Bulletin of the **S**ection of **L**ogic
Volume 54/2 (2025), pp. 157–205
https://doi.org/10.18778/0138-0680.2025.04

Norihiro Kamide

CUT-ELIMINATION AND NORMALIZATION THEOREMS FOR CONNEXIVE LOGICS OVER WANSING'S C[1]

Abstract

Gentzen-style sequent calculi and Gentzen-style natural deduction systems are introduced for a family (C-family) of connexive logics over Wansing's basic constructive connexive logic C. The C-family is derived from C by incorporating Peirce's law, the law of excluded middle, and the generalized law of excluded

[1]This paper is an extended and refined version of the preliminary short conference paper [16], which was published in the proceedings of the 11th International Conference on Non-Classical Logics – Theory and Applications (NCL'24). The results in Sections 4 and 5, concerning natural deduction systems with general elimination rules and certain results on the N-family, are new contributions in this paper. Additionally, this paper includes newly added detailed proofs and further remarks in various sections, including a new concluding remarks section, Section 6. Some errors in the normalization proofs presented in [16] have also been corrected in this paper.

Presented by: Michał Zawidzki
Received: December 22, 2024, **Received in revised form**: April 17, 2025,
Accepted: June 11, 2025, **Published online**: July 2, 2025

middle. Natural deduction systems with general elimination rules are also introduced for the C-family. Theorems establishing the equivalence between the proposed sequent calculi and natural deduction systems are demonstrated. Cut-elimination and normalization theorems are established for the proposed sequent calculi and natural deduction systems, respectively. Additionally, similar results are obtained for a family (N-family) of paraconsistent logics over Nelson's constructive four-valued logic N4.

Keywords: connexive logic, cut-elimination theorem, normalization theorem.

2020 Mathematical Subject Classification: 03B50, 03B53.

1. Introduction

Connexive logics are recognized as philosophically plausible paraconsistent contradictory logics [3, 22, 40, 43]. A distinguishing feature of connexive logics is their validation of the so-called Aristotle's theses: $\sim(\alpha\rightarrow\sim\alpha)$ and $\sim(\sim\alpha\rightarrow\alpha)$, and the so-called Boethius' theses: $(\alpha\rightarrow\beta)\rightarrow\sim(\alpha\rightarrow\sim\beta)$ and $(\alpha\rightarrow\sim\beta)\rightarrow\sim(\alpha\rightarrow\beta)$. On the one hand, the roots of connexive logics can be traced back to Aristotle and Boethius. On the other hand, modern perspectives on connexive logics were established by Angell [3] and McCall [22].

A basic constructive connexive logic referred to as C, considered a variant of Nelson's constructive four-valued logic N4 [2, 24, 19], was introduced by Wansing in [40]. Furthermore, C was extended by Wansing in [40] to introduce a constructive connexive modal logic, serving as a constructive connexive analogue of the smallest normal modal logic K. For further details on connexive logics, including C, refer to, for example, [3, 22, 40, 4, 17, 20, 43, 30, 26] and the references therein.

In this study, a unified Gentzen-style framework for proving cut-elimination and normalization theorems is employed to investigate several connexive logics over Wansing's C. The term "unified Gentzen-style framework" means that we can handle cut-elimination theorems in Gentzen-style sequent calculi and normalization theorems in Gentzen-style natural deduction systems uniformly, with equivalences between these calculi and systems. Additionally, natural deduction systems with general elimination rules are added to this framework.

The logics under consideration include Omori and Wansing's connexive logic C3 [30], Wansing's material connexive logic MC [43], and Cantwell's connexive logic CN [4]. The relationships among these logics can be summarized as follows: C3 is obtained from C by adding the law of excluded middle $\neg\alpha\vee\alpha$, MC is obtained from C by adding Peirce's law $((\alpha\to\beta)\to\alpha)\to\alpha$, and CN is obtained from C3 by adding Peirce's law.

On the one hand, Gentzen-style or G3-style sequent calculi for C, C3, CN and some intermediate logics between C and C3 have been introduced and investigated [40, 30, 6, 25], along with a Gentzen-style natural deduction system for the implicational fragment of C [13]. On the other hand, a unified Gentzen-style framework for C, C3, MC, and CN has not been established. Therefore, we construct such a framework in this study. This framework enables an integrated proof-theoretical treatment of these logics and establishes a natural correspondence between sequent calculi and natural deduction systems for them.

We now discuss some related works on sequent calculi for connexive logics. The cut-elimination theorem for a Gentzen-style sequent calculus, referred to as sC, was proved by Wansing in [40], although the name sC was not used by him. The cut-elimination theorems for G3-style sequent calculi, namely G3C and G3C3at for C and C3, respectively, were established by Omori and Wansing in [30]. In this context, G3C3at is a sequent calculus that incorporates the rule of atomic excluded middle (at-ex-middle) in place of the rule of excluded middle (ex-middle). The admissibility of (ex-middle) in G3C3at was also demonstrated by them. Consequently, the cut-elimination theorem for a G3-style sequent calculus, referred to as G3C3, which is obtained from G3C3at by replacing (at-ex-middle) with (ex-middle), was also demonstrated by them in [26]. Additionally, the first-order extensions of G3C, G3C3at, and G3C3 were also introduced and investigated by them. The systems G3C, G3C3at, and G3C3 were also used by Niki and Wansing in [26] to explore the provable contradictions of C and C3.

Several sequent calculi for some intermediate logics between C and C3 have recently been studied by Niki in [25]. A three-sided sequent calculus for CN, under the name CC/TTm, has recently been introduced and investigated by Égré et al. in [6]. A natural deduction system, NC2, and a two-sorted typed λ-calculus, 2λ, were introduced and investigated by Wansing in [42] for the bi-connexive propositional logic 2C. Natural deduction systems for two variants of connexive logics concerning non-classical interpretations of a certain kind between negation and implication were studied by Francez in [9]. In addition, some extensions of C were studied by Olkhovikov in [27, 28] and by Omori in [29], although these studies are not concerned with sequent calculus or natural deduction system.

The structure of this paper is as follows.

In Section 2, we introduce Gentzen-style sequent calculi for C, C3, MC, and CN, referred to as sC, sC3, sMC, and sCN, respectively. Furthermore, we prove the cut-elimination theorems for these calculi. The calculi sC3, sMC, and sCN are obtained from sC by adding the excluded middle rule (ex-middle), the Peirce rule (Peirce), and both (ex-middle) and (Peirce), respectively. Moreover, we introduce alternative Gentzen-style sequent calculi for MC and CN, referred to as sMC* and sCN*, respectively. These calculi are obtained from sC by adding the generalized excluded middle rule (g-ex-middle) and both (ex-middle) and (g-ex-middle), respectively. We then prove a theorem establishing the cut-free equivalence between sMC* (sCN*) and sMC (sCN, resp.), along with presenting the cut-elimination theorems for sMC* and sCN*.

In Section 3, we introduce Gentzen-style natural deduction systems for C, C3, MC, and CN, referred to as nC, nC3, nMC, and nCN, respectively. We then prove a theorem establishing equivalence between nC, (nC3, nMC, and nCN), and sC, (sC3, sMC*, and sCN*, resp.). Furthermore, we prove the normalization theorems for nC, nC3, nMC, and nCN.

In Section 4, we introduce natural deduction systems with general elimination rules [33, 37], for C, C3, MC, and CN, referred to as gC, gC3, gMC, and gCN, resp. We then prove a theorem establishing equivalence between gC, (gC3, gMC, and gCN), and sC, (sC3, sMC*, and sCN*, resp.). Furthermore, we prove the normalization theorems for gC, gC3, gMC, and gCN.

In Section 5, we show some similar results on a family (N-family) of paraconsistent logics over Nelson's constructive four-valued logic N4 [2, 24]. The N-family is obtained from N4 by adding Peirce's law, the law of excluded middle, and/or the law of generalized excluded middle.

In Section 6, we conclude this study and provide some final remarks.

2. Gentzen-style sequent calculi and cut-elimination theorems

Formulas of connexive logics are constructed using countably many propositional variables, the logical connectives $\wedge$ (conjunction), $\vee$ (disjunction), $\rightarrow$ (implication), and $\sim$ (connexive negation). We use small letters $p, q, ...$ to denote propositional variables, Greek small letters $\alpha, \beta, ...$ to denote formulas, and Greek capital letters $\Gamma, \Delta, ...$ to denote finite (possibly empty) sets of formulas. A *sequent* is an expression of the form $\Gamma \Rightarrow \gamma$. We use the expression $L \vdash S$ to represent the fact that a sequent S is provable in a sequent calculus L. We say that "a rule R of inference is *admissible* in a sequent calculus L" if the following condition is satisfied: For any instance

$$\frac{S_1 \cdots S_n}{S}$$

of R, if $L \vdash S_i$ for all i, then $L \vdash S$. Furthermore, we say that "R is *derivable* in L" if there is a derivation from $S_1, \cdots, S_n$ to S in L.

We introduce Gentzen-style sequent calculi LJ$^+$ [34], sC [40], sC3, sMC, and sCN for positive intuitionistic logic, C [40], C3 [30], MC [43], and CN [4], respectively.

Definition 2.1 (LJ$^+$, sC, sC3, sMC, and sCN).

1. LJ$^+$ is defined by the initial sequents and structural and logical inference rules of the following form, for any propositional variable p:

$$p, \Gamma \Rightarrow p \text{ (init1)}$$

$$\frac{\Gamma \Rightarrow \alpha \quad \alpha, \Sigma \Rightarrow \gamma}{\Gamma, \Sigma \Rightarrow \gamma} \text{ (cut)}$$

$$\frac{\Gamma \Rightarrow \alpha \quad \beta, \Delta \Rightarrow \gamma}{\alpha{\rightarrow}\beta, \Gamma, \Delta \Rightarrow \gamma}\ ({\rightarrow}\text{left}) \qquad \frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha{\rightarrow}\beta}\ ({\rightarrow}\text{right})$$

$$\frac{\alpha, \beta, \Gamma \Rightarrow \gamma}{\alpha{\wedge}\beta, \Gamma \Rightarrow \gamma}\ ({\wedge}\text{left}) \qquad \frac{\Gamma \Rightarrow \alpha \quad \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha{\wedge}\beta}\ ({\wedge}\text{right})$$

$$\frac{\alpha, \Gamma \Rightarrow \gamma \quad \beta, \Gamma \Rightarrow \gamma}{\alpha{\vee}\beta, \Gamma \Rightarrow \gamma}\ ({\vee}\text{left})$$

$$\frac{\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \alpha{\vee}\beta}\ ({\vee}\text{right1}) \qquad \frac{\Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha{\vee}\beta}\ ({\vee}\text{right2}).$$

2. sC is obtained from LJ$^+$ by adding the negated initial sequents and logical inference rules of the following form, for any propositional variable p:

$${\sim}p, \Gamma \Rightarrow {\sim}p \ (\text{init2})$$

$$\frac{\alpha, \Gamma \Rightarrow \gamma}{{\sim}{\sim}\alpha, \Gamma \Rightarrow \gamma}\ ({\sim}\text{left}) \qquad \frac{\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow {\sim}{\sim}\alpha}\ ({\sim}\text{right})$$

$$\frac{\Gamma \Rightarrow \alpha \quad {\sim}\beta, \Delta \Rightarrow \gamma}{{\sim}(\alpha{\rightarrow}\beta), \Gamma, \Delta \Rightarrow \gamma}\ ({\sim}{\rightarrow}\text{left}) \qquad \frac{\alpha, \Gamma \Rightarrow {\sim}\beta}{\Gamma \Rightarrow {\sim}(\alpha{\rightarrow}\beta)}\ ({\sim}{\rightarrow}\text{right})$$

$$\frac{{\sim}\alpha, \Gamma \Rightarrow \gamma \quad {\sim}\beta, \Gamma \Rightarrow \gamma}{{\sim}(\alpha{\wedge}\beta), \Gamma \Rightarrow \gamma}\ ({\sim}{\wedge}\text{left})$$

$$\frac{\Gamma \Rightarrow {\sim}\alpha}{\Gamma \Rightarrow {\sim}(\alpha{\wedge}\beta)}\ ({\sim}{\wedge}\text{right1}) \qquad \frac{\Gamma \Rightarrow {\sim}\beta}{\Gamma \Rightarrow {\sim}(\alpha{\wedge}\beta)}\ ({\sim}{\wedge}\text{right2})$$

$$\frac{{\sim}\alpha, {\sim}\beta, \Gamma \Rightarrow \gamma}{{\sim}(\alpha{\vee}\beta), \Gamma \Rightarrow \gamma}\ ({\sim}{\vee}\text{left}) \qquad \frac{\Gamma \Rightarrow {\sim}\alpha \quad \Gamma \Rightarrow {\sim}\beta}{\Gamma \Rightarrow {\sim}(\alpha{\vee}\beta)}\ ({\sim}{\vee}\text{right}).$$

3. sC3 is obtained from sC by adding the excluded middle rule of the form:

$$\frac{{\sim}\alpha, \Gamma \Rightarrow \gamma \quad \alpha, \Gamma \Rightarrow \gamma}{\Gamma \Rightarrow \gamma}\ (\text{ex-middle}).$$

4. sMC is obtained from sC by adding the Peirce rule of the form:

$$\frac{\alpha{\rightarrow}\beta, \Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \alpha}\ \text{(Peirce)}.$$

5. sCN is obtained from sC3 by adding (Peirce).

Remark 2.2.

1. It is known that single-succedent Gentzen-style sequent calculi for classical logic are obtained from Gentzen's sequent calculus LJ (or other variants such as the G3-style sequent calculus G3ip) for intuitionistic logic by adding one of (ex-middle), (Peirce), and their variants. These single-succeddent calculi have been studied by several researchers [5, 7, 10, 1, 36, 23, 12, 15]. For a survey on these calculi, see, for example, [12, 15].

2. (ex-middle), which corresponds to the law of excluded middle $\sim\alpha\vee\alpha$, was introduced and investigated by von Plato [36, 23], although the name (ex-middle) was not used by him. He showed that (ex-middle) can be restricted to the inference rule of the form:

$$\frac{\sim p, \Gamma \Rightarrow \gamma \quad p, \Gamma \Rightarrow \gamma}{\Gamma \Rightarrow \gamma}\ \text{(at-ex-middle)}$$

   where p is a propositional variable. Namely, (at-ex-middle) and (ex-middle) are equivalent over intuitionistic logic. He proved the cut-elimination theorems for some sequent calculi with (at-ex-middle) or (ex-middle).

3. (Peirce), which corresponds to the Peirce law $((\alpha{\rightarrow}\beta){\rightarrow}\alpha){\rightarrow}\alpha$, was introduced and investigated by Curry [5], Felscher [7], Gordeev [10], and Africk [1]. The cut-elimination theorem for LJ + (Peirce) was proved by them. Specifically, Africk [1] obtained a simple embedding-based proof of the cut-elimination theorem for LJ + (Peirce). The subformula property for a version of LJ + (Peirce) without the falsity

constant $\perp$ was shown by Gordeev. Specifically, he proved in [10] that β in (Peirce) can be restricted to a subformula of some formulas in (Γ, α).

4. Gentzen's LK for classical logic, LJ + (ex-middle), and LJ + (Peirce) are theorem-equivalent within the language $\{\wedge, \vee, \rightarrow, \neg, \perp\}$. However, sC3, sMC, and sCN (and their corresponding logics C3, MC, and CN) are not logically-equivalent. This fact will be formally shown in Theorem 2.7.

PROPOSITION 2.3. Let L be LJ$^+$, sC, sC3, sMC, or sCN. For any formula α and any set Γ of formulas, we have: $L \vdash \alpha, \Gamma \Rightarrow \alpha$.

PROOF: By induction on α. $\square$

PROPOSITION 2.4. Let L be LJ$^+$, sC, sC3, sMC, or sCN. The following rule is admissible in cut-free L:

$$\frac{\Gamma \Rightarrow \gamma}{\alpha, \Gamma \Rightarrow \gamma} \text{ (we)}.$$

PROOF: By induction on the proofs P of $\Gamma \Rightarrow \gamma$ of (we) in cut-free L. $\square$

The following cut-elimination theorems for LJ$^+$ and sC are well-known.

