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HILBERT ALGEBRAS WITH HILBERT-GALOIS CONNECTIONS II

Abstract

Hilbert algebra with a Hilbert-Galois connection, or HilGC-algebra, is a triple (A, f, g) where A is a Hilbert algebra, and f and g are unary maps on A such that $f(a) \leq b$ iff $a \leq g(b)$, and $g(a \to b) \leq g(a) \to g(b)$ for all $a, b \in A$. In this paper, we are going to prove that some varieties of HilGC-algebras are characterized by first-order conditions defined in the dual space and that these varieties are canonical. Additionally, we will also study and characterize the congruences of an HilGC-algebra through specific closed subsets of the dual space. This characterization will be applied to determine the simple algebras and subdirectly irreducible HilGC-algebras.

Keywords: Hilbert algebras, modal operators, Galois connection, canonical varieties, congruences.

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1. Introduction

This paper can be read as a continuation of [6] where we defined the notion of Hilbert-Galois algebra. Recall that an order-preserving connection in a Hilbert algebra A is a pair (f, g), where $f, g : A \to A$ are order-preserving maps such that $a \leq (g \circ f)(a)$ and $(f \circ g)(a) \leq a$, for $a \in A$ (see Definition 2.6). A Hilbert-Galois connection on A is an order-preserving connection (f, g) such that g is a Hilbert semi-homomorphism, i.e., $g(a \to b) \leq g(a) \to$

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$g(b)$, for all $a, b \in A$. A Hilbert algebra with a Hilbert-Galois connection, or HilGC-algebra, is a triple (A, f, g) where A is a Hilbert algebra and the pair (f, g) is a Hilbert-Galois connection. As shown in [6], the class of HilGC-algebras is a variety. Moreover, it was proved that there exists a topological duality between the category of HilGC-algebras and the class of Hilbert-Galois spaces. A Hilbert-Galois space is a structure $(X, \mathcal{T}_{\mathcal{K}}, R)$ where $(X, \mathcal{T}_{\mathcal{K}})$ is a Hilbert space (the dual space of a Hilbert algebra), and R is a binary relation on X satisfying certain conditions (see Definition 2.12).

In this paper we applied the representation developed in [6] to characterize some subvarieties of HilGC-algebras in terms of first-order conditions defined in the dual space. As consequence of this characterization, we show that these varieties are canonical. The duality given in [6] is applied to study the congruences of HilGC-algebras. We prove that the lattice of congruences of a HilGC-algebra (A, f, g) is isomorphic to the lattice of Galois implicative filters (Definition 4.1), and dually isomorphic to the lattice of certain closed subsets of the dual space of (A, f, g) called G-closed (Definition 4.3). The characterization is applied to study the simple and subdirectly irreducible HilGC-algebras.

2. Preliminaries

We assume that the reader is familiar with basic concepts with Hilbert algebras and with the duality between the category of Hilbert algebras and Hilbert homomorphisms, and the category of Hilbert spaces and H-functional relations [2, 3, 4, 5, 8]. Nevertheless, in this section we will recall the definitions, results and notations that will be needed in the rest of this paper.

Let $\langle X, \leq \rangle$ be a poset and consider the powerset $\mathcal{P}(X)$. Let $Y \subseteq X$. We say that Y is an *upset* (resp. *downset*) if $Y = \{x \in X : \exists y \in Y\ (y \leq x)\} = [Y)$ (resp. $Y = \{x \in X : \exists y \in Y\ (x \leq y)\} = (Y]$) . The set of *all upset* subsets of X is denoted by $\mathrm{Up}\,(X)$. The set complement of a subset $Y \subseteq X$ is denoted by Y^c.

The purely implicational subreducts of Heyting algebras are known in the literature as Hilbert algebras, or (positive) implication algebras [7, 8, 9].

Definition 2.1. A *Hilbert algebra* is an algebra $A = \langle A, \to, 1\rangle$ of type $(2,0)$ such that the following axioms hold in A:

1. $a \to a = 1$,
2. $1 \to a = a$,
3. $a \to (b \to c) = (a \to b) \to (a \to c)$,
4. $(a \to b) \to ((b \to a) \to a) = (b \to a) \to ((a \to b) \to b)$.

Hilbert algebras form a variety denoted by Hil. Every Hilbert algebra A has a natural order $\leq$ defined by $a \leq b$ iff $a \to b = 1$. Given a Hilbert algebra A and a sequence $a, a_1, \ldots, a_n \in A$, we define:

$$(a_1, \ldots, a_n; a) = \begin{cases} a_1 \to a & \text{if} \quad n = 1, \\ a_1 \to (a_2, \ldots, a_n; a) & \text{if} \quad n > 1. \end{cases}$$

A nonempty subset $F \subseteq A$ is an *implicative filter* of A if $1 \in F$, and if $a, a \to b \in F$ then $b \in F$. The set of all implicative filters of A is denoted by $\mathrm{Fi}(A)$. Note that every implicative filter of A is an upset of A. Let $S \subseteq A$. The implicative filter generated by S is $\langle S\rangle = \bigcap \{F \in \mathrm{Fi}(A) : S \subseteq F\}$. The deductive system generated by a subset $S \subseteq A$ can be characterized as the set

$$\langle S\rangle = \{a \in A : \exists \{a_1, \ldots, a_n\} \subseteq S : (a_1, \ldots, a_n; a) = 1\}.$$

The following result is proved in [2] and [9] and we will be useful in this paper:

Lemma 2.2. *Let* $A \in \mathrm{Hil}$. *Let* $F \in \mathrm{Fi}(A)$ *and* $a \in A$. *Then,*

$$F \vee \langle a\rangle = \langle F \cup \{a\}\rangle = \{b \in A : a \to b \in F\}.$$

Let $F \in \mathrm{Fi}(A) - \{A\}$. We will say that F is *irreducible* if for any $F_1, F_2 \in \mathrm{Fi}(A)$ such that $F = F_1 \cap F_2$, it follows that $F = F_1$ or $F = F_2$. The set of all irreducible implicative filters of a Hilbert algebra A is denoted by $X(A)$. A downset I of A is called an *order-ideal of* A if for all $a, b \in I$ there exists $c \in I$ such that $a \leq c$ and $b \leq c$. The set of all order-ideals of A is denoted by $\mathrm{Ido}(A)$.

The following is a Hilbert algebra analogue of Birkhoff's Prime Filter Theorem and it is proved in [3].

Theorem 2.3. *Let $A \in \mathrm{Hil}$. Let $F \in \mathrm{Fi}(A)$ and let $I \in \mathrm{Ido}(A)$ such that $F \cap I = \emptyset$. Then, there exists $x \in X(A)$ such that $F \subseteq x$ and $x \cap I = \emptyset$.*

Corollary 2.4. Let $A \in \mathrm{Hil}$. Then,

1. for all $a, b \in A$, if $a \nleq b$, then there exists $x \in X(A)$ such that $a \in x$ and $b \notin x$.
2. If $x \in X(A)$ and $a, b \notin x$, there exists $c \notin x$ such that $a, b \leq c$.
3. If $x \in X(A)$, then $a \to b \notin x$ iff there exists $y \in X(A)$ such that $x \subseteq y$, $a \in y$ and $b \notin y$.

