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SEMANTIC INCOMPLETENESS OF LIBERMAN ET AL. (2020)'S HILBERT-STYLE SYSTEMS FOR TERM-MODAL LOGICS WITH EQUALITY AND NON-RIGID TERMS

Abstract

In this paper, we prove the semantic incompleteness of some expansions of the Hilbert-style system for the minimal normal term-modal logic with equality and non-rigid terms that were proposed in Liberman et al. (2020) "Dynamic Term-modal Logics for First-order Epistemic Planning." Term-modal logic is a family of first-order modal logics having term-modal operators indexed with terms in the first-order language. While some first-order formula is valid over the corresponding class of frames in the involved Kripke semantics, it is not provable in those expansions. We show this fact by introducing a non-standard Kripke semantics which makes the meanings of constants and function symbols relative to the meanings of relation symbols combined with them. We also address an incorrect frame correspondence result given in Liberman et al. (2020).

Keywords: incompleteness, term-modal logic, first-order modal logic.

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1. Introduction

This paper is an extended version of Sawasaki [10]. In [10] we proved the semantic incompleteness of Liberman et al. [8]'s Hilbert-style system $\mathsf{H}\mathbf{K}$ for the minimal normal term-modal logic $\mathbf{K}$ with equality and non-rigid terms. In this paper, we further prove the semantic incompleteness of all the expansions that [8] has obtained from $\mathsf{H}\mathbf{K}$ by adding some of axioms $\mathsf{T}, \mathsf{D}, \mathsf{4}$ and $\mathsf{5}$. This paper also addresses an incorrect frame correspondence result given in [8].

Term-modal logic, developed by Thalmann [12] and Fitting et al. [4], is a family of first-order modal logics having term-modal operators $[t]$ indexed with terms t in the first-order language. In the language of term-modal logic, for example, $[x]P(x)$, $[f(x)]P(x)$ and $\forall x[f(x)]P(x)$ are formulas. Term-modal logic is more expressive than multi-modal propositional logic and has been applied to epistemic logic in e.g. [7, 8, 13] and deontic logic in e.g. [11, 5, 6]. Some other developments of term-modal logic have been overviewed e.g. in [8, pp. 22–24] and [5, pp. 48–50].

The logics developed in Liberman et al. [8] are first-order dynamic epistemic logics for epistemic planning, and term-modal logic is invoked as its underlying logic. Technically speaking, their term-modal logics are two-sorted normal term-modal logics of the constant domain with equality and non-rigid terms. They make their logics two-sorted because, while letting the domain of a model include both agents and objects, they read an epistemically interpreted term-modal operator K_t as "*agent* t knows." The language defined in [8] allows $K_t\varphi$ to be a formula only if t is a term for an agent, and thereby excludes the possibility that terms denoting objects appear in the argument of the term-modal operator. In [8, p. 17], $\mathsf{H}\mathbf{K}$ and its expansions with $\mathsf{T}, \mathsf{D}, \mathsf{4}$ and $\mathsf{5}$ are claimed to be strongly complete with respect to the class of all the corresponding frames. Most of these results were originally presented in [1]. Later, two issues concerning action models and reduction axioms were fixed in the erratum [9] of [8].

Unfortunately, $\mathsf{H}\mathbf{K}$ and its expansions are semantically incomplete. In particular, the valid first-order formula $x = c \to (P(x) \to P(c))$ is unprovable. To be more precise, for a set $\Gamma \subseteq \{\mathsf{T}, \mathsf{D}, \mathsf{4}, \mathsf{5}\}$, we can prove that this formula is valid over the class of all frames to which Γ corresponds but is not provable in the system $\mathsf{HK}\Gamma$ obtained from HK by adding Γ. To this end, in Section 3 we introduce a non-standard Kripke semantics which makes the meanings of constants and function symbols relative to the meanings of relation symbols combined with them.

This paper will proceed as follows. In Section 2 we first introduce the syntax in [8]. Since there are some minor defects on the definitions for type, we do this with some modifications. Then we introduce the Kripke semantics and the Hilbert-style systems given in [8], addressing an incorrect frame correspondence result. In Section 3 we prove the semantic incompleteness of the expansions of HK by introducing a non-standard Kripke semantics.

2. Syntax, Semantics and Hilbert-style Systems

We will first introduce the syntax presented in [8, pp. 3–4] with some modifications. The idea there is to define the notions of term and formula while assigning (sequences of) types "`agt`", "`obj`" or "`agt_or_obj`" to all symbols like variables or relation symbols. It is basically the same idea as in Enderton [2, Section 4.3], but there is an important difference. In the syntax of [8], not only `agt` or `obj` but also `agt_or_obj` may be assigned to the arguments of function symbols and relation symbols, so that $P(x)$ seems to be intended to become a formula even when x has type `agt` and P takes type `agt_or_obj`.

However, the original definitions 1–3 for the syntax seem to have two minor defects. First, the original definition 1 for type assignment and the original definition 2 for term are dependent upon one another, thus they are circular definitions. Second, while $P(x)$ seems to be intended to become a formula when x has type `agt` and P takes type `agt_or_obj`, it does not actually become a formula since the original definition 3 for formula requires that the type of x and the type of the argument of P must be the same. Accordingly, for example, $x = x$ cannot be a formula in any signature since the type of x is either `agt` or `obj` but the type of the arguments of $=$ is always `agt_or_obj`.

To amend the above two defects, we redefine the syntax in [8, pp. 3–4] as follows.

Definition 2.1 (Signature). Let $\texttt{Var}$ be a countably infinite set of *variables*, $\texttt{Cn}$ a countable set of *constants*, $\texttt{Fn}$ a countable set of *function symbols*, and $\texttt{Rel}$ a countable set of *relation symbols* containing the *equality symbol* $=$. Let $\langle \texttt{TYPE}, \preccurlyeq \rangle$ be also the ordered set of *types* where $\texttt{TYPE} = \{\, \texttt{agt}, \texttt{obj}, \texttt{agt\_or\_obj} \,\}$ and $\preccurlyeq$ is the reflexive ordering on $\texttt{TYPE}$ with $\texttt{agt} \preccurlyeq \texttt{agt\_or\_obj}$ and $\texttt{obj} \preccurlyeq \texttt{agt\_or\_obj}$, i.e.,

$$\preccurlyeq \;:=\; \{\, \langle \tau, \tau \rangle \mid \tau \in \texttt{TYPE} \,\} \cup \{\, \langle \texttt{agt}, \texttt{agt\_or\_obj} \rangle, \langle \texttt{obj}, \texttt{agt\_or\_obj} \rangle \,\}.$$

A *type assignment* $\texttt{t}\colon \texttt{Var} \cup \texttt{Cn} \cup \texttt{Fn} \cup \texttt{Rel} \to \bigcup_{n \in \mathbb{N}} \texttt{TYPE}^n$ is an assignment mapping

1. a variable x to a type $\texttt{t}(x) \in \{\, \texttt{agt}, \texttt{obj} \,\}$ such that both $\texttt{Var} \cap \texttt{t}^{-1}[\{\, \texttt{agt} \,\}]$ and $\texttt{Var} \cap \texttt{t}^{-1}[\{\, \texttt{obj} \,\}]$ are countably infinite, where $\texttt{t}^{-1}[X]$ is the inverse image of a set X;
2. a constant c to a type $\texttt{t}(c) \in \{\, \texttt{agt}, \texttt{obj} \,\}$;
3. a function symbol f to a sequence of types $\texttt{t}(f) \in \texttt{TYPE}^n \times \{\, \texttt{agt}, \texttt{obj} \,\}$ for some $n \in \mathbb{N}$;
4. the equality symbol $=$ to the sequence of types $\texttt{t}(=) = \langle \texttt{agt\_or\_obj}, \texttt{agt\_or\_obj} \rangle$;
5. a relation symbol P distinct from $=$ to a sequence of types $\texttt{t}(P) \in \texttt{TYPE}^n$ for some $n \in \mathbb{N}$.