THEOREM 2.5 (Cut-elimination for LJ$^+$ and sC [34, 40]). *Let L be* LJ$^+$ *or* sC*. The rule* (cut) *is admissible in cut-free L.*

We now show the cut-elimination theorems for sC3, sMC, and sCN.

THEOREM 2.6 (Cut-elimination for sC3, sMC, and sCN). *Let L be* sC3*,* sMC*, or* sCN*. The rule* (cut) *is admissible in cut-free L.*

PROOF (Sketch): We give a sketch of the proof.

• First, we show the cut-elimination theorem for sC3. It is known that the cut-elimination theorem for the G3-style sequent calculus G3C3 for C3, which has (ex-middle), holds [30]. Then, we can show the cut-free equivalence between G3C3 and sC3. Thus, from this equivalence and the cut-elimination theorem for G3C3, we obtain the cut-elimination theorem for sC3.

• Second, we show the cut-elimination theorem for sMC. It is known that the cut-elimination theorem for LJ + (Peirce) holds. This theorem was proved directly and indirectly by using the methods by Gordeev [10] and Africk [1]. Thus, the cut-elimination theorem for the negation-less fragment (i.e., LJ^+ + (Peirce)) of LJ + (Peirce) holds because LJ + (Peirce) is a conservative extension of LJ^+ + (Peirce) by the cut-elimination theorem for LJ + (Peirce). Then, we can show a theorem for embedding (cut-free) sMC into (cut-free) LJ^+ + (Peirce), and by using this theorem, we can show the cut-elimination theorem for sMC. We will show this in the following.

Prior to showing the embedding theorem, we introduce a translation of sMC to LJ^+ + (Peirce). Let Φ be a set of propositional variables and Φ' be the set $\{p' \mid p \in \Phi\}$ of propositional variables. Then, the language $\mathcal{L}_{\mathrm{MC}}$ of sMC is defined using Φ, $\wedge$, $\vee$, $\rightarrow$, and $\sim$. The language $\mathcal{L}_{\mathrm{Int+}}$ of LJ^+ is obtained from $\mathcal{L}_{\mathrm{MC}}$ by replacing $\sim$ with Φ'. A mapping f from $\mathcal{L}_{\mathrm{MC}}$ to $\mathcal{L}_{\mathrm{Int+}}$ is defined inductively by:

1. for any $p \in \Phi$, $f(p) := p$ and $f(\sim p) := p' \in \Phi'$,
2. $f(\alpha \sharp \beta) := f(\alpha) \sharp f(\beta)$ with $\sharp \in \{\wedge, \vee, \rightarrow\}$,
3. $f(\sim\sim\alpha) := f(\alpha)$,
4. $f(\sim(\alpha\wedge\beta)) := f(\sim\alpha)\vee f(\sim\beta)$,
5. $f(\sim(\alpha\vee\beta)) := f(\sim\alpha)\wedge f(\sim\beta)$,
6. $f(\sim(\alpha\rightarrow\beta)) := f(\alpha)\rightarrow f(\sim\beta)$.

An expression $f(\Gamma)$ denotes the result of replacing every occurrence of a formula α in Γ by an occurrence of $f(\alpha)$. We remark that a similar translation defined as above has been used by Gurevich [11], Rautenberg [32] and Vorob'ev [38] to embed Nelson's constructive logic [2, 24] into positive intuitionistic logic.

We then obtain the following theorem for embedding sMC into LJ^+ + (Peirce):

1. sMC $\vdash \Gamma \Rightarrow \gamma$ iff LJ^+ + (Peirce) $\vdash f(\Gamma) \Rightarrow f(\gamma)$,
2. sMC $-$ (cut) $\vdash \Gamma \Rightarrow \gamma$ iff LJ^+ + (Peirce) $-$ (cut) $\vdash f(\Gamma) \Rightarrow f(\gamma)$.

The proof of this theorem is almost the same as that for the theorem for embedding sC or a Gentzen-style sequent calculus for Nelson's paraconsistent four-valued logic N4 into LJ^+. For more information on these embedding theorems, see, for example, [18, 19, 17, 20, 14].

We are ready to prove of the cut-elimination theorem for sMC. Suppose that $\mathrm{sMC} \vdash \Gamma \Rightarrow \gamma$. Then, we have $\mathrm{LJ}^+ + (\mathrm{Peirce}) \vdash f(\Gamma) \Rightarrow f(\gamma)$ by the statement (1) of the theorem, and hence $\mathrm{LJ}^+ + (\mathrm{Peirce}) - (\mathrm{cut}) \vdash f(\Gamma) \Rightarrow f(\gamma)$ by the cut-elimination theorem for $\mathrm{LJ}^+ + (\mathrm{Peirce})$. Then, by the statement (2) of the theorem, we obtain $\mathrm{sMC} - (\mathrm{cut}) \vdash \Gamma \Rightarrow \gamma$.

• Finally, the cut-elimination theorem for sCN can be proved in a similar way as for sMC. □

Theorem 2.7 (Separation of C, C3, MC, and CN). *The logics* C*,* C3*,* MC*, and* CN *are not logically-equivalent.*

Proof: To show this, we use sC, sC3, sMC, sCN, and Theorem 2.6. Let p and q be distinct propositional variables. Then, we consider only the following facts:

1. $\Rightarrow ((p{\rightarrow}q){\rightarrow}p){\rightarrow}p$ is provable in cut-free sMC, but not provable in cut-free sC3,
2. $\Rightarrow {\sim}p{\vee}p$ is provable in cut-free sC3, but not provable in cut-free sMC.

The unprovabilities of these sequents are guaranteed by Theorem 2.6. We thus show the case for $\mathrm{sMC} - (\mathrm{cut}) \vdash\ \Rightarrow ((p{\rightarrow}q){\rightarrow}p){\rightarrow}p$ by:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p \Rightarrow p\ (\mathrm{init1}) \quad q \Rightarrow q\ (\mathrm{init1})}{p{\rightarrow}q, p \Rightarrow q}\ (\rightarrow\mathrm{left})}{p{\rightarrow}q \Rightarrow p{\rightarrow}q}\ (\rightarrow\mathrm{right}) \quad p \Rightarrow p\ (\mathrm{init1})}{p{\rightarrow}q, (p{\rightarrow}q){\rightarrow}p \Rightarrow p}\ (\rightarrow\mathrm{left})}{(p{\rightarrow}q){\rightarrow}p \Rightarrow p}\ (\mathrm{Peirce})}{\Rightarrow ((p{\rightarrow}q){\rightarrow}p){\rightarrow}p}\ (\rightarrow\mathrm{right})$$

and the case for $\mathrm{sC3} - (\mathrm{cut}) \vdash\ \Rightarrow {\sim}p{\vee}p$ by:

$$\dfrac{\dfrac{{\sim}p \Rightarrow {\sim}p\ (\mathrm{init2})}{{\sim}p \Rightarrow {\sim}p{\vee}p}\ (\vee\mathrm{right1}) \quad \dfrac{p \Rightarrow p\ (\mathrm{init1})}{p \Rightarrow {\sim}p{\vee}p}\ (\vee\mathrm{right2})}{\Rightarrow {\sim}p{\vee}p}\ (\mathrm{ex\text{-}middle}).$$

□

Next, we introduce alternative Gentzen-style sequent calculi sMC* and sCN* for MC and CN, respectively. These calculi will be used to prove the normalization theorems for the natural deduction systems nMC and nCN for MC and CN, respectively.

Definition 2.8 (sMC* and sCN*).

1. sMC* is obtained from sC by adding the generalized excluded middle rule of the form:

$$\dfrac{\alpha{\rightarrow}\beta, \Gamma \Rightarrow \gamma \quad \alpha, \Gamma \Rightarrow \gamma}{\Gamma \Rightarrow \gamma}\ \text{(g-ex-middle)}.$$

2. sCN* is obtained from sC3 by adding (g-ex-middle).

Remark 2.9.

1. (g-ex-middle), which corresponds to the generalized law of excluded middle $(\alpha{\rightarrow}\beta){\vee}\alpha$, was introduced and investigated by Kamide in [12], although the name (g-ex-middle) was not used by him. He proved the cut-elimination theorem for LJ + (g-ex-middle) using the method by Africk [1].

2. LJ + (g-ex-middle) is regarded as a sequent calculus for classical logic. Actually, (g-ex-middle) and (ex-middle) are equivalent over positive intuitionistic logic. (g-ex-middle) is regarded as a generalization of (ex-middle) if we assume the falsity constant $\perp$ and the definition ${\sim}\alpha := \alpha{\rightarrow}\perp$. (g-ex-middle) is also regarded as a generalization of (Peirce) and it was referred to as generalized Peirce rule (named (g-Peirce)) in [12].

3. The following is an example proof of $\Rightarrow (p{\rightarrow}q){\vee}p$ in cut-free sMC*:

$$\dfrac{\dfrac{\dfrac{\dfrac{p \Rightarrow p\ \text{(init1)} \quad q \Rightarrow q\ \text{(init1)}}{p{\rightarrow}q, p \Rightarrow q}\ (\rightarrow\text{left})}{p{\rightarrow}q \Rightarrow p{\rightarrow}q}\ (\rightarrow\text{right})}{p{\rightarrow}q \Rightarrow (p{\rightarrow}q){\vee}p}\ (\vee\text{right1}) \qquad \dfrac{p \Rightarrow p\ \text{(init1)}}{p \Rightarrow (p{\rightarrow}q){\vee}p}\ (\vee\text{right2})}{\Rightarrow (p{\rightarrow}q){\vee}p}\ \text{(g-ex-middle)}.$$

Proposition 2.10. Let L be sMC* or sCN*. For any formula α and any set Γ of formulas, we have: $L \vdash \alpha, \Gamma \Rightarrow \alpha$.

Proof: By induction on α. □

Proposition 2.11. Let L be sMC* or sCN*. The rule (we) is admissible in cut-free L.

Proof: Similar to the proof of Proposition 2.4. □

Theorem 2.12 (Equivalence between sMC (sCN) and sMC* (sCN*)). *Let L_1 and L_2 be the sequent calculi* sMC *and* sCN*, respectively. Let L_1^* and L_2^* be the sequent calculi* sMC* *and* sCN**, respectively. For any $i \in \{1, 2\}$, we have:*

1. $L_i \vdash \Gamma \Rightarrow \gamma$ *iff* $L_i^* \vdash \Gamma \Rightarrow \gamma$,
2. $L_i - (\text{cut}) \vdash \Gamma \Rightarrow \gamma$ *iff* $L_i^* - (\text{cut}) \vdash \Gamma \Rightarrow \gamma$.

Proof: We show only (2). The fact that $L_i - (\text{cut}) \vdash \Gamma \Rightarrow \gamma$ implies $L_i^* - (\text{cut}) \vdash \Gamma \Rightarrow \gamma$ is obvious because (Peirce) is an instance of (g-ex-middle). Thus, we show that $L_i^* - (\text{cut}) \vdash \Gamma \Rightarrow \gamma$ implies $L_i - (\text{cut}) \vdash \Gamma \Rightarrow \gamma$ by induction on the proofs P of $\Gamma \Rightarrow \gamma$ in $L_i^* - (\text{cut})$. We distinguish the cases according to the last inference of P and show only the following case.

Case (g-ex-middle): The last inference of P is of the form:

$$\dfrac{\begin{matrix}\vdots \\ \alpha{\to}\beta, \Gamma \Rightarrow \gamma\end{matrix} \quad \begin{matrix}\vdots \\ \alpha, \Gamma \Rightarrow \gamma\end{matrix}}{\Gamma \Rightarrow \gamma}\ \text{(g-ex-middle)}.$$

By induction hypotheses, we have: $L_i - (\text{cut}) \vdash \alpha{\to}\beta, \Gamma \Rightarrow \gamma$ and $L_i - (\text{cut}) \vdash \alpha, \Gamma \Rightarrow \gamma$. Then, we obtain the required fact:

$$\dfrac{\dfrac{\begin{matrix}\vdots\ \mathit{Ind.hyp.} \\ \alpha{\to}\beta, \Gamma \Rightarrow \gamma\end{matrix} \qquad \begin{matrix}\vdots\ \mathit{Ind.hyp.} \\ \alpha, \Gamma \Rightarrow \gamma\end{matrix}}{\gamma{\to}\alpha, \alpha{\to}\beta, \Gamma \Rightarrow \gamma}\ ({\to}\text{left})}{\alpha{\to}\beta, \Gamma \Rightarrow \gamma}\ \text{(Peirce)}.$$

□

THEOREM 2.13 (Cut-elimination for sMC* and sCN*). *Let L be* sMC* *or* sCN*. *The rule* (cut) *is admissible in cut-free L.*

PROOF: By Theorems 2.6 and 2.12. □

3. Gentzen-style natural deduction systems and normalization theorems

We now define Gentzen-style natural deduction systems NJ$^+$, nC, nC3, nMC, and nCN for positive intuitionistic logic, C, C3, MC, and CN, respectively. We use the notation $[\alpha]$ in the definitions of natural deduction systems to denote the discharged assumption (i.e., the formula α is a discharged assumption by the underlying logical inference rule).

DEFINITION 3.1 (NJ$^+$, nC, nC3, nMC, and nCN).

1. NJ$^+$ is defined as the logical inference rules of the form: [2]

$$\dfrac{\begin{matrix}[\alpha]\\ \vdots \\ \beta\end{matrix}}{\alpha{\to}\beta}\ (\to\mathrm{I}) \qquad \dfrac{\alpha{\to}\beta \quad \alpha}{\beta}\ (\to\mathrm{E})$$

$$\dfrac{\alpha \quad \beta}{\alpha\wedge\beta}\ (\wedge\mathrm{I}) \qquad \dfrac{\alpha\wedge\beta}{\alpha}\ (\wedge\mathrm{E1}) \qquad \dfrac{\alpha\wedge\beta}{\beta}\ (\wedge\mathrm{E2})$$

$$\dfrac{\alpha}{\alpha\vee\beta}\ (\vee\mathrm{I1}) \qquad \dfrac{\beta}{\alpha\vee\beta}\ (\vee\mathrm{I2}) \qquad \dfrac{\alpha\vee\beta \quad \begin{matrix}[\alpha]\\ \vdots \\ \gamma\end{matrix} \quad \begin{matrix}[\beta]\\ \vdots \\ \gamma\end{matrix}}{\gamma}\ (\vee\mathrm{E}).$$

[2]The discharge in $(\to\mathrm{I})$ can be vacuous. Namely, the following rule is an instance of $(\to\mathrm{I})$:

$$\dfrac{\beta}{\alpha{\to}\beta.}$$

2. nC is obtained from NJ$^+$ by adding the negated logical inference rules of the form:

$$\frac{\alpha}{{\sim}{\sim}\alpha}\ ({\sim}{\sim}\mathrm{I}) \qquad \frac{{\sim}{\sim}\alpha}{\alpha}\ ({\sim}{\sim}\mathrm{E})$$

$$\frac{\begin{array}{c}[\alpha]\\ \vdots\\ {\sim}\beta\end{array}}{{\sim}(\alpha{\rightarrow}\beta)}\ ({\sim}{\rightarrow}\mathrm{I}) \qquad \frac{{\sim}(\alpha{\rightarrow}\beta) \quad \alpha}{{\sim}\beta}\ ({\sim}{\rightarrow}\mathrm{E})$$

$$\frac{{\sim}\alpha}{{\sim}(\alpha\wedge\beta)}\ ({\sim}\wedge\mathrm{I1}) \qquad \frac{{\sim}\beta}{{\sim}(\alpha\wedge\beta)}\ ({\sim}\wedge\mathrm{I2})$$

$$\frac{{\sim}(\alpha\wedge\beta) \quad \begin{array}{c}[{\sim}\alpha]\\ \vdots\\ \gamma\end{array} \quad \begin{array}{c}[{\sim}\beta]\\ \vdots\\ \gamma\end{array}}{\gamma}\ ({\sim}\wedge\mathrm{E})$$

$$\frac{{\sim}\alpha \quad {\sim}\beta}{{\sim}(\alpha\vee\beta)}\ ({\sim}\vee\mathrm{I}) \qquad \frac{{\sim}(\alpha\vee\beta)}{{\sim}\alpha}\ ({\sim}\vee\mathrm{E1}) \qquad \frac{{\sim}(\alpha\vee\beta)}{{\sim}\beta}\ ({\sim}\vee\mathrm{E2}).$$

3. nC3 is obtained from nC by adding the rule of excluded middle of the form:

$$\frac{\begin{array}{c}[{\sim}\alpha]\\ \vdots\\ \gamma\end{array} \quad \begin{array}{c}[\alpha]\\ \vdots\\ \gamma\end{array}}{\gamma}\ (\mathrm{EM}).$$

4. nMC is obtained from nC by adding the generalized rule of excluded middle of the form:

$$\frac{\begin{array}{c}[\alpha{\rightarrow}\beta]\\ \vdots\\ \gamma\end{array} \quad \begin{array}{c}[\alpha]\\ \vdots\\ \gamma\end{array}}{\gamma}\ (\mathrm{GEM}).$$

5. nCN is obtained from nC3 by adding (GEM).