Let $\langle X, \mathcal{T} \rangle$ be a topological space. We recall that *the specialization dual order of* $\langle X, \mathcal{T} \rangle$ is the binary relation $\leq \subseteq X \times X$ defined by:

$$x \leq y \text{ iff } \forall W \in \mathcal{T}(x \notin W \text{ then } y \notin W). \tag{2.1}$$

If $\langle X, \mathcal{T} \rangle$ is T_0, then $\langle X, \leq \rangle$ is a poset. Now we define the Hilbert spaces as special T_0 topological spaces $\langle X, \mathcal{T}_{\mathcal{K}} \rangle$ having a base of compact sets $\mathcal{K}$. Recall also that a subset $Y \subseteq X$ is said to be irreducible when, for all closed sets $Y_1, Y_2 \subseteq X$, we have that $Y = Y_1 \cup Y_2$ entails $Y = Y_1$ or $Y = Y_2$. A space is *sober* when, for every irreducible closed set $Y \subseteq X$ there exists a unique $x \in X$ such that $Y = \mathrm{cl}(x)$. Let $D(X) := \{U : U^c \in \mathcal{K}\}$.

Definition 2.5. [4] An *H-space* is a topological space $\langle X, \mathcal{T}_{\mathcal{K}} \rangle$ such that:

(H1) $\mathcal{K}$ is a base of open and compact subsets for the topology $\mathcal{T}_{\mathcal{K}}$,

(H2) $U \Rightarrow V = (U \cap V^c]^c \in D(X)$, for all $U, V \in D(X)$,

(H3) $\langle X, \mathcal{T}_{\mathcal{K}} \rangle$ is sober.

If $\langle X, \mathcal{T}_{\mathcal{K}} \rangle$ is an H-space, then it is easy to see that $D(X) = \langle D(X), \Rightarrow, X \rangle$ is a Hilbert algebra.

Let $A \in \mathrm{Hil}$. Then A is isomorphic to the subalgebra $D(X(A)) = \{\varphi(a) : a \in A\}$ of the Hilbert algebra $\langle \mathrm{Up}(X(A)), \Rightarrow, X(A) \rangle$ via the map $\varphi : A \to \mathrm{Up}(X(A))$ defined by $\varphi(a) = \{x \in X(A) : a \in x\}$. From the results on representation for Hilbert algebras in [4] we have that $\langle X(A), \mathcal{T}_{\mathcal{K}_A} \rangle$ is an H-space where the family $\mathcal{K}_A = \{\varphi(a)^c : a \in A\}$ is a base of compact subsets.

Next we will review definitions and properties of Hilbert algebras with Hilbert-Galois connections introduced in [6].

DEFINITION 2.6. [6] Let $A \in$ Hil. A pair (f, g), where $f : A \to A$ and $g : A \to A$ are maps, is called *Hilbert-Galois connection* between in A if

1. $f(a) \leq b$ iff $a \leq g(b)$, for all $a, b \in A$.
2. $g(b \to c) \leq g(b) \to g(c)$, for all $b, c \in A$, i.e., g is a semi-homomorphism on A.

A triple $\langle A, f, g\rangle$ is a *Hilbert algebra with a Hilbert-Galois connection*, or a HilGC-algebra for short, if (f, g) is a Hilbert-Galois connection defined in A.

In [6] we give the following equational characterization of HilGC-algebras.

THEOREM 2.7. *[6] Let $A \in$ Hil and let $f : A \to A$ and $g : A \to A$ two maps. Then $\langle A, f, g\rangle$ is a HilGC-algebra if and only if the maps f and g satisfy the following conditions for all $a, b \in A$:*

(HilGC1) $g(1) = 1$.

(HilGC2) $f(a) \leq f((a \to b) \to b)$.

(HilGC3) $g(a \to b) \to (g(a) \to g(b)) = 1$.

(HilGC4) $a \to g(f(a)) = 1$.

(HilGC5) $f(g(a)) \to a = 1$.

In [6] we prove that f, g are monotonic maps. We denote by HilGC the variety of HilGC-algebras.

PROPOSITION 2.8. Let $\langle A, f, g\rangle$ be a HilGC-algebra. Then:

(1) If $F \in \mathrm{Fi}(A)$ then $g^{-1}(F) \in \mathrm{Fi}(A)$,

(2) If $x \in X(A)$ then $f^{-1}(x^c), (g(x^c)] \in \mathrm{Ido}(A)$.

PROOF: Item (1) and the affirmation $f^{-1}(x^c) \in \mathrm{Ido}(A)$, for each $x \in X(A)$, are proved in Proposition 14 of [6]. We prove that $(g(x^c)] \in \mathrm{Ido}(A)$. Assume that $x \in X(A)$ and let $a, b \in (g(x^c)]$. Then there exist $c, d \notin x$ such that $a \leq g(c)$ and $b \leq g(d)$, or equivalently, $f(a) \leq c$ and $f(b) \leq d$. Since $c, d \notin x$, by Corollary 2.4, there exists $e \notin x$ such that $c, d \leq e$. Thus, $f(a) \leq e$ and $f(b) \leq e$ and consequently, $a \leq g(e)$ and $b \leq g(e)$. Since $e \in x^c$ results, $g(e) \in (g(x^c)]$, and thus $(g(x^c)] \in \mathrm{Ido}(A)$. □

LEMMA 2.9. *[6, Lemma 21] Let* $\langle A, f, g\rangle \in \mathsf{HilGC}$. *Then*

1. *Let* $x \in X(A)$. *For all* $a \in A$, $g(a) \notin x$ *iff there exists* $y \in X(A)$ *such that* $g^{-1}(x) \subseteq y$ *and* $a \notin y$,
2. *Let* $x \in X(A)$. *For all* $a \in A$, $f(a) \in x$ *iff there exists* $y \in X(A)$ *such that* $y \subseteq f^{-1}(x)$ *and* $a \in y$.

We recall that a *IntGC-frame* is a relational structure $\langle X, \leq, R\rangle$ where $\langle X, \leq\rangle$ is a poset and $R \subseteq X \times X$ is a relation satisfying the condition

$$\leq^{-1} \circ R \circ \leq^{-1} \subseteq R. \tag{2.2}$$

We note that by the condition (2.2) and the reflexivility of $\leq^{-1}$, we have that $\leq^{-1} \circ R = R$ and $R \circ \leq^{-1} = R$.