The tuple $\langle \texttt{Var}, \texttt{Cn}, \texttt{Fn}, \texttt{Rel}, \texttt{t} \rangle$ is called a *signature*.

Definition 2.2 (Term of Type). Let $\langle \texttt{Var}, \texttt{Cn}, \texttt{Fn}, \texttt{Rel}, \texttt{t} \rangle$ be a signature. The set of *terms of types* is defined as follows.

1. any variable $x \in \texttt{Var}$ is a term of type $\texttt{t}(x)$.
2. any constant $c \in \texttt{Cn}$ is a term of type $\texttt{t}(c)$.
3. If $t_1, \ldots, t_n$ are terms of types $\tau_1, \ldots, \tau_n$ and f is a function symbol in $\texttt{Fn}$ such that $\texttt{t}(f) = \langle \tau'_1, \ldots, \tau'_n, \tau'_{n+1} \rangle$ and $\tau_i \preccurlyeq \tau'_i$, then $f(t_1, \ldots, t_n)$ is a term of type τ'_{n+1}.

For convenience, henceforth we use a type assignment $\mathtt{t}$ to mean its uniquely extended assignment by letting $\mathtt{t}(f(t_1, \dots, t_n)) = \tau$ for each term of the form $f(t_1, \dots, t_n)$ of type τ.

Definition 2.3 (Language). Let $\langle \mathtt{Var}, \mathtt{Cn}, \mathtt{Fn}, \mathtt{Rel}, \mathtt{t} \rangle$ be a signature. The *language* is the set of formulas φ defined in the following BNF.

$$\varphi ::= P(t_1, \dots, t_n) \mid \neg\varphi \mid \varphi \wedge \varphi \mid K_s\varphi \mid \forall x\varphi,$$

where $t_1, \dots, t_n, s$ are terms with $\mathtt{t}(s) = \mathtt{agt}$ and $P \in \mathtt{Rel}$ such that $\mathtt{t}(P) = \langle \tau_1, \dots, \tau_n \rangle$ and $\mathtt{t}(t_i) \preccurlyeq \tau_i$. Note here that P can be $=$.

As usual, we use the notations $t \neq s := \neg(t = s)$, $\varphi \to \psi := \neg(\varphi \wedge \neg\psi)$, $\exists x\varphi := \neg\forall x\neg\varphi$, $\bot := P \wedge \neg P$ for some fixed nullary relation symbol P, and $\top := \neg\bot$.

We believe that our definitions successfully capture what was intended in the original definitions 1–3. On top of these definitions, we will follow [8, p. 4] to define the notions of *free variable* and *bound variable* in a formula as usual, where the set of free variables in $K_t\varphi$ is defined as the union of the set of variables in t and the set of free variables in φ. For a variable x, terms t, s and a formula φ such that $\mathtt{t}(x) = \mathtt{t}(s)$ and no variables in s are bound variables in φ, we also define *substitutions* $t(s/x)$ and $\varphi(s/x)$ of s for x in t and φ in a usual manner, except that $(K_t\varphi)(s/x) = K_{t(s/x)}\varphi(s/x)$. Whenever we write $t(s/x)$ or $\varphi(s/x)$, we tacitly assume that $\mathtt{t}(x) = \mathtt{t}(s)$ and no variables in s are bound variables in φ. We also define the lengths of term and formula as usual.

Let us now introduce the Kripke semantics presented in [8, pp. 5–6].

Definition 2.4 (Frame, [8, Def. 4]). A *frame* is a tuple $F = \langle D, W, R \rangle$ where

1. $D := D_{\mathtt{agt\_or\_obj}} := D_{\mathtt{agt}} \sqcup D_{\mathtt{obj}}$ is the disjoint union of a non-empty set $D_{\mathtt{agt}}$ of *agents* and a non-empty set $D_{\mathtt{obj}}$ of *objects*;
2. W is a non-empty set of *worlds*;
3. R is a mapping that assigns to each agent $i \in D_{\mathtt{agt}}$ a binary relation R_i on W, i.e., $R\colon D_{\mathtt{agt}} \to \mathcal{P}(W \times W)$.

Definition 2.5 (Model, [8, Def. 5]). Let $\langle \texttt{Var}, \texttt{Cn}, \texttt{Fn}, \texttt{Rel}, \texttt{t}\rangle$ be a signature. A *model* is a tuple $M = \langle D, W, R, I\rangle$ where $\langle D, W, R\rangle$ is a frame and I is an *interpretation* that maps

1. a pair $\langle c, w\rangle$ of some $c \in \texttt{Cn}$ and $w \in W$ to an element $I(c, w) \in D_{\texttt{t}(c)}$;
2. a pair $\langle f, w\rangle$ of some $f \in \texttt{Fn}$ and $w \in W$ to a function $I(f, w)\colon (D_{\tau_1} \times \cdots \times D_{\tau_n}) \to D_{\tau_{n+1}}$, where $\texttt{t}(f) = \langle \tau_1, \ldots, \tau_n, \tau_{n+1}\rangle$;
3. a pair $\langle =, w\rangle$ of the equality symbol $=$ and some $w \in W$ to the set $I(=, w) = \{\, \langle d, d\rangle \mid d \in D_{\texttt{agt\_or\_obj}} \,\}$;
4. a pair $\langle P, w\rangle$ of some $P \in \texttt{Rel} \setminus \{=\}$ and $w \in W$ to a subset $I(P, w)$ of $D_{\tau_1} \times \cdots \times D_{\tau_n}$, where $\texttt{t}(P) = \langle \tau_1, \ldots, \tau_n\rangle$.