Remark 3.2.

1. (EM) and its restricted version (EM-at) with the propositional variable discharged assumptions were originally introduced by von Plato [36], and called there *Gem* (for *generalized excluded middle*) and *Gem-at*, respectively. He proved normalization theorems for systems with (EM) or (EM-at).

2. Using (EM) and (GEM), we can prove the formulas $\sim\alpha\vee\alpha$ and $(\alpha\to\beta)\vee\alpha$ by:

$$\dfrac{\dfrac{[\sim\alpha]^1}{\sim\alpha\vee\alpha}\,(\vee\mathrm{I}1) \qquad \dfrac{[\alpha]^1}{\sim\alpha\vee\alpha}\,(\vee\mathrm{I}2)}{\sim\alpha\vee\alpha}\,(\mathrm{EM})^1$$

$$\dfrac{\dfrac{[\alpha\to\beta]^1}{(\alpha\to\beta)\vee\alpha}\,(\vee\mathrm{I}1) \qquad \dfrac{[\alpha]^1}{(\alpha\to\beta)\vee\alpha}\,(\vee\mathrm{I}2)}{(\alpha\to\beta)\vee\alpha}\,(\mathrm{GEM})^1.$$

3. Using (GEM), we can prove the formula $((\alpha\to\beta)\to\alpha)\to\alpha$ by:

$$\dfrac{\dfrac{\dfrac{[(\alpha\to\beta)\to\alpha]^1 \quad [\alpha\to\beta]^2}{\alpha}\,(\to\mathrm{E}) \qquad [\alpha]^2}{\alpha}\,(\mathrm{GEM})^2}{((\alpha\to\beta)\to\alpha)\to\alpha}\,(\to\mathrm{I}^*)^1.$$

4. The following Peirce rule was introduced by Curry [5] and studied by Zimmermann [44]:

$$\dfrac{\begin{array}{c}[\alpha\to\beta]\\ \vdots\\ \alpha\end{array}}{\alpha}\,(\mathrm{PE}).$$

   Natural deduction systems with (PE) were considered by them for classical logic and the normalization theorems for these systems were proved by them.

5. (PE) is regarded as an instance of (GEM), and using (PE), we can prove the formula $((\alpha{\to}\beta){\to}\alpha){\to}\alpha$ by:

$$\dfrac{\dfrac{\dfrac{[(\alpha{\to}\beta){\to}\alpha]^1 \quad [\alpha{\to}\beta]^2}{\alpha}\,(\to\mathrm{E})}{\alpha}\,(\mathrm{PE})^2}{((\alpha{\to}\beta){\to}\alpha){\to}\alpha}\,(\to\mathrm{I})^1.$$

6. In this study, we do not consider the natural deduction systems nC + (PE) and nC + (EM) + (PE) because reduction conditions for nC + (PE) and nC + (EM) + (PE) cannot be defined in a similar and uniform way as for nMC and nCN. Moreover, we do not know if nC + (PE) (nC + (EM) + (PE)) and nMC (nCN, respectively) are logically-equivalent or not. Namely, we have not yet proved the equivalence (or difference) of nC + (PE) (nC + (EM) + (PE)) and nMC (nCN, respectively).

7. The $\{\to, \sim\}$-fragment of nC was introduced and investigated by Kamide in [13], wherein the strong normalization theorem for the fragment was proved.

Next, we define some notions for the natural deduction systems.

Definition 3.3. The inference rules ($\to$I), ($\wedge$I), ($\vee$I1), ($\vee$I2), ($\sim\sim$I), ($\sim\to$I), ($\sim\wedge$I1), ($\sim\wedge$I2), ($\sim\vee$I), (EM), and (GEM) are called *introduction rules*, and the inference rules ($\to$E), ($\wedge$E1), ($\wedge$E2), ($\vee$E), ($\sim\sim$E), ($\sim\to$E), ($\sim\wedge$E), ($\sim\vee$E1), and ($\sim\vee$E2) are called *elimination rules*. The notions of *major* and minor premises of the inference rules without (EM) and (GEM) are defined as usual. The notions of *derivation*, *(open and discharged) assumptions* of derivation, and *end-formula* of derivation are also defined as usual. Any derivation starts with an assumption α can be considered a derivation of α from itself. For a derivation $\mathcal{D}$, we use the expression oa($\mathcal{D}$) to denote the set of open assumptions of $\mathcal{D}$ and the expression end($\mathcal{D}$) to denote the end-formula of $\mathcal{D}$. A formula α is said to be *provable* in a natural deduction system N if there exists a derivation in N with no open assumptions whose end-formula is α.

Remark 3.4. There are no notions of major and minor premises of (EM) and (GEM). Namely, both the premises of (EM) and (GEM) are neither major nor minor premise. In this study, (EM) and (GEM) are treated as introduction rules.

Next, we define a reduction relation $\rhd$ on the set of derivations in the natural deduction systems. Prior to defining $\rhd$, we define some notions concerning $\rhd$.

DEFINITION 3.5. Let L be nC, nC3, nMC, or nCN. Let α be a formula occurring in a derivation $\mathcal{D}$ in L. Then, α is called a *maximum formula* in $\mathcal{D}$ if α satisfies the following conditions:

1. α is the conclusion of an introduction rule, ($\vee$E), or ($\sim\wedge$E),
2. α is the major premise of an elimination rule.

A derivation is said to be *normal* if it contains no maximum formula. The notion of substitution of derivations to assumptions is defined as usual. We assume that the set of derivations is closed under substitution.

DEFINITION 3.6 (Reduction relation). Let γ be a maximum formula in a derivation that is the conclusion of an inference rule R.

1. The definition of the reduction relation $\rhd$ at γ in nC is obtained by the following conditions.
   (a) R is ($\to$I) and γ is $\alpha\to\beta$:

$$\dfrac{\dfrac{\begin{matrix}[\alpha]\\ \vdots\ \mathcal{D}\\ \beta\end{matrix}}{\alpha\to\beta}\,(\to\mathrm{I}) \qquad \begin{matrix}\vdots\ \mathcal{E}\\ \alpha\end{matrix}}{\beta}\,(\to\mathrm{E}) \qquad \rhd \qquad \begin{matrix}\vdots\ \mathcal{E}\\ \alpha\\ \vdots\ \mathcal{D}\\ \beta.\end{matrix}$$

   (b) R is ($\wedge$I) and γ is $\alpha_1\wedge\alpha_2$:

$$\dfrac{\dfrac{\begin{matrix}\vdots\ \mathcal{D}_1\\ \alpha_1\end{matrix} \qquad \begin{matrix}\vdots\ \mathcal{D}_2\\ \alpha_2\end{matrix}}{\alpha_1\wedge\alpha_2}\,(\wedge\mathrm{I})}{\alpha_i}\,(\wedge\mathrm{E}i) \qquad \rhd \qquad \begin{matrix}\vdots\ \mathcal{D}_i\\ \alpha_i\end{matrix}$$

   where i is 1 or 2.

(c) R is ($\vee$I1) or ($\vee$I2) and γ is $\alpha_1\vee\alpha_2$:

$$\dfrac{\dfrac{\begin{matrix}\vdots\ \mathcal{D}\\ \alpha_i\end{matrix}}{\alpha_1\vee\alpha_2}\,(\vee\mathrm{I}i) \quad \begin{matrix}[\alpha_1]\\ \vdots\ \mathcal{E}_1\\ \delta\end{matrix} \quad \begin{matrix}[\alpha_2]\\ \vdots\ \mathcal{E}_2\\ \delta\end{matrix}}{\delta}\,(\vee\mathrm{E}) \qquad \rhd \qquad \begin{matrix}\vdots\ \mathcal{D}\\ \alpha_i\\ \vdots\ \mathcal{E}_i\\ \delta\end{matrix}$$

where i is 1 or 2.

(d) R is ($\vee$E):

$$\dfrac{\dfrac{\begin{matrix}\vdots\ \mathcal{D}_1\\ \alpha\vee\beta\end{matrix} \quad \begin{matrix}[\alpha]\\ \vdots\ \mathcal{D}_2\\ \gamma\end{matrix} \quad \begin{matrix}[\beta]\\ \vdots\ \mathcal{D}_3\\ \gamma\end{matrix}}{\gamma}\,(\vee\mathrm{E}) \quad \begin{matrix}\vdots\ \mathcal{E}_1\\ \delta_1\end{matrix} \quad \begin{matrix}\vdots\ \mathcal{E}_2\\ \delta_2\end{matrix}}{\delta}\,R'$$

$$\rhd \qquad \dfrac{\begin{matrix}\vdots\ \mathcal{D}_1\\ \alpha\vee\beta\end{matrix} \quad \dfrac{\begin{matrix}[\alpha]\\ \vdots\ \mathcal{D}_2\\ \gamma\end{matrix} \quad \begin{matrix}\vdots\ \mathcal{E}_1\\ \delta_1\end{matrix} \quad \begin{matrix}\vdots\ \mathcal{E}_2\\ \delta_2\end{matrix}}{\delta}\,R' \quad \dfrac{\begin{matrix}[\beta]\\ \vdots\ \mathcal{D}_3\\ \gamma\end{matrix} \quad \begin{matrix}\vdots\ \mathcal{E}_1\\ \delta_1\end{matrix} \quad \begin{matrix}\vdots\ \mathcal{E}_2\\ \delta_2\end{matrix}}{\delta}\,R'}{\delta}\,(\vee\mathrm{E})$$

where R' is an arbitrary inference rule, and both $\mathcal{E}_1$ and $\mathcal{E}_2$ are derivations of the minor premises of R' if they exist.

(e) R is ($\sim\sim$I), and γ is $\sim\sim\alpha$:

$$\dfrac{\dfrac{\begin{matrix}\vdots\ \mathcal{D}\\ \alpha\end{matrix}}{\sim\sim\alpha}\,(\sim\sim\mathrm{I})}{\alpha}\,(\sim\sim\mathrm{E}) \qquad \rhd \qquad \begin{matrix}\vdots\ \mathcal{D}\\ \alpha.\end{matrix}$$

(f) R is ($\sim\!\to$I) and γ is $\sim(\alpha\to\beta)$:

$$\dfrac{\dfrac{\begin{matrix}[\alpha]\\ \vdots\ \mathcal{D}\\ \sim\beta\end{matrix}}{\sim(\alpha\to\beta)}\,(\sim\!\to\mathrm{I}) \quad \begin{matrix}\vdots\ \mathcal{E}\\ \alpha\end{matrix}}{\sim\beta}\,(\sim\!\to\mathrm{E}) \qquad \rhd \qquad \begin{matrix}\vdots\ \mathcal{E}\\ \alpha\\ \vdots\ \mathcal{D}\\ \sim\beta.\end{matrix}$$

(g) R is $(\sim\wedge\mathrm{I1})$ or $(\sim\wedge\mathrm{I2})$ and γ is $\sim(\alpha_1\wedge\alpha_2)$:

$$\dfrac{\dfrac{\begin{matrix}\vdots\ \mathcal{D}\\ \sim\alpha_i\end{matrix}}{\sim(\alpha_1\wedge\alpha_2)}\,(\sim\wedge\mathrm{I1}) \qquad \begin{matrix}[\sim\alpha_1]\\ \vdots\ \mathcal{E}_1\\ \delta\end{matrix} \qquad \begin{matrix}[\sim\alpha_2]\\ \vdots\ \mathcal{E}_2\\ \delta\end{matrix}}{\delta}\,(\sim\wedge\mathrm{E}) \qquad \rhd \qquad \begin{matrix}\vdots\ \mathcal{D}\\ \sim\alpha_i\\ \vdots\ \mathcal{E}_i\\ \delta\end{matrix}$$

where i is 1 or 2.

(h) R is $(\sim\wedge\mathrm{E})$:

$$\dfrac{\dfrac{\begin{matrix}\vdots\ \mathcal{D}_1\\ \sim(\alpha\wedge\beta)\end{matrix} \qquad \begin{matrix}[\sim\alpha]\\ \vdots\ \mathcal{D}_2\\ \gamma\end{matrix} \qquad \begin{matrix}[\sim\beta]\\ \vdots\ \mathcal{D}_3\\ \gamma\end{matrix}}{\gamma}\,(\sim\wedge\mathrm{E}) \qquad \begin{matrix}\vdots\ \mathcal{E}_1\\ \delta_1\end{matrix} \qquad \begin{matrix}\vdots\ \mathcal{E}_2\\ \delta_2\end{matrix}}{\delta}\,R'$$

$$\rhd \qquad \dfrac{\begin{matrix}\vdots\ \mathcal{D}_1\\ \sim(\alpha\wedge\beta)\end{matrix} \qquad \dfrac{\begin{matrix}[\sim\alpha]\\ \vdots\ \mathcal{D}_2\\ \gamma\end{matrix} \quad \begin{matrix}\vdots\ \mathcal{E}_1\\ \delta_1\end{matrix} \quad \begin{matrix}\vdots\ \mathcal{E}_2\\ \delta_2\end{matrix}}{\delta}\,R' \qquad \dfrac{\begin{matrix}[\sim\beta]\\ \vdots\ \mathcal{D}_3\\ \gamma\end{matrix} \quad \begin{matrix}\vdots\ \mathcal{E}_1\\ \delta_1\end{matrix} \quad \begin{matrix}\vdots\ \mathcal{E}_2\\ \delta_2\end{matrix}}{\delta}\,R'}{\delta}\,(\sim\wedge\mathrm{E})$$

where R' is an arbitrary inference rule, and both $\mathcal{E}_1$ and $\mathcal{E}_2$ are derivations of the minor premises of R' if they exist.

(i) R is $(\sim\vee\mathrm{I})$ and γ is $\sim(\alpha_1\vee\alpha_2)$:

$$\dfrac{\dfrac{\begin{matrix}\vdots\ \mathcal{D}_1\\ \sim\alpha_1\end{matrix} \qquad \begin{matrix}\vdots\ \mathcal{D}_2\\ \sim\alpha_2\end{matrix}}{\sim(\alpha_1\vee\alpha_2)}\,(\sim\vee\mathrm{I})}{\sim\alpha_i}\,(\sim\vee\mathrm{E}i) \qquad \rhd \qquad \begin{matrix}\vdots\ \mathcal{D}_i\\ \sim\alpha_i\end{matrix}$$

where i is 1 or 2.

(j) The set of derivations is closed under $\rhd$.

2. The definition of the reduction relation $\rhd$ at γ in nC3 is obtained from the conditions for the reduction relation $\rhd$ at γ in nC by adding the following condition.