LEMMA 2.10. *Let* $\mathcal{F} = \langle X, \leq, R\rangle$ *be an IntGC-frame. Then,*

$$R^{-1}(x), R(x)^c \in \mathrm{Up}(X), \text{ for each } x \in X.$$

PROOF: Let $x \in X$. We prove that $R(x)^c \in \mathrm{Up}(X)$. Let $y \leq z$ and $y \in R(x)^c$. Suppose that $z \in R(x)$. As $(z, y) \in \leq^{-1}$ results $(x, y) \in R$ and so, $y \in R(x)$, which is an absurd. Thus, $z \in R(x)^c$. Similarly, we can prove that $R^{-1}(x) \in \mathrm{Up}(X)$. □

It is know that if $\langle X, \leq\rangle$ is a poset, then $\langle \mathrm{Up}(X), \cup, \cap, \Rightarrow, \emptyset, X\rangle$ is a Heyting algebra. Moreover, we define the operators $f_R : \mathrm{Up}(X) \to \mathrm{Up}(X)$, and $g_R : \mathrm{Up}(X) \to \mathrm{Up}(X)$ as

$$f_R(U) = \{x \in X : R(x) \cap U \neq \emptyset\} = R^{-1}(U), \tag{2.3}$$

and

$$g_R(U) = \left\{x \in X : R^{-1}(x) \subseteq U\right\}, \tag{2.4}$$

for each $U \in \mathrm{Up}(X)$, respectively. The condition (2.2) ensures that $A(\mathcal{F}) = \langle \mathrm{Up}(X), \cup, \cap, \Rightarrow, f_R, g_R, \emptyset, X\rangle$ is a Heyting-Galois algebra, and in particular is a Hilbert-Galois algebra (see [6] Example 19).

If $\langle A, f, g\rangle \in \mathsf{HilGC}$, then $\mathcal{F}(A) = \langle X(A), \subseteq, R_A\rangle$ is an IntGC-frame, where the relation $R_A \subseteq X(A) \times X(A)$ is defined by

$$(x, y) \in R_A \text{ iff } y \subseteq f^{-1}(x).$$

By [6, Lemma 24], the relation R_A can be also defined as

$$(x, y) \in R_A \text{ iff } g^{-1}(y) \subseteq x.$$

The following representation theorem for HilGC-algebras follows from the results given in [6].

THEOREM 2.11 (of Representation). *Let* $A = \langle A, f, g \rangle \in \mathsf{HilGC}$. *Then the map* $\varphi : A \to A(\mathcal{F}(A))$ *is an embedding. Thus,* A *is isomorphic to some subalgebra of* $A(\mathcal{F}(A))$.

Now we recall the dual topological spaces of HilGC-algebras.

DEFINITION 2.12. [6, Def. 22] $\langle X, \mathcal{T}_{\mathcal{K}}, R \rangle$ is a *Hilbert-Galois space*, or *HG-space*, if $\langle X, \mathcal{T}_{\mathcal{K}} \rangle$ is an H-space and

1. $R^{-1}(U) \in D(X)$, for all $U \in D(X)$,
2. $R(U^c)^c \in D(X)$, for all $U \in D(X)$,
3. $R^{-1}(x)$ is a closed subset of $\langle X, \mathcal{T}_{\mathcal{K}} \rangle$, for all $x \in X$.

In [6] was proved that if $\langle X, \mathcal{T}_{\mathcal{K}}, R \rangle$ is an HG-space, then $\langle D(X), \Rightarrow, f_R, g_R, X \rangle$ is a HilGC-algebra where the operators $f_R : D(X) \to D(X)$ and $g_R : D(X) \to D(X)$ are defined by (2.3) and (2.4), respectively. Moreover, the map $\varepsilon_X : X \to X(D(X))$ given by $\varepsilon(x) = \{U \in D(X) : x \in U\}$ is a homeomorphism such that $(x, y) \in R$ iff $(\varepsilon(x), \varepsilon(y)) \in R_{D(X)}$, for all $x, y \in X$. If $\langle A, f, g \rangle \in \mathsf{HilGC}$, then $\langle X(A), \mathcal{T}_{\mathcal{K}_A}, R_A \rangle$ is an HG-space such that the map $\varphi : A \to D(X(A))$ given by $\varphi(a) = \{x \in X(A) : a \in x\}$ is an isomorphism of HilGC-algebras. For more details on the duality between HG-spaces and HilGC-algebras see [6].

3. Some canonical subvarieties of HGC-algebras

By Theorem 2.11, any HilGC-algebra A is a subalgebra of the HilGC-algebra $A(\mathcal{F}(A))$. The algebra $A(\mathcal{F}(A))$ is known as the canonical extension or canonical embedding algebra of A. We shall say that a variety V of HilGC-algebras is *canonical* it it is closed under canonical extensions, i.e., if $A \in \mathsf{V}$ then $A(\mathcal{F}(A)) \in V$. The notion of canonical varieties is an algebraic formulation of the notion of canonical logics ([1]). In this section we prove that certain varieties of HilGC-algebras are canonical.

Remark 3.1. Let V be a variety of HilGC-algebras. Let

$$Fr(\mathsf{V}) = \{\mathcal{F}(A) : A \in \mathsf{V}\}$$

be the class of IntGC-frames associated to V. Let F be a class of IntGC-frames. Let $\mathrm{Alg}(\mathrm{F}) = \{A(\mathcal{F}) : \mathcal{F} \in \mathrm{F}\}$ be class of HilGC-algebras associated to F. We note that if F is a class of IntGC-frames such that $\mathrm{Alg}(\mathrm{F}) \subseteq \mathsf{V}$ and $Fr(\mathsf{V}) \subseteq \mathrm{F}$, then V is canonical. Indeed. If $A \in \mathsf{V}$, then $\mathcal{F}(A) \in Fr(\mathsf{V}) \subseteq \mathrm{F}$. So, $A(\mathcal{F}(A)) \in Alg(\mathrm{F}) \subseteq \mathsf{V}$, i.e. V is canonical.

Let A be a HilGC-algebra . We will write $A \models \alpha \leq \beta$ when the equation $\alpha \wedge \beta \approx \alpha$ is valid in A. In the following Theorem 3.2, we characterize some classes of IntGC-frames. In the Theorem 3.3 we prove that some varieties of HilGC-algebras are canonical.

THEOREM 3.2. *Let $\mathcal{F} = \langle X, \leq, R\rangle$ be an IntGC-frame. Then*

1. $A(\mathcal{F}) \models a \leq g(a)$ *iff* $A(\mathcal{F}) \models f(a) \leq a$ *iff* $R \subseteq \leq^{-1}$.
2. $A(\mathcal{F}) \models g(a) \leq a$ *iff* $A(\mathcal{F}) \models a \leq f(a)$ *iff R is reflexive.*
3. $A(\mathcal{F}) \models g(a) \leq g^2(a)$ *iff* $A(\mathcal{F}) \models f^2(a) \leq f(a)$ *iff R is transitive.*
4. $A(\mathcal{F}) \models g^2(a) \leq g(a)$ *iff* $A(\mathcal{F}) \models f(a) \leq f^2(a)$ *iff R is weakly dense, i.e.,* $R \subseteq R^2$.
5. $A(\mathcal{F}) \models f(a) \leq g(a)$ *iff* $A(\mathcal{F}) \models a \leq g^2(a)$ *iff* $A \models f^2(a) \leq a$ *iff*
$$\forall x \forall y \forall z((x, y) \in R \wedge (y, z) \in R \Rightarrow z \leq x).$$
6. $A(\mathcal{F}) \models g(a) \leq f(a)$ *iff* $R(x) \cap R^{-1}(x) \neq \emptyset$ *for all* $x \in X$.