Definition 2.6 (Valuation, [8, Def. 6, 7]). Let $\langle \texttt{Var}, \texttt{Cn}, \texttt{Fn}, \texttt{Rel}, \texttt{t}\rangle$ be a signature. A *valuation* is a mapping $\sigma\colon \texttt{Var} \to D$ such that $\sigma(x) \in D_{\texttt{t}(x)}$ and the valuation $\sigma[x \mapsto d]$ is the same valuation as σ except for assigning to a variable x an element $d \in D_{\texttt{t}(x)}$. Given a valuation σ, a world w and an interpretation I in a model, the extension $[\![t]\!]^{I,\sigma}_w$ of a term t is defined by $[\![x]\!]^{I,\sigma}_w = \sigma(x)$, $[\![c]\!]^{I,\sigma}_w = I(c, w)$, and $[\![f(t_1, \ldots, f_n)]\!]^{I,\sigma}_w = I(f, w)([\![t_1]\!]^{I,\sigma}_w, \ldots, [\![t_n]\!]^{I,\sigma}_w)$.

Definition 2.7 (Satisfaction, [8, Def. 8]). The *satisfaction* $M, w \models_\sigma \varphi$ of a formula φ at a world w in a model M under a valuation σ is defined as follows.

$$
\begin{array}{lll}
M, w \models_\sigma P(t_1, \ldots, t_n) & \text{iff} & \langle [\![t_1]\!]^{I,\sigma}_w, \ldots, [\![t_n]\!]^{I,\sigma}_w\rangle \in I(P, w) \quad (P \text{ can be } =)\\
M, w \models_\sigma \neg\varphi & \text{iff} & M, w \not\models_\sigma \varphi\\
M, w \models_\sigma \varphi \wedge \psi & \text{iff} & M, w \models_\sigma \varphi \quad \text{and} \quad M, w \models_\sigma \psi\\
M, w \models_\sigma \forall x\varphi & \text{iff} & M, w \models_{\sigma[x \mapsto d]} \varphi \quad \text{for all } d \in D_{\texttt{t}(x)}\\
M, w \models_\sigma K_t\varphi & \text{iff} & M, w' \models_\sigma \varphi \quad \text{for all } w' \in W \text{ such that}\\
 & & \langle w, w'\rangle \in R_{[\![t]\!]^{I,\sigma}_w}
\end{array}
$$

Definition 2.8 (Validity, [8, p. 25]). A formula φ is *valid over a frame* F if for all models M based on F, all worlds $w \in W$ and all valuations σ, it holds that $M, w \models_\sigma \varphi$. A formula φ is *valid over a class* $\mathbb{F}$ *of frames* if for all frames $F \in \mathbb{F}$, φ is valid over F.

Remark 2.9. Instead of the x-variant of a valuation σ used in [8], we adopted the notion of $\sigma[x \mapsto d]$ to give the satisfaction for $\forall x\varphi$. This change is just for the clarity of our proof and does not affect the satisfiability of formulas.

For ease of reference, henceforth we call this semantics *TML-semantics*.

To state precisely our result on the semantic incompleteness in the next section, we import the frame correspondence results in [8, p. 18] with some modifications. Let us define the notion of frame correspondence as usual and say that a frame $\langle W, D, R\rangle$ is reflexive, serial, transitive or euclidean if R_i is so for all $i \in D_{\mathtt{agt}}$. We then have the following frame correspondences.

Proposition 2.10. Let $\langle \mathtt{Var}, \mathtt{Cn}, \mathtt{Fn}, \mathtt{Rel}, \mathtt{t}\rangle$ be a signature and $x \in \mathtt{Var}$ with $\mathtt{t}(x) = \mathtt{agt}$.

1. $\mathsf{T} := \forall x(K_x\varphi \to \varphi)$ corresponds to the class of all reflexive frames.
2. $\mathsf{D} := \forall x \neg K_x \bot$ corresponds to the class of all serial frames.
3. $\mathsf{4} := \forall x(K_x\varphi \to K_xK_x\varphi)$ corresponds to the class of all transitive frames.
4. $\mathsf{5} := \forall x(\neg K_x\varphi \to K_x\neg K_x\varphi)$ corresponds to the class of all euclidean frames.

We can find this proposition in [8, p. 18] if $\mathtt{t}(x)$ is supposed to be $\mathtt{agt}$.

In addition to Proposition 2.10, however, [8] also makes the following claims at the same page, where $\#X$ denotes the cardinality of a set X:[1]

(a) $\mathsf{AO}^{\vec{x}_n}_y := \exists x_1 \cdots x_n((\bigwedge_{i<j\leq n} x_i \neq x_j) \wedge \forall y \bigvee_{i\leq n} y = x_i)$ corresponds to the class of all frames such that $\#D = n$;

(b) $\mathsf{A}^{\vec{x}_n}_y := \exists x_1 \cdots x_n((\bigwedge_{i\leq n} K_{x_i}\top) \wedge (\bigwedge_{i<j\leq n} x_i \neq x_j) \wedge \forall y(K_y\top \to \bigvee_{i\leq n} y = x_i))$ corresponds to the class of all frames such that $\#D_{\mathtt{agt}} = n$,

[1] In [8, p. 18], $\mathsf{AO}^{\vec{x}_n}_y$ and $\mathsf{A}^{\vec{x}_n}_y$ are called M and N, respectively.

Amongst these, (a) is false in the current two-sorted language. Consider a signature such that all of $x_1, \dots, x_n, y$ have type $\mathtt{agt}$. Then, it is easy to check that the validity of $\mathsf{AO}_y^{\vec{x}_n}$ over a frame entails $\#D_{\mathtt{agt}} = n$. Hence it follows from $D_{\mathtt{obj}} \neq \emptyset$ that $n < \#D_{\mathtt{agt}} + \#D_{\mathtt{obj}} = \#D$, which contradicts (a). The frame property to which $\mathsf{AO}_y^{\vec{x}_n}$ corresponds is actually as follows, where $|y|$ denotes the set $\{\, x_i \mid \mathtt{t}(y) = \mathtt{t}(x_i), 1 \leq i \leq n \,\}$ and $\overline{\tau}$ denotes the converse type of a type τ.

Proposition 2.11. Let $\langle \mathtt{Var}, \mathtt{Cn}, \mathtt{Fn}, \mathtt{Rel}, \mathtt{t} \rangle$ be a signature such that x_1, …, x_n, y are pairwise distinct variables. Then $\mathsf{AO}_y^{\vec{x}_n}$ corresponds to the class of all frames such that $\#D_{\mathtt{t}(y)} = \#|y|$ and $\#D_{\overline{\mathtt{t}(y)}} \geq (n - \#|y|)$.

Proof: Suppose that $\mathsf{AO}_y^{\vec{x}_n}$ is valid over a frame F. Taking a world w, an interpretation I and a valuation σ arbitrarily, we obtain $\langle F, I \rangle, w \models_\sigma \mathsf{AO}_y^{\vec{x}_n}$. We then have some pairwise distinct elements $d_1, \dots, d_{\#|y|} \in D_{\mathtt{t}(y)}$ and $d_{\#|y|+1}, \dots, d_n \in D_{\overline{\mathtt{t}(y)}}$ such that for all $d \in D_{\mathtt{t}(y)}$ the following disjunction holds: $d = d_1$ or $\cdots$ or $d = d_{\#|y|}$ or $d = d_{\#|y|+1}$ or $\cdots$ or $d = d_n$. Since $d \in D_{\mathtt{t}(y)}$ and $d_{\#|y|+1}, \dots, d_n \in D_{\overline{\mathtt{t}(y)}}$, this disjunction is equivalent to the disjunction that $d = d_1$ or $\cdots$ or $d = d_{\#|y|}$. Thus, $\#D_{\mathtt{t}(y)} = \#|y|$ and $\#D_{\overline{\mathtt{t}(y)}} \geq (n - \#|y|)$.

