formula A such that $A \notin \Delta$ (nontriviality);
- $\Delta \vdash_{\mathsf{An}\cap\mathsf{ECQPC}} A \Rightarrow A \in \Delta$;
- $\Delta \vdash_{\mathsf{An}\cap\mathsf{ECQPC}} A \vee B \Rightarrow \Delta \vdash_{\mathsf{An}\cap\mathsf{ECQPC}} A$ or $\Delta \vdash_{\mathsf{An}\cap\mathsf{ECQPC}} B$.

THEOREM 6.8 (Completeness of $\mathsf{An} \cap \mathsf{ECQPC}$). *If* $\Gamma \models_{\mathsf{An}\cap\mathsf{ECQ}} A$, *then* $\Gamma \vdash_{\mathsf{An}\cap\mathsf{ECQPC}} A$.

PROOF: Given $\Gamma \nvDash_{\mathsf{An}\cap\mathsf{ECQ}} A$, we construct a saturated set $\Gamma_0 \supset \Gamma$ such that $\Gamma_0 \nvDash_{\mathsf{An}\cap\mathsf{ECQ}} A$. Then the canonical model $\mathcal{M} = (W, \leq, F, G, \mathcal{V})$ with respect to Γ_0 is defined standardly. For F and G, we define $F := \{\Delta \mid \neg B \in \Delta \text{ for all } B\}$and $G := \{\Delta \mid \neg B \in \Delta \text{ for some } B\}$. It is sufficient to show $F' \cup G = W$, since the remaining case can be proven in the same way as [8].

Suppose $\Delta \notin G$, and so $\neg B \notin \Delta$ for any B. Hence $\neg\neg(p \to p) \notin \Delta$. By $\mathsf{An} \cap \mathsf{ECQPC} \vdash \neg\neg(p \to p) \vee (\neg(q \to q) \to C)$ for any formula C, $\neg(q \to q) \to C \in \Delta$. By nontriviality of saturated set, $\neg(q \to q) \notin \Delta'$ for any $\Delta' \geq \Delta$, and so we obtain $\Delta' \notin F$ for any $\Delta' \geq \Delta$, and so $\Delta \in F'$. □

We next show the soundness and completeness of AVQPC for the class of AVQPC-frames.

Theorem 6.9 (Soundness of AVQPC). *If* $\Gamma \vdash_{\mathsf{AVQPC}} A$*, then* $\Gamma \models_{\mathsf{AVQPC}} A$*.*

Proof: We show this statement by induction on the length of the deduction. Here we show only the cases for the negative axiom (AVQPC), since the remaining case can be proven in the same manner as [8]. Let $(\mathcal{F}, \mathcal{V})$ and $w \in W$ be arbitrary.

Suppose $(\mathcal{F}, \mathcal{V}), u \Vdash_{\mathsf{AVQ}} \neg(\neg(p \to p) \to q)$ for $u \geq w$. We want to show $u \in F$. By assumption, $(\mathcal{F}, \mathcal{V}), v \Vdash_{\mathsf{AVQ}} \neg(p \to p) \to q$ implies $v \in F$ for any $v \geq u$. If $(\mathcal{F}, \mathcal{V}), v \Vdash_{\mathsf{AVQ}} \neg(p \to p) \to q$, then $v \in F$. Otherwise, $(\mathcal{F}, \mathcal{V}), v' \Vdash_{\mathsf{AVQ}} \neg(p \to p)$ and $(\mathcal{F}, \mathcal{V}), v' \nVdash_{\mathsf{AVQ}} q$ for some $v' \geq v \geq u$. By the definition of AVQ-frame, $u \in F$. □

For the completeness, we need the following lemma.

Lemma 6.10. $\mathsf{MPC}_\neg \vdash \neg A \leftrightarrow (A \to \neg(p \to p))$.

Proof: $\mathsf{MPC}_\neg \vdash \neg A \to (A \to \neg(p \to p))$ is immediate from (NECQ). For the other direction, by (NECQ), $\mathsf{MPC}_\neg \vdash ((A \to \neg(p \to p)) \wedge A) \to (A \to \neg A)$, and so $\mathsf{MPC}_\neg \vdash (A \to \neg(p \to p)) \to (A \to \neg A)$. By (An), $\mathsf{MPC}_\neg \vdash (A \to \neg A) \to \neg A$, and so $\mathsf{MPC}_\neg \vdash (A \to \neg(p \to p)) \to \neg A$. □

Theorem 6.11 (Completeness of AVQPC). *If* $\Gamma \models_{\mathsf{AVQ}} A$*, then* $\Gamma \vdash_{\mathsf{AVQPC}} A$*.*

Proof: Given $\Gamma \nvdash_{\mathsf{AVQ}} A$, we construct a saturated set $\Gamma_0 \supset \Gamma$ such that $\Gamma_0 \nvdash_{\mathsf{AVQ}} A$. Then the canonical model $\mathcal{M} = (W, \leq, F, G, \mathcal{V})$ with respect to Γ_0 is defined standardly. For F and G, we define $F := \{\Delta \mid \neg B \in \Delta \text{ for all } B\}$ and $G := \{\Delta \mid \neg B \in \Delta \text{ for some } B\}$. It is sufficient to show $\forall \Delta' \geq \Delta[\Delta' \notin F']$ implies $\Delta \in F$ for all $\Delta \in W$, since the remaining case can be proven in the same manner as [8], and $G = W$ is obvious: Take any $\Delta \in W$ and suppose $\Delta' \notin F'$ for any $\Delta' \geq \Delta \cdots (\star)$. We want to show $\Delta \in F$. Suppose $\Delta \notin F$, and so $\neg B \notin \Delta$ for some B. By (Co), $\mathsf{AVQPC} \vdash (B \to (p \to p)) \to (\neg(p \to p) \to \neg B)$. Hence $\neg(p \to p) \notin \Delta$. Let $C_0, C_1, C_2 \ldots$ be an enumeration of all formulas. By (AVQ) and the previous lemma, $\mathsf{AVQPC} \vdash ((\neg(p \to p) \to C_0) \to \neg(p \to p)) \to \neg(p \to p)$, and so $(\neg(p \to p) \to C_0) \to \neg(p \to p) \notin \Delta$. Hence, there is a saturated $\Delta^0 \supseteq \Delta$ such that $\neg(p \to p) \to C_0 \in \Delta^0$ and $\neg(p \to p) \notin \Delta^0$. By repeating this operation, we can take a sequence of saturated set $\Delta^0 \subseteq \Delta^1 \subseteq \Delta^2 \subseteq \ldots$ and a saturated set $\bigcup_{i \in \omega} \Delta^i$. By the nontriviality of saturated set, $\neg(p \to p) \notin \Delta'$ for any $\Delta' \geq \bigcup_{i \in \omega} \Delta^i$, and so $\Delta' \notin F$ and $\cup_{i \in \omega} \Delta^i \in F'$. But this contradicts $(\star)$, and so $\Delta \in F$. $\square$

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