PROOF: We prove (1), (4), and (6). The others items are left to the reader.

(1) $\Rightarrow$) Let $(x, y) \in R$. Let $U = [y) \in \mathrm{Up}(X)$. As $y \in U \subseteq g_R(U)$, we get $R^{-1}(y) \subseteq U$. So, $x \in [y)$,i.e., $y \leq x$.

$\Leftarrow$) Assume that $R \subseteq \leq^{-1}$. Let $x \in U$ and let $y \in R^{-1}(x)$. So, $x \leq y$ and as $U \in \mathrm{Up}(X)$, $y \in U$. Thus, $U \subseteq g_R(U)$.

(4) $\Rightarrow$) Let $(x, y) \in R$. Suppose that $z \notin R(x)$ for all $z \in R^{-1}(y)$. Let $U = (x]^c \in \mathrm{Up}(X)$. We prove that $y \in g_R(g_R(U))$, i.e., $R^{-1}(y) \subseteq g_R(U)$. Let $w \in R^{-1}(y)$. We need to prove that $R^{-1}(w) \subseteq U = (x]^c$, i.e.,

$R^{-1}(w) \cap (x] = \emptyset$. On the contrary, we suppose that there exists $u \in R^{-1}(w)$ such that $u \leq x$. Thus, $(x, w) \in R$, which contradicts our assumption because $w \in R^{-1}(y)$. So, $y \in g_R(g_R(U)) \subseteq g_R(U)$ and consequently, $R^{-1}(y) \subseteq U = (x]^c$. Contradiction, because $x \in R^{-1}(y)$. The direction $\Leftarrow$) is easy and left to the reader.

(6) $\Rightarrow$) Let $x \in X$ and consider $U = R(x)^c$. Suppose that $R(x) \cap R^{-1}(x) = \emptyset$, then $R^{-1}(x) \subseteq R(x)^c$. So, $x \in g_R(U) \subseteq f_R(U)$, i.e., $R(x) \cap R^{-1}(x) \neq \emptyset$, which is a contradiction. Thus, $R(x) \cap R^{-1}(x) \neq \emptyset$.

$\Leftarrow$) Let $U \in \mathrm{Up}(X)$ and let $x \in g_R(U)$, i.e., $R^{-1}(x) \subseteq U$. By assumption, $R(x) \cap R^{-1}(x) \neq \emptyset$. So, there exists $y \in X$ such that $y \in R(x)$ and $y \in R^{-1}(x)$. Thus, $y \in R(x)$ and $y \in U$, and consequently, $x \in f_R(U)$. □

Theorem 3.3. *Let $A \in \mathsf{HilGC}$. Let $\langle X, \leq, R\rangle$ be the IntGC-frame of A. Then,*

1. $A \models a \leq g(a)$ *iff* $A \models f(a) \leq a$ *iff* $R \subseteq \subseteq^{-1}$.
2. $A \models g(a) \leq a$ *iff* $A \models a \leq f(a)$ *iff R is reflexive.*
3. $A \models g(a) \leq g^2(a)$ *iff* $A \models f^2(a) \leq f(a)$ *iff R is transitive.*
4. $A \models g^2(a) \leq g(a)$ *iff* $A \models f(a) \leq f^2(a)$ *iff R is weakly dense, i.e.,* $R \subseteq R^2$.
5. $A \models f(a) \leq g(a)$ *iff* $A \models a \leq g^2(a)$ *iff* $A \models f^2(a) \leq a$ *iff*
$$\forall x \forall y \forall z\, ((x, y) \in R \wedge (y, z) \in R \Rightarrow z \subseteq x).$$
6. $A \models g(a) \leq f(a)$ *iff* $R(x) \cap R^{-1}(x) \neq \emptyset$ *for all* $x \in X$.

Proof: We will prove only the assertions (2), (4) and (6). The other proofs are analogous.

(2) Assume that $g(a) \leq a$, for all $a \in A$. In particular, for $f(a) \in A$: $g(f(a)) \leq f(a)$ and by (HilGC4),

$$a \leq g(f(a)) \leq f(a).$$

Similarly, we prove that $A \models a \leq f(a)$ implies $A \models g(a) \leq a$.

To prove that R is reflexive showing that $g^{-1}(x) \subseteq x$, for every $x \in X$. Let $a \in g^{-1}(x)$, then $g\,(a) \in x$. By assumption, $a \in x$. Conversely, suppose that there exists $a \in A$ such that $g(a) \nleq a$. So, there exists $x \in X$ such that $g(a) \in x$ and $a \notin x$. That is, $g^{-1}(x) \nsubseteq x$ or equivalently, $(x, x) \notin R$.

(4) Assume that $g^2(a) \leq g(a)$ for all $a \in A$. Let $x, y \in X$ such that $(x, y) \in R$, i.e., $g^{-1}(y) \subseteq x$. By Proposition 2.8, $g^{-1}(y) \in \mathrm{Fi}(A)$ and $(g(x^c)] \in \mathrm{Ido}(A)$. Suppose that there exists $a \in g^{-1}(y) \cap (g(x^c)]$. So, $g(a) \in y$ and there exists $b \notin x$ such that $a \leq g(b)$. Thus, $g(a) \leq g^2(b) \leq g(b)$ and consequently, $g(b) \in y$, i.e., $b \in g^{-1}(y)$. By assumption, $b \in x$, which is impossible. So, $g^{-1}(y) \cap (g(x^c)] = \emptyset$ and by Theorem 2.3, there exists $z \in X$ such that $g^{-1}(y) \subseteq z$ and $z \cap (g(x^c)] = \emptyset$. Consequently, $g^{-1}(z) \cap g^{-1}(g(x^c)) = \emptyset$ and as $x^c \subseteq g^{-1}(g(x^c))$, we get $g^{-1}(z) \cap x^c = \emptyset$, i.e., $g^{-1}(z) \subseteq x$. Thus, we have that there exists $z \in X$ such that $(x, z) \in R$ and $(z, y) \in R$, this is, $(x, y) \in R^2$.

Conversely, suppose that there exists $a \in A$ such that $g^2(a) \nleq g(a)$. So, there exists $y \in X(A)$ such that $g^2(a) \in y$ and $g(a) \notin y$. By Lemma 2.9, there exists $x \in X$ such that $(x, y) \in R$ and $a \notin x$. By assumption, $(x, y) \in R^2$, this is, there exists $z \in A$ such that $(x, z) \in R$ and $(z, y) \in R$. So, $g^{-1}(z) \subseteq x$ and $g^{-1}(y) \subseteq z$. As $a \notin x$, we get that $g^2(a) \notin y$, which is a contradiction.

