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APPLICATIONS OF COMMUTATORS IN PSEUDO BL-ALGEBRAS

Abstract

In this paper, the concepts of commutativity relation, commutant of a subset, center and solvable pseudo BL-algebras are defined and their properties are investigated. The notion of n-Engel pseudo BL-algebras as a natural generalization of BL-algebras is introduced, and we discuss Engel pseudo BL-algebra, which is defined by left and right normed commutators.

Keywords: pseudo BL-algebra, commutativity relation, commutator, solvable, Engel element.

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1. Introduction

BL-algebras were introduced by Hájek in 1998, an algebraic semantics of basic fuzzy logic which is generated by continuous t-norms on the interval [0,1] and their residuals [4]. Georgescu et all introduced pseudo BL-algebra as a non-commutative extension of BL-algebra [7, 8]. The idea of pseudo BL-algebra originates not only in logic and algebra, but also in algebraic properties that come from the syntax of certain non-classical propositional logics and intuitionistic logic. Main properties of pseudo-BL algebras were studied in [7, 8]. Pseudo-BL algebras correspond to a pseudo-basic fuzzy logic (see [2]). It should be pointed out that the difference between BL-algebras and pseudo BL-algebras is in the commutativity of the $\odot$ operation and therefore, in this paper, we intend to focus on this topic and investigate this difference. We present a definition for the notion of commutator in pseudo BL-algebras. We define also the notion of Engel pseudo BL-algebras based on commutators, give several characterizations of it and prove that the class of commutative pseudo BL-algebras and 1-Engel pseudo BL-algebras are equal.

2. Preliminaries

In this section, we recall the concept of a pseudo-BL algebra and then present some definitions and properties which we will need in the next sections.

Definition 2.1. A BL-algebra is an algebra $(A, \vee, \wedge, \odot, \rightarrow, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ satisfying the following [4]:

(BL1) $(A, \vee, \wedge, 0, 1)$ is a bounded lattice;

(BL2) $(A, \odot, 1)$ is a commutative monoid;

(BL3) $\odot$ and $\rightarrow$ form an adjoint pair, i.e. $a \odot b \leq c$ iff $a \leq b \rightarrow c$;

(BL4) $a \wedge b = a \odot (a \rightarrow b)$;

(BL5) $(a \rightarrow b) \vee (b \rightarrow a) = 1$; For all $a, b, c \in A$.

DEFINITION 2.2. A pseudo BL-algebra is an algebra $(A, \vee, \wedge, \odot, \rightarrow, \rightsquigarrow, 0, 1)$ of type $(2, 2, 2, 2, 2, 0, 0)$ satisfying the following [7]:

(PSBL1) $(A, \vee, \wedge, 0, 1)$ is a bounded lattice;

(PSBL2) $(A, \odot, 1)$ is a monoid;

(PSBL3) $a \odot b \leq c$ iff $a \leq b \rightarrow c$ iff $b \leq a \rightsquigarrow c$, for all $a, b, c \in A$;

(PSBL4) $a \wedge b = (a \rightarrow b) \odot a = a \odot (a \rightsquigarrow b)$;

(PSBL5) $(a \rightarrow b) \vee (b \rightarrow a) = (a \rightsquigarrow b) \vee (b \rightsquigarrow a) = 1$, for all $a, b \in A$.

It is proved in [7] that commutative pseudo-BL algebras are BL-algebras. A pseudo BL-algebra A is nontrivial if and only if $0 \neq 1$.

For any $a \in A$, we define $a^- = a \rightarrow 0$ and $a^\sim = a \rightsquigarrow 0$. We define $a^0 = 1$ and $a^n = a^{n-1} \odot a$ for $n \in N$.

The following properties hold in any pseudo BL-algebra A and will be used in the sequel. See [7] for details.

(psbl-c_1) $a \leq b$ iff $a \rightarrow b = 1$ iff $a \rightsquigarrow b = 1$;

(psbl-c_2) $a \rightarrow a = a \rightsquigarrow a = 1$;

(psbl-c_3) $a \rightarrow 1 = a \rightsquigarrow 1 = 1$;

(psbl-c_4) $1 \rightarrow a = 1 \rightsquigarrow a = a$;

(psbl-c_5) $a \odot b \leq a \wedge b \leq a, b$;

(psbl-c_6) $a \rightarrow (b \rightarrow c) = (a \odot b) \rightarrow c$;

(psbl-c_7) $a \rightsquigarrow (b \rightsquigarrow c) = (b \odot a) \rightsquigarrow c$;

(psbl-c_8) $a \odot 0 = 0 \odot a = 0$;

(psbl-c_9) $a \odot 1 = 1 \odot a = a$;

(psbl-c_{10}) $a \odot a^\sim = a^- \odot a = 0$;

(psbl-c_{11}) $a \leq b$ implies $a \odot c \leq b \odot c$ and $c \odot a \leq c \odot b$;

(psbl-c_{12}) $a \leq b$ implies $b^{-} \leq a^{-}$ and $b^{\sim} \leq a^{\sim}$;

(psbl-c_{13}) $a \odot b = 0$ iff $a \leq b^{\sim}$ iff $b \leq a^{-}$;

(psbl-c_{14}) $a \leq a^{-\sim}$ and $a \leq a^{\sim -}$;

(psbl-c_{15}) $(a \wedge b) \odot c = (a \odot c) \wedge (b \odot c)$ and $c \odot (a \wedge b) = (c \odot a) \wedge (c \odot b)$;

(psbl-c_{16}) $(a \vee b) \odot c = (a \odot c) \vee (b \odot c)$ and $c \odot (a \vee b) = (c \odot a) \vee (c \odot b)$;

(psbl-c_{17}) $(\bigvee_{i \in I} b_i) \odot a = \bigvee_{i \in I}(b_i \odot a)$ and $a \odot (\bigvee_{i \in I} b_i) = \bigvee_{i \in I}(a \odot b_i)$;

(psbl-c_{18}) $(\bigwedge_{i \in I} b_i) \odot a = \bigwedge_{i \in I}(b_i \odot a)$ and $a \odot (\bigwedge_{i \in I} b_i) = \bigwedge_{i \in I}(a \odot b_i)$.

(psbl-c_{19}) if $a \vee b = 1$, then $a \odot b = a \wedge b$.

(psbl-c_{20}) if $a \vee b = 1$, then for each $n \geq 1$, $a^n \vee b^n = 1$.

(psbl-c_{21}) $(a \wedge b)^{\sim} = a^{\sim} \vee b^{\sim}$, $(a \wedge b)^{-} = a^{-} \vee b^{-}$.

(psbl-c_{22}) $(a \odot b)^{-} = a \to b^{-}$, $(a \odot b)^{\sim} = b \rightsquigarrow a^{\sim}$.

Let A be a pseudo BL-algebra. A filter of A is a nonempty subset F of A such that for all $a, b \in A$ [8]
$i)$ if $a, b \in F$, then $a \odot b \in F$; $ii)$ if $a \in F$ and $a \leq b$, then $b \in F$.
A filter F of A is proper if $F \neq A$. Let $X \subseteq A$. The filter of A generated by X will be denoted by $< X >$. We have that $< X >= \{a \in A : x_1 \odot x_2 \odot ... \odot x_n \leq a$ for some $n \in N$ and some $x_1, x_2, ..., x_n \in X\}$. For any $a \in A$, $< a >$ denotes the principal filter of A generated by $\{a\}$. Then, $< a >= \{b \in A : b^n \leq a$ for some $n \in N\}$. If F is a filter and $a \notin F$, we denote by $F(a) =< F \cup \{a\} >$. A filter H of A is called normal ([8]) if for every $a, b \in A$ we have the equivalence: $a \to b \in H$ iff $a \rightsquigarrow b \in H$. With any normal filter H of A we can associate a congruence relation $\equiv_H$ on A by defining $a \equiv_H b$ if and only if $(a \to b) \odot (b \to a) \in H$ if and only if $(a \rightsquigarrow b) \odot (b \rightsquigarrow a) \in H$. In [8] it is proved that the map $H \longrightarrow \equiv_H$ is an isomorphism between the lattice of normal filters of A and the lattice of congruences of A. If we denote by A/H the quotient set $A/\equiv_H$,

then A/H becomes a pseudo BL-algebra A/H with the natural operations induced from those of A. The function $f : A \longrightarrow B$ is a homomorphism of A to pseudo BL-algebra B iff it satisfies the following conditions, for every $a, b \in A$:
$f(0) = 0,\ f(a \odot b) = f(a) \odot f(b),\ f(a \to b) = f(a) \to f(b),\ f(a \rightsquigarrow b) = f(a) \rightsquigarrow f(b)$. It follows that, $f(1) = 1,\ f(a \wedge b) = f(a) \wedge f(b),\ f(a \vee b) = f(a) \vee f(b)$.
For any pseudo BL-algebra A, let us denote

$$G(A) = \{a \in A : a \odot a = a\}$$
$$M(A) = \{a \in A : a = (a^-)^\sim = (a^\sim)^-\}$$