For the other direction, suppose $\#D_{\mathtt{t}(y)} = \#|y|$ and $\#D_{\overline{\mathtt{t}(y)}} \geq (n - \#|y|)$. We show that $\langle F, I \rangle, w \models_\sigma \mathsf{AO}_y^{\vec{x}_n}$ for any interpretation I, any world w and any valuation σ. By our supposition we have some pairwise distinct elements $d_1, \dots, d_{\#|y|} \in D_{\mathtt{t}(y)}$ and $d_{\#|y|+1}, \dots, d_n \in D_{\overline{\mathtt{t}(y)}}$ such that $D_{\mathtt{t}(y)} = \{\, d_1, \dots, d_{\#|y|} \,\}$. Note here that $\mathtt{t}(x_{k_1}) = \cdots = \mathtt{t}(x_{k_{\#|y|}}) = \mathtt{t}(y)$ and $\mathtt{t}(x_{k_{\#|y|+1}}) = \cdots = \mathtt{t}(x_{k_n}) = \overline{\mathtt{t}(y)}$ for some variables $x_{k_1}, \dots, x_{k_n}$ coming from $x_1, \dots, x_n$. Letting $\sigma' = \sigma[x_{k_1} \mapsto d_1] \cdots [x_{k_n} \mapsto d_n]$ and $\sigma'' = \sigma[x_1 \mapsto \sigma'(x_1)] \cdots [x_n \mapsto \sigma'(x_n)]$, we have $\langle F, I \rangle, w \models_{\sigma''} \bigwedge_{i<j\leq n} x_i \neq x_j$. On top of that, we have $\langle F, I \rangle, w \models_{\sigma''} \forall y \bigvee_{i \leq \#|y|} y = x_{k_i}$ hence $\langle F, I \rangle, w \models_{\sigma''} \forall y \bigvee_{i \leq n} y = x_i$. Thus, $\langle F, I \rangle, w \models_\sigma \mathsf{AO}_y^{\vec{x}_n}$. □

On the other hand, (b) is true on some reading, but then just a corollary of Proposition 2.11. For, (b) can be true only if we presuppose on the meta-level that $\mathsf{A}_y^{\vec{x}_n}$ is a well-formed formula under a given signature Σ, which means after all to presuppose at the meta-level that all of $x_1, \ldots, x_n$ have type $\mathtt{agt}$ under Σ. Then $\mathsf{A}_y^{\vec{x}_n}$ is just an equivalent variant of $\mathsf{AO}_y^{\vec{x}_n}$ corresponding to the class of all frames such that $\#D_{\mathtt{agt}} = n$, as shown below.

Corollary 2.12. Let $\langle \mathtt{Var}, \mathtt{Cn}, \mathtt{Fn}, \mathtt{Rel}, \mathtt{t} \rangle$ be a signature such that $x_1, \ldots,$ x_n, y are pairwise distinct variables with $\mathtt{t}(x_1) = \cdots = \mathtt{t}(x_n) = \mathtt{t}(y) = \mathtt{agt}$. Then each of $\mathsf{AO}_y^{\vec{x}_n}$ and $\mathsf{A}_y^{\vec{x}_n}$ corresponds to the class of all frames such that $\#D_{\mathtt{agt}} = n$.

Thus we may treat $\mathsf{A}_y^{\vec{x}_n}$ as $\mathsf{AO}_y^{\vec{x}_n}$ with $\mathtt{t}(x_1) = \cdots = \mathtt{t}(x_n) = \mathtt{t}(y) = \mathtt{agt}$.

Axioms			
	all propositional tautologies		
UE	$\forall x\varphi \to \varphi(y/x)$	K	$K_t(\varphi \to \psi) \to (K_t\varphi \to K_t\psi)$
Id	$t = t$	BF†	$\forall x K_t \varphi \to K_t \forall x \varphi$
PS	$x = y \to (\varphi(x/z) \to \varphi(y/z))$	KNI	$x \neq y \to K_t x \neq y$
∃Id	$c = c \to \exists x(x = c)$		
DD	$x \neq y$ if $\mathtt{t}(x) \neq \mathtt{t}(y)$		
Inference rules			
MP	From φ and $\varphi \to \psi$, infer ψ		
KG	From φ, infer $K_t\varphi$		
UG‡	From $\varphi \to \psi$, infer $\varphi \to \forall x\psi$		

†: x does not occur in t and ‡: x is not free in φ.

Table 1: The Hilbert-style system $\mathsf{H}\mathbf{K}$ for term-modal logic $\mathbf{K}$

Finally, we introduce the Hilbert-style system $\mathsf{H}\mathbf{K}$ for the minimal normal term-modal logic $\mathbf{K}$ and some expansions of it presented in [8, pp. 17–18]. The Hilbert-style system $\mathsf{H}\mathbf{K}$ is defined as in Table 1. For its expansions, put $\mathrm{AX} = \{\mathsf{T}, \mathsf{D}, \mathsf{4}, \mathsf{5}\}$ throughout the paper. Given a set $\Gamma \subseteq \mathrm{AX}$, the Hilbert-style system $\mathsf{H}\mathbf{K}\Gamma$ is defined to be the system obtained from $\mathsf{H}\mathbf{K}$ by adding all axioms in Γ. The notion of proof in $\mathsf{H}\mathbf{K}\Gamma$

is defined as usual. It is easy to check that $\mathsf{HK}\mathrm{AX}$ and $\mathsf{HS5} \coloneqq \mathsf{HK}\{\mathsf{T}, \mathsf{5}\}$ have the same provability.

Remark 2.13. In [8, Theorem 3], the strong completeness of some expansions of $\mathsf{HK}\Gamma$ with $\mathsf{AO}_y^{\vec{x}_n}$ and $\mathsf{A}_{y'}^{\vec{x'}_{n'}}$ is also claimed. Since there are some signatures and some $n, n' \geq 1$ such that the addition of $\mathsf{AO}_y^{\vec{x}_n}$ and $\mathsf{A}_{y'}^{\vec{x'}_{n'}}$ make $\mathsf{HK}\Gamma$ inconsistent, in those cases we cannot prove the semantic incompleteness of those expansions. On the other hand, the side conditions of $\mathsf{AO}_y^{\vec{x}_n}$ and $\mathsf{A}_{y'}^{\vec{x'}_{n'}}$ to keep the consistency of $\mathsf{HK}\Gamma \cup \{\mathsf{AO}_y^{\vec{x}_n}, \mathsf{A}_{y'}^{\vec{x'}_{n'}}\}$ are somewhat complicated.[2] Thus, for simplicity we confine ourselves to the expansions with axioms in AX.