Now, assume that $f(a) \leq f^2(a)$ for all $a \in A$. Let $x, y \in X$ such that $(x, y) \in R$. We will prove that the implicative filter $g^{-1}(y)$ and the order-ideal $f^{-1}(x^c)$ of A are disjoints. On the contrary, suppose that there exists $a \in A$ such that $a \in g^{-1}(y)$ and $a \in f^{-1}(x^c)$, that is, $g(a) \in y$ and $f(a) \notin x$. As $y \subseteq f^{-1}(x)$, we have $f(g(a)) \in x$. By assumption, $f^2(g(a)) \in x$. On the other hand, by (HilGC5), $f(g(a)) \leq a$ and by monotony of f, $f^2(g(a)) \leq f(a)$. Since x is an upset of A, $f(a) \in x$, which is a contradiction. Thus, $g^{-1}(y) \cap f^{-1}(x^c) = \emptyset$ and so, there exists $z \in X$ such that $g^{-1}(y) \subseteq z$ and $f^{-1}(x^c) \cap z = \emptyset$, i.e., $z \subseteq f^{-1}(x)$, that is, $(z, y) \in R$ and $(x, z) \in R$. Conversely, suppose that there exists $a \in A$ such that $f(a) \nleq f^2(a)$. There exists $x \in X$ such that $f(a) \in x$ and $f^2(a) \notin x$. By Lemma 2.9, there exists $y \in X$ such that $(x, y) \in R$ and $a \in y$. By assumption, there exists $z \in X$ such that $(x, z) \in R$ and $(z, y) \in R$, i.e., $z \subseteq f^{-1}(x)$ and $y \subseteq f^{-1}(z)$. As $a \in y$, results $f(a) \in z$ and consequently, $f^2(a) \in x$, a contradiction.

(6) Let $g(a) \leq f(a)$ for all $a \in A$. Let $x \in X$. We will prove that $g^{-1}(x) \cap f^{-1}(x^c) = \emptyset$. Suppose the contrary. Let $a \in A$ such that $a \in g^{-1}(x)$ and $a \in f^{-1}(x^c)$. As $g(a) \in x$, by assumption we obtain $f(a) \in x$, which is impossible. Thus, there exists $y \in X$ such that $g^{-1}(x) \subseteq y$ and $y \subseteq f^{-1}(x)$. Consequently, $y \in R^{-1}(x) \cap R(x)$.

Now, assume that $R^{-1}(x) \cap R(x) \neq \emptyset$ for all $x \in X$ and suppose that there exists $a \in A$ such that $g(a) \nleq f(a)$. So, there exists $x \in X$ such that $g(a) \in x$ and $f(a) \notin x$. By assumption, there exists $y \in X$ such that $g^{-1}(x) \subseteq y$ and $y \subseteq f^{-1}(x)$. As $a \in g^{-1}(x)$, we obtain $f(a) \in x$, which is contradiction. □

We denote by V_Γ be the variety of HilGC-algebras generated by the set of equations Γ. Let us consider the set of equations . By Theorem 3.2 and Theorem 3.3 we have the following result.

THEOREM 3.4. *Any variety of HilGC-algebras V_{Γ_0} generated by a finite subset Γ_0 of the set of equations $\Gamma = \{\phi \wedge g(\phi) \approx \phi,\ \phi \wedge g(\phi) \approx g(\phi),\ g(\phi) \wedge g^2(\phi) \approx g(\phi),\ g^2(\phi) \wedge g(\phi) \approx g^2(\phi),\ f(\phi) \wedge g(\phi) \approx f(\phi), f(\phi) \wedge g(\phi) \approx g(\phi)\}$ is canonical.*

4. Congruences of HilGC-algebras

Let A be a Hilbert algebra. Let $\mathrm{Con}(A)$ be the lattice of congruences of A. It is known that the equivalence class

$$[1]_\theta = \{a \in A : (1, a) \in \theta\},$$

is an implicative filter. Moreover, if $F \in \mathrm{Fi}(A)$, then the binary relation θ_F defined by

$$(a, b) \in \theta_F \ \text{ iff } \ a \to b, b \to a \in F$$

is a congruence of A. A well-known result given by A. Diego [8] (see also [7] or [9]) ensures that $\mathrm{Con}(A)$ is isomorphic to the lattice of the implicative filters of A under inverse mappings $\theta \to [1]_\theta$ and $F \to \theta_F$.

There exists a bijective correspondence between implicative filters of a Hilbert algebra and closed subsets of the dual space of A ([4]). Let A be a Hilbert algebra and let $\langle X, \mathcal{T}\rangle$ its dual H-space. We denote by $\mathcal{C}(X)$ the lattice of closed subsets of $\langle X, \mathcal{T}\rangle$. If $F \in \mathrm{Fi}(A)$, then

$$\delta(F) = \{x \in X : F \subseteq x\} \in \mathcal{C}(X).$$

If $Y \in \mathcal{C}(X)$, then

$$\pi(Y) = \{a \in A : Y \subseteq \varphi(a)\} \in \mathrm{Fi}(A).$$

Moreover, if $Y \in \mathcal{C}(X)$ and $F \in \mathrm{Fi}(A)$ then, $\delta(\pi(Y)) = Y$ and $\pi(\delta(F)) = F$. Thus, there is a dual isomorphism between $\mathrm{Fi}(A)$ and $\mathcal{C}(X)$. Note that if $Y \in \mathcal{C}(X)$ then

$$\sigma(Y) = \left\{(a,b) \in A^2 : a \to b, b \to a \in \pi(Y)\right\}$$

is a congruence of A.

If L is a lattice, we denote by L^d the lattice with the dual order. To denote that two lattices L_1 and L_2 are isomorphic we will write $L_1 \cong L_2$. By the results given by A. Diego [8] (see also [9]) and the results given in [4], we have the following lattice isomorphisms

$$\mathrm{Con}(A) \cong \mathrm{Fi}(A) \cong \mathcal{C}(X)^d.$$

Let $A \in \mathsf{HilGC}$. An Hilbert congruence θ is called *G-congruence* if it is compatible with f and g, i.e., if $(a,b), (c,d) \in \theta$, then $(f(a), f(b)) \in \theta$, and $(g(a), g(b)) \in \theta$. We denote by $\mathrm{Con}_G(A)$ the set of all G-congruences of A.

Now, we will study the particular class of implicative filters in a HilGC-algebra A that are in bijective correspondence with its G-congruences.

Definition 4.1. Let $\langle A, f, g\rangle$ be a HilGC-algebra. Let $F \in \mathrm{Fi}(A)$. We said that F is a *Galois implicative filter, or* G-filter for short, if F satisfies the following proprieties:

(GF1) $a \in F$ implies $g(a) \in F$, i.e., $F \subseteq g^{-1}(F)$,

(GF2) $a \to b \in F$ implies $f(a) \to f(b) \in F$.

The set of all Galois implicative filters of a HilGC-algebra A ordered by inclusion will be denoted by $\mathrm{Fi}_G(A)$. It is almost trivial to prove that $\bigcap\{F_i : F_i \in \mathrm{Fi}_G(A)\} \in \mathrm{Fi}_G(A)$. Consequently, for every $S \subseteq A$ there exists the least G -filter containing S. Thus, given $S \subseteq A$, the set

$$\langle S\rangle_G = \bigcap\{F \in \mathrm{Fi}_G(A) : S \subseteq F\}$$

is called the G-filter generated by S. Note that $\langle\emptyset\rangle_G = \{1\}$ is the trivial G-filter. Moreover, since $\mathrm{Fi}_G(A)$ is closed under arbitrary intersections and contains the whole A, it is a complete lattice with respect to set inclusion whose meets coincide with set intersections and joins are G-filter generated by set unions of given G-filters.