Recall that an element $a \in A$ is called complemented if there is an element $b \in A$ such that $a \vee b = 1$ and $a \wedge b = 0$. The set of the complemented elements of the bounded lattice reduct of A is called the Boolean center of A and is denoted by $B(A)$. Clearly $0, 1 \in B(A)$. The elements of $B(A)$ are called Boolean elements of A. It is known that $B(A)$ is a Boolean algebra, with the operation induced by those of A together with the complementation operation given by the negation in A. Also it is straightforward that $B(A)$ is a subalgebra of the pseudo BL-algebra A. In [2] it proved that $B(A) = M(A) \bigcap G(A)$.

Definition 2.3 ([7]). A pseudo BL-algebra A is commutative iff $a \odot b = b \odot a$, for any $a, b \in A$.

Theorem 2.4 ([7]). *A pseudo BL-algebra A is commutative iff $a \to b = a \rightsquigarrow b$, for any $a, b \in A$. Any commutative pseudo BL-algebra A is a BL-algebra.*

From now on, $(A, \vee, \wedge, \odot, \to, \rightsquigarrow, 0, 1)$ or simply A is a pseudo BL-algebra, unless otherwise specified.

3. Commutativity and the Center in Pseudo BL-Algebras: Definitions and Properties

In this section, we introduce and study the commutativity relation in A. We define central elements and the center of A, and show that the center of A is a filter of A.

DEFINITION 3.1. For elements $a, b \in A$, if $a \odot b = b \odot a$, we write aCb and say that a commutes with b. The relation C is called the commutativity relation of A. It can be observe that the relation C is reflexive and symmetric, but not necessarily transitive. Furthermore, in every commutative pseudo BL-algebra A (and in any BL-algebra A), aCb holds for all $a, b \in A$.

The following simple example of a non-commutative pseudo BL-algebra will be used to characterize our concepts.

Example 3.2. Let $a, b, c, d \in R$, where R is the set of all real numbers. Define the order relation $\leq$ on R^2 by

$$(a, b) \leq (c, d) \Leftrightarrow a < c \text{ or } (a = c \text{ and } b \leq d).$$

For any $x, y \in R^2$, we define the operations $\vee$ and $\wedge$ as follows: $x \vee y = \max\{x, y\}$ and $x \wedge y = \min\{x, y\}$. Let

$$A = \{(\frac{1}{2}, b) \in R^2 : b \geq 0\} \cup \{(a, b) \in R^2 : \frac{1}{2} < a < 1\} \cup \{(1, b) \in R^2 : b \leq 0\}$$

For $(a, b), (c, d) \in A$, consider the operations

$$(a, b) \odot (c, d) = (\frac{1}{2}, 0) \vee (ac, bc + d)$$

$$(a, b) \to (c, d) = (\frac{1}{2}, 0) \vee [(\frac{c}{a}, \frac{d - b}{a}) \wedge (1, 0)]$$

$$(a, b) \rightsquigarrow (c, d) = (\frac{1}{2}, 0) \vee [(\frac{c}{a}, \frac{ad - bc}{a}) \wedge (1, 0)]$$

Then $(A, \vee, \wedge, \odot, \to, \rightsquigarrow, (\frac{1}{2}, 0), (1, 0))$ is a pseudo BL-algebra.

If, in this algebra, we choose $x = (1, -1)$ and $y = (1, -2)$, by routine calculations we obtain $(1, -1) \odot (1, -2) = (\frac{1}{2}, 0) \vee (1, -3) = (1, -3)$ and $(1, -2) \odot (1, -1) = (\frac{1}{2}, 0) \vee (1, -3) = (1, -3)$. Thus, xCy.

Also, if $x = (\frac{1}{2}, 2)$ and $y = (1, -2)$, then $(\frac{1}{2}, 2) \odot (1, -2) = (\frac{1}{2}, 0) \vee (\frac{1}{2}, 0) = (\frac{1}{2}, 0)$, and $(1, -2) \odot (\frac{1}{2}, 2) = (\frac{1}{2}, 0) \vee (\frac{1}{2}, 1) = (\frac{1}{2}, 1)$. Hence, x does not commute with y.

Lemma 3.3. *Let a and b be elements of A. Then*

1. aCa, $aC0$, *and* $aC1$.
2. aCb *if and only if* bCa.
3. *If* $a \vee b = 1$, *then* $a^n C b^n$ *for each positive integer* n.
4. *If* $a \wedge b = 0$, *then* $a^{\sim} C b^{\sim}$ *and* $a^{-} C b^{-}$.

Proof: 1. These properties are obvious by psbl-c_8 and psbl-c_9.
2. aCb if and only if $a \odot b = b \odot a$. This is equivalent to $b \odot a = a \odot b$, which by definition means bCa.
3. By psbl-c_{19}, if $a \vee b = 1$, then $a \odot b = a \wedge b = b \wedge a = b \odot a$, which implies aCb. Now, assume $a \vee b = 1$. By psbl-c_{20}, we have $a^n \vee b^n = 1$. Therefore, $a^n C b^n$ for every $n \geq 1$.
4. If $a \wedge b = 0$, then $(a \wedge b)^{\sim} = 0^{\sim} = 1$. By psbl-$c_{21}$, $(a \wedge b)^{\sim} = a^{\sim} \vee b^{\sim}$, so $a^{\sim} \vee b^{\sim} = 1$. Consequently, $a^{\sim} C b^{\sim}$. Also, $1 = 0^{-} = (a \wedge b)^{-} = a^{-} \vee b^{-}$. Thus, $a^{-} C b^{-}$. □

Theorem 3.4. *Let a, b, c be elements of A. If a commutes with b and c, then a also commutes with $b \odot c$, $b \wedge c$ and $b \vee c$.*

Proof: Assume aCb and aCc. This means $a \odot b = b \odot a$ and $a \odot c = c \odot a$. We need to show that $a \odot (b \odot c) = (b \odot c) \odot a$, $a \odot (b \wedge c) = (b \wedge c) \odot a$ and $a \odot (b \vee c) = (b \vee c) \odot a$.
We know that $\odot$ is associative, then $a \odot (b \odot c) = (a \odot b) \odot c = (b \odot a) \odot c = b \odot (a \odot c) = b \odot (c \odot a) = (b \odot c) \odot a$, which means $aC(b \odot c)$.
Using psbl-c_{15}, we have $a \odot (b \wedge c) = (a \odot b) \wedge (a \odot c)$. Since aCb and aCc, we can substitute $b \odot a$ for $a \odot b$ and $c \odot a$ for $a \odot c$. So $(a \odot b) \wedge (a \odot c) = (b \odot a) \wedge (c \odot a)$. By the commutative property of the operation $\wedge$ with respect to $\odot$ (which is implied by the properties of pseudo BL-algebras, specifically how $\wedge$ interacts with $\odot$), we get $(b \odot a) \wedge (c \odot a) = (b \wedge c) \odot a$ Thus, $a \odot (b \wedge c) = (b \wedge c) \odot a$, which means $aC(b \wedge c)$.

Similarly, using psbl-c_{16} we deduce $a \odot (b \vee c) = (a \odot b) \vee (a \odot c)$. Substituting as before $(a \odot b) \vee (a \odot c) = (b \odot a) \vee (c \odot a)$ Again, due to the interaction between $\vee$ and $\odot$ we have $(b \odot a) \vee (c \odot a) = (b \vee c) \odot a$. Thus, $a \odot (b \vee c) = (b \vee c) \odot a$, which means $aC(b \vee c)$. $\square$

Remark 3.5. An immediate question arises: Given two elements a and b of a pseudo BL-algebra, how can we determine if they commute? While direct computation using the definition is possible, it is not always the most efficient method. Instead, we can leverage properties such as, if $a \vee b = 1$, then aCb, and if $a \wedge b = 0$, then $a^{\sim}Cb^{\sim}$ and $a^{-}Cb^{-}$. Our aim is to discover and utilize such rules to streamline the commutativity check.

