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COMPARING SENSE AND DENOTATION IN BILATERALIST PROOF SYSTEMS FOR PROOFS AND REFUTATIONS

Abstract

In this paper a framework to distinguish in a Fregean manner between sense and denotation of λ-term-annotated derivations will be applied to a bilateralist sequent calculus displaying two derivability relations, one for proving and one for refuting. Therefore, a two-sorted typed λ-calculus will be used to annotate this calculus and a Dualization Theorem will be given, stating that for any derivable sequent expressing a proof, there is also a derivable sequent expressing a refutation and vice versa. By having joint λ-term annotations for proof systems in

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natural deduction and sequent calculus style, a comparison with respect to sense and denotation between derivations in those systems will be feasible, since the annotations elucidate the structural correspondences of the respective derivations. Thus, we will have a basis for determining in which cases, firstly, derivations expressing a proof vs. derivations expressing a refutation and, secondly, derivations in natural deduction vs. in sequent calculus can be identified and on which level.

Keywords: proof-theoretic semantics, bilateralism, bi-intuitionistic logic, meaning of proofs, proof identity, refutations.

1. Introduction

The philosophical background of this paper lies in a bilateralist conception of proof-theoretic semantics (PTS). In PTS it is assumed that the meaning of the logical connectives is given by the rules of inference governing them in some underlying proof system. *Bilateralist* PTS assumes that with our traditional proof systems this picture is incomplete: next to rules expressing something like assertion conditions for connectives, we also need to consider rules expressing something like their denial conditions. An important part of bilateralism is, then, to consider these dual notions to be on a par—not one as reducible to the other, as is traditionally often done. Rather than expressing the bilateralist idea in terms of speech acts, though, I will henceforth express it in terms of proof and refutation.[1]
It is for this reason that I will consider the specific bi-intuitionistic logic `2Int` [27, **?**] here. To wit, `2Int` has certain features which make it especially suitable from the standpoint of bilateralist PTS. Firstly, it displays bilateralism on the very fundamental level of having two derivability relations, one expressing provability and one refutability. This seems desirable from a bilateralist point of view if our concept of logic is that of being a consequence relation (not a set of theorems), which seems to me inherent in PTS. Next, based on these two derivability relations all connectives contained in `2Int` have a dual counterpart, i.e., importantly, there is a

[1] For more details and also overviews over PTS and bilateralism, see, e.g., [24] and [3, 4] respectively.

connective dual to implication expressing the dual derivability relation in the object language. Thus, the bilaterally desirable balance between these two counterparts is also evident within the connectives. Thirdly, `2Int` is constructive (also a feature inherent in the whole idea of PTS), in that it enjoys both the disjunction property, i.e., if $A \vee B$ is provable, either A is provable or B is provable, *and* the dual conjunction property, i.e., if $A \wedge B$ is refutable, either A is refutable or B is refutable.

The aim of the present paper is to apply a framework that was developed in [2] to distinguish in a Fregean manner sense and denotation of λ-term-annotated derivations to this bilateralist setting. On this basis, then, I will argue, firstly, for an identification of proofs and refutations on the level of denotations, not on the level of sense, though, and secondly, with respect to comparing sense and denotation between derivations in different kinds of proof systems, for a modification of what has been proposed in [2], which I think better grasps our underlying intuitions. The proof systems that are to be compared here are a natural deduction and a sequent calculus system for `2Int`.[2] The comparison will be feasible by a joint λ-term calculus, λ^{2Int}, developed in [5], for which the Curry-Howard correspondence, holding between the simply typed λ-calculus and natural deduction systems for intuitionistic logic, is extended in that a two-sorted typed λ-calculus is devised, which is suitable to annotate such bilateralist proof systems displaying two derivability relations.

Therefore, I will proceed as follows: Firstly, I will present the sequent calculus system annotated with terms from λ^{2Int} (Section 2.1). Then I will present a Dualization Theorem for the system stating that for any derivable sequent expressing a proof, there is also a derivable sequent expressing a refutation and vice versa (Section 2.2). After recapitulating then the philosophical motivation and reasoning on how to distinguish sense and denotation for derivations (Section 3.1), I will discuss how this framework can be extended to accommodate bilateralism and why it is reasonable on

[2]For the non-term-annotated versions of these systems, see [27] for the natural deduction system, `N2Int`, and [1] for the sequent calculus, `SC2Int`. In [29] and [6] one can find respectively a proof of a normal form theorem for `N2Int` and a proof of cut elimination for `SC2Int`.

this account to identify certain proofs and refutations (Section 3.2). Finally, I will look into what this means for comparisons between derivations in natural deduction and sequent calculus, i.e., which derivations can be identified here and on which level—just with respect to denotation or also with respect to sense (Section 3.3). A closer investigation in terms of how structural differences between these systems also yield differences for sense and denotation will motivate a modification of what should be considered as (sameness of) sense here.

2. A bilateralist sequent calculus for proofs and refutations

2.1. The sequent calculus $\texttt{SC2Int}_{\lambda}$

Let `Prop` be a countably infinite set of atomic formulas. Elements from `Prop` will be denoted ρ, σ, τ, ... etc. Formulas generated from `Prop` will be denoted A, B, C, ... etc. We use Γ, Δ,... for (possibly empty) multisets of formulas. The concatenation Γ, A stands for $\Gamma \cup \{A\}$. The language $\mathcal{L}_{2Int}$ of `2Int`, as given by Wansing, is defined in Backus-Naur form as follows:

$$A ::= \rho \mid \bot \mid \top \mid (A \wedge A) \mid (A \vee A) \mid (A \rightarrow A) \mid (A \prec A).$$

As can be seen, we have a non-standard connective in this language, namely the operator of co-implication $\prec$,[3] which acts as a dual to implication, just like conjunction and disjunction can be seen as dual connectives of each other. With that we are in the realm of so-called *bi-intuitionistic* logic, which is a conservative extension of intuitionistic logic by co-implication. Note that there is also a use of "bi-intuitionistic logic" in the literature to refer to a specific system, namely `BiInt`, also called "Heyting-Brouwer logic", developed by Rauszer [23] (see also [14, 20, 11, ?]). The understanding of co-implication there is as internalizing the preservation of non-truth from the conclusion to the premises in a valid inference. The system `2Int` uses the same language as `BiInt`, but the meaning of co-implication differs

[3] Sometimes also called "pseudo-difference", "subtraction" or "exclusion" and used with different symbols.

in that it internalizes the preservation of falsity from the premises to the conclusion in a dually valid inference [27, ?, ?]. Also, in `BiInt` we do not have two derivability relations and the system neither enjoys the disjunction property [23] nor the dual conjunction property [19], all of which are reasons for us to prefer `2Int` in this context.

What I will present here, is a term-annotated sequent calculus, which I will call $\mathtt{SC2Int}_\lambda$. Sequents are of the form $(\Gamma; \Delta) \vdash^* C$, which can be informally read as "From the assumed verification of all formulas in Γ and the assumed falsification of all formulas in Δ one can derive the verification (resp. falsification) of C for $* = +$ (resp. $* = -$)". Within the right introduction rules we need to distinguish whether the derivability relation expresses verification or falsification by using the superscripts $+$ and $-$. Within the left rules this is not necessary, but what is needed here instead is distinguishing an introduction of the principal formula into the *assumptions* (indexed by superscript a) from an introduction into the *counterassumptions* (indexed by superscript c). Thus, the set of *proof rules* in `SC2Int` consists of the rules marked with $+$ or with a, while the set of *refutation rules* consists of the rules marked with $-$ or with c. When a rule contains multiple occurrences of *, application of this rule requires that all such occurrences are instantiated in the same way, i.e. either as $+$ or as $-$.

In general, whenever the superscript * is used with a symbol, this is to indicate that the superscript can be either $+$ or $-$ (called *polarities*). When * is used multiple times within a symbol, this is meant to always denote the same polarity. In contrast, when † is used next to * in a symbol this means that it can—but does not have to—be of another polarity. Yet again, multiple † denote the same polarity, i.e., for example $\mathtt{case}\ r^*\{x^*.t^\dagger|y^*.s^\dagger\}^\dagger$ could either stand for $\mathtt{case}\ r^+\{x^+.t^+|y^+.s^+\}^+$, $\mathtt{case}\ r^-\{x^-.t^-|y^-.s^-\}^-$, $\mathtt{case}\ r^+\{x^+.t^-|y^+.s^-\}^-$, or $\mathtt{case}\ r^-\{x^-.t^+|y^-.s^+\}^+$ but not for, e.g., $\mathtt{case}\ r^+\{x^+.t^+|y^+.s^-\}^-$. Furthermore, we use '$\equiv$' to denote *syntactic identity* between terms or types.

DEFINITION 2.1. The set of *type symbols* (or just *types*) is the set of all formulas of $\mathcal{L}_{2Int}$. Let $\mathtt{Var}_{2Int}$ be a countably infinite set of two-sorted term variables. Elements from $\mathtt{Var}_{2Int}$ will be denoted x^*, y^*, z^*, x_1^*, x_2^* ... etc. The two-sorted terms generated from $\mathtt{Var}_{2Int}$ will be denoted

$t^*, r^*, s^*, t^*_1, t^*_2$, ... etc. The set $\mathtt{Term}_{2Int}$ can be defined in Backus-Naur form as follows:
$t ::= x^* \mid \mathtt{top}^+ \mid \mathtt{bot}^- \mid abort(t^*)^\dagger \mid \langle t^*, t^*\rangle^* \mid fst(t^*)^* \mid snd(t^*)^* \mid inl(t^*)^* \mid inr(t^*)^* \mid \mathtt{case}\, t^*\{x^*.t^\dagger | x^*.t^\dagger\}^\dagger \mid (\lambda x^*.t^*)^* \mid App(t^*, t^*)^* \mid \{t^+, t^-\}^* \mid \pi_1(t^*)^\dagger \mid \pi_2(t^*)^\dagger$.

Definition 2.2. A term $t^* \in \mathtt{Term}_{2Int}$ is called a *complex term* if $t^* \notin \mathtt{Var}_{2Int}$.

Definition 2.3. A *(type assignment) statement* is of the form $t : A$ with term t being the *subject* and type A the *predicate* of the statement. It is read "term t is of type A" or, in the 'proof-reading', "t is a proof of formula A".