(a) R is (EM) and γ is $\gamma_1{\rightarrow}\gamma_2$, $\gamma_1{\wedge}\gamma_2$, $\gamma_1{\vee}\gamma_2$, ${\sim}{\sim}\gamma'$, ${\sim}(\gamma_1{\rightarrow}\gamma_2)$, ${\sim}(\gamma_1{\wedge}\gamma_2)$, or ${\sim}(\gamma_1{\vee}\gamma_2)$:

$$\dfrac{\dfrac{\begin{matrix}[{\sim}\alpha]\\ \vdots\ \mathcal{D}_1\\ \gamma\end{matrix} \qquad \begin{matrix}[\alpha]\\ \vdots\ \mathcal{D}_2\\ \gamma\end{matrix}}{\gamma}\,(\mathrm{EM}) \qquad \begin{matrix}\vdots\ \mathcal{E}_1\\ \delta_1\end{matrix} \qquad \begin{matrix}\vdots\ \mathcal{E}_2\\ \delta_2\end{matrix}}{\delta}\,R'$$

$$\rhd \quad \dfrac{\dfrac{\begin{matrix}[{\sim}\alpha]\\ \vdots\ \mathcal{D}_1\\ \gamma\end{matrix} \qquad \begin{matrix}\vdots\ \mathcal{E}_1\\ \delta_1\end{matrix} \qquad \begin{matrix}\vdots\ \mathcal{E}_2\\ \delta_2\end{matrix}}{\delta}\,R' \qquad \dfrac{\begin{matrix}[\alpha]\\ \vdots\ \mathcal{D}_2\\ \gamma\end{matrix} \qquad \begin{matrix}\vdots\ \mathcal{E}_1\\ \delta_1\end{matrix} \qquad \begin{matrix}\vdots\ \mathcal{E}_2\\ \delta_2\end{matrix}}{\delta}\,R'}{\delta}\,(\mathrm{EM})$$

where R' is (${\rightarrow}$E), (${\wedge}$E1), (${\wedge}$E2), (${\vee}$E), (${\sim}{\sim}$E) (${\sim}{\rightarrow}$E), (${\sim}{\wedge}$E), (${\sim}{\vee}$E1), or (${\sim}{\vee}$E2), and both $\mathcal{E}_1$ and $\mathcal{E}_2$ are derivations of the minor premises of R' if they exist.

3. The definition of the reduction relation $\rhd$ at γ in nMC is obtained from the conditions for the reduction relation $\rhd$ at γ in nC by adding the following condition.

(a) R is (GEM) and γ is $\gamma_1{\rightarrow}\gamma_2$, $\gamma_1{\wedge}\gamma_2$, $\gamma_1{\vee}\gamma_2$, ${\sim}{\sim}\gamma'$, ${\sim}(\gamma_1{\rightarrow}\gamma_2)$, ${\sim}(\gamma_1{\wedge}\gamma_2)$, or ${\sim}(\gamma_1{\vee}\gamma_2)$:

$$\dfrac{\dfrac{\begin{matrix}[\alpha{\rightarrow}\beta]\\ \vdots\ \mathcal{D}_1\\ \gamma\end{matrix} \qquad \begin{matrix}[\alpha]\\ \vdots\ \mathcal{D}_2\\ \gamma\end{matrix}}{\gamma}\,(\mathrm{GEM}) \qquad \begin{matrix}\vdots\ \mathcal{E}_1\\ \delta_1\end{matrix} \qquad \begin{matrix}\vdots\ \mathcal{E}_2\\ \delta_2\end{matrix}}{\delta}\,R'$$

$$\rhd \quad \dfrac{\dfrac{\begin{matrix}[\alpha{\rightarrow}\beta]\\ \vdots\ \mathcal{D}_1\\ \gamma\end{matrix} \qquad \begin{matrix}\vdots\ \mathcal{E}_1\\ \delta_1\end{matrix} \qquad \begin{matrix}\vdots\ \mathcal{E}_2\\ \delta_2\end{matrix}}{\delta}\,R' \qquad \dfrac{\begin{matrix}[\alpha]\\ \vdots\ \mathcal{D}_2\\ \gamma\end{matrix} \qquad \begin{matrix}\vdots\ \mathcal{E}_1\\ \delta_1\end{matrix} \qquad \begin{matrix}\vdots\ \mathcal{E}_2\\ \delta_2\end{matrix}}{\delta}\,R'}{\delta}\,(\mathrm{GEM})$$

where R' is ($\rightarrow$E), ($\wedge$E1), ($\wedge$E2), ($\vee$E), ($\sim\sim$E) ($\sim\rightarrow$E), ($\sim\wedge$E), ($\sim\vee$E1), or ($\sim\vee$E2), and both $\mathcal{E}_1$ and $\mathcal{E}_2$ are derivations of the minor premises of R' if they exist.

4. The definition of the reduction relation $\rhd$ at γ in nCN is obtained from the conditions for the reduction relation $\rhd$ at γ in nC3 by adding the other conditions of nMC. Namely, it is defined as all the conditions for both nC3 and nMC.

Prior to proving the normalization theorems for nC, nC3, nMC, and nCN, we need the following lemma.

LEMMA 3.7. *Let* N_1*,* N_2*,* N_3*, and* N_4 *be* nC*,* nC3*,* nMC*, and* nCN*, respectively. Let* S_1*,* S_2*,* S_3*, and* S_4 *be* sC*,* sC3*,* sMC**, and* sCN**, respectively. For any* $i \in \{1, 2, 3, 4\}$*, the following hold.*

1. If $\mathcal{D}$ *is a derivation in* N_i *such that* $\mathrm{oa}(\mathcal{D}) = \Gamma$ *and* $\mathrm{end}(\mathcal{D}) = \beta$*, then* $S_i \vdash \Gamma \Rightarrow \beta$*,*

2. If $S_i - (\mathrm{cut}) \vdash \Gamma \Rightarrow \beta$*, then we can obtain a derivation* $\mathcal{D}'$ *in* N_i *such that*

   (a) $\mathrm{oa}(\mathcal{D}') \subseteq \Gamma$*,*

   (b) $\mathrm{end}(\mathcal{D}') = \beta$*,*

   (c) $\mathcal{D}'$ *is normal.*

PROOF:

1. We prove 1 by induction on the derivations $\mathcal{D}$ of N_i such that $\mathrm{oa}(\mathcal{D}) = \Gamma$ and $\mathrm{end}(\mathcal{D}) = \beta$. We distinguish the cases according to the last inference of $\mathcal{D}$. We show some cases.

   (a) Case (EM): $\mathcal{D}$ is of the form:

$$\dfrac{\begin{array}{c}[\sim\alpha]\Gamma_1 \\ \vdots\ \mathcal{D}_1 \\ \gamma\end{array} \qquad \begin{array}{c}[\alpha]\Gamma_2 \\ \vdots\ \mathcal{D}_2 \\ \gamma\end{array}}{\gamma}\ (\mathrm{EM})$$

where $\mathrm{oa}(\mathcal{D}) = \Gamma_1 \cup \Gamma_2$ and $\mathrm{end}(\mathcal{D}) = \gamma$. By induction hypothesis, we have $S_i \vdash {\sim}\alpha, \Gamma_1 \Rightarrow \gamma$ and $S_i \vdash \alpha, \Gamma_2 \Rightarrow \gamma$. Then, we obtain the required fact $S_i \vdash \Gamma_1, \Gamma_2 \Rightarrow \gamma$:

$$
\dfrac{\begin{array}{c} \vdots\ \mathit{Ind.\,hyp.} \\ {\sim}\alpha, \Gamma_1 \Rightarrow \gamma \\ \vdots\ (\mathrm{we}) \\ {\sim}\alpha, \Gamma_1, \Gamma_2 \Rightarrow \gamma \end{array} \qquad \begin{array}{c} \vdots\ \mathit{Ind.\,hyp.} \\ \alpha, \Gamma_2 \Rightarrow \gamma \\ \vdots\ (\mathrm{we}) \\ \alpha, \Gamma_1, \Gamma_2 \Rightarrow \gamma \end{array}}{\Gamma_1, \Gamma_2 \Rightarrow \gamma}\ (\text{ex-middle})
$$

where (we) is admissible in $S_i - (\mathrm{cut})$ by Propositions 2.4 and 2.11.

(b) Case (${\sim}{\sim}$E): $\mathcal{D}$ is of the form:

$$
\dfrac{\begin{array}{c} \Gamma \\ \vdots\ \mathcal{D}_1 \\ {\sim}{\sim}\alpha \end{array}}{\alpha}\ ({\sim}{\sim}\mathrm{E})
$$

where $\mathrm{oa}(\mathcal{D}) = \Gamma$ and $\mathrm{end}(\mathcal{D}) = \alpha$. By induction hypothesis, we have $S_i \vdash \Gamma \Rightarrow {\sim}{\sim}\alpha$. Then, we obtain the required fact $S_i \vdash \Gamma \Rightarrow \alpha$:

$$
\dfrac{\begin{array}{c} \vdots\ \mathit{Ind.\,hyp.} \\ \Gamma \Rightarrow {\sim}{\sim}\alpha \end{array} \qquad \dfrac{\begin{array}{c} \vdots\ \mathit{Prop.}2.3 \\ \alpha \Rightarrow \alpha \end{array}}{{\sim}{\sim}\alpha \Rightarrow \alpha}\ ({\sim}{\sim}\mathrm{left})}{\Gamma \Rightarrow \alpha}\ (\mathrm{cut}).
$$

(c) Case (${\sim}{\rightarrow}$I): We divide this case into two subcases.

i. Subcase 1: $\mathcal{D}$ is of the form:

$$
\dfrac{\begin{array}{c} \Gamma \\ \vdots\ \mathcal{D}' \\ {\sim}\beta \end{array}}{{\sim}(\alpha{\rightarrow}\beta)}\ ({\sim}{\rightarrow}\mathrm{I})
$$

where $\mathrm{oa}(\mathcal{D}) = \Gamma$ and $\mathrm{end}(\mathcal{D}) = {\sim}(\alpha{\rightarrow}\beta)$. By induction hypothesis, we have $S_i \vdash \Gamma \Rightarrow {\sim}\beta$. Then, we obtain that $S_i \vdash \Gamma \Rightarrow {\sim}(\alpha{\rightarrow}\beta)$:

$$\dfrac{\dfrac{\overset{\vdots\ Ind.\,hyp.}{\Gamma \Rightarrow \sim\beta}}{\alpha, \Gamma \Rightarrow \sim\beta}\ (\text{we})}{\Gamma \Rightarrow \sim(\alpha\to\beta)}\ (\sim\to\text{right})$$

where (we) is admissible in S_i − (cut) by Propositions 2.4 and 2.11.

ii. Subcase 2: $\mathcal{D}$ is of the form:

$$\dfrac{\overset{[\alpha]\ \Gamma}{\overset{\vdots\ \mathcal{D}'}{\sim\beta}}}{\sim(\alpha\to\beta)}\ (\sim\to\text{I})$$

where $\text{oa}(\mathcal{D}) = \Gamma$ and $\text{end}(\mathcal{D}) = \sim(\alpha\to\beta)$. By induction hypothesis, we have $S_i \vdash \alpha, \Gamma \Rightarrow \sim\beta$. Then, we obtain the required fact $S_i \vdash \Gamma \Rightarrow \sim(\alpha\to\beta)$:

$$\dfrac{\overset{\vdots\ Ind.\,hyp.}{\alpha, \Gamma \Rightarrow \sim\beta}}{\Gamma \Rightarrow \sim(\alpha\to\beta)}\ (\sim\to\text{right}).$$

(d) Case $(\sim\to\text{E})$: $\mathcal{D}$ is of the form:

$$\dfrac{\overset{\Gamma_1}{\overset{\vdots\ \mathcal{D}_1}{\sim(\alpha\to\beta)}} \quad \overset{\Gamma_2}{\overset{\vdots\ \mathcal{D}_2}{\alpha}}}{\sim\beta}\ (\sim\to\text{E})$$

where $\text{oa}(\mathcal{D}) = \Gamma_1 \cup \Gamma_2$ and $\text{end}(\mathcal{D}) = \sim\beta$. By induction hypotheses, we have $S_i \vdash \Gamma_1 \Rightarrow \sim(\alpha\to\beta)$ and $S_i \vdash \Gamma_2 \Rightarrow \alpha$. Then, we obtain the required fact $S_i \vdash \Gamma_1, \Gamma_2 \Rightarrow \sim\beta$:

$$\dfrac{\overset{\vdots\ Ind.\,hyp.}{\Gamma_2 \Rightarrow \alpha} \quad \dfrac{\overset{\vdots\ Ind.\,hyp.}{\Gamma_1 \Rightarrow \sim(\alpha\to\beta)} \quad \dfrac{\overset{\vdots\ Prop.2.3}{\alpha \Rightarrow \alpha} \quad \overset{\vdots\ Prop.2.3}{\sim\beta \Rightarrow \sim\beta}}{\sim(\alpha\to\beta), \alpha \Rightarrow \sim\beta}\ (\sim\to\text{left})}{\alpha, \Gamma_1 \Rightarrow \sim\beta}\ (\text{cut})}{\Gamma_1, \Gamma_2 \Rightarrow \sim\beta}\ (\text{cut}).$$

(e) Case ($\sim\wedge$E): $\mathcal{D}$ is of the form:

$$\dfrac{\begin{array}{c}\Gamma_1\\ \vdots\ \mathcal{D}_1\\ \sim(\alpha\wedge\beta)\end{array}\quad \begin{array}{c}[\sim\alpha]\Gamma_2\\ \vdots\ \mathcal{D}_2\\ \gamma\end{array}\quad \begin{array}{c}[\sim\beta]\Gamma_3\\ \vdots\ \mathcal{D}_3\\ \gamma\end{array}}{\gamma}\ (\sim\wedge\mathrm{E})$$

where $\mathrm{oa}(\mathcal{D}) = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ and $\mathrm{end}(\mathcal{D}) = \gamma$. By induction hypotheses, we have $S_i \vdash \Gamma_1 \Rightarrow \sim(\alpha\wedge\beta)$, $S_i \vdash \sim\alpha, \Gamma_2 \Rightarrow \gamma$, and $S_i \vdash \sim\beta, \Gamma_3 \Rightarrow \gamma$. Then, we obtain the required fact $S_i \vdash \Gamma_1, \Gamma_2, \Gamma_3 \Rightarrow \gamma$:

$$\dfrac{\begin{array}{c}\vdots\ \mathit{Ind.\,hyp.}\\ \Gamma_1 \Rightarrow \sim(\alpha\wedge\beta)\end{array}\quad \dfrac{\begin{array}{c}\vdots\ \mathit{Ind.\,hyp.}\\ \sim\alpha, \Gamma_2 \Rightarrow \gamma\\ \vdots\ (\mathrm{we})\\ \sim\alpha, \Gamma_2, \Gamma_3 \Rightarrow \gamma\end{array}\quad \begin{array}{c}\vdots\ \mathit{Ind.\,hyp.}\\ \sim\beta, \Gamma_3 \Rightarrow \gamma\\ \vdots\ (\mathrm{we})\\ \sim\beta, \Gamma_2, \Gamma_3 \Rightarrow \gamma\end{array}}{\sim(\alpha\wedge\beta), \Gamma_2, \Gamma_3 \Rightarrow \gamma}\ (\sim\wedge\mathrm{left})}{\Gamma_1, \Gamma_2, \Gamma_3 \Rightarrow \gamma}\ (\mathrm{cut})$$

where (we) is admissible in $S_i - (\mathrm{cut})$ by Propositions 2.4 and 2.11.

(f) Case ($\sim\vee$I): $\mathcal{D}$ is of the form:

$$\dfrac{\begin{array}{c}\Gamma_1\\ \vdots\ \mathcal{D}_1\\ \sim\alpha\end{array}\quad \begin{array}{c}\Gamma_2\\ \vdots\ \mathcal{D}_2\\ \sim\beta\end{array}}{\sim(\alpha\vee\beta)}\ (\sim\vee\mathrm{I})$$

where $\mathrm{oa}(\mathcal{D}) = \Gamma_1 \cup \Gamma_2$ and $\mathrm{end}(\mathcal{D}) = \sim(\alpha\vee\beta)$. By induction hypotheses, we have $S_i \vdash \Gamma_1 \Rightarrow \sim\alpha$ and $S_i \vdash \Gamma_2 \Rightarrow \sim\beta$. Then, we obtain the required fact $S_i \vdash \Gamma_1, \Gamma_2 \Rightarrow \sim(\alpha\vee\beta)$:

$$\dfrac{\begin{array}{c}\vdots\ \mathit{Ind.\,hyp.}\\ \Gamma_1 \Rightarrow \sim\alpha\\ \vdots\ (\mathrm{we})\\ \Gamma_1, \Gamma_2 \Rightarrow \sim\alpha\end{array}\quad \begin{array}{c}\vdots\ \mathit{Ind.\,hyp.}\\ \Gamma_2 \Rightarrow \sim\beta\\ \vdots\ (\mathrm{we})\\ \Gamma_1, \Gamma_2 \Rightarrow \sim\beta\end{array}}{\Gamma_1, \Gamma_2 \Rightarrow \sim(\alpha\vee\beta)}\ (\sim\vee\mathrm{right})$$

where (we) is admissible in $S_i - $ (cut) by Propositions 2.4 and 2.11.