It is also beneficial to be able to compute the commutant of a subset of a pseudo BL-algebra, as well as its individual elements.

Definition 3.6. If M is a subset of A, the set

$$\{a \in A : aCm \; \forall m \in M\}$$

is called the commutant of M in A and is denoted by $C(M)$. The elements of $C(A)$ are referred to as central elements, and $C(A)$ is called the center of A.

By utilizing psbl-c_8 and psbl-c_9, we deduce that $0, 1 \in C(A)$. We use the notation $C(a)$ to denote the commutant of the singleton set $\{a\}$. Obvious, $0, 1, a \in C(a)$.

Example 3.7. Consider the pseudo BL-algebra defined in Example 3.2. If $M = \{(1, -3), (1, -2)\}$, then through routine calculations, we find that $C(M) = \{(1, 0)\}$. For $x = (\frac{1}{2}, 2)$, we have $C(x) = \{(a, 4 - 4a) : 1 < a < 2\}$. This is because $(\frac{1}{2}, 2) \odot (a, 4 - 4a) = (\frac{1}{2}, 0) \vee (\frac{a}{2}, 4 - 2a)$ and $(a, 4 - 4a) \odot (\frac{1}{2}, 2) = (\frac{1}{2}, 0) \vee (\frac{a}{2}, 4 - 2a)$. Thus, $(2, 3)$ commutes with $(a, 4 - 4a)$. Furthermore, for any $(a, b) \in A$, $(a, b)C(\frac{1}{2}, 2)$ if and only if $b = 4 - 4a$.

Proposition 3.8. Let $a, b \in C(A)$. Then

1. $a \odot b \in C(A)$.
2. $a \wedge b \in C(A)$.
3. $a \vee b \in C(A)$.

PROOF: 1. Let $a, b \in C(A)$. Then $a \odot x = x \odot a$ and $b \odot x = x \odot b$ for each $x \in A$. Hence, $(a \odot b) \odot x = a \odot (b \odot x) = a \odot (x \odot b) = (a \odot x) \odot b = (x \odot a) \odot b = x \odot (a \odot b)$. Since $(a \odot b) \odot x = x \odot (a \odot b)$ for all $x \in A$, we conclude that $a \odot b \in C(A)$.

2. By psbl-c_{15}, we have $(a \wedge b) \odot x = (a \odot x) \wedge (b \odot x) = (x \odot a) \wedge (x \odot b) = x \odot (a \wedge b)$. Thus, $(a \wedge b) \odot x = x \odot (a \wedge b)$ for all $x \in A$, and hence $a \wedge b \in C(A)$.

3. By using psbl-c_{16}, we get $(a \vee b) \odot x = (a \odot x) \vee (b \odot x) = (x \odot a) \vee (x \odot b) = x \odot (a \vee b)$. Thus, $(a \vee b) \odot x = x \odot (a \vee b)$ for all $x \in A$, and hence $a \vee b \in C(A)$. $\square$

According to Proposition 3.8, the center of A is a subalgebra of A, but it is not a filter of A, in general. For instance, in Example 3.2, we have $C(A) = \{(\frac{1}{2}, 0), (1, 0)\}$ and $a = (\frac{1}{2}, 0) \leq (\frac{1}{2}, 2) = b$, and $a \in C(A)$ but $b \notin C(A)$.

LEMMA 3.9. *If $a \in C(A)$, then $a^- = a^\sim$.*

PROOF: If $a \in C(A)$, then $a \odot b = b \odot a$ for all $b \in A$. Thus, $a \odot a^- = a^- \odot a$ and $a \odot a^\sim = a^\sim \odot a$. By psbl-$c_{10}$, we obtain $a \odot a^- = a^\sim \odot a = 0$, so $(a \odot a^-)^\sim = 0^\sim = 1$ and $(a^\sim \odot a)^- = 0^- = 1$. Using psbl-$c_{21}$, we obtain $a^\sim \to a^- = 1$ and $a^- \rightsquigarrow a^\sim = 1$, which means $a^\sim \leq a^-$ and $a^- \leq a^\sim$. Hence, $a^- = a^\sim$. $\square$

If we choose $a = (1, -1)$ in the pseudo BL-algebra defined in Example 3.2, then we have $a^- = a^\sim = (\frac{1}{2}, 1)$. By routine calculus we obtain $(1, -1) \odot (\frac{1}{2}, 2) = (\frac{1}{2}, \frac{3}{2})$ and $(\frac{1}{2}, 2) \odot (1, -1) = (\frac{1}{2}, 1)$. Thus $a \notin C(A)$. Therefore the converse of Lemma 3.9 is not true, generally.

PROPOSITION 3.10. Let H be a normal filter of A and $a \in A \setminus H$. If $a \in C(A)$, then

(i) $< a >= \{x \in A : a \leq x\}$ is a normal filter of A.

(ii) $H(a)$ is a normal filter of A, where $H(a) =< H \cup \{a\} >$.

PROOF: (i) Suppose that $a \in C(A)$. If $x, y \in A$ such that $x \to y \in < a >$, then

$$\begin{aligned} a \leq x \to y &\Longleftrightarrow a \odot x \leq y \\ &\Longleftrightarrow x \odot a \leq y \\ &\Longleftrightarrow a \leq x \rightsquigarrow y \\ &\Longleftrightarrow x \rightsquigarrow y \in < a > . \end{aligned}$$

That is, $< a >$ is a normal filter of A.

(ii) Since $a \in C(A)$, $a \odot b = b \odot a$ for all $b \in A$. Let $x, y \in A$ such that $x \to y \in H(a)$. Then there exists $h \in H$ such that

$$\begin{aligned} h \odot a \leq x \to y &\Longleftrightarrow (h \odot a) \odot x \leq y \\ &\Longleftrightarrow h \odot (a \odot x) \leq y \\ &\Longleftrightarrow h \odot (x \odot a) \leq y \\ &\Longleftrightarrow (h \odot x) \odot a \leq y. \end{aligned}$$

Since $h \in H$ and H is a normal filter, there exists $h^{'} \in H$ such that $x \odot h^{'} = h \odot x$. We obtain

$$(x \odot h^{'}) \odot a \leq y \Longleftrightarrow x \odot (h^{'} \odot a) \leq y \qquad \Longleftrightarrow h^{'} \odot a \leq x \rightsquigarrow y.$$

Therefore, $x \rightsquigarrow y \in H(a)$. Analogously, $x \rightsquigarrow y \in H(a)$ implies that $x \to y \in H(a)$. Hence, $H(a)$ is a normal filter of A. □

PROPOSITION 3.11. $B(A) \subseteq G(A) \subseteq C(A)$.

PROOF: We know that $B(A) = M(A) \cap G(A)$, hence $B(A) \subseteq G(A)$. If $a \in G(A)$, then $a \odot b = a \wedge b = b \wedge a = b \odot a$ for all $b \in A$. Therefore, aCb, for all $b \in A$. Hence, $a \in C(A)$. □

Remark 3.12. Since in any BL-algebra, the $\odot$ operation is commutative, $C(A) = A$ for any BL-algebra A. However, there exist BL-algebras A such that $G(A)$ is a proper subset of A. Hence, $G(A) \neq C(A)$ in general.

PROPOSITION 3.13. The algebraic structure $(C(A), \wedge, \vee, \odot, \to, 0, 1)$ forms a BL-algebra.