We are thus using a type-system *à la Curry*, in which the terms are not typed, in the sense that the types are part of the term's structure, but are *assigned* types. We will write $t[s]$ to indicate that s is a subterm of t. If we want to express that term t can (but need not) contain one of s or r as subterms, we write $t[s|r]$. Substitution is expressed by $t[s/x]$, meaning that in term t every free occurrence of x is substituted by s and $t[s/x|r/y]$ means that, if applicable (i.e., if both x and y are free in t), there are simultaneous substitutions of x by s and y by r (see, e.g., rule $\wedge L^a$ below). The usual capture-avoiding requirements for variable substitution are to be observed. We use the same notation with respect to replacement of *terms* (not term variables) with other terms (which will be important in the formulation of the Generation Lemma), i.e., t[s/r] to indicate replacement of subterms of the same type within terms.

Definition 2.4. We write that there is a *derivation* $\Rightarrow_{SC2Int_\lambda} (\Gamma; \Delta) \vdash^* t : A$ in $\mathtt{SC2Int}_\lambda$ if the sequent $(\Gamma; \Delta) \vdash^* t : A$ can be produced as the conclusion of instances of applications of the following rules:[4]

[4]The subscript of $\Rightarrow$ will be omitted henceforth unless there is a possibility for confusion. Also, note that on the left hand of the sequent sign only variables are typed. This corresponds to formulas that are assumptions in natural deduction being types of variables.

`SC2Int`

$$\frac{}{(\Gamma, x^+ : \rho; \Delta) \vdash^+ x^+ : \rho}\ Rf^+ \qquad \frac{}{(\Gamma; \Delta, x^- : \rho) \vdash^- x^- : \rho}\ Rf^-$$

$$\frac{}{(\Gamma; \Delta) \vdash^- \texttt{bot}^- : \bot}\ \bot R^- \qquad \frac{}{(\Gamma, x^+ : \bot; \Delta) \vdash^* abort(x^+)^* : C}\ \bot L^a$$

$$\frac{}{(\Gamma; \Delta) \vdash^+ \texttt{top}^+ : \top}\ \top R^+ \qquad \frac{}{(\Gamma; \Delta, x^- : \top) \vdash^* abort(x^-)^* : C}\ \top L^c$$

$$\frac{(\Gamma; \Delta) \vdash^+ s^+ : A \qquad (\Gamma; \Delta) \vdash^+ t^+ : B}{(\Gamma; \Delta) \vdash^+ \langle s^+, t^+\rangle^+ : A \wedge B}\ \wedge R^+$$

$$\frac{(\Gamma, x^+ : A, y^+ : B; \Delta) \vdash^* s^* : C}{(\Gamma, z^+ : A \wedge B; \Delta) \vdash^* s[fst(z^+)^+/x^+|snd(z^+)^+/y^+]^* : C}\ \wedge L^a$$

$$\frac{(\Gamma; \Delta) \vdash^- t^- : A}{(\Gamma; \Delta) \vdash^- inl(t^-)^- : A \wedge B}\ \wedge R_1^- \qquad \frac{(\Gamma; \Delta) \vdash^- t^- : B}{(\Gamma; \Delta) \vdash^- inr(t^-)^- : A \wedge B}\ \wedge R_2^-$$

$$\frac{(\Gamma; \Delta, x^- : A) \vdash^* s^* : C \quad (\Gamma; \Delta, y^- : B) \vdash^* t^* : C}{(\Gamma; \Delta, z^- : A \wedge B) \vdash^* \texttt{case}\ z^-\{x^-.s^*|y^-.t^*\}^* : C}\ \wedge L^c$$

$$\frac{(\Gamma; \Delta) \vdash^+ t^+ : A}{(\Gamma; \Delta) \vdash^+ inl(t^+)^+ : A \vee B}\ \vee R_1^+ \qquad \frac{(\Gamma; \Delta) \vdash^+ t^+ : B}{(\Gamma; \Delta) \vdash^+ inr(t^+)^+ : A \vee B}\ \vee R_2^+$$

$$\frac{(\Gamma, x^+ : A; \Delta) \vdash^* s^* : C \quad (\Gamma, y^+ : B; \Delta) \vdash^* t^* : C}{(\Gamma, z^+ : A \vee B; \Delta) \vdash^* \texttt{case}\ z^+\{x^+.s^*|y^+.t^*\}^* : C}\ \vee L^a$$

$$\frac{(\Gamma; \Delta) \vdash^- s^- : A \qquad (\Gamma; \Delta) \vdash^- t^- : B}{(\Gamma; \Delta) \vdash^- \langle s^-, t^-\rangle^- : A \vee B}\ \vee R^-$$

$$\frac{(\Gamma; \Delta, x^- : A, y^- : B) \vdash^* s^* : C}{(\Gamma; \Delta, z^- : A \vee B) \vdash^* s[fst(z^-)^-/x^- | snd(z^-)^-/y^-]^* : C} \vee L^c$$

$$\frac{(\Gamma, x^+ : A; \Delta) \vdash^+ t^+ : B}{(\Gamma; \Delta) \vdash^+ (\lambda x^+.t^+)^+ : A \rightarrow B} \rightarrow R^+$$

$$\frac{(\Gamma, x^+ : A \rightarrow B; \Delta) \vdash^+ t^+ : A \qquad (\Gamma, y^+ : B; \Delta) \vdash^* s^* : C}{(\Gamma, x^+ : A \rightarrow B; \Delta) \vdash^* s[App(x^+, t^+)^+/y^+]^* : C} \rightarrow L^a$$

$$\frac{(\Gamma; \Delta) \vdash^+ s^+ : A \qquad (\Gamma; \Delta) \vdash^- t^- : B}{(\Gamma; \Delta) \vdash^- \{s^+, t^-\}^- : A \rightarrow B} \rightarrow R^-$$

$$\frac{(\Gamma, x^+ A; \Delta, y^- : B) \vdash^* s^* : C}{(\Gamma; \Delta, z^- : A \rightarrow B) \vdash^* s[\pi_1(z^-)^+/x^+ | \pi_2(z^-)^-/y^-]^* : C} \rightarrow L^c$$

$$\frac{(\Gamma; \Delta) \vdash^+ s^+ : A \qquad (\Gamma; \Delta) \vdash^- t^- : B}{(\Gamma; \Delta) \vdash^+ \{s^+, t^-\}^+ : A \prec B} \prec R^+$$

$$\frac{(\Gamma, x^+ : A; \Delta, y^- : B) \vdash^* s^* : C}{(\Gamma, z^+ : A \prec B; \Delta) \vdash^* s[\pi_1(z^+)^+/x^+ | \pi_2(z^+)^-/y^-]^* : C} \prec L^a$$

$$\frac{(\Gamma; \Delta, x^- : A) \vdash^- t^- : B}{(\Gamma; \Delta) \vdash^- (\lambda x^-.t^-)^- : B \prec A} \prec R^-$$

$$\frac{(\Gamma; \Delta, x^- : B \prec A) \vdash^- t^- : A \qquad (\Gamma; \Delta, y^- : B) \vdash^* s^* : C}{(\Gamma; \Delta, x^- : B \prec A) \vdash^* s[App(x^-, t^-)^-/y^-]^* : C} \prec L^c$$

The rules Rf^+ and Rf^- can be generalized to instances with arbitrary formulas, not only atomic formulas [6]. The following structural rules of weakening, contraction, and cut can be shown to be admissible in $\texttt{SC2Int}_\lambda$:

$$\frac{(\Gamma;\Delta)\vdash^* t^* : C}{(\Gamma, x^+ : A;\Delta)\vdash^* t^* : C}\ W^a \qquad \frac{(\Gamma;\Delta)\vdash^* t^* : C}{(\Gamma;\Delta, x^- : A)\vdash^* t^* : C}\ W^c$$

$$\frac{(\Gamma, x^+ : A, y^+ : A;\Delta)\vdash^* t^* : C}{(\Gamma, x^+ : A;\Delta)\vdash^* t[x^+/y^+]^* : C}\ C^a \qquad \frac{(\Gamma;\Delta, x^- : A, y^- : A)\vdash^* t^* : C}{(\Gamma;\Delta, x^- : A)\vdash^* t[x^-/y^-]^* : C}\ C^c$$

$$\frac{(\Gamma;\Delta)\vdash^+ t^+ : D \qquad (\Gamma', x^+ : D;\Delta')\vdash^* s^* : C}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^* s[t^+/x^+]^* : C}\ Cut^a$$

$$\frac{(\Gamma;\Delta)\vdash^- t^- : D \qquad (\Gamma';\Delta', x^- : D)\vdash^* s^* : C}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^* s[t^-/x^-]^* : C}\ Cut^c$$

DEFINITION 2.5. The *height* of a derivation is the greatest number of successive applications of rules in it. The zero-premise rules Rf^+, Rf^-, $\top R^+$, $\bot R^-$, $\bot L^a$, and $\top L^c$ have height 0.

The following lemma shows how terms of a certain form are typed and we need it to prove our Dualization Theorem. The terminology and presentation of the following is to a great extent in the style of [7] and [25]. The lemma is divided into five parts, each corresponding to a set of rules starting with the group of zero-premise rules and then going on with the rules for the four connectives.

LEMMA 2.6 (Generation Lemma for $\texttt{SC2Int}_\lambda$).

1. *Zero-premise rules*

   1.1 For every x, if $\Rightarrow (\Gamma;\Delta)\vdash^+ x^+ : A$, *then* $(x^+ : A)\in\Gamma$.

   1.2 For every x, if $\Rightarrow (\Gamma;\Delta)\vdash^- x^- : A$, *then* $(x^- : A)\in\Delta$.

   1.3 If $\Rightarrow (\Gamma;\Delta)\vdash^+ \texttt{top}^+ : A$, *then* $A\equiv\top$.

   1.4 If $\Rightarrow (\Gamma;\Delta)\vdash^- \texttt{bot}^- : A$, *then* $A\equiv\bot$.

   1.5 If $\Rightarrow (\Gamma;\Delta)\vdash^* abort(x^+)^* : A$, *then* $(x^+ : \bot)\in\Gamma$.

   1.6 If $\Rightarrow (\Gamma;\Delta)\vdash^* abort(x^-)^* : A$, *then* $(x^- : \top)\in\Delta$.

2. $\to$-rules

2.1 *If* $\Rightarrow (\Gamma;\Delta) \vdash^+ (\lambda x^+.t^+)^+ : C$, *then* $\exists A, B[\Rightarrow (\Gamma, x^+ : A;\Delta) \vdash^+ t^+ : B \;\&\; C \equiv A \to B]$.