Before going to the next section, we note that UE and PS are particularly relevant to the semantic incompleteness of $\mathsf{HK}\Gamma$. As remarked in Fagin et al. [3, pp. 88–89], the ordinary first-order axioms $\forall x\varphi \to \varphi(t/x)$ and $t = s \to (\varphi(t/z) \to \varphi(s/z))$ are not valid in Kripke semantics for first-order modal logic where constants or function symbols are interpreted as non-rigid. Probably because of this reason, [8] instead adopted UE and PS that are the variable-restricted versions of these ordinary first-order axioms. The problem is that UE and PS or their combinations with other axioms are not sufficient to derive a valid formula $x = c \to (P(x) \to P(c))$ over the class of all frames.

3. Semantic Incompleteness

In this section, for all $\Gamma \subseteq \mathrm{AX}$ we prove the semantic incompleteness of $\mathsf{HK}\Gamma$ by showing that $x = c \to (P(x) \to P(c))$ is valid over the class of all frames to which Γ corresponds in the TML-semantics but not provable in $\mathsf{HK}\Gamma$. For the former, since any valid formula over the class of all frames is also valid over the class of all frames to which Γ corresponds, it is sufficient to show that $x = c \to (P(x) \to P(c))$ is valid over the class of all frames in the TML-semantics. As expected, it is straightforward to show this fact.

[2] The following two side conditions are at least necessary: (1) if $\mathtt{t}(y) = \mathtt{t}(y')$ then $\#|y| = \#|y'|$; (2) if $\mathtt{t}(y) \neq \mathtt{t}(y')$ then $\#|y| \geq (n' - \#|y'|)$ and $\#|y'| \geq (n - \#|y|)$.

PROPOSITION 3.1. Let $\langle \mathtt{Var}, \mathtt{Cn}, \mathtt{Fn}, \mathtt{Rel}, \mathtt{t} \rangle$ be a signature, $x \in \mathtt{Var}$, $c \in \mathtt{Cn}$ and $P \in \mathtt{Rel}$ with $\mathtt{t}(P) = \langle \mathtt{agt\_or\_obj} \rangle$. A formula $x = c \to (P(x) \to P(c))$ is valid over the class of all frames in the TML-semantics.

For the latter, since any unprovable formula in **HS5** is also unprovable in **HK**Γ, it is sufficient to show that $x = c \to (P(x) \to P(c))$ is not provable in **HS5**. To this end, we introduce a new semantics in which **HS5** is sound with respect to the class of all reflexive, symmetric and transitive frames but $x = c \to (P(x) \to P(c))$ is not valid over this class of frames.

DEFINITION 3.2 (Non-standard Model). Let $\langle \mathtt{Var}, \mathtt{Cn}, \mathtt{Fn}, \mathtt{Rel}, \mathtt{t} \rangle$ be a signature. A *non-standard model* is a tuple $N = \langle D, W, R, J \rangle$ where $\langle D, W, R \rangle$ is a frame in the sense of Definition 2.4 and J is an interpretation that maps

1. a triple $\langle c, w, X \rangle$ of some $c \in \mathtt{Cn}$, some $w \in W$ and some $X \subseteq D^n$ for some $n \in \mathbb{N}$ to an element $J(c, w, X) \in D_{\mathtt{t}(c)}$;

2. a triple $\langle f, w, X \rangle$ of some $f \in \mathtt{Fn}$, some $w \in W$ and some $X \subseteq D^n$ for some $n \in \mathbb{N}$ to a function $J(f, w, X) \colon (D_{\tau_1} \times \cdots \times D_{\tau_n}) \to D_{\tau_{n+1}}$, where $\mathtt{t}(f) = \langle \tau_1, \ldots, \tau_{n+1} \rangle$;

3. a pair $\langle =, w \rangle$ of the equality symbol $=$ and some $w \in W$ to the set $J(=, w) = \{ \langle d, d \rangle \mid d \in D_{\mathtt{agt\_or\_obj}} \}$;

4. a pair $\langle P, w \rangle$ of some $P \in \mathtt{Rel} \setminus \{=\}$ and some $w \in W$ to a subset $J(P, w)$ of $D_{\tau_1} \times \cdots \times D_{\tau_n}$, where $\mathtt{t}(P) = \langle \tau_1, \ldots, \tau_n \rangle$.

Here is the intuition. A subset X of D^n is a set of sequences consisting of either/both of agents and objects. Thus, the set X mentioned in the meanings $J(c, w, X)$ and $J(f, w, X)$ of a constant c and a function symbol f can serve as the meaning of a relation symbol. This trick enables us to make the meanings of constants and function symbols relative to the meanings of relation symbols combined with them.

We then define the notion of satisfaction of formulas in non-standard models. In what follows, we use the same notion of valuation as in the TML-semantics and define the extension $[\![t]\!]^{J,\sigma}_{w,X}$ of a term t in a given non-standard model similarly by letting $[\![x]\!]^{J,\sigma}_{w,X} = \sigma(x)$, $[\![c]\!]^{J,\sigma}_{w,X} = J(c, w, X)$ and $[\![f(t_1, \ldots, t_n)]\!]^{J,\sigma}_{w,X} = J(f, w, X)([\![t_1]\!]^{J,\sigma}_{w,X}, \ldots, [\![t_n]\!]^{J,\sigma}_{w,X})$.

DEFINITION 3.3 (Satisfaction in Non-standard Model). The *satisfaction* $N, w \models_\sigma \varphi$ of a formula φ at a world w in a non-standard model N under a valuation σ is defined as follows.

$$
\begin{array}{lll}
N, w \models_\sigma P(t_1, \dots, t_n) & \text{iff} & \langle [\![t_1]\!]^{J,\sigma}_{w,J(P,w)}, \dots, [\![t_n]\!]^{J,\sigma}_{w,J(P,w)} \rangle \in J(P, w) \\
 & & (P \text{ can be } =) \\
N, w \models_\sigma \neg\varphi & \text{iff} & N, w \not\models_\sigma \varphi \\
N, w \models_\sigma \varphi \wedge \psi & \text{iff} & N, w \models_\sigma \varphi \quad \text{and} \quad N, w \models_\sigma \psi \\
N, w \models_\sigma \forall x \varphi & \text{iff} & N, w \models_{\sigma[x \mapsto d]} \varphi \quad \text{for all } d \in D_{\mathtt{t}(x)} \\
N, w \models_\sigma K_t \varphi & \text{iff} & N, w' \models_\sigma \varphi \quad \text{for all } w' \in W \text{ such that} \\
 & & \langle w, w' \rangle \in R_{[\![t]\!]^{J,\sigma}_{w,\emptyset}}
\end{array}
$$

What we should pay attention here is the satisfactions of atomic formula $P(t_1, \dots, t_n)$ and term-modal formula $K_t\varphi$. In the satisfaction of $P(t_1, \dots, t_n)$ in non-standard models, the meaning $[\![t_i]\!]^{J,\sigma}_{w,J(P,w)}$ of each t_i in $P(t_1, \dots, t_n)$ is determined by the interpretation J, the valuation σ, the world w and *the meaning* $J(P, w)$ *of the relation symbol* P *combined with terms* $t_1, \dots, t_n$. Thus, as explained in the following Example 3.4, the meaning of a constant c occurring in $P(c)$ could be different from that of c occurring in $Q(c)$.