PROOF: $(C(A), \wedge, \vee, 0, 1)$ is a bounded lattice. From the properties of pseudo BL-algebras and the definition of the center, it is known that

$0, 1 \in C(A)$. Furthermore, if $x, y \in C(A)$, then $x \wedge y \in C(A)$ and $x \vee y \in C(A)$. This implies that $C(A)$ inherits the lattice structure and is bounded by 0 and 1.
$(C(A), \odot, 1)$ is a commutative monoid. For any $x, y, z \in C(A)$, since $x, y, z \in A$, we have $(x \odot y) \odot z = x \odot (y \odot z)$ because $\odot$ is associative in A. $1 \in C(A)$ by definition of the center. For any $x \in C(A)$, $x \odot 1 = x$ because 1 is the identity element in A. For any $x, y \in C(A)$, by the definition of the center $C(A)$, x commutes with all elements of A. Therefore, x commutes with y, which means $x \odot y = y \odot x$. This holds for all $x, y \in C(A)$. Thus, $(C(A), \odot, 1)$ is a commutative monoid. The operations $\odot$ and $\to$ form an adjoint pair. Since this property holds for all elements in A, it certainly holds for all elements in $C(A)$. $x \wedge y = x \odot (x \to y)$ for all $x, y \in C(A)$. Since $x, y \in C(A)$, they are also elements of A. The property $x \wedge y = x \odot (x \to y)$ holds in A, then this property holds for elements in $C(A)$. $(x \to y) \vee (y \to x) = 1$ for all $x, y \in C(A)$. Similar to the previous point, since $x, y \in C(A)$, they are elements of A. The property $(x \to y) \vee (y \to x) = 1$ holds in A. Given these points, especially the commutativity inherited from the definition of the center, $(C(A), \wedge, \vee, \odot, \to, 0, 1)$ fulfills all the requirements of a BL-algebra. □

LEMMA 3.14. *Suppose that H is a normal filter of A and $a, b \in A$. Then:*

1. $C(0) = C(1) = A$.
2. $a \in C(b)$ *if and only if* $b \in C(a)$.
3. $C(a)/H \subseteq C(a/H)$.

PROOF:

1. Since $aC0$ and $aC1$ for all $a \in A$, $C(0) = C(1) = A$.
2. $a \in C(b)$ if and only if aCb if and only if bCa if and only if $b \in C(a)$.
3. Let $b/H \in C(a)/H$. Then $b \in C(a)$ and hence bCa. Therefore, $b \odot a = a \odot b$. Then $a/H \odot b/H = (a \odot b)/H = (b \odot a)/H = b/H \odot a/H$, so $a/HCb/H$. Hence, $b/H \in C(a/H)$.

□

THEOREM 3.15. *Suppose that $a, b \in A$. Then*

1. $a \in C(A)$ *if and only if* $C(a) = A$.
2. $C(A) = \bigcap_{a \in A} C(a)$.
3. $C(a) \subseteq C(b)$ *if and only if* $b \in C(C(a))$.
4. $C(A) \subseteq C(a)$ *for every* $a \in A$.
5. A *is commutative if and only if* $C(a) = A$, *for every* $a \in A$.
6. $C(a) = C(b)$ *if and only if* $C(C(a)) = C(C(b))$.

PROOF:

1. Let $a \in C(A)$. Then for all $b \in A$, we obtain aCb. Let $x \in A$, then aCx. Therefore, $x \in C(a)$, that is, $A \subseteq C(a)$. We also have $C(a) \subseteq A$. Hence, $C(a) = A$.

   Conversely, let $C(a) = A$. Since $C(a) = \{b \in A : aCb\} = A$, it follows that $a \in C(A)$.

2. Let $a \in C(A)$. Then aCb for every $b \in A$. So $b \in C(a)$ for every $a \in A$. That means $b \in \bigcap_{a \in A} C(a)$. Therefore, $C(A) \subseteq \bigcap_{a \in A} C(a)$.

   Conversely, let $b \in \bigcap_{a \in A} C(a)$. Then $b \in C(a)$ for every $a \in A$. Hence, aCb for every $a \in A$. Then $b \in C(A)$. So $\bigcap_{a \in A} C(a) \subseteq C(A)$.

3. Let $C(a) \subseteq C(b)$. Then for any $x \in C(a)$, $x \in C(b)$. Hence, xCb for any $x \in C(a)$. Therefore, $b \in C(C(a))$.

   Conversely, let $b \in C(C(a))$ and $x \in C(a)$. Therefore, xCb. Hence, $x \in C(b)$. Then $C(a) \subseteq C(b)$.

4. We know that $C(A) = \bigcap_{a \in A} C(a)$, it follows that $C(A) \subseteq C(a)$ for any $a \in A$.

5. A is commutative if and only if $C(A) = A$ if and only if $A = \bigcap_{a \in A} C(a)$ if and only if $A = C(a)$ for every $a \in A$.

6. Obviously, if $C(a) = C(b)$, then $C(C(a)) = C(C(b))$.

Conversely, let $C(C(a)) = C(C(b))$. Since $a \in C(C(a))$, $a \in C(C(b))$. Then $C(b) \subseteq C(a)$. Similarly, since $b \in C(C(b))$, $C(a) \subseteq C(b)$. Therefore, $C(a) = C(b)$. □

THEOREM 3.16. *For a subset M of A, $C(M) = \bigcap_{m \in M} C(m)$.*

PROOF:

$$\begin{aligned} x \in C(M) &\Leftrightarrow \forall m \in M, xCm \\ &\Leftrightarrow \forall m \in M, x \in C(m) \\ &\Leftrightarrow x \in \bigcap_{m \in M} C(m). \end{aligned}$$

□

In the following theorems, some of the properties of the operator C, such as symmetry and monotonicity, are examined.

THEOREM 3.17. *For subsets M and N of A, it holds that $M \subseteq C(C(M))$, and that $M \subseteq N$ implies $C(N) \subseteq C(M)$.*

PROOF: Suppose that $x \in M$. It is sufficient to show that for all $m \in C(M)$, xCm. If m is an arbitrary element of $C(M)$, then by Definition 3.6, mCn for all $n \in M$. Then xCm. Hence, $x \in C(C(M))$ and so $M \subseteq C(C(M))$. For the proof of the second part, let $M \subseteq N$ and $x \in C(N)$. Then xCn for all $n \in N$. Therefore, xCm for all $m \in M$. Hence, $x \in C(M)$. Thus, $C(N) \subseteq C(M)$. □

THEOREM 3.18. *Suppose that M and N are two subsets of A. Then*

(i) $M \subseteq C(N)$ if and only if $N \subseteq C(M)$.

(ii) $C(C(C(M))) = C(M)$.

PROOF: (i) Let $M \subseteq C(N)$, and let $n \in N$. To show that $n \in C(M)$, we must show that for all $m \in M$, nCm. Suppose that m is an arbitrary element of M. Therefore, $m \in C(N)$. Then by definition, mCn for every $n \in N$. Hence, $n \in C(M)$. By symmetry, if $N \subseteq C(M)$, then we see $M \subseteq C(N)$.

(ii) Since $M \subseteq C(C(M))$, it follows that $C(C(C(M))) \subseteq C(M)$. We also obtain $C(M) \subseteq C(C(C(M)))$ by putting $C(M)$ instead of M in Theorem 3.17. Hence, $C(M) = C(C(C(M)))$. □

4. Commutators relative to $\rightarrow$ and $\rightsquigarrow$

This section delves into the concept of commutators within pseudo BL-algebras. Our main objective is to introduce and examine the properties of these commutators, which play a crucial role in understanding the structure and behavior of these algebraic structures.

Definition 4.1. Suppose a and b are elements of A. We define the commutators of a and b relative to $\rightarrow$ and $\rightsquigarrow$ as follows:

- $com_{\rightarrow}(a, b) = (a \odot b) \rightarrow (b \odot a)$
- $com_{\rightsquigarrow}(a, b) = (b \odot a) \rightsquigarrow (a \odot b)$

The *commutator* of a and b, denoted by $[a, b]$, is defined as their meet:

$$[a, b] = com_{\rightarrow}(a, b) \wedge com_{\rightsquigarrow}(a, b).$$

For elements $a_1, a_2, ..., a_n \in A$ with $n \geq 2$, the *commutator* of weight n is defined recursively as $[a_1, a_2, ..., a_n] = [[a_1, ..., a_{n-1}], a_n]$, with the convention that $[a_1] = a_1$.

Example 4.2. In Example 3.2, let $a = (\frac{3}{4}, -2)$ and $b = (1, -2)$. Then,

- $com_{\rightarrow}(a, b) = (1, 0)$
- $com_{\rightsquigarrow}(a, b) = (1, -\dfrac{3}{8})$
- $[a, b] = com_{\rightarrow}(a, b) \wedge com_{\rightsquigarrow}(a, b) = (1, 0) \wedge (1, -\dfrac{3}{8}) = (1, -\dfrac{3}{8})$.