2.2 *If* $\Rightarrow (\Gamma, x^+ : D;\Delta) \vdash^* s[App(x^+, t^+)^+]^* : C$, *then* $\exists A, B[\Rightarrow (\Gamma, x^+ : A \to B;\Delta) \vdash^+ t^+ : A \;\&\; \Rightarrow (\Gamma, y^+ : B;\Delta) \vdash^* s[y^+/App(x^+, t^+)^+]^* : C \;\&\; D \equiv A \to B]$.

2.3 *If* $\Rightarrow (\Gamma;\Delta) \vdash^- \{s^+, t^-\}^- : C$, *then* $\exists A, B[\Rightarrow (\Gamma;\Delta) \vdash^+ s^+ : A \;\&\; \Rightarrow (\Gamma;\Delta) \vdash^- t^- : B \;\&\; C \equiv A \to B]$.

2.4 *If* $\Rightarrow (\Gamma;\Delta, z^- : D) \vdash^* s[\pi_1(z^-)^+|\pi_2(z^-)^-]^* : C$, *then* $\exists A, B[\Rightarrow (\Gamma, x^+ : A;\Delta, y^- : B) \vdash^* s[x^+/\pi_1(z^-)^+|y^-/\pi_2(z^-)^-]^* : C \;\&\; D \equiv A \to B]$.

3. $\prec$-rules

3.1 *If* $\Rightarrow (\Gamma;\Delta) \vdash^+ \{s^+, t^-\}^+ : C$, *then* $\exists A, B[\Rightarrow (\Gamma;\Delta) \vdash^+ s^+ : B \;\&\; \Rightarrow (\Gamma;\Delta) \vdash^- t^- : A \;\&\; C \equiv B \prec A]$.

3.2 *If* $\Rightarrow (\Gamma, z^+ : D;\Delta) \vdash^* s[\pi_1(z^+)^+|\pi_2(z^+)^-]^* : C$, *then* $\exists A, B[\Rightarrow (\Gamma, x^+ : A;\Delta, y^- : B) \vdash^* s[x^+/\pi_1(z^+)^+|y^-/\pi_2(z^+)^-]^* : C \;\&\; D \equiv A \prec B]$.

3.3 *If* $\Rightarrow (\Gamma;\Delta) \vdash^- (\lambda x^-.t^-)^- : C$, *then* $\exists A, B[\Rightarrow (\Gamma;\Delta, x^- : A) \vdash^- t^- : B \;\&\; C \equiv B \prec A]$.

3.4 *If* $\Rightarrow (\Gamma;\Delta, x^- : D) \vdash^* s[App(x^-, t^-)^-]^* : C$, *then* $\exists A, B[\Rightarrow (\Gamma;\Delta, x^- : B \prec A) \vdash^- t^- : A \;\&\; \Rightarrow (\Gamma;\Delta, y^- : B) \vdash^* s[y^-/App(x^-, t^-)^-]^* : C \;\&\; D \equiv B \prec A]$.

4. $\wedge$-rules

4.1 *If* $\Rightarrow (\Gamma;\Delta) \vdash^+ \langle s^+, t^+\rangle^+ : C$, *then* $\exists A, B[\Rightarrow (\Gamma;\Delta) \vdash^+ s^+ : A \;\&\; \Rightarrow (\Gamma;\Delta) \vdash^+ t^+ : B \;\&\; C \equiv A \wedge B]$.

4.2 *If* $\Rightarrow (\Gamma, z^+ : D;\Delta) \vdash^* s[fst(z^+)^+|snd(z^+)^+]^* : C$, *then* $\exists A, B[\Rightarrow (\Gamma, x^+ : A, y^+ : B;\Delta) \vdash^* s[x^+/fst(z^+)^+|y^+/snd(z^+)^+]^* : C \;\&\; D \equiv A \wedge B]$.

4.3 *If* $\Rightarrow (\Gamma;\Delta) \vdash^- inl(t^-)^- : C$, *then* $\exists A, B[\Rightarrow (\Gamma;\Delta) \vdash^- t^- : A$ & $C \equiv A \wedge B]$.

4.4 *If* $\Rightarrow (\Gamma;\Delta) \vdash^- inr(t^-)^- : C$, *then* $\exists A, B[\Rightarrow (\Gamma;\Delta) \vdash^- t^- : B$ & $C \equiv A \wedge B]$.

4.5 *If* $\Rightarrow (\Gamma;\Delta, z^- : D) \vdash^* \texttt{case}\ z^-\{x^-.s^*|y^-.t^*\}^* : C$, *then* $\exists A, B[\Rightarrow (\Gamma;\Delta, x^- : A) \vdash^* s^* : C$ & $(\Gamma;\Delta, y^- : B) \vdash^* t^* : C$ & $D \equiv A \wedge B]$.

5. $\vee$*-rules*

5.1 *If* $\Rightarrow (\Gamma;\Delta) \vdash^+ inl(t^+)^+ : C$, *then* $\exists A, B[\Rightarrow (\Gamma;\Delta) \vdash^+ t^+ : A$ & $C \equiv A \vee B]$.

5.2 *If* $\Rightarrow (\Gamma;\Delta) \vdash^+ inr(t^+)^+ : C$, *then* $\exists A, B[\Rightarrow (\Gamma;\Delta) \vdash^+ t^+ : B$ & $C \equiv A \vee B]$.

5.3 *If* $\Rightarrow (\Gamma, z^+ : D;\Delta) \vdash^* \texttt{case}\ z^+\{x^+.s^*|y^+.t^*\}^* : C$, *then* $\exists A, B[\Rightarrow (\Gamma, x^+ : A;\Delta) \vdash^* s^* : C$ & $(\Gamma, y^+ : B;\Delta) \vdash^* t^* : C$ & $D \equiv A \vee B]$.

5.4 *If* $\Rightarrow (\Gamma;\Delta) \vdash^- \langle s^-, t^-\rangle^- : C$, *then* $\exists A, B[\Rightarrow (\Gamma;\Delta) \vdash^- s^- : A$ & $\Rightarrow (\Gamma;\Delta) \vdash^- t^- : B$ & $C \equiv A \vee B]$.

5.5 *If* $\Rightarrow (\Gamma;\Delta, z^- : D) \vdash^* s[fst(z^-)^-|snd(z^-)^-]^* : C$, *then* $\exists A, B[\Rightarrow (\Gamma;\Delta, x^- : A, y^- : B) \vdash^* s[x^-/fst(z^-)^-|y^-/snd(z^-)^-]^* : C$ & $D \equiv A \vee B]$.

PROOF: Trivial by induction on the height n of the derivation and the definition of the rules given above for $\texttt{SC2Int}_\lambda$. $\square$

Note, that due to the existence of rules involving substitution in $\texttt{SC2Int}_\lambda$ as opposed to the term-annotated natural deduction calculus, $\texttt{N2Int}_\lambda$, there is a difference: For $\texttt{SC2Int}_\lambda$ it is not, unlike for $\texttt{N2Int}_\lambda$, always possible to read off the term generated at a certain step in the derivation which rule was applied last. However, this does not matter for the proof of the Generation Lemma, since it suffices that if there is a derivation of height $n+1$ of the form given on the left side of the implication, then—although other ways of deriving it may be possible—a derivation of height n of the form given on the right side of the implication is one of the possible ways, which it is due to the definition of the rules.

Let us now consider the reductions available in our framework. For their definition the definition of a *compatible* relation is needed. Since for λ^{2Int} we need many clauses for the inductive definition, which can be inquired in detail in [5, Appendix 1], I think it suffices here to say that a "compatible relation 'respects' the syntactic constructions" [25, p. 12] of the terms, i.e., let $\mathcal{R}$ be a compatible relation on $\mathtt{Term_{2Int}}$, then for all $t, r, s \in \mathtt{Term_{2Int}}$: if $t\mathcal{R}r$, then $(\lambda x^*.t^*)^*\mathcal{R}(\lambda x^*.r^*)^*$, $App(t^*, s^*)^*\mathcal{R}App(r^*, s^*)^*$, $App(s^*, t^*)^*\mathcal{R}App(s^*, r^*)^*$, etc.

Definition 2.7 (Reductions).
1. The least compatible relation $\rightsquigarrow_{1\beta}$ on $\mathtt{Term_{2Int}}$ satisfying the following clauses is called *β-reduction*:

$$App((\lambda x^*.t^*)^*, s^*)^* \rightsquigarrow_{1\beta} t[s^*/x^*]^*$$
$$\pi_1(\{s^+, t^-\}^*)^+ \rightsquigarrow_{1\beta} s^+ \quad \pi_2(\{s^+, t^-\}^*)^- \rightsquigarrow_{1\beta} t^-$$
$$fst(\langle s^*, t^*\rangle^*)^* \rightsquigarrow_{1\beta} s^* \quad snd(\langle s^*, t^*\rangle^*)^* \rightsquigarrow_{1\beta} t^*$$
$$\mathtt{case}\ inl(r^*)^*\{x^*.s^\dagger|y^*.t^\dagger\}^\dagger \rightsquigarrow_{1\beta} s[r^*/x^*]^\dagger$$
$$\mathtt{case}\ inr(r^*)^*\{x^*.s^\dagger|y^*.t^\dagger\}^\dagger \rightsquigarrow_{1\beta} t[r^*/y^*]^\dagger$$

2. For all clauses the term on the left of $\rightsquigarrow_{1\beta}$ is called *β-redex*, while the term on the right is its *contractum*.
3. The relation $\rightsquigarrow_{\beta}$ (multi-step β-reduction) is the transitive and reflexive closure of $\rightsquigarrow_{1\beta}$.

Theorem 2.8 (Subject Reduction Theorem for λ^{2Int}). *If* $\Rightarrow (\Gamma; \Delta) \vdash^* t^* : C$ *and* $t \rightsquigarrow_{\beta} t'$, *then there is a derivation such that* $\Rightarrow (\Gamma'; \Delta') \vdash^* t'^* : C$ *for* $\Gamma' \subseteq \Gamma$ *and* $\Delta' \subseteq \Delta$.

The proof follows straightforward from having a proof of cut elimination for the system.[5] From now on we will omit the superscripts of subterms in the cases where the superscript of the whole term clearly determines the other polarities, i.e., instead of, e.g., $(\lambda x^+.t^+)^+$ writing $(\lambda x.t)^+$ suffices.
Due to the structure of term formation for $\vee L^a$ and $\wedge L^c$ (they do not work, like the other left introduction rules, with substitution of terms within terms), we also need further *permutation conversions*. These are different,

[5] To be more precise, these cases correspond to the cases in which the cut formula is principal in both premises of cut, see [6, pp. 231–235].

though, from the ones for the natural deduction system, which is due to the structural differences between normalization and cut elimination proofs. In order to cover all the cases of permutation procedures occurring in a proof of cut elimination, we need the following definition.[6]

Definition 2.9 (Permutation conversions).