Example 3.4. Let $lewis \in \mathtt{Cn}$ with $\mathtt{t}(lewis) = \mathtt{agt}$ and $SL, CF \in \mathtt{Rel}$ with $\mathtt{t}(SL) = \mathtt{t}(CF) = \langle \mathtt{agt} \rangle$, and consider a non-standard model such that $J(SL, w) = \{i \in D_{\mathtt{agt}} \mid i \text{ is one of the authors of } \textit{Symbolic Logic}\}$, $J(CF, w) = \{ i \in D_{\mathtt{agt}} \mid i \text{ is the author of } \textit{Counterfactuals} \}$, $J(lewis, w, J(SL, w))$ is C. I. Lewis and $J(lewis, w, J(CF, w))$ is D. Lewis. Then, the meaning $J(lewis, w, J(SL, w))$ of *lewis* occurring in $SL(lewis)$ is different from the meaning $J(lewis, w, J(CF, w))$ of *lewis* occurring in $CF(lewis)$. Note that, although $J(lewis, w, J(SL, w)) \in J(SL, w)$ in the above non-standard model, we can technically assign to $J(lewis, w, J(SL, w))$ D. Lewis to have a non-standard model such that $J(lewis, w, J(SL, w)) \notin J(SL, w)$.

On the other hand, because the meaning $[\![t]\!]^{J,\sigma}_{w,\emptyset}$ of t in K_t is determined independently of the meaning of any relation symbol, the satisfaction of $K_t\varphi$ in non-standard model is in effect the same as the satisfaction of $K_t\varphi$ in models of the TML-semantics. By this fact we can validate axioms K and BF in this semantics.

The notions of validity is defined as in the TML-semantics. For ease of reference, henceforth we call this semantics *non-standard semantics*.

Now it is easy to see that $x = c \to (P(x) \to P(c))$ is not valid over the class of all reflexive, symmetric and transitive frames in the non-standard semantics.

PROPOSITION 3.5. Let $\langle \mathtt{Var}, \mathtt{Cn}, \mathtt{Fn}, \mathtt{Rel}, \mathtt{t}\rangle$ be a signature, $x \in \mathtt{Var}$, $c \in \mathtt{Cn}$ with $\mathtt{t}(x) = \mathtt{t}(c)$ and $P \in \mathtt{Rel}$ with $\mathtt{t}(P) = \langle \mathtt{agt\_or\_obj}\rangle$. A formula $x = c \to (P(x) \to P(c))$ is not valid over the class of all reflexive, symmetric and transitive frames in the non-standard semantics.

PROOF: We may assume $\mathtt{t}(x) = \mathtt{t}(c) = \mathtt{agt}$ without loss of generality. Let $N = \langle D, W, R, J\rangle$ be a non-standard model such that $w \in W$, R_i is reflexive, symmetric and transitive for all $i \in D_{\mathtt{agt}}$, $D_{\mathtt{agt}} = \{\alpha, \beta\}$, $J(c, w, \{\langle d, d\rangle \mid d \in D_{\mathtt{agt\_or\_obj}}\}) = \alpha$, $J(c, w, \{\alpha\}) = \beta$ and $J(P, w) = \{\alpha\}$. Let σ be also a valuation such that $\sigma(x) = \alpha$. Since $[\![x]\!]^{J,\sigma}_{w,J(=,w)} = \sigma(x) = \alpha = J(c, w, \{\langle d, d\rangle \mid d \in D_{\mathtt{agt\_or\_obj}}\}) = J(c, w, J(=, w)) = [\![c]\!]^{J,\sigma}_{w,J(=,w)}$, we have $N, w \models_\sigma x = c$. It is also easy to see $N, w \models_\sigma P(x)$. However, since $[\![c]\!]^{J,\sigma}_{w,J(P,w)} = J(c, w, J(P, w)) = J(c, w, \{\alpha\}) = \beta$, it fails that $N, w \models_\sigma P(c)$. Therefore $x = c \to (P(x) \to P(c))$ is not valid over the class of all reflexive, symmetric and transitive frames in the non-standard semantics. □

On the other hand, we can use the following lemmas to prove the soundness of $\mathbf{HS5}$ in the non-standard semantics, and thereby obtain the unprovability of $x = c \to (P(x) \to P(c))$ in $\mathbf{HS5}$.

LEMMA 3.6. *Let $\langle \mathtt{Var}, \mathtt{Cn}, \mathtt{Fn}, \mathtt{Rel}, \mathtt{t}\rangle$ be a signature and $x, y \in \mathtt{Var}$ with $\mathtt{t}(x) = \mathtt{t}(y)$. Let $\langle D, W, R, J\rangle$ be also a non-standard model, w a world, X a subset of D^n for some $n \in \mathbb{N}$ and σ a valuation. For all terms t,*

$$[\![t(y/x)]\!]^{J,\sigma}_{w,X} \quad = \quad [\![t]\!]^{J,\sigma[x\mapsto\sigma(y)]}_{w,X}.$$

Proof: By induction on the length of terms. □

Lemma 3.7. *Let* $\langle \mathtt{Var}, \mathtt{Cn}, \mathtt{Fn}, \mathtt{Rel}, \mathtt{t}\rangle$ *be a signature,* $x, y \in \mathtt{Var}$ *with* $\mathtt{t}(x) = \mathtt{t}(y)$ *and* $N = \langle D, W, R, J\rangle$ *a non-standard model. For all worlds* w*, all valuations* σ *and all formulas* φ*,*

$$N, w \models_\sigma \varphi(y/x) \quad \textit{iff} \quad N, w \models_{\sigma[x\mapsto\sigma(y)]} \varphi.$$

Proof: By induction on the length of formulas. □

Theorem 3.8 (Soundness). *If* φ *is provable in* **HS5***, then* φ *is valid over the class of all reflexive, symmetric and transitive frames in the non-standard semantics.*

Proof: It is sufficient to prove that all axioms in **HS5** are valid and that all inference rules preserve validity. Since the proof of the latter is done as usual, we see only the former.