2. We prove 2 by induction on the derivations $\mathcal{D}$ of $\Gamma \Rightarrow \beta$ in $S_i -$ (cut). We distinguish the cases according to the last inference of $\mathcal{D}$. We show some cases.

   (a) Case (init2): $\mathcal{D}$ is of the form:

   $$\sim p, \Gamma \Rightarrow \sim p \text{ (init2)}.$$

   In this case, we obtain a required normal derivation $\mathcal{D}'$ by:

   $$\sim p$$

   where $\mathrm{oa}(\mathcal{D}') = \{\sim p\} \subseteq \{\sim p\} \cup \Gamma$ and $\mathrm{end}(\mathcal{D}') = \sim p$.

   (b) Case (ex-middle): $\mathcal{D}$ is of the form:

   $$\dfrac{\begin{array}{c}\vdots\ \mathcal{D}_1 \\ \sim\alpha, \Gamma \Rightarrow \gamma\end{array} \quad \begin{array}{c}\vdots\ \mathcal{D}_2 \\ \alpha, \Gamma \Rightarrow \gamma\end{array}}{\Gamma \Rightarrow \gamma} \text{ (ex-middle)}.$$

   By induction hypotheses, we have normal derivations $\mathcal{E}_1$ and $\mathcal{E}_2$ in N_i of the form:

   $$\begin{array}{c}(\sim\alpha, \Gamma)^* \\ \vdots\ \mathcal{E}_1 \\ \gamma\end{array} \qquad \begin{array}{c}(\alpha, \Gamma)^* \\ \vdots\ \mathcal{E}_2 \\ \gamma\end{array}$$

   where $\mathrm{oa}(\mathcal{E}_1) = (\{\sim\alpha\} \cup \Gamma)^* \subseteq \{\sim\alpha\} \cup \Gamma$, $\mathrm{oa}(\mathcal{E}_2) = (\{\alpha\} \cup \Gamma)^* \subseteq \{\alpha\} \cup \Gamma$, $\mathrm{end}(\mathcal{E}_1) = \gamma$, and $\mathrm{end}(\mathcal{E}_2) = \gamma$. Then, we distinguish the cases according to $(\{\sim\alpha\} \cup \Gamma)^*$ and $(\{\alpha\} \cup \Gamma)^*$. We consider the following cases: (1) $\sim\alpha \notin (\{\sim\alpha\} \cup \Gamma)^*$, (2) $\alpha \notin (\{\alpha\} \cup \Gamma)^*$, and (3) $\sim\alpha \in (\{\sim\alpha\} \cup \Gamma)^*$ and $\alpha \in (\{\alpha\} \cup \Gamma)^*$.

      i. Subcase (1): We obtain a required normal derivation $\mathcal{D}'$ in N_i by:

      $$\begin{array}{c}\Gamma^* \\ \vdots\ \mathcal{E}_1 \\ \gamma\end{array}$$

      where $\mathrm{oa}(\mathcal{D}') = \Gamma^* \subseteq \Gamma$ and $\mathrm{end}(\mathcal{D}') = \gamma$.

ii. Subcase (2): We obtain a required normal derivation $\mathcal{D}'$ in N_i by:

$$\begin{array}{c} \Gamma^* \\ \vdots\ \mathcal{E}_2 \\ \gamma \end{array}$$

where $\mathrm{oa}(\mathcal{D}') = \Gamma^* \subseteq \Gamma$ and $\mathrm{end}(\mathcal{D}') = \gamma$.

iii. Subcase (3): We obtain a required normal derivation $\mathcal{D}'$ in N_i by:

$$\dfrac{\begin{array}{c} ([{\sim}\alpha]\ \Gamma)^* \\ \vdots\ \mathcal{E}_1 \\ \gamma \end{array} \qquad \begin{array}{c} ([\alpha]\ \Gamma)^* \\ \vdots\ \mathcal{E}_2 \\ \gamma \end{array}}{\gamma}\ (\mathrm{EM})$$

where $\mathrm{oa}(\mathcal{D}') = \Gamma^* \subseteq \Gamma$ and $\mathrm{end}(\mathcal{D}') = \gamma$.

(c) Case (${\sim}{\sim}$left): $\mathcal{D}$ is of the form:

$$\dfrac{\begin{array}{c} \vdots\ \mathcal{E} \\ \alpha, \Gamma \Rightarrow \gamma \end{array}}{{\sim}{\sim}\alpha, \Gamma \Rightarrow \gamma}\ ({\sim}{\sim}\mathrm{left}).$$

By induction hypothesis, we have a normal derivation $\mathcal{E}'$ in N_i of the form:

$$\begin{array}{c} (\alpha, \Gamma)^* \\ \vdots\ \mathcal{E}' \\ \gamma \end{array}$$

where $\mathrm{oa}(\mathcal{E}') = (\{\alpha\} \cup \Gamma)^* \subseteq \{\alpha\} \cup \Gamma$ and $\mathrm{end}(\mathcal{E}') = \gamma$. In what follows, we show only the case for $(\{\alpha\} \cup \Gamma)^* \equiv \{\alpha\} \cup \Gamma$. In this case, we obtain a required normal derivation $\mathcal{D}'$ in N_i by:

$$\begin{array}{c} \dfrac{{\sim}{\sim}\alpha}{\alpha}\ ({\sim}{\sim}\mathrm{E}) \quad \Gamma \\ \vdots\ \mathcal{E}' \\ \gamma \end{array}$$

where $\mathrm{oa}(\mathcal{D}') = \{{\sim}{\sim}\alpha\} \cup \Gamma$ and $\mathrm{end}(\mathcal{D}') = \gamma$.

(d) Case ($\sim\!\to$left): $\mathcal{D}$ is of the form:

$$\dfrac{\begin{array}{c}\vdots\ \mathcal{D}_1\\ \Gamma \Rightarrow \alpha\end{array} \quad \begin{array}{c}\vdots\ \mathcal{D}_2\\ \sim\beta, \Delta \Rightarrow \gamma\end{array}}{\sim(\alpha\to\beta), \Gamma, \Delta \Rightarrow \gamma}\ (\sim\!\to\text{left}).$$

By induction hypotheses, we have normal derivations $\mathcal{E}_1$ and $\mathcal{E}_2$ in N_i of the form:

$$\begin{array}{c}\Gamma^*\\ \vdots\ \mathcal{E}_1\\ \alpha\end{array} \qquad \begin{array}{c}(\sim\beta, \Delta)^*\\ \vdots\ \mathcal{E}_2\\ \gamma\end{array}$$

where $\mathrm{oa}(\mathcal{E}_1) = \Gamma^* \subseteq \Gamma$, $\mathrm{end}(\mathcal{E}_1) = \alpha$, $\mathrm{oa}(\mathcal{E}_2) = (\{\beta\} \cup \Delta)^* \subseteq \{\beta\} \cup \Delta$, and $\mathrm{end}(\mathcal{E}_2) = \gamma$. In what follows, we show only the case for $\Gamma^* \equiv \Gamma$ and $(\{\beta\} \cup \Delta)^* \equiv \{\beta\} \cup \Delta$. In this case, we obtain a required normal derivation $\mathcal{D}'$ in N_i by:

$$\begin{array}{c}\dfrac{\sim(\alpha\to\beta) \quad \begin{array}{c}\Gamma\\ \vdots\ \mathcal{E}_1\\ \alpha\end{array}}{\sim\beta}\ (\sim\!\to\mathrm{E}) \qquad \Delta\\ \vdots\ \mathcal{E}_2\\ \gamma\end{array}$$

where $\mathrm{oa}(\mathcal{D}') = \{\sim(\alpha\to\beta)\} \cup \Gamma \cup \Delta$ and $\mathrm{end}(\mathcal{D}') = \gamma$.

(e) Case ($\sim\wedge$left): $\mathcal{D}$ is of the form:

$$\dfrac{\begin{array}{c}\vdots\ \mathcal{D}_1\\ \sim\alpha, \Gamma \Rightarrow \gamma\end{array} \quad \begin{array}{c}\vdots\ \mathcal{D}_2\\ \sim\beta, \Gamma \Rightarrow \gamma\end{array}}{\sim(\alpha\wedge\beta), \Gamma \Rightarrow \gamma}\ (\sim\wedge\text{left}).$$

By induction hypotheses, we have normal derivations $\mathcal{E}_1$ and $\mathcal{E}_2$ in N_i of the form:

$$\begin{array}{c}(\sim\alpha, \Gamma)^*\\ \vdots\ \mathcal{E}_1\\ \gamma\end{array} \qquad \begin{array}{c}(\sim\beta, \Gamma)^*\\ \vdots\ \mathcal{E}_2\\ \gamma\end{array}$$

where $\mathrm{oa}(\mathcal{E}_1) = (\{\sim\alpha\} \cup \Gamma)^* \subseteq \{\sim\alpha\} \cup \Gamma$, $\mathrm{oa}(\mathcal{E}_2) = (\{\sim\beta\} \cup \Gamma)^* \subseteq \{\sim\beta\} \cup \Gamma$, and $\mathrm{end}(\mathcal{E}_1) = \mathrm{end}(\mathcal{E}_2) = \gamma$. In what follows, we show only the case for $(\{\sim\alpha\} \cup \Gamma)^* \equiv \{\sim\alpha\} \cup \Gamma$ and $(\{\sim\beta\} \cup \Gamma)^* \equiv \{\sim\beta\} \cup \Gamma$. In this case, we obtain a required normal derivation $\mathcal{D}'$ in N_i by:

$$\dfrac{\sim(\alpha\wedge\beta) \qquad \begin{array}{c}[\sim\alpha]\Gamma \\ \vdots\ \mathcal{E}_1 \\ \gamma\end{array} \qquad \begin{array}{c}[\sim\beta]\Gamma \\ \vdots\ \mathcal{E}_2 \\ \gamma\end{array}}{\gamma}\ (\sim\wedge\mathrm{E})$$

where $\mathrm{oa}(\mathcal{D}') = \{\sim(\alpha\wedge\beta)\} \cup \Gamma$ and $\mathrm{end}(\mathcal{D}') = \gamma$.

(f) Case ($\sim\vee$right): $\mathcal{D}$ is of the form:

$$\dfrac{\begin{array}{c}\vdots\ \mathcal{D}_1 \\ \Gamma \Rightarrow \sim\alpha\end{array} \qquad \begin{array}{c}\vdots\ \mathcal{D}_2 \\ \Gamma \Rightarrow \sim\beta\end{array}}{\Gamma \Rightarrow \sim(\alpha\vee\beta)}\ (\sim\vee\mathrm{right}).$$

By induction hypotheses, we have normal derivations $\mathcal{E}_1$ and $\mathcal{E}_2$ in N_i of the form:

$$\begin{array}{c}\Gamma^* \\ \vdots\ \mathcal{E}_1 \\ \sim\alpha\end{array} \qquad\qquad \begin{array}{c}\Gamma^* \\ \vdots\ \mathcal{E}_2 \\ \sim\beta\end{array}$$

where $\mathrm{oa}(\mathcal{E}_1) = \mathrm{oa}(\mathcal{E}_2) = \Gamma^* \subseteq \Gamma$, $\mathrm{end}(\mathcal{E}_1) = \sim\alpha$, and $\mathrm{end}(\mathcal{E}_2) = \sim\beta$. In what follows, we show only the case for $\Gamma^* \equiv \Gamma$. In this case, we obtain a required normal derivation $\mathcal{D}'$ in N_i by:

$$\dfrac{\begin{array}{c}\Gamma \\ \vdots\ \mathcal{E}_1 \\ \sim\alpha\end{array} \qquad \begin{array}{c}\Gamma \\ \vdots\ \mathcal{E}_2 \\ \sim\beta\end{array}}{\sim(\alpha\vee\beta)}\ (\sim\vee\mathrm{I})$$

where $\mathrm{oa}(\mathcal{D}') = \Gamma$ and $\mathrm{end}(\mathcal{D}') = \sim(\alpha\vee\beta)$. $\square$

We then obtain the following theorems.

THEOREM 3.8 (Equivalence between nC-family and sC-family). *Let N_1, N_2, N_3, and N_4 be* nC, nC3, nMC, *and* nCN, *respectively. Let S_1, S_2, S_3, and S_4 be* sC, sC3, sMC*, *and* sCN*, *respectively. For any formula α and any $i \in \{1, 2, 3, 4\}$, $S_i \vdash\ \Rightarrow \alpha$ iff α is provable in N_i.*

PROOF: Taking $\emptyset$ as Γ in Lemma 3.7, we obtain the claim. □

THEOREM 3.9 (Normalization for nC, nC3, nMC, and nCN). *Let N be* nC, nC3, nMC, *or* nCN. *All derivations in N are normalizable. More precisely, if a derivation $\mathcal{D}$ in N is given, then we can obtain a normal derivation $\mathcal{D}'$ in N such that* $\mathrm{oa}(\mathcal{D}') \subseteq \mathrm{oa}(\mathcal{D})$ *and* $\mathrm{end}(\mathcal{D}') = \mathrm{end}(\mathcal{D})$.

PROOF: Let N_1, N_2, N_3, and N_4 be nC, nC3, nMC, and nCN, respectively. Let S_1, S_2, S_3, and S_4 be sC, sC3, sMC*, and sCN*, respectively. Let i be 1, 2, 3, or 4. Suppose that a derivation $\mathcal{D}$ in N_i is given, and assume that $\mathrm{oa}(\mathcal{D}) = \Gamma$ and $\mathrm{end}(\mathcal{D}) = \beta$. Then, by Lemma 3.7 (1), we obtain $S_i \vdash \Gamma \Rightarrow \beta$. By the cut-elimination theorem for S_i (i.e., Theorems 2.5, 2.6, and 2.13), we obtain $S_i - (\mathrm{cut}) \vdash \Gamma \Rightarrow \beta$. Then, by Lemma 3.7 (2), we can obtain a normal derivation $\mathcal{D}'$ in N_i such that $\mathrm{oa}(\mathcal{D}') \subseteq \mathrm{oa}(\mathcal{D})$ and $\mathrm{end}(\mathcal{D}') = \mathrm{end}(\mathcal{D})$. □

4. Natural deduction systems with general elimination rules and normalization theorems

We define natural deduction systems with general elimination rules, referred to as gNJ$^+$, gC, gC3, gMC, and gCN, for positive intuitionistic logic, C, C3, MC, and CN, respectively. For these systems, we use the same notations and notions as those for the previously introduced systems.

DEFINITION 4.1 (gNJ$^+$, gC, gC3, gMC, and gCN).

1. gNJ$^+$ is obtained from NJ$^+$ by replacing ($\to$E), ($\wedge$E1), and ($\wedge$E2) with the general elimination rules of the form:

$$\dfrac{\alpha{\to}\beta \quad \alpha \quad \begin{array}{c}[\beta]\\ \vdots\\ \gamma\end{array}}{\gamma}\ ({\to}\mathrm{GE}) \qquad \dfrac{\alpha{\wedge}\beta \quad \begin{array}{c}[\alpha,\beta]\\ \vdots\\ \gamma\end{array}}{\gamma}\ ({\wedge}\mathrm{GE}).$$

2. gC is obtained from gNJ$^+$ by adding (${\sim}{\sim}$E), (${\sim}{\to}$E), (${\sim}{\vee}$E1), (${\sim}{\vee}$E2), and the negated general elimination rules of the form:

$$\dfrac{{\sim}{\sim}\alpha \quad \begin{array}{c}[\alpha]\\ \vdots\\ \gamma\end{array}}{\gamma}\ ({\sim}{\sim}\mathrm{GE}) \qquad \dfrac{{\sim}(\alpha{\to}\beta) \quad \alpha \quad \begin{array}{c}[{\sim}\beta]\\ \vdots\\ \gamma\end{array}}{\gamma}\ ({\sim}{\to}\mathrm{GE})$$

$$\dfrac{{\sim}(\alpha \vee \beta) \quad \begin{array}{c}[{\sim}\alpha,{\sim}\beta]\\ \vdots\\ \gamma\end{array}}{\gamma}\ ({\sim}{\vee}\mathrm{GE}).$$

3. gC3 is obtained from gC by adding (EM).

4. gMC is obtained from gC by adding (GEM).

5. gCN is obtained from gC3 by adding (GEM).