1. The least compatible relation $\rightsquigarrow_{1p}$ on $\mathtt{Term}_{\mathtt{2Int}}$ satisfying the following clauses is called *permutation conversion*:

   $r[\mathtt{case}\ z^*\{x^*.s^\dagger|y^*.t^\dagger\}]^+ \rightsquigarrow_{1p} \mathtt{case}\ z^*\{x^*.r[s^\dagger]^+|y^*.r[t^\dagger]^+\}$

   $r[\mathtt{case}\ z^*\{x^*.s^\dagger|y^*.t^\dagger\}]^- \rightsquigarrow_{1p} \mathtt{case}\ z^*\{x^*.r[s^\dagger]^-|y^*.r[t^\dagger]^-\}$

2. For all clauses the term on the left of $\rightsquigarrow_{1p}$ is called *p-redex*.

3. The relation $\rightsquigarrow_p$ (multi-step permutation conversion) is the transitive and reflexive closure of $\rightsquigarrow_{1p}$.

Definition 2.10 (Normal form).

A term $t \in \mathtt{Term}_{\mathtt{2Int}}$ is said to be in *normal form* iff t does not contain any β- or p-redex.

2.2. Duality in λ^{2Int}

As mentioned above, I want to motivate an account here that ultimately yields an identification of the denotation of certain proofs and refutations. In order to make this explicit on the formal level, I will define dualities in λ^{2Int} and prove on this basis a Dualization Theorem, which will show the close relation between proofs and refutations in this system.

---

[6] These correspond to the cases in which the cut formula is not principal in the left premise of cut and the last rule used to derive the left premise is $\vee L^a$ or $\wedge L^c$, see [6, p. 219f.]. All other cases are unproblematic because both cut and the other left introduction rules work with substitution operations, which means that permutation procedures for cut elimination will not change the term of the derived formulas. Note that due to cut elimination type preservation straightforwardly also holds for the permutation conversions. I expressed only type preservation for the β-reductions in form of a 'Subject Reduction Theorem' because this terminology is conventional.

Definition 2.11. We will define a duality function d mapping types to their dual types, terms to their dual terms and contexts to their dual contexts as follows:[7]

1. $d(\rho) = \rho$
2. $d(\top) = \bot$
3. $d(\bot) = \top$
4. $d(A \wedge B) = d(A) \vee d(B)$
5. $d(A \vee B) = d(A) \wedge d(B)$
6. $d(A \to B) = d(B) \prec d(A)$
7. $d(A \prec B) = d(B) \to d(A)$
8. $d(x^*) = x^d$
9. $d(\texttt{top}^+) = \texttt{bot}^-$
10. $d(\texttt{bot}^-) = \texttt{top}^+$
11. $d(abort(t^*)^\dagger) = abort(d(t^*))^d$
12. $d(\langle t^*, s^* \rangle^*) = \langle d(t^*), d(s^*) \rangle^d$
13. $d(inl(t^*)^*) = inl(d(t^*))^d$
14. $d(inr(t^*)^*) = inr(d(t^*))^d$
15. $d((\lambda x^*.t^*)^*) = (\lambda d(x^*).d(t^*))^d$
16. $d(\{t^+, s^-\}^*) = \{d(s^-), d(t^+)\}^d$
17. $d(fst(t^*)^*) = fst(d(t^*))^d$
18. $d(snd(t^*)^*) = snd(d(t^*))^d$
19. $d(\texttt{case } r^*\{x^*.s^\dagger | y^*.t^\dagger\}^\dagger) = \texttt{case } d(r^*)\{d(x^*).d(s^\dagger) | d(y^*).d(t^\dagger)\}^d$
20. $d(App(s^*, t^*)^*) = App(d(s^*), d(t^*))^d$
21. $d(\pi_1(t^*)^\dagger) = \pi_2(d(t^*))^d$
22. $d(\pi_2(t^*)^\dagger) = \pi_1(d(t^*))^d$
23. $d((\Gamma; \Delta)) = (d(\Delta); d(\Gamma))$,with $d(\Delta) = \{d(t^*) \mid t^* \in \Delta\}$, resp. for $d(\Gamma)$.

The following theorem then states that whenever we have a derivable sequent expressing a provability (refutability) relation, we can construct a

[7]The superscript ^{d} is used to indicate the dual polarity of whatever polarity * stands for in its respective dual version.

derivation of its dual sequent expressing a refutability (provability) relation with the same height in our system.

THEOREM 2.12 (Dualization). *If* $\Rightarrow (\Gamma; \Delta) \vdash^* t^* : A$ *with a height of derivation at most* n*, then* $\Rightarrow d(\Gamma; \Delta) \vdash^d d(t^*) : d(A)$ *(called its* dual sequent*) with a height of derivation at most* n*.*

PROOF: By induction on the height of derivation n using the Generation Lemma.
If $n = 0$, then one of these six cases holds:

1. $\Rightarrow (\Gamma, x^+ : \rho; \Delta) \vdash^+ x^+ : \rho$
2. $\Rightarrow (\Gamma; \Delta, x^- : \rho) \vdash^- x^- : \rho$
3. $\Rightarrow (\Gamma; \Delta) \vdash^+ \texttt{top}^+ : \top$
4. $\Rightarrow (\Gamma; \Delta) \vdash^- \texttt{bot}^- : \bot$
5. $\Rightarrow (\Gamma, x^+ : \bot; \Delta) \vdash^* abort(x^+)^* : C$
6. $\Rightarrow (\Gamma; \Delta, x^- : \top) \vdash^* abort(x^-)^* : C$

In case 1 the dual derivation is $\Rightarrow (d(\Delta); d(\Gamma), x^- : \rho) \vdash^- x^- : \rho$.
In case 2 the dual derivation is $\Rightarrow (d(\Delta), x^+ : \rho; d(\Gamma)) \vdash^+ x^+ : \rho$.
In case 3 the dual derivation is $(d(\Delta); d(\Gamma)) \vdash^- \texttt{bot}^- : \bot$.
In case 4 the dual derivation is $(d(\Delta); d(\Gamma)) \vdash^+ \texttt{top}^+ : \top$.
In case 5 the dual derivation is $\Rightarrow (d(\Delta); d(\Gamma), x^- : \top) \vdash^* abort(x^-)^* : C$.
In case 6 the dual derivation is $\Rightarrow (d(\Delta), x^+ : \bot; d(\Gamma)) \vdash^* abort(x^+)^* : C$.
All dual derivations can be trivially constructed with a height of $n = 0$.
Assume height-preserving dualization up to derivations of height at most n.[8]

If $\Rightarrow (\Gamma; \Delta) \vdash^+ \langle s^+, t^+ \rangle^+ : A \wedge B$, resp. $\Rightarrow (\Gamma; \Delta) \vdash^- \langle s^-, t^- \rangle^- : A \vee B$, is of height $n + 1$, then (by Generation Lemma 4.1, resp. 5.4) we have $\Rightarrow (\Gamma; \Delta) \vdash^+ s^+ : A$ and $\Rightarrow (\Gamma; \Delta) \vdash^+ t^+ : B$, resp. $\Rightarrow (\Gamma; \Delta) \vdash^- s^- : A$ and $\Rightarrow (\Gamma; \Delta) \vdash^- t^- : B$, with height at most n.
Then by inductive hypothesis $\Rightarrow (d(\Delta); d(\Gamma)) \vdash^- d(s^+) : d(A)$ and $\Rightarrow (d(\Delta); d(\Gamma)) \vdash^- d(t^+) : d(B)$, resp. $\Rightarrow (d(\Delta); d(\Gamma)) \vdash^+ d(s^-) : d(A)$ and $\Rightarrow (d(\Delta); d(\Gamma)) \vdash^+ d(t^-) : d(B)$, are of height at most n as well.

[8] Since the proof for the Dualization Theorem for the natural deduction system $\texttt{N2Int}_\lambda$ is given in full form in [5] and proceeds in essentially the same manner, I will only show two cases here to give a sketch of how the proof works.

By application of $\vee R^-$, resp. $\wedge R^+$, we can construct a derivation of height $n+1$ s.t. $\Rightarrow (d(\Delta); d(\Gamma)) \vdash^- \langle d(s^+), d(t^+)\rangle^- : d(A) \vee d(B)$, resp. $\Rightarrow (d(\Delta); d(\Gamma)) \vdash^+ \langle d(s^-), d(t^-)\rangle^+ : d(A) \wedge d(B)$.

If $\Rightarrow (\Gamma; \Delta) \vdash^+ fst(z^+)^+ : C$, resp. $\Rightarrow (\Gamma; \Delta) \vdash^- fst(z^-)^- : C$, is of height $n+1$, then (by Generation Lemma 4.2, resp. 5.5) $\Gamma = \Gamma' \cup z^+ : A \wedge B$, resp. $\Delta = \Delta' \cup z^- : A \vee B$, and we have $\Rightarrow (\Gamma, x^+ : A, y^+ : B; \Delta) \vdash^+ x^+ : C$, resp. $\Rightarrow (\Gamma; \Delta, x^- : A, y^- : B) \vdash^- x^- : C$, with height at most n.

Then by inductive hypothesis $\Rightarrow (d(\Delta); d(\Gamma'), x^- : A, y^- : B) \vdash^- x^- : d(C)$, resp. $\Rightarrow (d(\Delta'), x^+ : A, y^+ : B; d(\Gamma)) \vdash^+ x^+ : d(C)$, is of height at most n as well.

By application of $\vee L^c$, resp. $\wedge L^a$, we can construct a derivation of height $n+1$ s.t. $\Rightarrow (d(\Delta); d(\Gamma'), z^- : A \vee B) \vdash^- fst(z^-)^- : d(C)$, resp. $\Rightarrow (d(\Delta'), z^+ : A \wedge B; d(\Gamma)) \vdash^+ fst(z^+)^+ : d(C)$.