Similarly, if $a = (1, -4)$ and $b = (\frac{3}{4}, 8)$, then

- $com_{\rightarrow}(a, b) = (1, -\dfrac{4}{3})$

- $com_{\rightsquigarrow}(a,b) = (1,0)$
- $[a,b] = com_{\rightarrow}(a,b) \wedge com_{\rightsquigarrow}(a,b) = (1,-\dfrac{4}{3}) \wedge (1,0) = (1,-\dfrac{4}{3})$.

Remark 4.3. Referring to Example 3.2, for $a = (1,-4)$ and $b = (\frac{3}{4},8)$, we have:

- $com_{\rightarrow}(a,b) = (1,-\dfrac{4}{3})$
- $com_{\rightarrow}(b,a) = (1,0)$
- $com_{\rightsquigarrow}(a,b) = (1,0)$
- $com_{\rightsquigarrow}(b,a) = (1,-1)$

Therefore,

- $[a,b] = com_{\rightarrow}(a,b) \wedge com_{\rightsquigarrow}(a,b) = (1,-\dfrac{4}{3}) \wedge (1,0) = (1,-\dfrac{4}{3})$
- $[b,a] = com_{\rightarrow}(b,a) \wedge com_{\rightsquigarrow}(b,a) = (1,0) \wedge (1,-1) = (1,-1)$.

Thus, in general, $[a,b] \neq [b,a]$ in pseudo BL-algebras.

Next we establish some properties of the commutators.

THEOREM 4.4. *For any $a, b \in A$, we have*

(i) $com_{\rightarrow}(a,a) = com_{\rightsquigarrow}(a,a) = 1$, *hence* $[a,a] = 1$;

(ii) $com_{\rightarrow}(a,1) = com_{\rightsquigarrow}(a,1)$, *hence* $[a,1] = 1$;

(iii) $com_{\rightarrow}(1,a) = com_{\rightsquigarrow}(1,a) = 1$, *hence* $[1,a] = 1$;

(iv) $com_{\rightarrow}(a,0) = com_{\rightsquigarrow}(a,0) = 1$, *hence* $[a,0] = 1$;

(v) $com_{\rightarrow}(0,a) = com_{\rightsquigarrow}(0,a)$, *hence* $[0,a] = 1$;

(vi) $com_{\rightarrow}(\bar{a},a) = com_{\rightsquigarrow}(a,\bar{a}) = 1$;

(vii) $com_{\rightarrow}(a,\tilde{a}) = com_{\rightsquigarrow}(\tilde{a},a) = 1$;

(viii) $com_{\rightarrow}(a,b),\ com_{\rightsquigarrow}(a,b) \geq b$, *hence* $[a,b] \geq b$.

PROOF: The proof of (i), (ii), ..., (vi) are obvious.

(vii) $com_{\to}(\bar{a}, a) = (\bar{a} \odot a) \to (a \odot \bar{a}) = 0 \to (a \odot \bar{a}) = 1$ and $com_{\rightsquigarrow}(\tilde{a}, a) = (a \odot \tilde{a}) \rightsquigarrow (\tilde{a} \odot a) = 0 \rightsquigarrow (a \odot \tilde{a}) = 1$.

(viii) $b \to com_{\to}(a, b) = b \to ((a \odot b) \to (b \odot a)) = (b \odot (a \odot b)) \to (b \odot a) = 1$. Therefore $b \leq com_{\to}(a, b)$. Also $b \rightsquigarrow com_{\rightsquigarrow}(a, b) = b \rightsquigarrow ((b \odot a) \rightsquigarrow (a \odot b)) = ((b \odot a) \odot b) \rightsquigarrow (a \odot b) = 1$. Then $b \leq com_{\rightsquigarrow}(a, b)$. So $b \leq [a, b]$. □

LEMMA 4.5. *Suppose H is a normal filter of A and $a, b \in A$. Then*

- $com_{\to}(a, b)/H = com_{\to}(a/H, b/H)$,
- $com_{\rightsquigarrow}(a, b)/H = com_{\rightsquigarrow}(a/H, b/H)$.

PROOF: We have

$$\begin{aligned} com_{\to}(a, b)/H &= ((a \odot b) \to (b \odot a))/H \\ &= (a \odot b)/H \to (b \odot a)/H \\ &= (a/H \odot b/H) \to (b/H \odot a/H) \\ &= com_{\to}(a/H, b/H). \end{aligned}$$

and similarly,

$$\begin{aligned} com_{\rightsquigarrow}(a, b)/H &= ((b \odot a) \rightsquigarrow (a \odot b))/H \\ &= (b \odot a)/H \rightsquigarrow (a \odot b)/H \\ &= (b/H \odot a/H) \rightsquigarrow (a/H \odot b/H) \\ &= com_{\rightsquigarrow}(a/H, b/H). \end{aligned}$$

□

PROPOSITION 4.6. For elements $a, b \in A$, aCb holds if and only if $[a, b] = 1$.

PROOF: If aCb holds, then $a \odot b = b \odot a$. Therefore,

$$com_{\to}(a, b) = (a \odot b) \to (b \odot a) = (a \odot b) \to (a \odot b) = 1$$
$$com_{\rightsquigarrow}(a, b) = (b \odot a) \rightsquigarrow (a \odot b) = (a \odot b) \rightsquigarrow (a \odot b) = 1$$

hence $[a, b] = com_{\to}(a, b) \wedge com_{\rightsquigarrow}(a, b) = 1 \wedge 1 = 1$.

Conversely, if $[a, b] = 1$, then $com_{\to}(a, b) = com_{\rightsquigarrow}(a, b) = 1$. From $com_{\to}(a, b) = 1$, we deduce $a \odot b \leq b \odot a$ and $com_{\rightsquigarrow}(a, b) = 1$ implies $b \odot a \leq a \odot b$. Thus $a \odot b = b \odot a$, and hence aCb holds. □

LEMMA 4.7. *For elements $a, b \in A$*

1. *If $a \vee b = 1$, then $[a, b] = 1$,*
2. *If $b \leq \bar{a}$, then $com_{\rightsquigarrow}(a, b) = 1$,*
3. *If $b \leq \tilde{a}$, then $com_{\rightarrow}(a, b) = 1$,*
4. *$[a \to b, b \to a] = [a \rightsquigarrow b, b \rightsquigarrow a] = 1$.*

PROOF:

1. Suppose $a \vee b = 1$. By psbl-c$_{19}$, $a \odot b = a \wedge b = b \wedge a = b \odot a$. Hence $com_{\rightarrow}(a, b) = com_{\rightsquigarrow}(a, b) = 1$, and so $[a, b] = 1$.
2. Let $b \leq \bar{a}$. Then $b \odot a = 0$ (psbl-c$_{13}$). So $com_{\rightsquigarrow}(a, b) = (b \odot a) \rightsquigarrow (a \odot b) = 0 \rightsquigarrow (a \odot b) = 1$.
3. Let $b \leq \tilde{a}$. Then $a \odot b = 0$ (psbl-c$_{13}$). So $com_{\rightarrow}(a, b) = (a \odot b) \to (b \odot a) = 0 \to (b \odot a) = 1$.
4. By ps-BL$_5$, $(a \to b) \vee (b \to a) = (a \rightsquigarrow b) \vee (b \rightsquigarrow a) = 1$. So, by part (1), $[a \to b, b \to a] = [a \rightsquigarrow b, b \rightsquigarrow a] = 1$ for all $a, b \in A$. $\square$

PROPOSITION 4.8. Let $a, b_i \in A, (i \in I)$. If $[a, b_i] = 1$ for all $i \in I$, then $[a, \bigvee_{i \in I} b_i] = 1$ and $[a, \bigwedge_{i \in I} b_i] = 1$, whenever the arbitrary meets and joins exist.