Remark 4.2. The rules (${\to}$GE), (${\wedge}$GE), (${\sim}{\sim}$GE), (${\sim}{\to}$GE), and (${\sim}{\vee}$GE) are referred to as general elimination rules. These rules are considered as elimination rules. For more information on general elimination rules and systems incorporating them, see [33, 37, 23].

Next, we define reduction relations on the set of derivations in natural deduction systems with general elimination rules. We employ the same notions, such as the maximum formula and normal derivation, as those used for nC, nC3, nMC, or nCN.

Definition 4.3 (Reduction relation). Let γ be a maximum formula in a derivation that is the conclusion of an inference rule R.

1. The definition of the reduction relation $\triangleright$ at γ in gC is obtained by the following conditions.

(a) R is $(\to\mathrm{I})$ and γ is $\alpha\to\beta$:

$$\dfrac{\dfrac{\begin{array}{c}[\alpha]\\ \vdots\ \mathcal{D}\\ \beta\end{array}}{\alpha\to\beta}\,(\to\mathrm{I}) \qquad \begin{array}{c}\vdots\ \mathcal{E}_1\\ \alpha\end{array} \qquad \begin{array}{c}[\beta]\\ \vdots\ \mathcal{E}_2\\ \gamma\end{array}}{\gamma}\,(\to\mathrm{GE}) \qquad \rhd \qquad \begin{array}{c}\vdots\ \mathcal{E}_1\\ \alpha\\ \vdots\ \mathcal{D}\\ \beta\\ \vdots\ \mathcal{E}_2\\ \gamma.\end{array}$$

(b) R is $(\wedge\mathrm{I})$ and γ is $\alpha_1\wedge\alpha_2$:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \mathcal{D}_1\\ \alpha\end{array} \quad \begin{array}{c}\vdots\ \mathcal{D}_2\\ \beta\end{array}}{\alpha\wedge\beta}\,(\wedge\mathrm{I}) \qquad \begin{array}{c}[\alpha,\beta]\\ \vdots\ \mathcal{E}\\ \gamma\end{array}}{\gamma}\,(\wedge\mathrm{GE}) \qquad \rhd \qquad \begin{array}{c}\begin{array}{cc}\vdots\ \mathcal{D}_1 & \vdots\ \mathcal{D}_2\\ \alpha & \beta\end{array}\\ \vdots\ \mathcal{E}\\ \gamma.\end{array}$$

(c) R is $(\vee\mathrm{I1})$ or $(\vee\mathrm{I2})$ and γ is $\alpha_1\vee\alpha_2$: The same condition as the one defined in Definition 3.6.

(d) R is $(\vee\mathrm{E})$: The same condition as the one defined in Definition 3.6.

(e) R is $({\sim}{\sim}\mathrm{I})$, and γ is ${\sim}{\sim}\alpha$:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \mathcal{D}\\ \alpha\end{array}}{{\sim}{\sim}\alpha}\,({\sim}{\sim}\mathrm{I}) \qquad \begin{array}{c}[\alpha]\\ \vdots\ \mathcal{E}\\ \gamma\end{array}}{\gamma}\,({\sim}{\sim}\mathrm{GE}) \qquad \rhd \qquad \begin{array}{c}\vdots\ \mathcal{D}\\ \alpha\\ \vdots\ \mathcal{E}\\ \gamma.\end{array}$$

(f) R is $({\sim}{\to}\mathrm{I})$ and γ is ${\sim}(\alpha\to\beta)$:

$$\dfrac{\dfrac{\begin{array}{c}[\alpha]\\ \vdots\ \mathcal{D}\\ {\sim}\beta\end{array}}{{\sim}(\alpha\to\beta)}\,({\sim}{\to}\mathrm{I}) \qquad \begin{array}{c}\vdots\ \mathcal{E}_1\\ \alpha\end{array} \qquad \begin{array}{c}[{\sim}\beta]\\ \vdots\ \mathcal{E}_2\\ \gamma\end{array}}{\gamma}\,({\sim}{\to}\mathrm{GE}) \qquad \rhd \qquad \begin{array}{c}\vdots\ \mathcal{E}_1\\ \alpha\\ \vdots\ \mathcal{D}\\ {\sim}\beta\\ \vdots\ \mathcal{E}_2\\ \gamma.\end{array}$$

(g) R is $({\sim}{\wedge}\mathrm{I1})$ or $({\sim}{\wedge}\mathrm{I2})$ and γ is ${\sim}(\alpha_1\wedge\alpha_2)$: The same condition as the one defined in Definition 3.6.

(h) R is ($\sim\wedge$E): The same condition as the one defined in Definition 3.6.

(i) R is ($\sim\vee$I) and γ is $\sim(\alpha_1\vee\alpha_2)$:

$$\dfrac{\dfrac{\begin{matrix}\vdots\ \mathcal{D}_1\\ \sim\alpha\end{matrix}\quad \begin{matrix}\vdots\ \mathcal{D}_2\\ \sim\beta\end{matrix}}{\sim(\alpha\vee\beta)}\,(\sim\vee\mathrm{I}) \qquad \begin{matrix}[\sim\alpha,\sim\beta]\\ \vdots\ \mathcal{E}\\ \gamma\end{matrix}}{\gamma}\,(\sim\vee\mathrm{GE}) \qquad \rhd \qquad \begin{matrix}\begin{matrix}\vdots\ \mathcal{D}_1\\ \sim\alpha\end{matrix}\quad \begin{matrix}\vdots\ \mathcal{D}_2\\ \sim\beta\end{matrix}\\ \vdots\ \mathcal{E}\\ \gamma.\end{matrix}$$

(j) The set of derivations is closed under $\rhd$.

2. The definition of the reduction relation $\rhd$ at γ in gC3 is obtained from the conditions for the reduction relation $\rhd$ at γ in gC by adding the following condition.

(a) R is (EM) and γ is $\gamma_1\to\gamma_2$, $\gamma_1\wedge\gamma_2$, $\gamma_1\vee\gamma_2$, $\sim\sim\gamma'$, $\sim(\gamma_1\to\gamma_2)$, $\sim(\gamma_1\wedge\gamma_2)$, or $\sim(\gamma_1\vee\gamma_2)$:

$$\dfrac{\dfrac{\begin{matrix}[\sim\alpha]\\ \vdots\ \mathcal{D}_1\\ \gamma\end{matrix}\quad \begin{matrix}[\alpha]\\ \vdots\ \mathcal{D}_2\\ \gamma\end{matrix}}{\gamma}\,(\mathrm{EM}) \quad \begin{matrix}\vdots\ \mathcal{E}_1\\ \delta_1\end{matrix}\quad \begin{matrix}\vdots\ \mathcal{E}_2\\ \delta_2\end{matrix}}{\delta}\,R'$$

$$\rhd \quad \dfrac{\dfrac{\begin{matrix}[\sim\alpha]\\ \vdots\ \mathcal{D}_1\\ \gamma\end{matrix}\quad \begin{matrix}\vdots\ \mathcal{E}_1\\ \delta_1\end{matrix}\quad \begin{matrix}\vdots\ \mathcal{E}_2\\ \delta_2\end{matrix}}{\delta}\,R' \quad \dfrac{\begin{matrix}[\alpha]\\ \vdots\ \mathcal{D}_2\\ \gamma\end{matrix}\quad \begin{matrix}\vdots\ \mathcal{E}_1\\ \delta_1\end{matrix}\quad \begin{matrix}\vdots\ \mathcal{E}_2\\ \delta_2\end{matrix}}{\delta}\,R'}{\delta}\,(\mathrm{EM})$$

where R' is ($\to$GE), ($\wedge$GE), ($\vee$E), ($\sim\sim$GE) ($\sim\to$GE), ($\sim\wedge$E), or ($\sim\vee$GE), and both $\mathcal{E}_1$ and $\mathcal{E}_2$ are derivations of the minor premises of R' if they exist.

3. The definition of the reduction relation $\rhd$ at γ in gMC is obtained from the conditions for the reduction relation $\rhd$ at γ in gC by adding the following condition.

(a) R is (GEM) and γ is $\gamma_1{\to}\gamma_2$, $\gamma_1{\wedge}\gamma_2$, $\gamma_1{\vee}\gamma_2$, ${\sim}{\sim}\gamma'$, ${\sim}(\gamma_1{\to}\gamma_2)$, ${\sim}(\gamma_1{\wedge}\gamma_2)$, or ${\sim}(\gamma_1{\vee}\gamma_2)$:

$$\dfrac{\dfrac{\begin{matrix}[\alpha{\to}\beta]\\ \vdots\ \mathcal{D}_1\\ \gamma\end{matrix} \quad \begin{matrix}[\alpha]\\ \vdots\ \mathcal{D}_2\\ \gamma\end{matrix}}{\gamma}\,(\mathrm{GEM}) \quad \begin{matrix}\vdots\ \mathcal{E}_1\\ \delta_1\end{matrix} \quad \begin{matrix}\vdots\ \mathcal{E}_2\\ \delta_2\end{matrix}}{\delta}\,R'$$

$$\rhd \quad \dfrac{\dfrac{\begin{matrix}[\alpha{\to}\beta]\\ \vdots\ \mathcal{D}_1\\ \gamma\end{matrix} \quad \begin{matrix}\vdots\ \mathcal{E}_1\\ \delta_1\end{matrix} \quad \begin{matrix}\vdots\ \mathcal{E}_2\\ \delta_2\end{matrix}}{\delta}\,R' \quad \dfrac{\begin{matrix}[\alpha]\\ \vdots\ \mathcal{D}_2\\ \gamma\end{matrix} \quad \begin{matrix}\vdots\ \mathcal{E}_1\\ \delta_1\end{matrix} \quad \begin{matrix}\vdots\ \mathcal{E}_2\\ \delta_2\end{matrix}}{\delta}\,R'}{\delta}\,(\mathrm{GEM})$$

where R' is $({\to}\mathrm{GE})$, $({\wedge}\mathrm{GE})$, $({\vee}\mathrm{E})$, $({\sim}{\sim}\mathrm{GE})$ $({\sim}{\to}\mathrm{GE})$, $({\sim}{\wedge}\mathrm{E})$, or $({\sim}{\vee}\mathrm{GE})$, and both $\mathcal{E}_1$ and $\mathcal{E}_2$ are derivations of the minor premises of R' if they exist.

4. The definition of the reduction relation $\rhd$ at γ in gCN is obtained from the conditions for the reduction relation $\rhd$ at γ in gC3 by adding the other conditions of gMC.

LEMMA 4.4. *Let G_1, G_2, G_3, and G_4 be* gC, gC3, gMC, *and* gCN, *respectively. Let S_1, S_2, S_3, and S_4 be* sC, sC3, sMC*, *and* sCN*, *respectively. For any $i \in \{1, 2, 3, 4\}$, the following hold.*

1. *If $\mathcal{D}$ is a derivation in G_i such that* $\mathrm{oa}(\mathcal{D}) = \Gamma$ *and* $\mathrm{end}(\mathcal{D}) = \beta$, *then* $S_i \vdash \Gamma \Rightarrow \beta$,

2. *If $S_i -$ (cut) $\vdash \Gamma \Rightarrow \beta$, then we can obtain a derivation $\mathcal{D}'$ in G_i such that*

   (a) $\mathrm{oa}(\mathcal{D}') \subseteq \Gamma$,

   (b) $\mathrm{end}(\mathcal{D}') = \beta$,

   (c) $\mathcal{D}'$ is normal.

PROOF:

1. We prove 1 by induction on the derivations $\mathcal{D}$ of G_i such that oa($\mathcal{D}$) $= \Gamma$ and end($\mathcal{D}$) $= \beta$. We distinguish the cases according to the last inference of $\mathcal{D}$. We show some cases.

   (a) Case ($\sim\sim$E): $\mathcal{D}$ is of the form:

$$\dfrac{\begin{matrix}\Gamma_1 \\ \vdots\ \mathcal{D}_1 \\ \sim\sim\alpha\end{matrix} \qquad \begin{matrix}[\alpha]\ \Gamma_2 \\ \vdots\ \mathcal{D}_2 \\ \gamma\end{matrix}}{\gamma}\ (\sim\sim\text{GE})$$

   where $\text{oa}(\mathcal{D}) = \Gamma_1 \cup \Gamma_2$ and $\text{end}(\mathcal{D}) = \gamma$.

   i. Subcase (1): The discharge of ($\sim\sim$GE) is vacuous. In this case, by induction hypothesis, we have $S_i \vdash \Gamma_2 \Rightarrow \gamma$. We then obtain the required fact by:

$$\begin{matrix} \vdots\ \textit{Ind. hyp.} \\ \Gamma_2 \Rightarrow \gamma \\ \vdots\ (\text{we}) \\ \sim\sim\alpha, \Gamma_1, \Gamma_2 \Rightarrow \gamma \end{matrix}$$

   where (we) is admissible in cut-free S_i by Proposition 2.4.

   ii. Subcase (2): α is the discharged assumption of ($\sim\sim$GE). In this case, by induction hypothesis, we have $S_i \vdash \alpha, \Gamma_2 \Rightarrow \gamma$. We then obtain the required fact by:

$$\dfrac{\begin{matrix} \vdots\ \textit{Ind. hyp.} \\ \alpha, \Gamma_2 \Rightarrow \gamma \\ \vdots\ (\text{we}) \\ \alpha, \Gamma_1, \Gamma_2 \Rightarrow \gamma \end{matrix}}{\sim\sim\alpha, \Gamma_1, \Gamma_2 \Rightarrow \gamma}\ (\sim\sim\text{left})$$

   where (we) is admissible in cut-free S_i by Proposition 2.4.

(b) Case ($\sim\!\to$GE): $\mathcal{D}$ is of the form:

$$\dfrac{\begin{array}{c}\Gamma_1\\ \vdots\ \mathcal{D}_1\\ \sim(\alpha\to\beta)\end{array}\quad \begin{array}{c}\Gamma_2\\ \vdots\ \mathcal{D}_2\\ \alpha\end{array}\quad \begin{array}{c}[\sim\beta]\ \Gamma_3\\ \vdots\ \mathcal{D}_3\\ \gamma\end{array}}{\gamma}\ (\sim\!\to\text{GE})$$

where $\mathrm{oa}(\mathcal{D}) = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ and $\mathrm{end}(\mathcal{D}) = \gamma$.

i. Subcase (1): The discharge of ($\sim\!\to$GE) is vacuous. In this case, by induction hypothesis, we have $S_i \vdash \Gamma_3 \Rightarrow \gamma$. We then obtain the required fact by:

$$\begin{array}{c}\vdots\ \textit{Ind. hyp.}\\ \Gamma_3 \Rightarrow \gamma\\ \vdots\ (\text{we})\\ \Gamma_1, \Gamma_2, \Gamma_3 \Rightarrow \gamma\end{array}$$

where (we) is admissible in cut-free S_i by Proposition 2.4.

ii. Subcase (2): $\sim\beta$ is the discharged assumption of ($\sim\!\to$GE). In this case, by induction hypotheses, we have the following: $S_i \vdash \Gamma_1 \Rightarrow \sim(\alpha\to\beta)$, $S_i \vdash \Gamma_2 \Rightarrow \alpha$, and $S_i \vdash \sim\beta, \Gamma_3 \Rightarrow \gamma$. We then obtain the required fact by:

$$\dfrac{\begin{array}{c}\vdots\ \textit{Ind. hyp.}\\ \Gamma_1 \Rightarrow \sim(\alpha\to\beta)\end{array}\quad \dfrac{\begin{array}{c}\vdots\ \textit{Ind. hyp.}\\ \Gamma_2 \Rightarrow \alpha\end{array}\quad \begin{array}{c}\vdots\ \textit{Ind. hyp.}\\ \sim\beta, \Gamma_3 \Rightarrow \gamma\end{array}}{\sim(\alpha\to\beta), \Gamma_2, \Gamma_3 \Rightarrow \gamma}\ (\sim\!\to\text{left})}{\Gamma_1, \Gamma_2, \Gamma_3 \Rightarrow \gamma}\ (\text{cut}).$$

(c) Case ($\sim\!\vee$GE): $\mathcal{D}$ is of the form:

$$\dfrac{\begin{array}{c}\Gamma_1\\ \vdots\ \mathcal{D}_1\\ \sim(\alpha\vee\beta)\end{array}\quad \begin{array}{c}[\sim\alpha, \sim\beta]\ \Gamma_2\\ \vdots\ \mathcal{D}_2\\ \gamma\end{array}}{\gamma}\ (\sim\!\vee\text{GE})$$

where $\mathrm{oa}(\mathcal{D}) = \Gamma_1 \cup \Gamma_2$ and $\mathrm{end}(\mathcal{D}) = \gamma$.

i. Subcase (1): The discharge of ($\sim\!\vee$GE) is vacuous. In this case, by induction hypothesis, we have $S_i \vdash \Gamma_2 \Rightarrow \gamma$. We then obtain the required fact by:

$$\dfrac{\dfrac{\vdots\ \textit{Ind. hyp.}}{\Gamma_2 \Rightarrow \gamma}}{\vdots\ \text{(we)}}\\ \Gamma_1, \Gamma_2 \Rightarrow \gamma$$

where (we) is admissible in cut-free S_i by Proposition 2.4.

ii. Subcase (2): $\sim\alpha$ and/or $\sim\beta$ is (are) the discharged assumption(s) of ($\sim\vee$GE). We show only the case that $\sim\beta$ is only the discharged assumption of ($\sim\vee$GE). In this case, by induction hypotheses, we have $S_i \vdash \Gamma_1 \Rightarrow \sim(\alpha\vee\beta)$ and $S_i \vdash \sim\beta, \Gamma_2 \Rightarrow \gamma$. We then obtain the required fact by:

$$\dfrac{\dfrac{\vdots\ \textit{Ind. hyp.}}{\Gamma_1 \Rightarrow \sim(\alpha\vee\beta)} \quad \dfrac{\dfrac{\dfrac{\vdots\ \textit{Ind. hyp.}}{\sim\beta, \Gamma_2 \Rightarrow \gamma}}{\sim\alpha, \sim\beta, \Gamma_2 \Rightarrow \gamma}\ \text{(we)}}{\sim(\alpha\vee\beta), \Gamma_2 \Rightarrow \gamma}\ (\sim\vee\text{left})}{\Gamma_1, \Gamma_2 \Rightarrow \gamma}\ \text{(cut)}$$

where (we) is admissible in cut-free S_i by Proposition 2.4.