Proposition 4.2. Let $A = \langle A, f, g\rangle$ be a HilGC-algebra. Then,

$$\mathrm{Fi}_G(A) \cong \mathrm{Con}_G(A).$$

Proof: We need to prove that $[1]_\theta \in \mathrm{Fi}_G(A)$, for each $\theta \in \mathrm{Con}_G(A)$ and that θ_F is a G-congruence of A, for each $F \in \mathrm{Fi}_G(A)$. Let $\theta \in \mathrm{Con}_G(A)$. So, $[1]_\theta \in \mathrm{Fi}(A)$. We prove that $[1]_\theta$ satisfies the conditions of Definition 4.1. Let $a, b \in A$.

(GF1) Let $a \in [1]_\theta$, i.e., $(1, a) \in \theta$. As $\theta \in \mathrm{Con}_G(A)$, $(g(1), g(a)) = (1, g(a)) \in \theta$. Thus, $g(a) \in [1]_\theta$.

(GF2) Let $a \to b \in [1]_\theta$, i.e., $(1, a \to b) \in \theta$. Thus,

$$(1 \to b, (a \to b) \to b) = (b, (a \to b) \to b) \in \theta.$$

As $\theta \in \mathrm{Con}_G(A)$, $(f(b), f((a \to b) \to b)) \in \theta$ and so,

$$(f(a) \to f(b), f(a) \to f((a \to b) \to b)) \in \theta.$$

By (HilGC2), $(f(a) \to f(b), 1) \in \theta$, that is, $f(a) \to f(b) \in [1]_\theta$.

Now, assume that $F \in \mathrm{Fi}_G(A)$. Then θ_F is a Hilbert congruence. Let $(a, b) \in \theta_F$, that is, $a \to b, b \to a \in F$. By (GF2), $f(a) \to f(b), f(b) \to f(a) \in F$ and consequently, $(f(a), f(b)) \in \theta_F$. On the other hand, by (GF1), $g(a \to b), g(b \to a) \in F$. Since g is a semi-homomorphism, $g(a \to b) \le g(a) \to g(b)$ and as F is an upset of A, we get $g(a) \to g(b) \in F$. Analogously, we have $g(b) \to g(a) \in F$ and so, $(g(a), g(b)) \in \theta_F$. Thus, θ_F is an G-congruence of A. □

4.1. *G*-closed

Now we are going to prove that the lattice of G-filters of a HilGC-algebra $\langle A, f, g\rangle$ is dually isomorphic to the lattice of certain closed sets of the dual space of $\langle A, f, g\rangle$.

Let X be a set and R a binary relation defined on X. Let Y be a subset of X. Let $R^{-1}(Y) = \bigcup \{R^{-1}(y) : y \in Y\}$.

We recall that if $\langle X, \mathcal{T}_\mathcal{K}\rangle$ is an H-space, then every closed subset of X is an upset of X, i.e., for $Y \in \mathcal{C}(X)$ we have that $x \le y$ and $x \in Y$ implies $y \in Y$, where we recall that the order $\le$ is given by (2.1).

Definition 4.3. Let $\langle X, \mathcal{T}_{\mathcal{K}}, R\rangle$ be an HG-space and let $Y \in \mathcal{C}(X)$. We shall say that Y is a *G-closed* if Y satisfies the following conditions:

(G1) $R^{-1}(Y) \subseteq Y$.

(G2) $\max(R(x)) \subseteq Y$, for all $x \in Y$.

The family of all G-closed subsets of an HG-space $\langle X, \mathcal{T}_{\mathcal{K}}, R\rangle$ is denoted by $\mathcal{C}_G(X)$. It is clear that X and $\emptyset$ are trivially G-closed subsets of an HG-space $\langle X, \mathcal{T}_{\mathcal{K}}, R\rangle$ and it is easy to check from the above definition that the $\mathcal{C}_G(X)$ is closed under arbitrary intersections and that the union of any finite family of G-closed subsets is again an G-closed set. So, we can conclude that the set of all G-closed subsets of $\langle X, \mathcal{T}_{\mathcal{K}}, R\rangle$ is a complete sublattice of $\mathcal{P}(X)$ which shall be denoted also by $\mathcal{C}_G(X)$.

Lemma 4.4. *Let $\langle A, f, g\rangle$ be a HilGC-algebra and let $\langle X, \mathcal{T}, R\rangle$ be its dual HG-space. Then, for all $x \in X$ and $a \in A$ such that $f(a) \in x$ there exists $z \in \max(R(x))$ such that $a \in z$*

Proof: Let $x \in X$ and let $a \in A$ such that $f(a) \in x$. We consider the following family of implicative filters of A:

$$\mathcal{F} = \left\{D \in \mathrm{Fi}(A) : D \subseteq f^{-1}(x) \text{ and } a \in D\right\}.$$

We prove that $\mathcal{F} \neq \emptyset$. As $f(a) \in x$, by Lemma 2.9 there exists $y \in X$ such that $y \subseteq f^{-1}(x)$ and $a \in y$. We see that every chain in $\mathcal{F}$ has an upper bound in $\mathcal{F}$. Let $C = \{D_i\}_{i\in I}$ be a chain of elements of $\mathcal{F}$. Consider $P = \bigcup_{i\in I} \{D_i : D_i \in C\}$. As $D_i \subseteq P$ for every $i \in I$, P is an upper bound of C. We will prove that $P \in \mathcal{F}$. As $D_i \in \mathrm{Fi}(A)$ for all $i \in I$, $1 \in D_i$ for all $i \in I$ and so, $1 \in P$. Let $b, b \to c \in P$. So, there are $i, j \in I$ such that $b \in D_i$ and $b \to c \in D_j$. Without loss of generality, we may assume that $i \leq j$ and so, that $D_i \subseteq D_j$. Thus, $b, b \to c \in D_j$ and as $D_j \in \mathrm{Fi}(A)$, $c \in D_j$. Thus, $c \in P$ and consequently, $P \in \mathrm{Fi}(A)$. On the other hand. As for all $i \in I$, we get $D_i \subseteq f^{-1}(x)$ and $a \in D_i$, so we have

$$P = \bigcup_{i\in I} \{D_i : D_i \in C\} \subseteq f^{-1}(x) \text{ and } a \in P,$$