PROOF: Let $[a, b_i] = 1$ for every $i \in I$. Then, by Proposition $efd1$, aCb_i for all $i \in I$. By psbl-c$_{17}$ and psbl-c$_{18}$, we have

$$a \odot (\bigwedge_{i \in I} b_i) = \bigwedge_{i \in I} (a \odot b_i) = \bigwedge_{i \in I} (b_i \odot a) = (\bigwedge_{i \in I} b_i) \odot a$$

$$a \odot (\bigvee_{i \in I} b_i) = \bigvee_{i \in I} (a \odot b_i) = \bigvee_{i \in I} (b_i \odot a) = (\bigvee_{i \in I} b_i) \odot a$$

Therefore, $aC(\bigwedge_{i \in I} b_i)$ and $aC(\bigvee_{i \in I} b_i)$. Applying Proposition 4.6 again, we obtain $[a, (\bigwedge_{i \in I} b_i)] = 1$ and $[a, (\bigvee_{i \in I} b_i)] = 1$. $\square$

5. Engel elements and Engel sets in pseudo BL-algebras

In this section, using the notion of commutator of two elements in a pseudo BCI-algebra, we introduce and study Engel sets, which are defined by left and right normed commutators. Several properties of these sets are then discussed.

Suppose that a and b are elements of A. For a non-negative integer n, we define the *n-Engel left normed commutator* $[a, {}_n b]$ inductively as follows:

$$[a, {}_0 b] = a$$
$$[a, {}_n b] = [[a, {}_{n-1} b], b] \quad \text{for } n \geq 1$$

Similarly, the *n-Engel right normed commutator* $[{}_n a, b]$ of the pair (a, b) is defined by induction:

$$[{}_0 a, b] = b$$
$$[{}_n a, b] = [a, [{}_{n-1} a, b]] \quad \text{for } n \geq 1$$

Note that $[a, {}_1 b] = [{}_1 a, b] = [a, b]$.

For a positive integer k, an element $a \in A$ is called a *right k-Engel element* of A if $[a, {}_k b] = 1$, for all $b \in A$. An element $a \in A$ is called a *right Engel element* if it is right k-Engel for some non-negative integer k. We denote by $R(A)$ and $R_k(A)$ the set of right Engel elements and right k-Engel elements, respectively.

$$R_k(A) = \{a \in A \mid [a, {}_k b] = 1, \forall b \in A\}$$
$$R(A) = \bigcup_{k \in \mathbb{N}} R_k(A)$$

Observe that the variable element b appears on the right of the bracket. If k can be chosen independently of b, then a is a right k-Engel element of A. Left Engel elements are defined analogously.

For a positive integer k, an element $a \in A$ is called a *left k-Engel element* of A, whenever $[b, {}_k a] = 1$, for all $b \in A$. Also, a is said to be a *left Engel* *element* of A if it is left k-Engel for some non-negative integer k. We denote by $L(A)$ and $L_k(A)$ the set of left Engel elements and left k-Engel elements, respectively.

$$L_k(A) = \{a \in A \mid [b, {}_k a] = 1, \forall b \in A\}$$
$$L(A) = \bigcup_{k \in \mathbb{N}} L_k(A)$$

In $L_k(A)$ and $L(A)$, the variable element is on the left of the bracket. Furthermore, since $[a, 0] = [0, a] = [a, 1] = [1, a] = 1$ for every $a \in A$, we have $0, 1 \in R(A) \cap L(A)$.

An element $a \in A$ that is both a left and right Engel element is said to be an *Engel element*. The set of all Engel elements of A is denoted by $En(A)$. Obviously, 0 is an Engel element in any pseudo BL-algebra.

THEOREM 5.1. *If a is in the center of A, then a is a right Engel element.*

PROOF: Let $a \in C(A)$. By definition of the center, aCb for all $b \in A$. This implies that $[a, b] = 1$ for all $b \in A$. Therefore, a is a right k-Engel element for $k = 1$, and thus $a \in R_1(A) \subseteq R(A)$. $\square$

LEMMA 5.2. *If A is commutative, then every $a \in A$ is an Engel element.*

PROOF: Let A be commutative and let $a \in A$. By the definition of a commutative algebra, $a \odot b = b \odot a$ for all $b \in A$. This means aCb for all $b \in A$. Consequently, $[a, b] = 1$ for all $b \in A$. Thus, a is a right 1-Engel element ($a \in R_1(A)$) and also a left 1-Engel element ($a \in L_1(A)$). Therefore, a is an Engel element of A. $\square$

COROLLARY 5.3. Any element a of a BL-algebra A is an Engel element.

PROOF: In a BL-algebra, it is a known property that $a \odot b = b \odot a$ for all $a, b \in A$. This means that every BL-algebra is commutative. By Lemma 5.2, if A is commutative, then any $a \in A$ is an Engel element. $\square$

LEMMA 5.4. *i) $L_1(A) \subseteq L_2(A) \subseteq \cdots \subseteq L_n(A) \subseteq \cdots \subseteq L(A)$.*
ii) $R_1(A) \subseteq R_2(A) \subseteq \cdots \subseteq R_n(A) \subseteq \cdots \subseteq R(A)$.

PROOF: i) Let $a \in L_1(A)$. Then for all $b \in A$, we have $[b, a] = 1$. This means $[b, {}_1 a] = 1$. Now, consider $[b, {}_2 a]$,

$$[b, {}_2a] = [[b, {}_1a], a] = [1, a] = 1.$$

Thus, $a \in L_2(A)$, which shows that $L_1(A) \subseteq L_2(A)$. By induction, assume $L_n(A) \subseteq L_{n+1}(A)$ for some $n \geq 1$. Let $a \in L_n(A)$, so $[b, {}_na] = 1$ for all $b \in A$. Then

$$[b, {}_{n+1}a] = [[b, {}_na], a] = [1, a] = 1.$$

This implies $a \in L_{n+1}(A)$, so $L_n(A) \subseteq L_{n+1}(A)$. Finally, by the definition of $L(A)$, it is clear that $L_n(A) \subseteq L(A) = \bigcup_{k \in \mathbb{N}} L_k(A)$ for all $n \in \mathbb{N}$.

ii) The proof for right Engel elements is analogous to the proof for left Engel elements. If $a \in R_1(A)$, then $[a, b] = 1$ for all $b \in A$. Then $[a, {}_2b] = [[a, {}_1b], b] = [1, b] = 1$. Thus, $a \in R_2(A)$, which shows $R_1(A) \subseteq R_2(A)$. The induction proceeds similarly. □

PROPOSITION 5.5. Let f be a homomorphism from A to a pseudo BL-algebra B, and let $a, b \in A$. Then $f[a, {}_nb] = [f(a), {}_nf(b)]$ for every $n \in \mathbb{N}$.

PROOF: Suppose $a, b \in A$ and $n \in \mathbb{N}$. We proceed by induction on n. If $n = 1$, then

$$\begin{aligned} f(com_{\to}(a, b)) &= f((a \odot b) \to (b \odot a)) \\ &= (f(a) \odot f(b)) \to (f(b) \odot f(a)) \\ &= com_{\to}(f(a), f(b)) \end{aligned}$$

and

$$\begin{aligned} f(com_{\rightsquigarrow}(a, b)) &= f((b \odot a) \rightsquigarrow (a \odot b)) \\ &= (f(b) \odot f(a)) \rightsquigarrow (f(a) \odot f(b)) \\ &= com_{\rightsquigarrow}(f(a), f(b)). \end{aligned}$$

Therefore,

$$\begin{aligned} f[a, b] = {} & f((com_{\to}(a, b)) \wedge (com_{\rightsquigarrow}(a, b))) \\ = {} & (f(com_{\to}(a, b)) \wedge f(com_{\rightsquigarrow}(a, b))) \\ = {} & com_{\to}(f(a), f(b)) \wedge com_{\rightsquigarrow}(f(a), f(b)) \\ = {} & [f(a), f(b)]. \end{aligned}$$

Thus, $f([a,b]) = [f(a), f(b)]$. Assume the hypothesis holds for n, i.e., $f([a,{}_n b]) = [f(a),{}_n f(b)]$. Now, by the inductive hypothesis,

$$\begin{aligned} f([a,{}_{n+1} b]) &= f([[a,{}_n b], b]) \\ &= [f([a,{}_n b]), f(b)] \\ &= [[f(a),{}_n f(b)], f(b)] \\ &= [f(a),{}_{n+1} f(b)]. \end{aligned}$$

Hence, the result holds for all positive integers n. □

Proposition 5.6. $L(A) = A$ if and only if $R(A) = A$ if and only if $En(A) = A$.