2. We prove 2 by induction on the derivations $\mathcal{D}$ of $\Gamma \Rightarrow \beta$ in $S_i -$ (cut). We distinguish the cases according to the last inference of $\mathcal{D}$. We show some cases.

   (a) Case ($\sim\sim$left): $\mathcal{D}$ is of the form:

$$\dfrac{\dfrac{\vdots\ \mathcal{E}}{\alpha, \Gamma \Rightarrow \gamma}}{\sim\sim\alpha, \Gamma \Rightarrow \gamma}\ (\sim\sim\text{left}).$$

   By induction hypothesis, we have a normal derivation $\mathcal{E}'$ in G_i of the form:

$$\begin{array}{c} (\alpha, \Gamma)^* \\ \vdots\ \mathcal{E}' \\ \gamma \end{array}$$

where $\mathrm{oa}(\mathcal{E}') = (\{\alpha\} \cup \Gamma)^* \subseteq \{\alpha\} \cup \Gamma$ and $\mathrm{end}(\mathcal{E}') = \gamma$. In what follows, we show only the case for $(\{\alpha\} \cup \Gamma)^* \equiv \{\alpha\} \cup \Gamma$. In this case, we obtain a required normal derivation $\mathcal{D}'$ in G_i by:

$$\dfrac{{\sim}{\sim}\alpha \qquad \begin{array}{c}[\alpha]\ \Gamma \\ \vdots\ \mathcal{E}' \\ \gamma\end{array}}{\gamma}\ ({\sim}{\sim}\mathrm{GE})$$

where $\mathrm{oa}(\mathcal{D}') = \{{\sim}{\sim}\alpha\} \cup \Gamma$ and $\mathrm{end}(\mathcal{D}') = \gamma$.

(b) Case (${\sim}{\to}$left): $\mathcal{D}$ is of the form:

$$\dfrac{\begin{array}{c}\vdots\ \mathcal{D}_1 \\ \Gamma \Rightarrow \alpha\end{array} \qquad \begin{array}{c}\vdots\ \mathcal{D}_2 \\ {\sim}\beta, \Delta \Rightarrow \gamma\end{array}}{{\sim}(\alpha{\to}\beta), \Gamma, \Delta \Rightarrow \gamma}\ ({\sim}{\to}\mathrm{left}).$$

By induction hypotheses, we have normal derivations $\mathcal{D}_1'$ and $\mathcal{D}_2'$ in G_i of the form:

$$\begin{array}{c}\Gamma^* \\ \vdots\ \mathcal{D}_1' \\ \alpha\end{array} \qquad \begin{array}{c}({\sim}\beta, \Delta)^* \\ \vdots\ \mathcal{D}_2' \\ \gamma\end{array}$$

where $\mathrm{oa}(\mathcal{E}_1) = \Gamma^* \subseteq \Gamma$, $\mathrm{end}(\mathcal{E}_1) = \alpha$, $\mathrm{oa}(\mathcal{E}_2) = (\{{\sim}\beta\} \cup \Delta)^* \subseteq \{{\sim}\beta\} \cup \Delta$, and $\mathrm{end}(\mathcal{E}_2) = \gamma$. In what follows, we show only the case for $\Gamma^* \equiv \Gamma$ and $(\{{\sim}\beta\} \cup \Delta)^* \equiv \{{\sim}\beta\} \cup \Delta$. In this case, we obtain a required normal derivation $\mathcal{D}'$ in G_i by:

$$\dfrac{{\sim}(\alpha{\to}\beta) \quad \begin{array}{c}\Gamma \\ \vdots\ \mathcal{D}_1' \\ \alpha\end{array} \qquad \begin{array}{c}[{\sim}\beta]\ \Delta \\ \vdots\ \mathcal{D}_2' \\ \gamma\end{array}}{\gamma}\ ({\sim}{\to}\mathrm{GE})$$

where $\mathrm{oa}(\mathcal{D}') = \{{\sim}(\alpha{\to}\beta)\} \cup \Gamma \cup \Delta$ and $\mathrm{end}(\mathcal{D}') = \gamma$.

(c) Case (${\sim}{\vee}$left): $\mathcal{D}$ is of the form:

$$\dfrac{\begin{array}{c}\vdots\ \mathcal{E} \\ {\sim}\alpha, {\sim}\beta, \Gamma \Rightarrow \gamma\end{array}}{{\sim}(\alpha{\vee}\beta), \Gamma \Rightarrow \gamma}\ ({\sim}{\vee}\mathrm{left}).$$

By induction hypothesis, we have a normal derivation $\mathcal{E}'$ in G_i of the form:

$$\begin{array}{c} (\sim\alpha, \sim\beta, \Gamma)^* \\ \vdots\ \mathcal{E}' \\ \gamma \end{array}$$

where $\mathrm{oa}(\mathcal{E}') = (\{\sim\alpha, \sim\beta\} \cup \Gamma)^* \subseteq \{\sim\alpha, \sim\beta\} \cup \Gamma$ and $\mathrm{end}(\mathcal{E}') = \gamma$. In what follows, we show only the case for $(\{\sim\alpha, \sim\beta\} \cup \Gamma)^* \equiv \{\sim\alpha, \sim\beta\} \cup \Gamma$. In this case, we obtain a required normal derivation $\mathcal{D}'$ in G_i by:

$$\dfrac{\sim(\alpha\vee\beta) \qquad \begin{array}{c} [\sim\alpha, \sim\beta]\ \Gamma \\ \vdots\ \mathcal{E}' \\ \gamma \end{array}}{\gamma}\ (\sim\vee\mathrm{GE})$$

where $\mathrm{oa}(\mathcal{D}') = \{\sim(\alpha\vee\beta)\} \cup \Gamma$ and $\mathrm{end}(\mathcal{D}') = \gamma$. □

THEOREM 4.5 (Equivalence between gC-family and sC-family). *Let G_1, G_2, G_3, and G_4 be* gC, gC3, gMC, *and* gCN, *respectively. Let S_1, S_2, S_3, and S_4 be* sC, sC3, sMC*, *and* sCN*, *respectively. For any formula α and any $i \in \{1, 2, 3, 4\}$, $S_i \vdash\ \Rightarrow \alpha$ iff α is provable in G_i.*

PROOF: By Lemma 4.4. □

THEOREM 4.6 (Equivalence between gC-family and nC-family). *Let G_1, G_2, G_3, and G_4 be* gC, gC3, gMC, *and* gCN, *respectively. Let N_1, N_2, N_3, and N_4 be* nC, nC3, nMC, *and* nCN, *respectively. For any formula α and any $i \in \{1, 2, 3, 4\}$, α is provable in G_i iff α is provable in N_i.*

PROOF: By Theorems 3.8 and 4.5. □

THEOREM 4.7 (Normalization for gC, gC3, gMC, and gCN). *Let G be* gC, gC3, gMC, *or* gCN. *All derivations in G are normalizable.*

PROOF: Similar to the proof of Theorem 3.9. We use Lemma 4.4 and Theorems 2.5, 2.6, and 2.13. □

5. Systems and theorems for N-family

5.1. Gentzen-style sequent calculi and cut-elimination theorems

A Gentzen-style sequent calculus sN4 for N4 is obtained from sC by replacing ($\sim\to$left) and ($\sim\to$right) with the negated logical inference rules of the form:

$$\frac{\alpha, \sim\beta, \Gamma \Rightarrow \gamma}{\sim(\alpha\to\beta), \Gamma \Rightarrow \gamma}\ (\sim\to\text{left}^\star) \qquad \frac{\Gamma \Rightarrow \alpha \quad \Gamma \Rightarrow \sim\beta}{\Gamma \Rightarrow \sim(\alpha\to\beta)}\ (\sim\to\text{right}^\star)$$

which correspond to the axiom scheme $\sim(\alpha\to\beta) \leftrightarrow \alpha\wedge\sim\beta$. For more information on this calculus, see, for example, [18, 19].

We then obtain the following Gentzen-style sequent calculi for the N-family in a similar way as for the C-family:

1. sN4e = sN4 + (ex-middle),
2. sN4p = sN4 + (Peirce),
3. sN4ep = sN4 + (ex-middle) + (Peirce),
4. sN4g = sN4 + (g-ex-middle),
5. sN4eg = sN4 + (ex-middle) + (g-ex-middle).

Then, we obtain the cut-elimination theorems for these calculi, the cut-free equivalence between sN4p (sN4ep) and sN4g (sN4eg, resp.), and the separation theorem for the N4-based logics that correspond to sN4, sN4e, sN4p, and sN4ep.

Remark 5.1. As presented in [15], a single-succedent Gentzen-style sequent calculus for classical logic is obtained from a single-succedent Gentzen-style sequent calculus for N4 by adding (ex-middle) and the structural and logical inference rules of the following form:

$$\frac{\Gamma \Rightarrow}{\Gamma \Rightarrow \alpha}\ (\text{we-right}) \qquad \frac{\Gamma \Rightarrow \sim\alpha \quad \Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \gamma}\ (\text{explosion}).$$

Namely, sN4e + (we-right) + (explosion) becomes a sequent calculus for classical logic, although the definition of sequent should be modified as $\Gamma \Rightarrow \gamma$ where γ is a formula or the empty set. Additionally, we remark that sC + (explosion) becomes a sequent calculus for trivial logic (i.e., it is a meaningless logic).

5.2. Gentzen-style natural deduction systems and normalization theorems

A Gentzen-style natural deduction system nN4 for N4 is obtained from nC by replacing ($\sim\!\to$I), and ($\sim\!\to$E) with the negated logical inference rules of the form:

$$\frac{\alpha \quad {\sim}\beta}{{\sim}(\alpha{\to}\beta)}\ ({\sim}{\to}\mathrm{I}^\star) \qquad \frac{{\sim}(\alpha{\to}\beta)}{\alpha}\ ({\sim}{\to}\mathrm{E1}^\star) \qquad \frac{{\sim}(\alpha{\to}\beta)}{{\sim}\beta}\ ({\sim}{\to}\mathrm{E2}^\star)$$

which correspond to the axiom scheme ${\sim}(\alpha{\to}\beta) \leftrightarrow \alpha{\wedge}{\sim}\beta$. For more information on this system, see, for example, [31, 39, 13].

We then obtain the following Gentzen-style natural deduction systems for the N-family in a similar way as for the C-family:

1. nN4e = nN4 + (EM),
2. nN4g = nN4 + (GEM),
3. nN4eg = nN4 + (EM) + (GEM).

Of course, we must introduce the appropriate reduction relations with respect to these systems, modifying the reduction relations with respect to nC, nC3, nMC, and nCN. For example, we must introduce the following reduction conditions for the case when R is ($\sim\!\to$I) and γ is ${\sim}(\alpha{\to}\beta)$, instead of the condition for the same case in nC, nC3, nMC, and nCN:

1. Subcase 1:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \mathcal{D}_1\\ \alpha\end{array} \quad \begin{array}{c}\vdots\ \mathcal{D}_2\\ {\sim}\beta\end{array}}{{\sim}(\alpha{\to}\beta)}\ ({\sim}{\to}\mathrm{I}^\star)}{\alpha}\ ({\sim}{\to}\mathrm{E1}^\star) \qquad \rhd \qquad \begin{array}{c}\vdots\ \mathcal{D}_1\\ \alpha.\end{array}$$

2. Subcase 2:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \mathcal{D}_1\\ \alpha\end{array}\qquad\begin{array}{c}\vdots\ \mathcal{D}_2\\ {\sim}\beta\end{array}}{{\sim}(\alpha{\to}\beta)}\,({\sim}{\to}\mathrm{I}^\star)}{{\sim}\beta}\,({\sim}{\to}\mathrm{E2}^\star)\qquad\rhd\qquad\begin{array}{c}\vdots\ \mathcal{D}_2\\ {\sim}\beta.\end{array}$$

Then, we obtain the normalization theorems for these systems and the equivalence between nN4-family and sN4-family.

Remark 5.2. Natural deduction systems for a family of many-valued logics including Nelson's constructive three-valued logic N3 [2, 24] has recently been introduced and investigated by Kürbis and Petrukhin in [21]. For more information on natural deduction systems for N4, N3, and related logics, see, for example, [31, 39, 35, 13, 41].

5.3. Natural deduction systems with general elimination rules and normalization theorems

A natural deduction system with general elimination rules, gN4, for N4 is obtained from nN4 by replacing (→E), (∧E1), (∧E2), (∼∼E), (∼∨E1), (∼∨E2), (∼→E1⋆), and (∼→E2⋆) with (→GE), (∧GE), (∼∼GE), (∼∨GE), and the negated general elimination rule of the form:

$$\dfrac{{\sim}(\alpha{\to}\beta)\qquad\begin{array}{c}[\alpha,{\sim}\beta]\\ \vdots\\ \gamma\end{array}}{\gamma}\,({\sim}{\to}\mathrm{GE}^\star).$$

For more information on this system, see, [13].

We then obtain the following natural deduction systems with general elimination rules, for the N-family in a similar way as for the C-family:

1. gN4e = gN4 + (EM),
2. gN4g = gN4 + (GEM),
3. gN4eg = gN4 + (EM) + (GEM).

Of course, we must introduce the appropriate reduction relations with respect to these systems, modifying the reduction relations with respect to nN4, nN4e, nN4g, and nN4eg. For example, we must introduce the following reduction condition for the case when R is $({\sim}{\to}\mathrm{I})$ and γ is ${\sim}(\alpha{\to}\beta)$, instead of the condition for the same case in nN4, nN4e, nN4g, and nN4eg:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \mathcal{D}_1\\ \alpha\end{array} \qquad \begin{array}{c}\vdots\ \mathcal{D}_2\\ {\sim}\beta\end{array}}{{\sim}(\alpha{\to}\beta)}\,({\sim}{\to}\mathrm{I}^\star) \qquad \begin{array}{c}[\alpha,{\sim}\beta]\\ \vdots\ \mathcal{E}\\ \gamma\end{array}}{\gamma}\,({\sim}{\to}\mathrm{GE}^\star) \qquad \rhd \qquad \begin{array}{c}\begin{array}{c}\vdots\ \mathcal{D}_1\\ \alpha\end{array} \qquad \begin{array}{c}\vdots\ \mathcal{D}_2\\ {\sim}\beta\end{array}\\ \vdots\ \mathcal{E}\\ \gamma.\end{array}$$

Then, we obtain the normalization theorems for these systems and the equivalence between gN4-family and sN4-family. Additionally, we also obtain the equivalence between gN4-family and nN4-family.