- For any propositional tautology, its validity is obvious since the non-standard semantics gives the ordinary satisfactions for $\neg$ and $\wedge$.
- For UE, i.e., $\forall x\varphi \to \varphi(y/x)$, suppose $N, w \models_\sigma \forall x\varphi$. Then we have $N, w \models_{\sigma[x\mapsto\sigma(y)]} \varphi$. Thus by Lemma 3.7 $N, w \models_\sigma \varphi(y/x)$ holds, as required.
- For Id, i.e., $t = t$, its validity is obvious.
- For PS, i.e., $x = y \to (\varphi(x/z) \to \varphi(y/z))$, its validity is shown by induction on φ.
  - For φ being of the form $P(t_1, \dots, t_n)$, suppose $N, w \models_\sigma x = y$ and $N, w \models_\sigma P(t_1, \dots, t_n)(x/z)$. Since

    $$\langle [\![t_1(x/z)]\!]^{J,\sigma}_{w,J(P,w)}, \dots, [\![t_n(x/z)]\!]^{J,\sigma}_{w,J(P,w)}\rangle \in J(P, w),$$

    we can use $\sigma(x) = \sigma(y)$ and Lemma 3.6 to obtain

    $$\langle [\![t_1(y/z)]\!]^{J,\sigma}_{w,J(P,w)}, \dots, [\![t_n(y/z)]\!]^{J,\sigma}_{w,J(P,w)}\rangle \in J(P, w).$$

    Thus $N, w \models_\sigma P(t_1, \dots, t_n)(y/z)$.

- For φ being of the forms $\neg\psi$ or $\psi \wedge \gamma$, the proof is straightforward.
- For φ being of the form $\forall z'\psi$, notice first that $z' \neq x$ and $z' \neq y$ since x, y are assumed not to be bound in φ whenever we write $\varphi(x/z)$ and $\varphi(y/z)$. Suppose $N, w \models_\sigma x = y$ and $N, w \models_\sigma (\forall z'\psi)(x/z)$. If $z' = z$, obviously $N, w \models_\sigma (\forall z'\psi)(y/z)$. If $z' \neq z$, then we have $N, w \models_\sigma \forall z'\psi(x/z)$ thus $N, w \models_{\sigma[z' \mapsto d]} \psi(x/z)$ for all $d \in D_{\mathtt{t}(z')}$. Since we have $N, w \models_{\sigma[z' \mapsto d]} x = y$ for all $d \in D_{\mathtt{t}(z')}$, by the inductive hypothesis we obtain $N, w \models_{\sigma[z' \mapsto d]} \psi(y/z)$ for all $d \in D_{\mathtt{t}(z')}$. Therefore, $N, w \models_\sigma (\forall z'\psi)(y/z)$.
- For φ being of the form $K_t\psi$, suppose $N, w \models_\sigma x = y$ and $N, w \models_\sigma (K_t\psi)(x/z)$. Then $N, w' \models_\sigma \psi(x/z)$ for all $w' \in W$ such that $\langle w, w' \rangle \in R_{[\![t(x/z)]\!]^{J,\sigma}_{w,\emptyset}}$. Now, we have $N, w' \models_\sigma x = y$ for all $w' \in W$, and $[\![t(x/z)]\!]^{J,\sigma}_{w,\emptyset} = [\![t(y/z)]\!]^{J,\sigma}_{w,\emptyset}$ by $\sigma(x) = \sigma(y)$ and Lemma 3.6. So by the inductive hypothesis we obtain $N, w' \models_\sigma \psi(y/z)$ for all $w' \in W$ such that $\langle w, w' \rangle \in R_{[\![t(y/z)]\!]^{J,\sigma}_{w,\emptyset}}$. Thus, $N, w \models_\sigma (K_t\psi)(y/z)$.

- For $\exists$Id, i.e., $c = c \to \exists x(x = c)$, notice that $N, w \models_{\sigma[x \mapsto J(c,w,J(=,w))]} x = c$. Then we have $N, w \models_\sigma \exists x(x = c)$ hence $N, w \models_\sigma c = c \to \exists x(x = c)$, as required.
- For DD, i.e., $x \neq y$ if $\mathtt{t}(x) \neq \mathtt{t}(y)$, suppose $\mathtt{t}(x) \neq \mathtt{t}(y)$ and let N, w and σ be arbitrary. By the definition of valuation, each of $\sigma(x)$ and $\sigma(y)$ is in $D_{\mathtt{t}(x)}$ and $D_{\mathtt{t}(y)}$, respectively. Since $\mathtt{t}(x) \neq \mathtt{t}(y)$, $D_{\mathtt{t}(x)}$ and $D_{\mathtt{t}(y)}$ must be disjoint. Thus $N, w \models_\sigma x \neq y$, as required.
- For K, i.e., $K_t(\varphi \to \psi) \to (K_t\varphi \to K_t\psi)$, suppose $N, w \models_\sigma K_t(\varphi \to \psi)$ and $N, w \models_\sigma K_t\varphi$. Let w' be any world such that $\langle w, w' \rangle \in R_{[\![t]\!]^{J,\sigma}_{w,\emptyset}}$. Then we have $N, w' \models_\sigma \varphi \to \psi$ and $N, w' \models_\sigma \varphi$. Thus $N, w' \models_\sigma \psi$, as required.
- For BF, i.e., $\forall x K_t\varphi \to K_t\forall x\varphi$ for x not occurring in t, suppose $N, w \models_\sigma \forall x K_t\varphi$. To show $N, w \models_\sigma K_t\forall x\varphi$, let w' be any world such that $\langle w, w' \rangle \in R_{[\![t]\!]^{J,\sigma}_{w,\emptyset}}$ and take any $d \in D_{\mathtt{t}(x)}$. By our supposition, we have $N, w \models_{\sigma[x \mapsto d]} K_t\varphi$. Now $[\![t]\!]^{J,\sigma}_{w,\emptyset} = [\![t]\!]^{J,\sigma[x \mapsto d]}_{w,\emptyset}$ holds since x does

not occur in t. Thus by $\langle w, w' \rangle \in R_{[\![t]\!]^{J,\sigma[x\mapsto d]}_{w,\emptyset}}$, we have $N, w' \models_{\sigma[x\mapsto d]} \varphi$, as required.

- For KNI, i.e., $x \neq y \to K_t x \neq y$, suppose $N, w \models_\sigma x \neq y$. By definition, obviously $N, w' \models_\sigma x \neq y$ for all worlds w'. Thus $N, w \models K_t x \neq y$, as required.
- For T, i.e., $\forall x(K_x\varphi \to \varphi)$, suppose $N, w \models_{\sigma[x\mapsto d]} K_x\varphi$ for any $d \in D_{\mathtt{t}(x)}$. Since $\langle w, w \rangle \in R_{\sigma[x\mapsto d](x)}$ by the reflexivity of the frame of N, we have $N, w \models_{\sigma[x\mapsto d]} \varphi$, as required.
- For 5, i.e., $\forall x(\neg K_x\varphi \to K_x\neg K_x\varphi)$, suppose $N, w \models_{\sigma[x\mapsto d]} \neg K_x\varphi$ for any $d \in D_{\mathtt{t}(x)}$. To show $N, w \models_{\sigma[x\mapsto d]} K_x\neg K_x\varphi$, it is sufficient to show $N, v \models_{\sigma[x\mapsto d]} \neg K_x\varphi$ for any world v such that $\langle w, v \rangle \in R_{\sigma[x\mapsto d](x)}$. Now by our supposition we have some world u such that $\langle w, u \rangle \in R_{\sigma[x\mapsto d](x)}$ and $N, u \not\models_{\sigma[x\mapsto d]} \varphi$. Since $\langle v, u \rangle \in R_{\sigma[x\mapsto d](x)}$ by the euclideaness of the frame of N, we have $N, v \models_{\sigma[x\mapsto d]} \neg K_x\varphi$, as required.