and so, $P \in \mathcal{F}$. Thus, every chain in $\mathcal{F}$ has an upper bound in $\mathcal{F}$ and by Zorn's Lemma, there is $m \in \max(\mathcal{F})$ and so, $m \subseteq f^{-1}(x)$ and $a \in m$. Now, we shall prove that $m \in X$. Let $a, b \in A$ such that $a, b \notin m$. We consider the implicative filters $F_a = \langle m \cup \{a\}\rangle$ and $F_b = \langle m \cup \{b\}\rangle$. As m is maximal of $\mathcal{F}$ and $m \subset F_a$, we get that $F_a \nsubseteq f^{-1}(x)$. Analogously, $F_b \nsubseteq f^{-1}(x)$. So, there exist $c, d \in A$ such that $c \in F_a$, $d \in F_b$ and $c, d \notin f^{-1}(x)$. By Lemma 2.2, $a \to c, b \to d \in m$. As $f(c), f(d) \notin x$ and $x \in X$, by Corollary 2.4, there exists $k \notin x$ such that $f(c) \leq k$ and $f(d) \leq k$, or equivalently, $c \leq g(k)$ and $d \leq g(k)$. Thus, $a \to c \leq a \to g(k)$ and $b \to d \leq b \to g(k)$, and consequently, $a \to g(k), b \to g(k) \in m$. Now, we will prove that $g(k) \notin m$. On the contrary. Suppose that $g(k) \in m \subseteq f^{-1}(x)$. So, $f(g(k)) \in x$ and by (HilGC5), results $k \in x$, which is impossible. So, for $a, b \notin m$ there exists $g(k) \notin m$ such that $a \to g(k), b \to g(k) \in m$. Thus, $m \in X$ and consequently, $m \in R(x)$.

We have proved that for all implicative filters of A belonging to $\mathcal{F}$ there exists $m \in X$ such that m is a maximal element of them. In particular, we can affirm that this happens if we consider only irreducible filters. This is, for all irreducible implicative filters of A belonging to $\mathcal{F}$ there exists $m \in X$ such that m is a maximal element of them. Thus, if $f(a) \in x$ then there exists $z \in \max(R(x))$ such that $a \in z$. □

The next result gives a characterization of the G-congruences applying the duality given in [6] for the HilGC-algebras.

Proposition 4.5. Let $\langle A, f, g\rangle$ be a HilGC-algebra and let $\langle X, \mathcal{T}, R\rangle$ its dual HG-space. Then,

$$\mathcal{C}_G(X)^d \cong \mathrm{Con}_G(A).$$

Proof: Consider the map $\sigma : \mathcal{C}_G(X) \to \mathrm{Con}_G(A)$ given by:

$$\sigma(Y) = \{(a, b) \in A^2 : a \to b, b \to a \in \pi(Y)\}.$$

We recall that $\sigma(Y)$ is a Hilbert congruence. Let $a, b \in A$ such that $(a, b) \in \sigma(Y)$, i.e., $a \to b, b \to a \in \pi(Y)$, i.e., $Y \subseteq \varphi(a \to b) = \varphi(a) \Rightarrow \varphi(b)$ and $Y \subseteq \varphi(b) \Rightarrow \varphi(a)$. We prove that $(f(a), f(b)) \in \sigma(Y)$, i.e.,

$$Y \subseteq \varphi(f(a)) \Rightarrow \varphi(f(b)) \text{ and } Y \subseteq \varphi(f(b)) \Rightarrow \varphi(f(a)).$$

First, we take $x \in Y$ and we will show that $[x) \cap \varphi(f(a)) \subseteq \varphi(f(b))$. Let $y \in X$ such that $y \in [x) \cap \varphi(f(a))$. So, $x \subseteq y$ and $f(a) \in y$. Since $f(a) \in y$, by Lemma 4.4, there exists $z \in \max(R(y))$ such that $a \in z$. On the other hand, as $Y \in \mathcal{C}(X)$, Y is an upset of X and so, $y \in Y$. Consequently, $\max(R(y)) \subseteq Y$ because $Y \in \mathcal{C}_G(X)$. Thus, $z \in Y \subseteq \varphi(a) \Rightarrow \varphi(b)$, that is, $[z) \cap \varphi(a) \subseteq \varphi(b)$. As $z \in [z) \cap \varphi(a)$, we obtain $z \in \varphi(b)$ and so, $f(b) \in y$. We have proved that $Y \subseteq \varphi(f(a)) \Rightarrow \varphi(f(b))$. By a similar argument we can prove that $Y \subseteq \varphi(f(b)) \Rightarrow \varphi(f(a))$.

To prove that $(g(a), g(b)) \in \sigma(Y)$, we show that

$$Y \subseteq \varphi(g(a)) \Rightarrow \varphi(g(b)) \text{ and } Y \subseteq \varphi(g(b)) \Rightarrow \varphi(g(a)).$$

Suppose that $Y \nsubseteq \varphi(g(a)) \Rightarrow \varphi(g(b))$. So, there exists $x \in Y$ such that $x \notin \varphi(g(a)) \Rightarrow \varphi(g(b))$, i.e., $[x) \cap \varphi(g(a)) \nsubseteq \varphi(g(b))$. Hence, there exists $z \in X$ such that $z \in [x) \cap \varphi(g(a))$ and $z \notin \varphi(g(b))$. As $g(b) \notin z$, by Lemma 2.9, there exists $w \in X$ such that $g^{-1}(z) \subseteq w$ and $b \notin w$. As $g(a) \in z$, we have that $a \in w$, i.e., $w \in \varphi(a)$. Moreover, since $x \in Y$ and $Y \in \mathrm{Up}(X)$, we have $z \in Y$. Thus, $w \in R^{-1}(Y)$ and as $Y \in \mathcal{C}_G(X)$, $w \in Y$. By assumption, $Y \subseteq \varphi(a) \Rightarrow \varphi(b)$, and so, $[w) \cap \varphi(a) \subseteq \varphi(b)$. Since $w \in [w) \cap \varphi(a)$, we have $w \in \varphi(b)$, which is a contradiction. Then, we have proved that $Y \subseteq \varphi(g(a)) \Rightarrow \varphi(g(b))$. Analogously, we prove that $Y \subseteq \varphi(g(b)) \Rightarrow \varphi(g(a))$. Thus, $\sigma(Y) \in \mathrm{Con}_G(A)$, for each $Y \in \mathcal{C}_G(X)$ and consequently σ is well defined.

Let $Y, W \in \mathcal{C}_G(X)$. It is clear that if $Y \subseteq W$ then $\pi(W) \subseteq \pi(Y)$ and consequently, $\sigma(W) \subseteq \sigma(Y)$. To prove that σ is one-to-one, assume that $\sigma(W) = \sigma(Y)$ and suppose that $Y \neq W$. Without loss of generality, we assume that $Y \nsubseteq W$, i.e., there exists $x \in Y$ such that $x \notin W$. As W is a closed subset of $\langle X, \mathcal{T}\rangle$, there exists $a \in A$ such that $W \subseteq \varphi(a)$ and $x \notin \varphi(a)$. Thus, $a = 1 \to a, 1 = a \to 1 \in \pi(W)$ and consequently, $(1, a) \in \sigma(W) = \sigma(Y)$. So, $a \to 1, 1 \to a \in \pi(Y)$. Thus $a \in \pi(Y)$ and so, $Y \subseteq \varphi(a)$. As $x \in Y$, we have $a \in x$, which is a contradiction.

It remains to prove that σ is onto. Let $\theta \in \mathrm{Con}_G(A)$. We recall that $Y = \delta([1]_\theta) \in \mathcal{C}(X)$. We prove that Y satisfies the two conditions of Definition 4.3.