6. Concluding remarks

In this study, we introduced the Gentzen-style sequent calculi sC, sC3, sMC, and sCN for the C-family: C, C3, MC, and CN, respectively. We proved the cut-elimination theorems for these calculi. We also introduced alternative Gentzen-style sequent calculi sMC* and sCN* for MC and CN, respectively. We then proved the theorem for cut-free equivalence between sMC* (sCN*) and sMC (sCN, respectively) and the cut-elimination theorem for sMC* and sCN*.

Furthermore, we introduced the Gentzen-style natural deduction systems nC, nC3, nMC, and nCN for the C-family. We then proved the normalization theorems for nC, nC3, nMC, and nCN. Additionally, we introduced the natural deduction systems with general elimination rules, gC, gC3, gMC, and gCN for the C-family. We then proved the normalization theorems for gC, gC3, gMC, and gCN.

Additionally, we have shown similar results for the N-family of paraconsistent logics based on Nelson's constructive four-valued logic N4. Thus, we have demonstrated that the proposed proof-theoretic framework can handle a wide range of non-classical logics, including connexive and paraconsistent logics. We have established a unified method for proving the cut-elimination and normalization theorems for these logics. In particular, we have obtained computational interpretations for the standard connexive logics C, C3, MC, and CN using the proposed natural deduction systems.

We remark that similar results can also be obtained for the family of paraconsistent logics with the conflation connective [8]. Specifically, we can consider a Gentzen-style sequent calculus for such a logic, derived from sC by replacing ($\sim\to$left) and ($\sim\to$right) with negated logical inference rules of the form:

$$\frac{\Gamma \Rightarrow {\sim}\alpha \quad {\sim}\beta, \Delta \Rightarrow \gamma}{{\sim}(\alpha{\to}\beta), \Gamma, \Delta \Rightarrow \gamma}\ ({\sim}{\to}\mathrm{left}^c) \qquad \frac{{\sim}\alpha, \Gamma \Rightarrow {\sim}\beta}{\Gamma \Rightarrow {\sim}(\alpha{\to}\beta)}\ ({\sim}{\to}\mathrm{right}^c)$$

where ($\sim\to$left^c) and ($\sim\to$right^c) correspond to the axiom ${\sim}(\alpha{\to}\beta) \leftrightarrow {\sim}\alpha{\to}{\sim}\beta$ of conflation. This conflation axiom is generally denoted as $-(\alpha{\to}\beta) \leftrightarrow -\alpha{\to}-\beta$ using the conflation connective $-$ [8], which serves as the logical counterpart of the conflation operator used in the algebraic structure of bilattices.

In future work, we intend to obtain similar results for extended intermediate connexive logics with the intuitionistic negation connective $\neg$ or the absurdity constant $\perp$. These extended connexive logics include $\mathrm{C}^{\perp}$, $\mathrm{C}_3^{\perp}$, and Niki's intermediate connexive logics, C_{po}^{ab} and C_{we}^{ab} [25]. As discussed by Niki, Gentzen-style sequent calculi for C_{po}^{ab} and C_{we}^{ab} are considered to include logical inference rules, referred to as the potential omniscience rule and weak negation rules, respectively, of the form:

$$\frac{{\sim}\alpha, \Gamma \Rightarrow \quad \alpha, \Gamma \Rightarrow}{\Gamma \Rightarrow}\ (\text{po-omni}) \qquad \frac{{\sim}\alpha, \Gamma \Rightarrow \gamma \quad \alpha, \Gamma \Rightarrow}{\Gamma \Rightarrow \gamma}\ (\text{we-neg})$$

which correspond to the axiom $\neg\neg({\sim}\alpha\vee\alpha)$ of potential omniscience and the axiom $\neg\alpha{\to}{\sim}\alpha$ of weak negation, respectively. The corresponding natural deduction rules for (po-omni) and (we-neg) can be considered respectively of the form:

$$\dfrac{\begin{matrix}[\sim\alpha] \\ \vdots \\ \bot\end{matrix} \quad \begin{matrix}[\alpha] \\ \vdots \\ \bot\end{matrix}}{\bot}\ (\mathrm{PO}) \qquad \dfrac{\begin{matrix}[\sim\alpha] \\ \vdots \\ \gamma\end{matrix} \quad \begin{matrix}[\alpha] \\ \vdots \\ \bot\end{matrix}}{\gamma}\ (\mathrm{WN}).$$

where $\neg\alpha$ can be defined as $\neg\alpha := \alpha{\to}\bot$ using $\bot$.

In other future work, from a technical perspective, we aim to prove the strong normalization and Church-Rosser theorems for the proposed natural deduction systems nC, nC3, nMC, nCN, gC, gC3, gMC, gCN, nN4e, nN4g, nN4eg, gN4e, gN4g, gN4eg, as well as for natural deduction systems for the intermediate connexive logics, including $\mathrm{C}^{\bot}$, $\mathrm{C}_3^{\bot}$, C_{po}^{ab}, and C_{wn}^{ab}. However, these issues remain unresolved at present.

Acknowledgements. I would like to thank the anonymous referees for their valuable comments. I would also like to thank Heinrich Wansing and the members of the Contradictory Logics Seminar at Ruhr University Bochum for their valuable comments and discussions, especially during the 1st Workshop on Contradictory Logics, held in Bochum from December 6 to 8, 2023. I would also like to thank the attendees of the 11th International Conference on Non-Classical Logics – Theory and Applications (NCL'24) and the 9th Workshop on Connexive Logics, both held in conjunction in Lodz from September 5 to 9, 2024, for their valuable discussions. Additionally, I would like to thank the anonymous referees of NCL'24 for their valuable comments on the early version of this paper.

References

[1] H. Africk, *Classical logic, intuitionistic logic, and the Peirce rule*, **Notre Dame Journal of Formal Logic**, vol. 33(2) (1992), pp. 229–235, DOI: https://doi.org/10.1305/ndjfl/1093636101.

[2] A. Almukdad, D. Nelson, *Constructible falsity and inexact predicates*, **Journal of Symbolic Logic**, vol. 49(1) (1984), pp. 231–233, DOI: https://doi.org/10.2307/2274105.

[3] R. Angell, *A propositional logics with subjunctive conditionals*, **Journal of Symbolic Logic**, vol. 27 (1962), pp. 327–343, DOI: https://doi.org/10.2307/2964651.

[4] J. Cantwell, *The logic of conditional negation*, **Notre Dame Journal of Formal Logic**, vol. 49 (2008), pp. 245–260, DOI: https://doi.org/10.1215/00294527-2008-010.

[5] H. B. Curry, **Foundations of mathematical logic**, McGraw-Hill, New York (1963).

[6] P. Égré, L. Rossi, J. Sprenger, *De finettian logics of indicative conditionals Part II: Proof theory and algebraic semantics*, **Journal of Philosophical Logic**, vol. 50 (2021), pp. 215–247, DOI: https://doi.org/10.1007/S10992-020-09572-7.

[7] W. Felscher, *Kombinatorische Konstruktionen mit Beweisen und Schnittelimination*, [in:] J. Diller, G. H. Müller (eds.), ⊨ **ISILC Proof Theory Symposion**, Springer Berlin Heidelberg, Berlin, Heidelberg (1975), pp. 119–151.

[8] M. Fitting, *Bilattices and the semantics of logic programming*, **The Journal of Logic Programming**, vol. 11(2) (1991), pp. 91–116, DOI: https://doi.org/10.1016/0743-1066(91)90014-G.

[9] N. Francez, *Natural deduction for two connexive logics*, **IfCoLog Journal of Logics and their Applications**, vol. 3(3) (2016), pp. 479–504.

[10] L. Gordeev, *On cut elimination in the presence of Peirce rule*, **Archiv für Mathematische Logik und Grundlagenforsch**, vol. 26 (1987), pp. 147–164, DOI: https://doi.org/10.1007/BF02017499.

[11] Y. Gurevich, *Intuitionistic logic with strong negation*, **Studia Logica 36**, vol. 36 (1977), pp. 49–59.

[12] N. Kamide, *Cut-free single-succedent systems revisited*, **Bulletin of the Section of Logic**, vol. 34(3) (2005), pp. 165–175.

[13] N. Kamide, *Natural deduction systems for Nelson's paraconsistent logic and its neighbors*, **Journal of Applied Non-Classical Logics**, vol. 15(4) (2005), pp. 405–435, DOI: https://doi.org/10.3166/JANCL.15.405-435.

[14] N. Kamide, *Embedding friendly first-order paradefinite and connexive logics*, **Journal of Philosophical logic**, vol. 51(5) (2022), pp. 1055–1102, DOI: https://doi.org/10.1007/S10992-022-09659-3.

[15] N. Kamide, *Rules of explosion and excluded middle: Constructing a unified single-succedent Gentzen-style framework for classical, paradefinite, paraconsistent, and paracomplete logics*, **Journal of Logic, Language and Information**, vol. 33(2) (2024), pp. 143–178, DOI: https://doi.org/10.1007/s10849-024-09416-6.

[16] N. Kamide, *Unified Gentzen approach to connexive logics over Wansing's C*, [in:] A. Indrzejczak, M. Zawidzki (eds.), **Proceedings of the 11th International Conference on Non-Classical Logics – Theory and Applications (NCL'24)**, vol. 415 of Electronic Proceedings in Theoretical Computer Science (2024), pp. 214–228, DOI: https://doi.org/10.4204/EPTCS.415.19.

[17] N. Kamide, H. Wansing, *Connexive modal logic based on positive S4*, [in:] J.-Y. Beziau, M. E. Coniglio (eds.), **Logic without Frontiers: Festschrift for Walter Alexandre Carnielli on the occasion of his 60th Birthday**, vol. 17 of Tribute Series, College Publications, London (2011), pp. 389–410.

[18] N. Kamide, H. Wansing, *Proof theory of Nelson's paraconsistent logic: A uniform perspective*, **Theoretical Computer Science**, vol. 415 (2012), pp. 1–38, DOI: https://doi.org/10.1016/J.TCS.2011.11.001.

[19] N. Kamide, H. Wansing, **Proof theory of N4-related paraconsistent logics**, vol. 54 of Studies in Logic, College Publications, London (2015), DOI: https://doi.org/10.1007/s11225-017-9729-9.

[20] N. Kamide, H. Wansing, *Completeness of connexive Heyting-Brouwer logic*, **IFCoLog Journal of Logics and their Applications**, vol. 3(3) (2016), pp. 441–466.

[21] N. Kürbis, Y. Petrukhin, *Normalisation for some quite interesting many-valued logics*, **Logic and Logical Philosophy**, vol. 30(3) (2021), pp. 493–534.

[22] S. McCall, *Connexive implication*, **Journal of Symbolic Logic**, vol. 31 (1966), pp. 415–433, DOI: https://doi.org/10.2307/2270458.

[23] S. Negri, J. von Plato, **Structural Proof Theory**, Cambridge University Press, Cambridge (2001), DOI: https://doi.org/10.1017/CBO9780511527340.

[24] D. Nelson, *Constructible falsity*, **Journal of Symbolic Logic**, vol. 14 (1949), pp. 16–26, DOI: https://doi.org/10.2307/2268973.

[25] S. Niki, *Intuitionistic views on connexive constructible falsity*, **Journal of Applied Logics**, vol. 11(2) (2024), pp. 125–157.

[26] S. Niki, H. Wansing, *On the provable contradictions of the connexive logics C and C3*, **Journal of Philosophical Logic**, vol. 52(5) (2023), pp. 1355–1383, DOI: https://doi.org/10.1007/S10992-023-09709-4.

[27] G. K. Olkhovikov, *On a new three-valued paraconsistent logic*, [in:] **Logic of Law and Tolerance**, Ural State University Press, Yekaterinburg (2002), pp. 96–113, it was translated by T.M. Ferguson in IfCoLog Journal of Logics and their Applications 3(3), pp. 317–334, 2016.

[28] G. K. Olkhovikov, *A complete, correct, and independent axiomatization of the first-order fragment of a three-valued paraconsistent logic*, **IfCoLog Journal of Logics and their Applications**, vol. 3(3) (2016), pp. 335–340.

[29] H. Omori, *From paraconsistent logic to dialetheic logic*, [in:] H. Andreas, P. Verdee (eds.), **Logical Studies of Paraconsistent Reasoning in Science and Mathematics**, Springer, Berlin (2016), pp. 111–134, DOI: https://doi.org/10.1007/978-3-319-40220-8_8.

[30] H. Omori, H. Wansing, *An extension of connexive logic C*, [in:] N. Olivetti, R. Verbrugge, S. Negri, G. Sandu (eds.), **Advances in Modal Logic 13**, College Publications, London (2020).

[31] D. Prawitz, **Natural deduction: a proof-theoretical study**, Almqvist and Wiksell, Stockholm (1965), DOI: https://doi.org/10.2307/2271676.

[32] W. Rautenberg, **Klassische und nicht-klassische Aussagenlogik**, Vieweg+Teubner Verlag, Wiesbaden (1979), DOI: https://doi.org/10.1007/978-3-322-85796-5.

[33] P. Schroeder-Heister, *A natural extension of natural deduction*, **Journal of Symbolic Logic**, vol. 49(4) (1984), pp. 1284–1300, DOI: https://doi.org/10.2307/2274279.

[34] M. E. Szabo (ed.), **Collected papers of Gerhard Gentzen**, Studies in Logic and the Foundations of Mathematics, North-Holland (1969), DOI: https://doi.org/10.2307/2272429.

[35] A. M. Tamminga, K. Tanaka, *A natural deduction system for first degree entailment*, **Notre Dame Journal of Formal Logic**, vol. 40(2) (1999), pp. 258–272, DOI: https://doi.org/10.1305/NDJFL/1038949541.

[36] J. von Plato, *Proof theory of full classical propositional logic* (1999), manuscript, 16 pages.

[37] J. von Plato, *Natural deduction with general elimination rules*, **Archive for Mathematical Logic**, vol. 40(7) (2001), pp. 541–567, DOI: https://doi.org/10.1007/s001530100091.

[38] N. Vorob'ev, *A constructive propositional calculus with strong negation (in Russian)*, **Doklady Akademii Nauk SSSR**, vol. 85 (1952), pp. 465–468.

[39] H. Wansing, **The logic of information structures**, vol. 681 of Lecture Notes in Artificial Intelligence, Springer, Berlin, Heidelberg (1993), DOI: https://doi.org/10.1007/3-540-56734-8.

[40] H. Wansing, *Connexive modal logic*, [in:] R. Schmidt, I. Pratt-Hartmann, M. Reynolds, H. Wansing (eds.), **Advances in Modal Logic 5**, King's College Publications, London (2005), pp. 367–385.

[41] H. Wansing, *Falsification, natural deduction and bi-intuitionistic logic*, **Journal of Logic and Computation**, vol. 26(1) (2016), pp. 425–450, DOI: https://doi.org/10.1093/LOGCOM/EXT035.

[42] H. Wansing, *Natural deduction for bi-connexive logic and a two-sorted typed λ-calculus*, **IfCoLog Journal of Logics and their Applications**, vol. 3(3) (2016), pp. 413–440.

[43] H. Wansing, *Connexive Logic*, [in:] E. N. Zalta, U. Nodelman (eds.), **The Stanford Encyclopedia of Philosophy**, Summer 2023 ed., Metaphysics Research Lab, Stanford University (2023).

[44] E. Zimmermann, *Peirce's rule in natural deduction*, **Theoretical Computer Science**, vol. 275 (2002), pp. 561–574, DOI: https://doi.org/10.1016/S0304-3975(01)00296-1.

Norihiro Kamide
Nagoya City University
Graduate School of Data Science
Yamanohata 1, Mizuho-cho, Mizuho-ku, Nagoya
467-8501 Aichi, Japan
e-mail: drnkamide08@kpd.biglobe.ne.jp

Funding information: This research was supported by JSPS KAKENHI Grant Number 23K10990.

Conflict of interests: None.

Ethical considerations: The Author assures of no violations of publication ethics and takes full responsibility for the content of the publication.

Declaration regarding the use of GAI tools: Not used.