The other cases work analogously. □

In order to illustrate what is stated by the Dualization Theorem, let us take a look at an example now and consider the following derivation:

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{$\scriptstyle \top R^+$}
\UnaryInfC{$(x^+ : \rho \prec \sigma; \emptyset) \vdash^+ \mathtt{top}^+ : \top$}
\AxiomC{}
\RightLabel{$\scriptstyle Rf^+$}
\UnaryInfC{$(z^+ : \rho; \emptyset) \vdash^+ z^+ : \rho$}
\RightLabel{$\scriptstyle W^c$}
\UnaryInfC{$(z^+ : \rho; z_1^- : \sigma) \vdash^+ z^+ : \rho$}
\RightLabel{$\scriptstyle \prec L^a$}
\UnaryInfC{$(x^+ : \rho \prec \sigma; \emptyset) \vdash^+ \pi_1(x^+)^+ : \rho$}
\AxiomC{}
\RightLabel{$\scriptstyle Rf^-$}
\UnaryInfC{$(\emptyset; z_1^- : \sigma) \vdash^- z_1^- : \sigma$}
\RightLabel{$\scriptstyle W^a$}
\UnaryInfC{$(z^+ : \rho; z_1^- : \sigma) \vdash^- z_1^- : \sigma$}
\RightLabel{$\scriptstyle \prec L^a$}
\UnaryInfC{$(x^+ : \rho \prec \sigma; \emptyset) \vdash^- \pi_2(x^+)^- : \sigma$}
\RightLabel{$\scriptstyle \to R^-$}
\BinaryInfC{$(x^+ : \rho \prec \sigma; \emptyset) \vdash^- \{\pi_1(x^+)^+, \pi_2(x^+)^-\}^- : \rho \to \sigma$}
\RightLabel{$\scriptstyle \prec R^+$}
\BinaryInfC{$(x^+ : \rho \prec \sigma; \emptyset) \vdash^+ \{\mathtt{top}^+, \{\pi_1(x^+)^+, \pi_2(x^+)^-\}^-\}^+ : \top \prec (\rho \to \sigma)$}
\RightLabel{$\scriptstyle \to R^+$}
\UnaryInfC{$\vdash^+ (\lambda x^+.\{\mathtt{top}^+, \{\pi_1(x^+)^+, \pi_2(x^+)^-\}^-\}^+)^+ : (\rho \prec \sigma) \to (\top \prec (\rho \to \sigma))$}
\end{prooftree}
$$

Now we dualize the end-term and the formula by our duality function d yielding the following:

$$d((\lambda x^+.\{\mathtt{top}^+, \{\pi_1(x^+)^+, \pi_2(x^+)^-\}^-\}^+)^+) = (\lambda x^-.\{\{\pi_1(x^-)^+, \pi_2(x^-)^-\}^+, \mathtt{bot}^-\}^-)^-$$

$$d((\rho \prec \sigma) \to (\top \prec (\rho \to \sigma))) = ((\sigma \prec \rho) \to \bot) \prec (\sigma \to \rho)$$

We can now construct a derivation of the dual sequent

$\vdash^- (\lambda x^-.\{\{\pi_1(x^-)^+, \pi_2(x^-)^-\}^+, \mathtt{bot}^-\}^-)^- : ((\sigma \prec \rho) \to \bot) \prec (\sigma \to \rho)$:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(z_1^+ : \sigma; \emptyset) \vdash^+ z_1^+ : \sigma}\,{\scriptstyle Rf^+}}{(z_1^+ : \sigma; z^- : \rho) \vdash^+ z_1^+ : \sigma}\,{\scriptstyle W^c}}{(\emptyset; x^- : \sigma \to \rho) \vdash^+ \pi_1(x^-)^+ : \sigma}\,{\scriptstyle \to L^c} \quad \dfrac{\dfrac{\dfrac{}{(\emptyset; z^- : \rho) \vdash^- z^- : \rho}\,{\scriptstyle Rf^-}}{(z_1^+ : \sigma; z^- : \rho) \vdash^- z^- : \rho}\,{\scriptstyle W^a}}{(\emptyset; x^- : \sigma \to \rho) \vdash^- \pi_2(x^-)^- : \rho}\,{\scriptstyle \to L^c}}{(\emptyset; x^- : \sigma \to \rho) \vdash^+ \{\pi_1(x^-)^+, \pi_2(x^-)^-\}^+ : \sigma \prec \rho}\,{\scriptstyle \prec R^+} \quad \dfrac{}{(\emptyset; x^- : \sigma \to \rho) \vdash^- \texttt{bot}^- : \bot}\,{\scriptstyle \bot R^-}}{\dfrac{(\emptyset; x^- : \sigma \to \rho) \vdash^- \{\{\pi_1(x^-)^+, \pi_2(x^-)^-\}^+, \texttt{bot}^-\}^- : (\sigma \prec \rho) \to \bot}{\vdash^- (\lambda x^-.\{\{\pi_1(x^-)^+, \pi_2(x^-)^-\}^+, \texttt{bot}^-\}^-)^- : ((\sigma \prec \rho) \to \bot) \prec (\sigma \to \rho)}\,{\scriptstyle \prec R^-}}\,{\scriptstyle \to R^-}
$$

The close relation between proofs and refutations is literally 'visible' in that these derivations look like the mirrored version of each other with respect to the construction of the proof tree. At each step we have the respective dual sequents derived according to the respective dual rules. That is why I want to lay down an account which identifies them with respect to their denotation, i.e., their underlying construction, although in one case it is delivered as a derivation of a provable sequent and in the other as one of a refutable sequent.

3. Sense and denotation in bilateralist proof systems

3.1. Philosophical background for unilateralist systems

In [26] and [2] related frameworks are developed, considering the background of proof-theoretic semantics, about how we can distinguish in a Fregean style between sense and denotation of proofs. Considering, firstly, that in this tradition proofs are seen as interesting objects of study in their own right and, secondly, the simple observation that there can be different ways to deliver a derivation from the same premises to the same conclusion – both within one (kind of) proof system and in different ones – it seems only natural to ask questions, like "Do these derivations represent the same underlying proof or not?" or "Are these derivations synonymous?".[9] The former question, then, would be concerned with derivations' *identity*, i.e.,

[9]We will distinguish for these purposes (as it is also done in the literature concerned with these questions, e.g., in [15, **?**, **?**, **?**]) between a *proof* as the underlying object (conceived of as a mental entity in line of the intuitionistic tradition) and a *derivation* as its respective linguistic representation.

sameness of *denotation*, while the latter would be concerned with derivations' meaning, i.e., *synonymy* would be sameness of *sense*.[10]
Since the setting in which these questions are considered (see, e.g., [21, p. 257ff.]) is usually natural deduction systems, a standard example that can be given are two derivations, one being in non-normal form and the other being in its respective normal form. Then, so Prawitz argues, since the derivation in normal form is the most *direct* way of representing the proof, this can be seen as representing the underlying proof object, i.e., the denotation, best. Thus, for two derivations sharing the same normal form the denotation would be the same, even though one may be represented in non-normal form. What would differ in this case would be the sense, though, because the way the denotation is represented essentially differs. While the point about normal form, denotation and proof identity is mentioned at several places in the literature (next to Prawitz see, e.g., [17, p. 101f.], [8] or [25, p. 83ff.]), a conception of *sense* for proofs is rarely found in the standard literature on this topic.[11] A notable exception is [26], where Tranchini gives a convincing approach on how such a distinction could be usefully applied in the context of PTS with respect to distinguishing 'well-behaved' derivations as opposed to paradoxical and also `tonk`-containing derivations. He argues that a derivation can only have sense if all the rules applied in it have reductions available (as opposed to, e.g., rules for `tonk`), since the reductions are what transfers a derivation into its normal form, i.e., its denotation, and thus, the reductions are the way to get to the denotation of proofs, which seems to fit nicely a Fregean conception of sense.
While with Tranchini's account it can be decided whether or not a derivation has sense, nothing more is said on what constitutes the sense in a way that would make it possible to decide whether or not two derivations sharing the same denotation also have the same sense, i.e., are to be considered *synonymous*. In [2], then, this is provided by adopting Tranchini's criterion

[10] The expressions 'meaning' and 'sense' are used interchangeably throughout the paper as it is usual convention.

[11] An exception is [10]: the notion of sense is mentioned but not further developed here.

for a derivation to have sense, while at the same time further developing an account of sense of derivations. For such an account we ultimately have to move to a setting with λ-term-annotated proof systems because this is making the structure of derivations as precise as is needed for such distinctions. On such a Fregean framework annotating proof systems with λ-terms can be seen as corresponding to something like transforming natural language into a formal language. Such a precisification is needed in order to apply these Fregean notions sensibly. The *denotation* of a derivation in such systems is then referred to by the λ-term annotating the conclusion of the derivation (called the *end-term*), which is a conception that is well in accordance with the discussion in the literature mentioned above. The novelty of the approach lies in giving a concrete definition of the *sense* of a derivation, which is argued to be represented by the set of all λ-terms occurring within the derivation. The reasoning is that these reflect the operations that are used to build the derivation and thus, they can be seen as encoding a procedure that takes us to the denotation, since the procedure finally yields the end-term.

Two questions should be clarified here a bit further. For one, why should a set of terms encoding a procedure taking us to the denotation be considered as reflecting the sense and for another, why should a *set* of terms encoding a procedure taking us to the denotation be considered as reflecting the sense? To the former question, I think this can be justified because such an interpretation of Fregean sense can be found in several seminal pieces in the literature, for example Dummett uses the phrase of a "procedure" to determine the denotation multiple times [9, pp. 232, 323, 636] and says that "names with different senses but the same referent correspond to different routes leading to the same destination" [9, p. 96]. Other examples are Girard [10, p. 2] speaking of sense as "a sequence of *instructions*" or Horty [13, pp. 66–69] speaking of "senses as procedures". To the second question, I think it is fair to ask why we should exactly consider the set of terms, why not the multi-set or a structured sequence? There is both a technical and a philosophical motivation for it. One worry that was raised by an anonymous reviewer is that sets rather erase the structure of the derivation, not showing which terms are tighter connected by the inference rules. I don't think this is the case, though, because the set of terms will let us read off the way the end-term is constructed. This is precisely why we need more than just the end-term. At least in sequent calculus, there will be ways to derive the same end-term with different ways of derivation. But the specific way taken in a derivation will be revealed by the other terms contained in the set that in our opinion reflects the sense. Another worry might be that with the use of sets indeed we do not track the use of (discharging) multiple assumptions, for instance. I think this is a justified worry if we would use a system of linear logic, for example, where this must be tracked carefully and multiple assumption discharge is forbidden. For such systems it would make sense to go for multi-sets instead of sets but in our setting this would actually lead to too much distinction between senses than seems philosophically justified.