By the above argument the proof has completed. □

THEOREM 3.9. *Let* $\Sigma = \langle \mathtt{Var}, \mathtt{Cn}, \mathtt{Fn}, \mathtt{Rel}, \mathtt{t} \rangle$ *be a signature,* $x \in \mathtt{Var}$, $c \in \mathtt{Cn}$ *with* $\mathtt{t}(x) = \mathtt{t}(c)$ *and* $P \in \mathtt{Rel}$ *with* $\mathtt{t}(P) = \langle \mathtt{agt\_or\_obj} \rangle$. *A formula* $x = c \to (P(x) \to P(c))$ *is not provable in* **HS5**.

PROOF: If $x = c \to (P(x) \to P(c))$ is provable in **HS5**, then by the soundness (Theorem 3.8) it must be valid over the class of all reflexive, symmetric and transitive frames in the non-standard semantics, which contradicts Proposition 3.5. □

We can now get the semantic incompleteness of **HK**Γ contradicting Theorem 3 in [8], as follows.

THEOREM 3.10 (Semantic Incompleteness of **HK**Γ). *Let* Γ ⊆ AX. *The Hilbert-style system* **HK**Γ *is semantically incomplete with respect to the class of all frames to which* Γ *corresponds in the TML-semantics, i.e., there exists some formula* φ *such that* φ *is valid over the class of all frames to which* Γ *corresponds in the TML-semantics, but not provable in* **HK**Γ.

PROOF: By Proposition 3.1 it follows that $x = c \to (P(x) \to P(c))$ is valid over the class of all frames to which Γ corresponds in the TML-semantics. On the other hand, by Theorem 3.9 it follows that $x = c \to (P(x) \to P(c))$ is not provable in $\mathsf{H}\mathbf{K}\Gamma$. □

COROLLARY 3.11 ([10, Corollary 1]). The Hilbert-style system $\mathsf{H}\mathbf{K}$ is semantically incomplete with respect to the class of all frames in the TML-semantics.

4. Conclusion

In this paper, for a set $\Gamma \subseteq \mathrm{AX} = \{\mathsf{T}, \mathsf{D}, \mathsf{4}, \mathsf{5}\}$, we proved that Liberman et al.[8]'s Hilbert-style system $\mathsf{H}\mathbf{K}\Gamma$ for the term-modal logic $\mathbf{K}\Gamma$ with equality and non-rigid terms is semantically incomplete with respect to the class of all frames to which Γ corresponds (Theorem 3.10). We also corrected the frame property to which [8] claims that axiom $\mathsf{AO}_y^{\vec{x}_n}$ corresponds (Proposition 2.11).

We make two remarks here. The first remark is that [8] also fails to prove the decidability of $\mathsf{H}\mathbf{K}\{\, \mathsf{AO}_y^{\vec{x}_n}, \mathsf{A}_{y'}^{\vec{x'}_{n'}} \,\}$ with $n' < n$ and $\mathtt{t}(x'_1) = \cdots = \mathtt{t}(x'_{n'}) = \mathtt{t}(y') = \mathtt{agt}$ ([8, item 1 of Proposition 7]). Let $\mathbb{F}$ be the class of all frames to which $\{\, \mathsf{AO}_y^{\vec{x}_n}, \mathsf{A}_{y'}^{\vec{x'}_{n'}} \,\}$ corresponds. The proof in [8] depends on a claim that $D_{\mathtt{obj}}$ is finite for any frame in $\mathbb{F}$. However, this is not the case in some signatures. A simple counterexample is the case in which $\mathtt{t}(x_1) = \cdots = \mathtt{t}(x_{n'}) = \mathtt{t}(y) = \mathtt{agt}$ and $\mathtt{t}(x_{n'+1}) = \cdots = \mathtt{t}(x_n) = \mathtt{obj}$. Then Proposition 2.11 tells us only that every frame in $\mathbb{F}$ satisfies $\#D_{\mathtt{agt}} = n'$ and $\#D_{\mathtt{obj}} \geq (n - n')$, so we can find a frame in $\mathbb{F}$ such that $D_{\mathtt{obj}}$ is infinite.

The second remark is that, as the problematic first-order formula $x = c \to (P(x) \to P(c))$ suggests, the semantic incompleteness of $\mathsf{H}\mathbf{K}\Gamma$ is irrelevant to its term-modal or two-sorted aspects. To make the point clear, let $\mathcal{L}$ be a first-order modal language having equality, constants, function symbols and only the ordinary non-indexed modal operator $\Box$ as its modal operators. Say that the semantics for first-order modal logic (the FOML-semantics for short) is the Kripke semantics of the constant domain given to $\mathcal{L}$ in which the accessibility relation is a binary relation on worlds, and constants and function symbols are interpreted relative to worlds. Using a semantics similar to the non-standard semantics introduced in Section 3, we can in fact prove that the Hilbert-style system naturally obtained from $\mathsf{HK}\Gamma$ by changing from the two-sorted term-modal language to $\mathcal{L}$ becomes semantically incomplete with respect to the corresponding class of frames in the FOML-semantics. The question to be asked here is how we can obtain a sound and complete Hilbert-style system with respect to this class of frames in the FOML-semantics. To the best of our knowledge, such a Hilbert-style system seems to have never been provided together with a detailed proof in the literature.[3]

A further direction to be pursued is to give sound and complete Hilbert-style systems for term-modal logics including **K** with equality and non-rigid terms. Such systems, for example, might be obtained as slight modifications of the system given in Fagin et al. [3, p. 90]. Another further direction that might be worth studying is to apply the non-standard semantics to the analysis of natural language. As Example 3.4 suggests, it is reasonable to see $J(P, w)$ in $J(c, w, J(P, w))$ as a kind of *context* uniquely determining the denotation of a constant c at a world w. Thus, the non-standard semantics could be used for a semantics capturing the context-dependency of the denotations of nouns in natural language.

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[3] As for a sound and complete proof system with respect to the multi-modal FOML-semantics with the epistemic accessibility relation for each agent, Fagin et al. [3, p. 90] offered a Hilbert-style system having two first-order principles $A(t/x) \to \exists x A$ and $t = s \to (A(t/z) \leftrightarrow A(s/z))$ as axioms with a restriction that t, s must be variables if A has any occurrence of an (non term-modal) epistemic operator K_a. However, the proof of this system's completeness is omitted.

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