Proof: Suppose that $A = L(A)$. Then $A = \{a \in A : \forall b \in A\, \exists n \in \mathbb{N} \text{ s.t } [b,{}_n a] = 1\}$. That is, for all $a \in A$ and for every $b \in A$ there exists $n \in \mathbb{N}$ such that $[b,{}_n a] = 1$. Substituting a for b and b for a, we have that for any $b \in A$, there exists a positive integer n such that $[a,{}_n b] = 1$ for all $a \in A$. Hence $A = R(A)$. The proof that $A = R(A)$ implies $A = L(A)$ is similar.

Now, let $R(A) = A$ and $a \in A = R(A)$. Then, by definition of $R(A)$, for all $b \in A$, there exists $n \in \mathbb{N}$ such that $[a,{}_n b] = 1$. So, for each $b \in A$ there exists $n \in \mathbb{N}$ such that $[b,{}_n a] = 1$. Thus $a \in R(A) \cap L(A)$ and hence $A \subseteq En(A)$. Since $En(A) \subseteq A$ we have $A = En(A)$.

If $En(A) = A$, then $R(A) \cap L(A) = A$. So $R(A) = L(A) = A$. □

Now, we introduce the concept of Engel pseudo BL-algebras and study it in detail.

Definition 5.7. A pseudo BL-algebra A in which all elements are Engel is said to be an Engel pseudo BL-algebra. We call the number $\sup\{n : [a,{}_n b] = 1, (a,b) \in A \times A\}$ the Engel degree of A.

By Proposition 5.6 a pseudo BL-algebra A is an Engel pseudo BL-algebra, if there exists $n \in \mathbb{N}$, such that $[a,{}_n b] = 1$, for any $a, b \in A$. Also a subset M of A is called an Engel set if, for all $a, b \in M$, there is a non-negative integer n such that $[a,{}_n b] = 1$.

Example 5.8. The pseudo BL-algebra in Example 3.2 is not Engel, because if we set $a = (1, -4)$ and $b = (\frac{3}{4}, 8)$, then $[a, b] = (1, -\frac{4}{3})$ and $[a, {}_2b] = (1, -\frac{4}{9})$. For any $n \in \mathbb{N}$, we obtain $[a, {}_nb] = (1, -\frac{4}{3^n}) \neq (1, 0)$.

Corollary 5.9. $A = L_n(A)$ if and only if $A = R_n(A)$.

Proof: $A = L_n(A)$ if and only if $\{a \in A : \forall b \in A \ \ [b, {}_na] = 1\} = A$ if and only if for all $a, b \in A$, $[b, {}_na] = 1$ if and only if for any $a, b \in A$, $[a, {}_nb] = 1$ if and only if $A = R_n(A)$. □

Definition 5.10. A is called n-Engel if $A = L_n(A)$ or equivalently $A = R_n(A)$.

Since $L_1(A) \subseteq L_2(A) \subseteq \cdots \subseteq L_n(A) \subseteq \cdots \subseteq L(A)$, if A is an n-Engel pseudo BL-algebra, then A is m-Engel for any $m \geq n$.

Theorem 5.11. *1-Engel pseudo BL-algebras are exactly commutative pseudo BL-algebras.*

Proof: A is commutative if and only if for any $a, b \in A$, aCb if and only if $[a, b] = 1$ for any $a, b \in A$ if and only if A is 1-Engel. □

Proposition 5.12. Let $f : A \longrightarrow B$ be a homomorphism between two pseudo BL-algebras. If A is Engel, then $f(A)$ is also Engel.

Proof: Suppose that $y_1, y_2 \in f(A)$. Then, for some $a, b \in A$ we have $y_1 = f(a)$ and $y_2 = f(b)$. Since A is Engel, there exists $n \in \mathbb{N}$ such that $[a, {}_nb] = 1$. Hence $[y_1, {}_ny_2] = [f(a), {}_nf(b)] = f([a, {}_nb]) = f(1) = 1$. So $f(A)$ is Engel. □

6. Solvable pseudo BL-algebras

In mathematics, specifically in group theory, a solvable group is a group that can be constructed from abelian groups using extensions. At this point, it is possible to mimic the solvability conditions of group theory.

Let $A_1, A_2, \ldots, A_n$ be nonempty subsets of A. Define the commutator of A_1 and A_2 to be

$$[A_1, A_2] = \langle \{[a_1, a_2] \mid a_1 \in A_1, a_2 \in A_2\} \rangle$$

More generally, for $n \geq 2$, we define $[A_1, \ldots, A_n] = [[A_1, \ldots, A_{n-1}], A_n]$.

When $A_1 = A_2 = A$, $[A, A]$ is the filter of A generated by all the commutators of A, denoted by A'. For any $n \in \mathbb{N}$, define $A^{(n)} = [A^{(n-1)}, A^{(n-1)}]$.

THEOREM 6.1. *Let H be a normal filter of A. Then A/H is commutative if and only if $A' \subseteq H$.*

PROOF: Let H be a normal filter of A. Then A/H is commutative if and only if $a/H \odot b/H = b/H \odot a/H$ for all $a, b \in A$, which is equivalent to $(a \odot b)/H = (b \odot a)/H$. This holds if and only if $(a \odot b) \to (b \odot a) \in H$ and $(b \odot a) \rightsquigarrow (a \odot b) \in H$ for all $a, b \in A$, which is equivalent to $com_{\to}(a, b) \in H$ and $com_{\rightsquigarrow}(a, b) \in H$ for all $a, b \in A$. This is equivalent to $[a, b] \in H$ for all $a, b \in A$. Thus, A/H is commutative if and only if $A' \subseteq H$. $\square$

Hence, if A' is a normal filter of A, then A' is the unique smallest normal filter H of A such that A/H is commutative.

THEOREM 6.2. *The following conditions are equivalent:*

(i) A *is commutative.*

(ii) $C(A) = A$.

(iii) $A' = \{1\}$.

PROOF: (i $\Leftrightarrow$ ii) A is commutative if and only if $a \odot b = b \odot a$ for all $a, b \in A$, which is equivalent to aCb for all $a, b \in A$. This holds if and only if $C(A) = A$.

(ii $\Leftrightarrow$ iii) We know that $C(A) = \{a \in A \mid aCb \text{ for all } b \in A\} = \{a \in A \mid [a, b] = 1 \text{ for all } b \in A\}$. Since $C(A) = A$, for every $a, b \in A$, we have $[a, b] = 1$. Therefore, $A' = [A, A] = \langle\{[a_1, a_2] \mid a_1 \in A, a_2 \in A\}\rangle = \langle\{1\}\rangle = \{1\}$.

Conversely, if $A' = \{1\}$, then $[a, b] = 1$ for every $a, b \in A$. Hence, aCb for every $a, b \in A$. Then $C(A) = \{a \in A \mid aCb \text{ for every } b \in A\} = A$. $\square$

COROLLARY 6.3. The following conditions on A are equivalent:

(i) A is commutative.

(ii) $A' = \{1\}$.

(iii) A is 1-Engel.

(iv) $A = C(A)$.

It is clear that $R_0(A) = L_0(A) = \{1\}$ and $R_1(A) = L_1(A) = C(A)$. So x is a left 1-Engel or right 1-Engel element if and only if x is an element of $C(A)$, and A is 1-Engel if and only if A is commutative.

PROPOSITION 6.4. If $f : A \longrightarrow B$ is a homomorphism from A to a pseudo BL-algebra B, and A_1, A_2 are nonempty subsets of A, then

(i) $f([A_1, A_2]) = [f(A_1), f(A_2)]$.

(ii) $f(A^{(n)}) = (f(A))^{(n)}$, for any positive integer $n \in \mathbb{N}$.

PROOF: i) Let $y \in f([A_1, A_2])$. Then there exists $x \in [A_1, A_2]$ such that $y = f(x)$. So $x = [a_1, a_2]$ for some $a_1 \in A_1$ and $a_2 \in A_2$, and hence $y = f(x) = f([a_1, a_2]) = [f(a_1), f(a_2)] \in [f(A_1), f(A_2)]$. Therefore $f([A_1, A_2]) \subseteq [f(A_1), f(A_2)]$.