In general, there are two advantages to this approach of distinguishing sense and denotation of derivations by using term-annotated proof systems. Firstly, we can distinguish identity of derivations on a more fine-grained level, namely not only when it comes to sameness of denotation, i.e., what we will call derivational *identity*, but also concerning sameness of sense, which we will refer to as derivational *synonymy*. Considering the reasoning about proof identity in PTS mentioned above, it seems reasonable, then, in our system to let identity between two derivations hold modulo their end-terms belonging to the same equivalence class induced by the set of conversions considered for the system, i.e., in $\mathtt{SC2Int}_\lambda$ the β-reductions and the permutation conversions. Synonymy, on the other hand, being the more fine-grained notion, only holds modulo α-conversions, i.e., renaming of bound variables, of the terms of the respective sense-denoting sets. We will see in section 3.2 below some examples supporting these conceptions. The second advantage of this approach is that the λ-term annotations allow us a seemingly easier comparison between natural deduction (ND) and sequent calculus systems (SC), since we can simply look at the terms contained in the derivations instead of having to figure out the structural correspondences. Thus, we can compare sense and denotation not only within one (kind of) proof system but also over different kinds of proofs systems.

Finally, by extending this framework to deal with bilateralist settings, we can do even more. Firstly, we can compare what is reasonable as notions for sense and denotation considering bilateralist vs. unilateralist settings and, secondly, also what the outcome looks like when comparing different kinds of bilateralist proof systems, i.e., here $\mathtt{N2Int}_{\lambda}$ and $\mathtt{SC2Int}_{\lambda}$. Concerning the latter question, I will propose (cf. Section 3.3) some slight changes to the suggestions made in [2], though, about what seems plausible to consider as sense and (non-)synonymy when comparing ND vs. SC systems. These suggestions for formal changes of the definition of sense and thus also of synonymy will be motivated by philosophical reasoning, though.

3.2. Extending the framework for bilateralism

But firstly, let us briefly consider what is supposed to be our understanding of sense and denotation of derivations in a bilateralist setting.[12] The definitions mentioned above still hold, i.e., the denotation of a derivation is referred to by its λ-end-term, while the sense is to be reflected by the set of λ-terms occurring within the derivation. What differs from the unilateralist setting is that we extend our concept of what establishes *identity* of proofs, i.e., in which cases we should think of two derivations representing the same underlying object. Our proposal is to identify denotation not only over end-terms that are obtained from each other by β- and permutation conversions but also by our duality function. This is motivated by the fact that in proof systems annotated with terms of λ^{2Int} whenever a derivation in the form of proving (refuting) a formula, resp. of a sequent expressing a provability (refutability) relation, can be delivered in the system, it is possible to give a corresponding derivation in the form of refuting (proving) the dual formula, resp. of the dual sequent, which can be proven by the Dualization Theorem. Since the construction of proofs and refutations can be conducted in essentially the same manner, i.e., with dual operations at each inferential step, they can be seen as mirroring representations of

[12]For a more detailed version of this argument, see [5].

the same 'derivational object'[13]. This yields an identification of proofs and refutations, at least on the denotational level, which can also be found in the traditional literature, e.g., in [18] and [16], in which something like refutation or falsification are taken to be concepts just as important and primitive as usually proof or verification. The sense, though, would of course differ in those cases of corresponding proofs and refutations, since the way the derivational object is represented is essentially different, via *proving* vs. via *refuting* something.

Consider the following exemplary derivations to see how this is supposed to work (for now, just considering SC-derivations):

$$\dfrac{\dfrac{}{(x^+ : \rho; \emptyset) \vdash^+ x^+ : \rho}\,Rf^+}{\vdash^+ (\lambda x.x)^+ : \rho \to \rho}\,{\to}R^+ \qquad \dfrac{\dfrac{}{(x^+ : \sigma; \emptyset) \vdash^+ x^+ : \sigma}\,Rf^+}{\vdash^+ (\lambda x.x)^+ : \sigma \to \sigma}\,{\to}R^+ \qquad \dfrac{\dfrac{}{(y^+ : \sigma; \emptyset) \vdash^+ y^+ : \sigma}\,Rf^+}{\vdash^+ (\lambda y.y)^+ : \sigma \to \sigma}\,{\to}R^+$$

$$\dfrac{\dfrac{}{(\emptyset; x^- : \rho) \vdash^- x^- : \rho}\,Rf^-}{\vdash^- (\lambda x.x)^- : \rho \prec \rho}\,{\prec}R^- \qquad \dfrac{\dfrac{}{(\emptyset; x^- : \sigma) \vdash^- x^- : \sigma}\,Rf^-}{\vdash^- (\lambda x.x)^- : \sigma \prec \sigma}\,{\prec}R^- \qquad \dfrac{\dfrac{}{(\emptyset; y^- : \sigma) \vdash^- y^- : \sigma}\,Rf^-}{\vdash^- (\lambda y.y)^- : \sigma \prec \sigma}\,{\prec}R^-$$

According to our framework about sense and denotation outlined above, all these derivations would be considered as having the same denotation, i.e., the underlying derivational object is identical in all these cases. For this it does not matter that different formulas are derived because what we are interested in is not the denotation of the formulas but of the derivation, i.e., the structure of the construction is decisive here. While the derivations on the respective vertical axes as well as those standing diagonally to each other have the same denotation because their end-terms can be obtained from each other by our duality function, they differ in sense, though. The sense of derivations is sensitive to the polarities that occur within the derivation because proving vs. refuting something seems a crucially different way of representation. The situation is different for the derivations on the horizontal axes: these do not only have the same denotation but also the *same sense*. Note that it is very much in accordance to Frege's distinction on this matter that there are different signs (i.e., here different

[13] I use this terminology here instead of the more usual 'proof object' to avoid a unilateralist connotation of favoring proofs over refutations.

variables for terms and for formulas) involved. Since we are concerned with the meaning of *derivations* (not formulas or propositions etc.), it should not make a difference which atomic formulas are chosen as long as the derived formula is structurally the same. I think it can be considered as an advantage of this framework that we do not get a collapse between signs and sense because this could mean that our notion of sense is too fine-grained. Thus, just like in Frege's considerations, there are cases where the difference in signs is negligible for the sense, namely, speaking in terms of type theory now, when the *principal types* of the terms involved are the same, i.e., the most general type that can be assigned to a term.[14] So, although the signs occurring in two derivations can be different, this will have no effect on them having different senses as long as the principal types of all terms occurring within the derivations are the same.

Thus, what leads to a difference in sense in our system is a difference in the principal types of the terms or a difference in the polarities. Note, that in both cases this inherently philosophical reasoning yields formal choices. The former is essentially the reason why for such a framework Curry-style typing can be considered as favorable over a Church-style typing. In the latter system each term is usually uniquely typed, i.e., we would get a collapse of signs and sense: Since the sense is constituted by the terms occurring in a derivation, a differently typed term would automatically lead to a different sense. With Curry-style typing we get the (in our opinion) more intuitive result that for the meaning of derivations it is irrelevant whether $p \to p$ or $q \to q$ is derived, as long as they are structurally derived in the same way. With respect to the polarities, it is also when considering distinctions in sense, i.e., a philosophical reason, that we see why the terms need to display polarities in such a system: Stripping them off the terms, would result in all of the above derivations not only being identical when it comes to their denotation but also when it comes to their sense, i.e., they would all have to be considered synonymous. It seems very reasonable, though, to argue that the way of inference is essentially different when proving something vs. refuting something, i.e., that the sense should be distinguished here.

[14] For example, for the term $(\lambda x.x)^+$ its types could be $p \to p$, $q \to q$, $(p \to q) \to (p \to q)$, etc., while its principal type would be $A \to A$.

3.3. Comparing derivations in natural deduction and in sequent calculus

In [2] it is argued that there can be the same denotation *and* the same sense, i.e., identity *and* synonymy between ND- and SC-derivations. An objection that has been voiced against this being possible *in principle* is that SC is a meta-linguistic expression of ND. On such a conception it may make sense to argue that they are incomparable with respect to meaning, in the sense of being on different levels and thus, never able to share the same meaning. However, although SC can surely from a historical perspective of how and why it was developed by Gentzen be seen as a meta-version of ND, I still think it is nowadays also justified to see it as a calculus in its own right. While certainly different linguistic expressions are derived, in one case formulas, in the other sequents, the point here is exactly to make a case for there being good reasons to see these in certain cases just as different linguistic representations, i.e., in Fregean terms: as a difference in the signs that are used, not more.

Let us now consider some differences between applying this framework of sense and denotation for derivations to natural deduction vs. to sequent calculi. Two features are important to consider here, which are, firstly, the effect of applications of structural rules, especially cut, in SC and secondly, that SC is much more flexible with respect to changing the order of rule applications. As mentioned above, I think that this context requires some tweaking of the definition of sense and synonymy. In a nutshell, it is the following I want to propose: Instead of saying that the sense of a derivation is represented by the set of *all* λ-terms occurring within the derivation, it seems more sensible to argue that it is represented by the set of *all complex* λ-terms occurring within the derivation.[15] Philosophically this makes sense because these reflect the operations that are used to build the derivation, and formally this tweaking allows us to retain a conception of synonymy

[15] Although this is a modification of what was proposed in [2], it is worth mentioning that if we would apply this modified definition to what was argued for in that paper, this would not change those former results. The exemplary derivations considered there were not as fine-grained as the ones here and thus, did not show the features that here motivate the modification.

of derivations to the extent that *prima facie*[16] corresponding ND- and SC-derivations are considered to be synonymous. Of course, this means to exclude the assumptions from making up the meaning of a derivation, which may strike us as an odd move: Surely, assumptions (open or closed) are part of a derivation, so why shouldn't they also be part of its meaning? There could be a lot more said about this, but suffice it here to point to the following: In systems, which disallow certain discharge conventions (vacuous or multiple), it surely makes sense to track the use of assumptions more closely and consider the meaning of the derivation as sensitive to it. In the systems considered in *this* paper, however, it really does not make a significant difference, since the assumptions implicitly still show by the variables being part of the complex terms.

3.3.1. Difference in sense because of structural rules

Let us consider the following derivations in $\texttt{N2Int}_\lambda$ and $\texttt{SC2Int}_\lambda$ to give an example of why I think there is a slight modification needed in the conception of sense and synonymy (in ND the single lines denote the proof relation that is expressed by $\vdash^+$ in SC and the double lines the refutation relation expressed by $\vdash^-$ in SC):

ND_1

$$\dfrac{\dfrac{\dfrac{[y^+ : \rho \wedge \rho]}{fst(y)^+ : \rho}\ {\scriptstyle \wedge E_1}}{inl(fst(y))^+ : \rho \vee \rho}\ {\scriptstyle \vee I_1}}{(\lambda y.inl(fst(y)))^+ : (\rho \wedge \rho) \rightarrow (\rho \vee \rho)}\ {\scriptstyle \rightarrow I}$$

Sense: $\{fst(y)^+, inl(fst(y))^+, (\lambda y.inl(fst(y)))^+\}$

[16]In the sense that the, again, *prima facie* corresponding rules, i.e., right (left) introduction rules in SC and introduction (elimination) rules in ND, are applied in the same order. Of course, it has to be kept in mind that any correspondence between these systems is not one-to-one but one-to-many, i.e., for an ND-derivation there can be possibly many SC-derivations.