(G1) Let $x \in X$ such that $x \in R^{-1}(Y)$. So, there exists $y \in Y$ such that $(x, y) \in R$, that it, $[1]_\theta \subseteq y$ and $g^{-1}(y) \subseteq x$. We prove that $x \in Y$ showing that $[1]_\theta \subseteq x$. Let $a \in [1]_\theta$. As $\theta \in \mathrm{Con}_G(A)$, by Proposition 4.2, $[1]_\theta \in \mathrm{Fi}_G(A)$ and thus, $g(a) \in [1]_\theta \subseteq y$. Consequently, $a \in x$. We have proved that $R^{-1}(Y) \subseteq Y$.

(G2) Suppose that there exists $x \in Y$ such that $\max(R(x)) \nsubseteq Y$. So, there exist $x, z \in X$ such that $[1]_\theta \subseteq x$, $z \in \max(R(x))$ and $z \notin Y$, i.e., $[1]_\theta \nsubseteq z$. So, there exists $a \in A$ such that $a \in [1]_\theta$ and $a \notin z$. We consider the ideal $f^{-1}(x^c)$ and the implicative filter $\langle z \cup \{a\}\rangle$ of A, and we will prove that there are disjoint. Conversely, suppose that there exists $b \in A$ such that $b \in \langle z \cup \{a\} \rangle \cap f^{-1}(x^c)$. By Lemma 2.2, $a \to b \in z$ and $f(b) \notin x$. By the assumption, $z \in R(x)$, i.e., $z \subseteq f^{-1}(x)$ and so, $f(a \to b) \in x$. On the other hand, as $(1, a) \in \theta$, we obtain $(b, a \to b) \in \theta$. As $\theta \in \mathrm{Con}_G(A)$, we have $(f(b), f(a \to b)) \in \theta$ and so, $(1, f(a \to b) \to f(b)) \in \theta$. Thus, $f(a \to b) \to f(b) \in [1]_\theta \subseteq x$ and since $f(a \to b) \in x$, we get $f(b) \in x$, which is a contradiction. Thus, $\langle z \cup \{a\} \rangle \cap f^{-1}(x^c) = \emptyset$ and consequently, there exists $y \in X$ such that $z \subseteq y$, $a \in y$ and $y \cap f^{-1}(x^c) = \emptyset$, that is, $y \subseteq f^{-1}(x)$, i.e., $y \in R(x)$. As $z \in \max(R(x))$ results $y = z$. Thus, $a \in z$, which contradicts our assumption. So, $\max(R(x)) \subseteq Y$ for all $x \in Y$.

Finally, we prove that $\sigma(Y) = \theta$. Let $a, b \in A$. Then,

$$\begin{aligned}(a,b) \in \sigma(Y) \quad &\text{iff} \quad a \to b, b \to a \in \pi(Y) = \pi(\delta([1]_\theta)) = [1 \\ &\text{iff} \quad (a,b) \in \theta_{[1]_\theta} = \theta\end{aligned}$$ □

By Propositions 4.2 and 4.5, we have the following result.

Corollary 4.6. Let $\langle A, f, g \rangle$ be a HilGC-algebra and let let $\langle X, \mathcal{T}, R \rangle$ its dual HG-space. Then,

$$\mathrm{Con}_G(A) \cong \mathrm{Fi}_G(A) \cong \mathcal{C}_G(X)^d.$$

Simple and subdirectly irreducibles algebras

We are going to apply the topological characterization of the G-congruences to give a characterization of the simple algebras and subdirectly irreducible algebras.

Let us recall that an algebra A is *subdirectly irreducible* if and only if there exists the smallest non trivial congruence relation θ in A. A particular case are the simple algebras, A is *simple* if and only if A has only two congruence relations.

Let $\langle A, f, g \rangle$ be a HilGC-algebra and let $\langle X, \mathcal{T}, R \rangle$ its dual space. By Propositions 4.5 and 4.2, we can affirm that a HilGC-algebra $\langle A, f, g \rangle$ is subdirectly irreducible if and only if there exists the smallest non-trivial Galois implicative filter of A iff in the dual HG-space $\langle X, \mathcal{T}, R \rangle$ there exists the largest $Y \in \mathcal{C}_G(X) - \{X, \emptyset\}$. Moreover, $\langle A, f, g \rangle$ is simple iff $\mathrm{Fi}_G(A) = \{\{1\}, A\}$ iff $\mathcal{C}_G(X) = \{\emptyset, X\}$.

Let $\langle X, \mathcal{T}, R\rangle$ be an HG-space. As the family $\mathcal{C}_G(X)$ is closed under arbitrary intersections, we can define for each $x \in X$ the set

$$Y_x = \bigcap \{Y \in \mathcal{C}_G(X) : x \in Y\} \in \mathcal{C}_G(X).$$

Note that Y_x is the smallest G-closed set containing the element x.

Now, we can characterize the simple and subdirectly irreducible HilGC-algebras.

THEOREM 4.7. *Let A be a HilGC-algebra and $\langle X, \mathcal{T}, R\rangle$ its dual HG-space. Then:*

1. *A is simple iff $Y_x = X$, for each $x \in X$.*
2. *A is subdirectly irreducible iff $\{x \in X : Y_x \neq X\} \in \mathcal{C}_G(X) - \{X\}$.*

PROOF: (1) Assume that A is simple. So, $\mathcal{C}_G(X) = \{\emptyset, X\}$. Let $x \in X$. As $x \in Y_x$, $Y_x \neq \emptyset$ and since $Y_x \in \mathcal{C}_G(X)$, we have that $Y_x = X$. Reciprocally. Let $Z \in \mathcal{C}_G(X)$ and suppose that $Z \neq \emptyset$. So, there exists $x \in X$ such that $x \in Z$. Thus, $X = Y_x \subseteq Z \subseteq X$. So, $Z = X$ and consequently, A is simple.

(2) Consider the set

$$W = \{x \in X : Y_x \neq X\}.$$

Assume that A is subdirectly irreducible and let V be the largest element of $\mathcal{C}_G(X) - \{X\}$. We will prove that $V = W$. Let $x \in X$ such that $x \in V$. As Y_x is the smallest G-closed set containing the element x, $Y_x \subseteq V \neq X$ and hence, $x \in W$. To prove the other inclusion, we take $x \in W$, i.e., $Y_x \neq X$. Thus, $Y_x \in \mathcal{C}_G(X) - \{X\}$, and so, $Y_x \subseteq V$. As $x \in Y_x$, we obtain $x \in V$. Thus, $W = V \in \mathcal{C}_G(X) - \{X\}$.

Reciprocally, assume that $W \in \mathcal{C}_G(X) - \{X\}$. We will prove that W is the largest element of $\mathcal{C}_G(X) - \{X\}$. Suppose that there exists $Z \in \mathcal{C}_G(X)$ such that $Z \nsubseteq W$. So, there exists $x \in Z$ such that $x \notin W$, this is, $Y_x = X$. Thus, $X = Y_x \subseteq Z$ and so, $Z = X$. □

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