Conversely, let $y \in [f(A_1), f(A_2)]$. Then there exist $y_1 \in f(A_1)$ and $y_2 \in f(A_2)$ such that $y = [y_1, y_2]$. Thus, $y_1 = f(a_1)$ and $y_2 = f(a_2)$ for some $a_1 \in A_1$ and $a_2 \in A_2$. Hence $y = [y_1, y_2] = [f(a_1), f(a_2)] = f([a_1, a_2]) \in f([A_1, A_2])$. Whence $[f(A_1), f(A_2)] \subseteq f([A_1, A_2])$. Therefore $f([A_1, A_2]) = [f(A_1), f(A_2)]$.

ii) We prove by induction on n. If $n = 1$, then $f(A^{(1)}) = f(A') = f([A, A]) = [f(A), f(A)] = (f(A))'$. Now assume that $f(A^{(k)}) = (f(A))^{(k)}$ for some $k \in \mathbb{N}$. Then $f(A^{(k+1)}) = f([A^{(k)}, A^{(k)}]) = [f(A^{(k)}), f(A^{(k)})] = [(f(A))^{(k)}, (f(A))^{(k)}] = (f(A))^{(k+1)}$. □

PROPOSITION 6.5. Let H be a normal filter of A. Then A/H is Engel pseudo BL-algebra if and only if A is Engel pseudo BL-algebra.

PROOF: Assume A is an Engel pseudo BL-algebra. Then by definition, for any $a, b \in A$, there exists $n \in \mathbb{N}$ such that $[a,_n b] = 1$. We claim that $[a/H,_n b/H] = [a,_n b]/H$. We prove this by induction on n. For $n = 1$:

$$\begin{aligned}
[a/H, b/H] &= (com_{\rightarrow}(a/H, b/H) \wedge com_{\rightsquigarrow}(a/H, b/H)) \\
&= (com_{\rightarrow}(a, b)/H) \wedge (com_{\rightsquigarrow}(a, b)/H) \\
&= (com_{\rightarrow}(a, b) \wedge com_{\rightsquigarrow}(a, b))/H \\
&= [a, b]/H
\end{aligned}$$

Now, assume that $[a/H, {}_k b/H] = [a, {}_k b]/H$ for any positive integer k. Then

$$\begin{aligned} [a/H, {}_{k+1} b/H] &= [[a/H, {}_k b/H], b/H] \\ &= [[a, {}_k b]/H, b/H] \\ &= [[a, {}_k b], b]/H \\ &= [a, {}_{k+1} b]/H \end{aligned}$$

Thus, the claim holds for every positive integer n. Since A is Engel, there exists $n \in \mathbb{N}$ such that $[a, {}_n b] = 1$ for all $a, b \in A$. Then $[a/H, {}_n b/H] = [a, {}_n b]/H = 1/H$. So A/H is an Engel pseudo BL-algebra.

Conversely, assume A/H is Engel. If a is an arbitrary element of A, then $a/H \in A/H$. Therefore, for every $b \in A$, there exists a positive integer n such that $[a/H, {}_n b/H] = 1/H$. As shown by the claim, $[a/H, {}_n b/H] = [a, {}_n b]/H$. So $[a, {}_n b]/H = 1/H$, which implies $[a, {}_n b] \in H$. Since H is Engel, there exists $m \in \mathbb{N}$ such that $[[a, {}_n b], {}_m b] = 1$. This means $[a, {}_{n+m} b] = 1$. Therefore, A is an Engel pseudo BL-algebra. □

Definition 6.6. A is called solvable if there exists $n \in \mathbb{N}$ such that $A^{(n)} = \{1\}$.

Theorem 6.7. *Let f be an isomorphism from a pseudo BL-algebra A to a pseudo BL-algebra B, and let $n \in \mathbb{N}$. Then*

i) A is an n-Engel pseudo BL-algebra if and only if B is an n-Engel pseudo BL-algebra.

ii) A is solvable if and only if B is solvable.

Proof: i) Suppose A is n-Engel. Let h, k be arbitrary elements of B. Since f is an isomorphism, there exist unique $a, b \in A$ such that $f(a) = h$ and $f(b) = k$. Since A is n-Engel, we have $[a, {}_n b] = 1$. Applying f to this equation, we get $f([a, {}_n b]) = f(1)$. Since f is a homomorphism, $f([a, {}_n b]) = [f(a), {}_n f(b)]$. Also, $f(1) = 1_B$, the identity element of B. Thus, $[h, {}_n k] = 1_B$. This shows that B is an n-Engel pseudo BL-algebra.

Conversely, suppose B is an n-Engel pseudo BL-algebra. Let $a, b \in A$. Then $f(a), f(b) \in B$. Since B is n-Engel, we have $[f(a), {}_n f(b)] = 1_B$. Applying the inverse isomorphism f^{-1} to this equation, we get $f^{-1}([f(a), {}_n f(b)]) = f^{-1}(1_B)$. Since f is a homomorphism, $f^{-1}([f(a), {}_n f(b)]) =$

$[f^{-1}(f(a)),{}_n f^{-1}(f(b))]$. Also, $f^{-1}(1_B) = 1_A$, the identity element of A. Thus, $[a,{}_n b] = 1_A$. This shows that A is an n-Engel pseudo BL-algebra.

ii) Since f is an isomorphism from A to B, we have $f(A) = B$. From Proposition 6.4 (ii), $f(A^{(n)}) = (f(A))^{(n)} = B^{(n)}$ for every positive integer n. If A is solvable, then there exists $m \in \mathbb{N}$ such that $A^{(m)} = \{1_A\}$. Applying f, we get $f(A^{(m)}) = f(\{1_A\})$. Since f is a homomorphism, $f(\{1_A\}) = \{f(1_A)\} = \{1_B\}$. Thus, $f(A^{(m)}) = B^{(m)} = \{1_B\}$. This shows that B is a solvable pseudo BL-algebra.

Conversely, if B is solvable, then there exists $m \in \mathbb{N}$ such that $B^{(m)} = \{1_B\}$. Since $f(A^{(m)}) = B^{(m)}$, we have $f(A^{(m)}) = \{1_B\}$. Since f is an isomorphism, $f(A^{(m)}) = \{1_B\}$ implies $A^{(m)} = f^{-1}(\{1_B\}) = \{1_A\}$. This shows that A is solvable. □

THEOREM 6.8. *Let H be a normal filter of a pseudo BL-algebra A. If H and A/H are solvable pseudo BL-algebras, then A is a solvable pseudo BL-algebra.*

PROOF: Let $f : A \to A/H$ be the natural homomorphism. Since A/H is solvable, there exists a positive integer n such that $(A/H)^{(n)} = \{H\}$. By Proposition 6.4 (ii), $(A/H)^{(n)} = f(A^{(n)})$. Thus, $f(A^{(n)}) = \{H\}$. By the definition of the kernel of a homomorphism, $H = \ker(f)$. Therefore, $f(A^{(n)}) = \{H\}$ implies that $A^{(n)} \subseteq \ker(f) = H$. Since H is solvable, there exists a positive integer k such that $H^{(k)} = \{1_A\}$. Since $A^{(n)} \subseteq H$, we have $(A^{(n)})^{(k)} \subseteq H^{(k)} = \{1_A\}$. The $(n+k)$-th derived chain of A is $A^{(n+k)} = (A^{(n)})^{(k)}$. Therefore, $A^{(n+k)} \subseteq \{1_A\}$. Since $A^{(n+k)}$ is a subset of A containing 1_A, and it must be closed under the commutator operation, the only possibility is $A^{(n+k)} = \{1_A\}$. This shows that A is solvable. □

7. Conclusions

In this paper, we have introduced the concepts of commutativity relations, commutators, and Engel elements in pseudo BL-algebras, and we have investigated several of their properties. To further develop the theory of pseudo BL-algebras, one promising direction would be to investigate the Engel degree of these algebras and to establish relationships between various subclasses. For example, 1-Engel BCI-algebras are strictly commutative pseudo BL-algebras. We hope that this work will contribute to further studies in this area. We believe that the results presented here, along with forthcoming works, can pave the way for a bright future in the theory of pseudo BL-algebras. A major goal of Engel theory in pseudo BL-algebras could be stated as follows: to identify conditions on A that ensure $L(A)$ and $R(A)$ are filters, if such conditions exist.

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