SC$_1$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(z^+ : \rho; \emptyset) \vdash^+ z^+ : \rho}\ Rf^+}{(z^+ : \rho, x^+ : \rho; \emptyset) \vdash^+ z^+ : \rho}\ W^a}{(y^+ : \rho \wedge \rho; \emptyset) \vdash^+ fst(y)^+ : \rho}\ \wedge L^a}{(y^+ : \rho \wedge \rho; \emptyset) \vdash^+ inl(fst(y))^+ : \rho \vee \rho}\ \vee R_1^+}{\vdash^+ (\lambda y.inl(fst(y)))^+ : (\rho \wedge \rho) \to (\rho \vee \rho)}\ \to R^+$$

Sense: $\{fst(y)^+, inl(fst(y))^+, (\lambda y.inl(fst(y)))^+\}$

SC$_2$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(z^+ : \rho; \emptyset) \vdash^+ z^+ : \rho}\ Rf^+}{(z^+ : \rho, x^+ : \rho; \emptyset) \vdash^+ z^+ : \rho}\ W^a}{(y^+ : \rho \wedge \rho; \emptyset) \vdash^+ fst(y)^+ : \rho}\ \wedge L^a \quad \dfrac{\dfrac{}{(z^+ : \rho; \emptyset) \vdash^+ z^+ : \rho}\ Rf^+}{(z^+ : \rho; \emptyset) \vdash^+ inl(z)^+ : \rho \vee \rho}\ \vee R_1^+}{(y^+ : \rho \wedge \rho; \emptyset) \vdash^+ inl(fst(y))^+ : \rho \vee \rho}\ Cut^a}{\vdash^+ (\lambda y.inl(fst(y)))^+ : (\rho \wedge \rho) \to (\rho \vee \rho)}\ \to R^+$$

Sense: $\{fst(y)^+$, $\mathbf{inl(z)}^+$, $inl(fst(y))^+, (\lambda y.inl(fst(y)))^+\}$

SC$_3$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(z^+ : \rho; \emptyset) \vdash^+ z^+ : \rho}\ Rf^+}{(z^+ : \rho, x^+ : \rho; \emptyset) \vdash^+ z^+ : \rho}\ W^a \quad \dfrac{}{(z^+ : \rho; \emptyset) \vdash^+ z^+ : \rho}\ Rf^+}{(z^+ : \rho, x^+ : \rho; \emptyset) \vdash^+ z^+ : \rho}\ Cut^a}{(y^+ : \rho \wedge \rho; \emptyset) \vdash^+ fst(y)^+ : \rho}\ \wedge L^a}{(y^+ : \rho \wedge \rho; \emptyset) \vdash^+ inl(fst(y))^+ : \rho \vee \rho}\ \vee R_1^+}{\vdash^+ (\lambda y.inl(fst(y)))^+ : (\rho \wedge \rho) \to (\rho \vee \rho)}\ \to R^+$$

Sense: $\{fst(y)^+, inl(fst(y))^+, (\lambda y.inl(fst(y)))^+\}$

If we would take the sense of derivations to be given by the set of all terms occurring in it, the derivations which would be considered synonymous here would be just SC$_1$ and SC$_3$. ND$_1$ and SC$_2$, on the other hand, would be considered different in sense both from the other two and also from each other. The now modified conception of sense, though, yields synonymy between ND_1, SC_1 and SC_3. Let us take a bit of a closer look at these derivations to see why this makes sense.

On the former account of sense, ND_1 would not be synonymous with SC_1 and SC_3 because the sequent calculus derivations contain more term variables than the natural deduction one. This difference is due to the fact that in ND_1 the derivation starts from an assumption in the form of a complex formula. In ND this is possible while still having an operation on that formula, namely in the form of an elimination rule. Of course, in SC, though, the corresponding operation is also an introduction rule, i.e., we cannot start with this complex formula but we need to introduce it first if we want the corresponding operation to be part of the derivation.[17] So, this difference is just due to the structural differences that are inherent in these kinds of proof systems, one having introduction and elimination rules, one only having introduction rules. Another example would be the correspondence between vacuous discharge in ND and weakening in SC, as in the exemplary derivations below. Of course, in this case the outcome in term variables will be different, since this is what the 'vacuous' is basically saying: the assumption does not really appear in the derivation, whereas in SC, it is the other way around, we are intentionally introducing it.

ND_2

$$\dfrac{\dfrac{[z^- : \rho]}{(\lambda z.z)^- : \rho \prec \rho}\ {\prec}I^d}{(\lambda y.\lambda z.z)^- : (\rho \prec \rho) \prec \sigma}\ {\prec}I^d$$

SC_4

$$\dfrac{\dfrac{\dfrac{\dfrac{}{(\emptyset; z^- : \rho) \vdash^- z^- : \rho}\ Rf^-}{(\emptyset; z^- : \rho, y^- : \sigma) \vdash^- z^- : \rho}\ W^c}{(\emptyset; y^- : \sigma) \vdash^- (\lambda z.z)^- : \rho \prec \rho}\ {\prec}R^-}{\vdash^- (\lambda y.\lambda z.z)^- : (\rho \prec \rho) \prec \sigma}\ {\prec}R^-$$

The step of weakening is of course not explicitly necessary, we could simply start with the conclusion of this step as the conclusion of the zero-premise

[17] To be clear, this is not to say that we cannot start with a complex formula in the reflexivity rule. Although our rule formulation is with atomic formulas in this rule, it is easily provable that the generalized rule version is admissible in our system. However, starting with a complex formula here, e.g., $p \wedge p$, would result in deriving a very different sequent of course.

rule Rf^-, but this is just to make explicit that here we need this appearance of y^- as a counterassumption, i.e., the set of terms is extended by this variable as opposed to the set of terms of ND_2. To give a philosophical motivation why it makes sense to disregard the term variables with respect to sense and thus, to consider these derivations as synonymous, it can be argued that the underlying operation, which is allowing monotonicity, is the same. It is just expressed differently in the syntax, much like there are languages which differ very much in how they express the same content syntactically. In German the sentence "I am going to school" is expressed—much like in English—as a concatenation of separated words (in that order) for the subject, the predicate and the object of the sentence (here in form of again separated words in form of a pronoun, a verb, a preposition and a noun): "Ich gehe zur Schule". In Turkish this sentence would be "Okula gidiyorum", ending, as it is typical for such a sentence, with the predicate, which is receiving possibly multiple suffixes expressing for example the tense (in this case '-yor-') or the case of the subject ('-um'), i.e., expressing the 'I' in that sentence, which does not have to be given by an explicit word (although this exists, too: 'ben'). The object in Turkish, on the other hand, is usually preceding the predicate, as it is here the case, and what is usually expressed by prepositions in German and English is here again expressed by suffixes: 'okul' meaning 'school', '-a' expressing a movement directed *toward* something. So, in these cases, where the sets of terms contained in the derivation only differ in that the one of an SC-derivation has more term *variables* than the one of the corresponding ND-derivation, it might make sense not to consider them as different in sense but to say that the way of inference, i.e., the meaning, is essentially the same, it is just expressed differently in the syntax.

Another question that may arise is what philosophical motivation can be given to consider SC_1 and SC_3 as synonymous but SC_2 not. Formally, we can just point to the set of terms of SC_3 being exactly the same as the one of SC_1, while this is not the case for SC_2. But on the other hand, both SC_2 and SC_3 contain an application of Cut^a, so one could think that this should bring them closer to one another in meaning, as opposed to the cut-free SC_1. However, the difference between these derivations is that the application of cut is permuted upward in SC_3 as opposed to the application in SC_2. Of course, one could say that this would be a reason to consider *these* two derivations as synonymous because they only differ in where cut is applied, which does not sound like much of a difference. But this is only prima facie so. If we take a closer look at what is happening due to the cut applications, we see that they *are* essentially different: In SC_2 cut is applied after the application of logical rules, while in SC_3 only after *structural* rules. From a PTS-standpoint, it is the logical rules in a sequent calculus which are considered to be meaning-giving for the connectives, while the meaning of whole derivations is to be composed by the meaning of what they contain. Thus, taking the operations occurring in a derivation to be what constitutes the meaning of derivations, made explicit for us by the complex λ-terms (i.e., terms that are the result of applying the *logical* rules), it makes sense to argue for these two points: (1) an application of cut to a complex term leads to a difference in meaning, and (2) an application of cut to a term variable does not lead to a difference in meaning. The sense-denoting sets of SC_1 and SC_3 are exactly the same, even if one derivation is with and one without cut. In other words, none of the operations in the derivation are affected by it. With this we have an example showing why it does not seem correct what is stated in [2, p. 589], namely that "cut does not need to create a non-normal term, [...] but still any application of cut will necessarily change the sense of a derivation as opposed to its cut-free form". We can revise this as follows: Applying cut will make a difference in the sense-denoting set iff cut is applied *after* the application of a logical rule. This is what happens in SC_2: Cut is here applied at a step where the terms involved have already been operated on, i.e., it is not only variables that are cut out by this application but terms carrying information about the way of inference so far. The information is not completely lost, of course, since applying cut is expressed at the level of terms in form of a substitution operation but the information is now built in the derivation in a different way.

To sum up how structural rules can affect the meaning of a derivation: In our calculus, contraction and weakening can only be applied on the left side of the sequent, and thus, can only make a difference in the term *variables*.

Therefore, applications of these rules are inconsequential for the meaning of the derivation. Applications of cut, on the other hand, can—but need not necessarily—lead to a difference in the complex terms that are reflecting logical operations occurring in the derivation. Therefore, applications of cut can—but need not necessarily—lead to a difference in meaning of the derivations in question.

3.3.2. Difference in sense due to rule permutations in SC

Finally, I will briefly consider the differences between SC and ND when it comes to the order of rule applications and the effect of this on differences concerning sense and denotation. Therefore, I will show some examples to illustrate the general reasoning and that this, too, applies in the bilateralist setting. Take our exemplary derivation from Section 2.2 again, here with the set of terms occurring within the derivation and thus, representing the sense, below:

$$\dfrac{\dfrac{\dfrac{}{(x^+ : \rho \prec \sigma; \emptyset) \vdash^+ \mathtt{top}^+ : \top}\;{\scriptstyle \top R^+} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{(z^+ : \rho; \emptyset) \vdash^+ z^+ : \rho}\;{\scriptstyle Rf^+}}{(z^+ : \rho; z_1^- : \sigma) \vdash^+ z^+ : \rho}\;{\scriptstyle W^c}}{(x^+ : \rho \prec \sigma; \emptyset) \vdash^+ \pi_1(x^+)^+ : \rho}\;{\scriptstyle \prec L^a} \qquad \dfrac{\dfrac{\dfrac{}{(\emptyset; z_1^- : \sigma) \vdash^- z_1^- : \sigma}\;{\scriptstyle Rf^-}}{(z^+ : \rho; z_1^- : \sigma) \vdash^- z_1^- : \sigma}\;{\scriptstyle W^a}}{(x^+ : \rho \prec \sigma; \emptyset) \vdash^- \pi_2(x^+)^- : \sigma}\;{\scriptstyle \prec L^a}}{(x^+ : \rho \prec \sigma; \emptyset) \vdash^- \{\pi_1(x^+)^+, \pi_2(x^+)^-\}^- : \rho \to \sigma}\;{\scriptstyle \to R^-}}{(x^+ : \rho \prec \sigma; \emptyset) \vdash^+ \{\mathtt{top}^+, \{\pi_1(x^+)^+, \pi_2(x^+)^-\}^-\}^+ : \top \prec (\rho \to \sigma)}\;{\scriptstyle \prec R^+}}{\vdash^+ (\lambda x^+.\{\mathtt{top}^+, \{\pi_1(x^+)^+, \pi_2(x^+)^-\}^-\}^+)^+ : (\rho \prec \sigma) \to (\top \prec (\rho \to \sigma))}\;{\scriptstyle \to R^+}$$

Sense: $\{\mathtt{top}^+, \pi_1(x^+)^+, \pi_2(x^+)^-, \{\pi_1(x^+)^+, \pi_2(x^+)^-\}^-,$
$\{\mathtt{top}^+, \{\pi_1(x^+)^+, \pi_2(x^+)^-\}^-\}^+, (\lambda x^+.\{\mathtt{top}^+, \{\pi_1(x^+)^+, \pi_2(x^+)^-\}^-\}^+)^+\}$

The corresponding derivation in ND is the following:

$$\dfrac{\dfrac{\dfrac{}{\mathtt{top}^+ : \top}\;{\scriptstyle \top I} \qquad \dfrac{\dfrac{[x^+ : \rho \prec \sigma]}{\pi_1(x^+)^+ : \rho}\;{\scriptstyle \prec E_1} \qquad \dfrac{[x^+ : \rho \prec \sigma]}{\pi_2(x^+)^- : \sigma}\;{\scriptstyle \prec E_2}}{\{\pi_1(x^+)^+, \pi_2(x^+)^-\}^- : \rho \to \sigma}\;{\scriptstyle \to I^d}}{\{\mathtt{top}^+, \{\pi_1(x^+)^+, \pi_2(x^+)^-\}^-\}^+ : \top \prec (\rho \to \sigma)}\;{\scriptstyle \prec I}}{(\lambda x^+.\{\mathtt{top}^+, \{\pi_1(x^+)^+, \pi_2(x^+)^-\}^-\}^+)^+ : (\rho \prec \sigma) \to (\top \prec (\rho \to \sigma))}\;{\scriptstyle \to I}$$

As can be seen, next to having the same denotation, which can be seen by the end-term being the same, this derivation has—under our revised notion of sense—the same sense as the derivation in SC above, since the *complex* terms occurring within the derivations are exactly the same. This reflects that the same operations are used to conduct the derivations *and* they are applied in the same order. However, in SC often there are different derivations of the same sequent possible, i.e., still yielding the same end-term, while in ND this is not possible. This is due to the left introduction rules in SC, which are more flexible as to when and in which order they are applied in a derivation. Thus, in this case here two other derivations are possible by downwards permutation of the application of $\prec L^a$, namely the following, again with their sense-denoting sets spelled out beneath them:

$$\dfrac{\dfrac{\dfrac{}{(x^+ : \rho \prec \sigma; \emptyset) \vdash^+ \texttt{top}^+ : \top}\,{\scriptstyle \top R^+} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{(z^+ : \rho; \emptyset) \vdash^+ z^+ : \rho}\,{\scriptstyle Rf^+}}{(z^+ : \rho; z_1^- : \sigma) \vdash^+ z^+ : \rho}\,{\scriptstyle W^c} \qquad \dfrac{\dfrac{}{(\emptyset; z_1^- : \sigma) \vdash^- z_1^- : \sigma}\,{\scriptstyle Rf^-}}{(z^+ : \rho; z_1^- : \sigma) \vdash^- z_1^- : \sigma}\,{\scriptstyle W^a}}{(z^+ : \rho; z_1^- : \sigma) \vdash^- \{z^+, z_1^-\}^- : \rho \to \sigma}\,{\scriptstyle \to R^-}}{(x^+ : \rho \prec \sigma; \emptyset) \vdash^- \{\pi_1(x^+)^+, \pi_2(x^+)^-\}^- : \rho \to \sigma}\,{\scriptstyle \prec L^a}}{(x^+ : \rho \prec \sigma; \emptyset) \vdash^+ \{\texttt{top}^+, \{\pi_1(x^+)^+, \pi_2(x^+)^-\}^-\}^+ : \top \prec (\rho \to \sigma)}\,{\scriptstyle \prec R^+}}{\vdash^+ (\lambda x^+.\{\texttt{top}^+, \{\pi_1(x^+)^+, \pi_2(x^+)^-\}^-\}^+)^+ : (\rho \prec \sigma) \to (\top \prec (\rho \to \sigma))}\,{\scriptstyle \to R^+}$$

Sense: $\{\texttt{top}^+, \{z^+, z_1^-\}^-, \{\pi_1(x^+)^+, \pi_2(x^+)^-\}^-,$
$\{\texttt{top}^+, \{\pi_1(x^+)^+, \pi_2(x^+)^-\}^-\}^+, (\lambda x^+.\{\texttt{top}^+, \{\pi_1(x^+)^+, \pi_2(x^+)^-\}^-\}^+)^+\}$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{(z^+ : \rho; z_1^- : \sigma) \vdash^+ \texttt{top}^+ : \top}\,{\scriptstyle \top R^+} \qquad \dfrac{\dfrac{\dfrac{}{(z^+ : \rho; \emptyset) \vdash^+ z^+ : \rho}\,{\scriptstyle Rf^+}}{(z^+ : \rho; z_1^- : \sigma) \vdash^+ z^+ : \rho}\,{\scriptstyle W^c} \qquad \dfrac{\dfrac{}{(\emptyset; z_1^- : \sigma) \vdash^- z_1^- : \sigma}\,{\scriptstyle Rf^-}}{(z^+ : \rho; z_1^- : \sigma) \vdash^- z_1^- : \sigma}\,{\scriptstyle W^a}}{(z^+ : \rho; z_1^- : \sigma) \vdash^- \{z^+, z_1^-\}^- : \rho \to \sigma}\,{\scriptstyle \to R^-}}{(z^+ : \rho; z_1^- : \sigma) \vdash^+ \{\texttt{top}^+, \{z^+, z_1^-\}^-\}^+ : \top \prec (\rho \to \sigma)}\,{\scriptstyle \prec R^+}}{(x^+ : \rho \prec \sigma; \emptyset) \vdash^+ \{\texttt{top}^+, \{\pi_1(x^+)^+, \pi_2(x^+)^-\}^-\}^+ : \top \prec (\rho \to \sigma)}\,{\scriptstyle \prec L^a}}{\vdash^+ (\lambda x^+.\{\texttt{top}^+, \{\pi_1(x^+)^+, \pi_2(x^+)^-\}^-\}^+)^+ : (\rho \prec \sigma) \to (\top \prec (\rho \to \sigma))}\,{\scriptstyle \to R^+}$$

Sense: $\{\texttt{top}^+, \{z^+, z_1^-\}^-, \{\texttt{top}^+, \{z^+, z_1^-\}^-\}^+,$
$\{\texttt{top}^+, \{\pi_1(x^+)^+, \pi_2(x^+)^-\}^-\}^+, (\lambda x^+.\{\texttt{top}^+, \{\pi_1(x^+)^+, \pi_2(x^+)^-\}^-\}^+)^+\}$

So, according to our underlying framework all of these derivations would have the same denotation (as would, just as a reminder, the derivations ending on the corresponding dual term) but only the first SC-derivation and the ND-derivation could be considered synonymous, since the complex terms occurring within the second and third derivation differ from the first one and also from each other. This is in line with our philosophical reasoning, too, namely that the way of inference is simply different by means of those rule permutations.

4. Conclusion

In this paper, I applied an account to make a Fregean distinction between sense and denotation of derivations to a bilateralist setting in the form of a sequent calculus for the bi-intuitionistic logic `2Int`, displaying proofs and refutations. For this account, devised in [2], λ-term-annotated proof systems are considered on the basis of which the concepts of sense and denotation, and correspondingly of synonymy and identity, are defined. The denotation of a derivation in a system with λ-term assignment is referred to by the end-term of the derivation. The sense of a derivation is represented by the set of all complex λ-terms of the derivation because these reflect the way of inference that is taken within the derivation by encoding the applied operations. Identity between derivations means sameness of denotation, i.e, referring to the same proof object, and holds modulo their end-terms belonging to the same equivalence class induced by the set of β- and permutation conversions. This account is extended in this paper by firstly using a two-sorted typed λ-calculus to annotate the sequent calculus for `2Int` and defining a duality function for the system. With this at hand, the bilateralist desideratum of having proofs and refutations on a par can be made explicit in form of a Dualization Theorem, stating that whenever we have a derivation of a sequent expressing a provability (refutability) relation, we can construct a derivation of its dual sequent expressing a refutability (provability) relation. On account of the bilateralist setting, I then argued for an extension of the notion of identity in that it should also hold for end-terms that can be obtained from one another by our duality function. Philosophically, this means that, in such a bilateralist system, proofs and refutations should be considered identical with respect to their underlying construction, i.e., their denotation, while their sense (being more fine-grained) on the other hand, needs to be distinguished. Finally, I compared the sequent calculus to a corresponding natural deduction system and showed which derivations between these different kinds of proof systems can on this account be considered as identical or even synonymous, i.e., as having the same sense.

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