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SOME ADDITIONAL AXIOMS FOR T-NORMAL LOGICS. DEFINING K45, KB4, KD45 AND S5 WITHOUT USING MODAL RULES

Abstract

The paper studies extensions of t-normal logics $\mathbf{S0.5}^{\circ}$ and **S0.5** obtained by means of some axioms of normal logics. We will prove determination theorems for these extensions by appropriate Kripke-style models. It will allow us to obtain the determinations of the logics **K45**, **KB4** (= **KB5**), **KD45** and **S5** without using modal rules.

Keywords: modal logic, t-normal logics, Kripke-style semantics.

Introduction

The definition of modal t-normal logics differs from the definition of normal logics in that we only take the necessity of classical tautologies instead of the rule of necessitation. The first such logic, **S0.5**, was defined by E. J. Lemmon in [3]. The smallest t-normal logic, $\mathbf{S0.5}^{\circ}$, was studied by R. Routley in [8]. In [5, 6, 7], we explored various types of t-normal logics and their location in the lattice of modal logics. The Lemmon's logic **S0.5** is the extension of $\mathbf{S0.5}^{\circ}$ by the following formula:

$$\Box p \supset p \tag{T}$$

The following formulas are theses of **S0.5**:

$$p \supset \Diamond p \tag{$\mathtt{T_d}$}$$

$$\Box p \supset \Diamond p \tag{D}$$

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This paper studies extensions of t-normal logics $\mathbf{S0.5}^{\circ}$ and $\mathbf{S0.5}$ using axioms known from normal logics: (D), (T) and the following[1]

$$\Box p \supset \Box\Box p \tag{4}$$

$$\Diamond\Diamond p \supset \Diamond p \tag{$4_{\mathtt{d}}$}$$

$$p \supset \Box\Diamond p \tag{B}$$

$$\Diamond\Box p \supset p \tag{$\mathtt{B}_{\mathtt{d}}$}$$

$$\Diamond p \supset \Box\Diamond p \tag{5}$$

$$\Diamond\Box p \supset \Box p \tag{$5_{\mathtt{d}}$}$$

It is known that dual versions are not needed in normal logics, i.e., formulas without the lower subscript 'd' are sufficient (or vice versa). For t-normal logics, the dual and non-dual versions of a given formula are independent.

As additional axioms for $\mathbf{S0.5}^{\circ}$ and $\mathbf{S0.5}$, we will also use the following formulas:

$$\Diamond\Box p \supset \Diamond\Diamond p \tag{$\mathtt{D}^{\mathtt{m}}$}$$

$$\Box\Box p \supset \Box\Diamond p \tag{$\mathtt{D}^{\mathtt{m}}_{\mathtt{d}}$}$$

$$\Diamond\Box p \supset \Diamond p \tag{$\mathtt{T}^{\mathtt{m}}$}$$

$$\Box p \supset \Box\Diamond p \tag{$\mathtt{T}^{\mathtt{m}}_{\mathtt{d}}$}$$

The names of the above formulas say that we obtain them from (D), (T) and ($\mathtt{T}_{\mathtt{d}}$), respectively, through the monotonicity rule and duality used for normal logics. So ($\mathtt{T}^{\mathtt{m}}$), ($\mathtt{T}^{\mathtt{m}}_{\mathtt{d}}$) $\in$ **KT** and ($\mathtt{D}^{\mathtt{m}}$), ($\mathtt{D}^{\mathtt{m}}_{\mathtt{d}}$) $\in$ **KD** $\subsetneq$ **KT**. These formulas are independent for t-normal logics.

Section 1 provides the necessary facts about modal logic. Following [4], we write that the normal logics **K45**, **KB4** (= **KB5**) and **KD45** are determined by the suitable classes of simplified Kripke-style models (which refers to the known fact that the class of universal Kripke models determines the logic **S5**). We end this section with a definition of t-normal modal logics, distinguishing very weak t-normal logics as those that are not closed under the replacement of tautological equivalents. We will notice that, unlike for normal logics, there is a significant difference between t-normal logics that are built in the set of formulas with two primary modal connectives '$\Box$' and '$\Diamond$' and that are built in the set with only the first of them (i.e. $\Diamond := \neg\Box\neg$).

[1] In [5, 6, 7] were explored various kinds of t-normal logics with additional axioms from sets $\Box\Phi$, where $\Phi \subseteq \mathbf{S0.5}$.

In Section 2, we present a syntactic and semantic analysis of four basic very weak t-normal logics: $\mathbf{S0.5}^\circ$, $\mathbf{S0.5}^\circ[\mathtt{D}]$, $\mathbf{S0.5}^\circ[\mathtt{T^q}]$, $\mathbf{S0.5}$. Unlike previous papers [5, 6, 7], this research will be presented in the set For, i.e., with two primitive modal connectives: '$\Box$' and '$\Diamond$'. We will use specific examples to show the difference that occurs when these logics are built in the set $\mathrm{For}_\Box$. Furthermore, following [5], in the Appendix, we will present an analysis of canonical models and completeness theorems for $\mathbf{S0.5}^\circ$, $\mathbf{S0.5}^\circ[\mathtt{D}]$, $\mathbf{S0.5}^\circ[\mathtt{T^q}]$ and $\mathbf{S0.5}$ built-in For with respect to suitable classes of Kripke-style models. This will also be used in the next section, where we analyze extensions of these logics with additional axioms.

In Section 3, we explore other t-normal logics with additional axioms, which we provided on page 512. For these logics, we give determination theorems with respect to the suitable classes of Kripke-style models. Thanks to this, we find the dependencies between the considered extensions of $\mathbf{S0.5}$ and $\mathbf{S0.5}^\circ$. We also provide what the equivalents of these logics in the set $\mathrm{For}_\Box$ would look like.

In [4] for the logics **K45**, **KB4** (= **KB5**) and **KD45** are given the determination theorems by suitable classes of simplified Kripke-style models. Using these theorems, the determination of the logic **S5** by the class of universal Kripke models, and the facts obtained in Section 3, in Section 4 we will prove that $\mathbf{K45} = \mathbf{S0.5}^\circ[\mathtt{4},\mathtt{4_d},\mathtt{5},\mathtt{5_d}]$, $\mathbf{KB4} = \mathbf{S0.5}^\circ[\mathtt{B},\mathtt{4},\mathtt{4_d},\mathtt{5},\mathtt{5_d}]$, $\mathbf{KD45} = \mathbf{S0.5}^\circ[\mathtt{D},\mathtt{4},\mathtt{4_d},\mathtt{5},\mathtt{5_d}]$ and $\mathbf{S5} = \mathbf{S0.5}^\circ[\mathtt{T},\mathtt{4},\mathtt{4_d},\mathtt{5},\mathtt{5_d}]$. Thus, we will show that these normal logics are definable without modal rules.

1. Normal and t-normal modal logics

1.1. Formulas, PL-tautologies and modal logics

Formulas. Modal propositional formulas with two modal connectives are built in the standard way from propositional letters (or *atoms*) from the set $\mathrm{At} := \{p, q, p_1, p_2, p_3, \ldots\}$), the Boolean propositional connectives '$\neg$', '$\vee$', '$\wedge$', '$\supset$' and '$\equiv$' (for negation, conjunction, disjunction, and material implication and equivalence, respectively) the modal connectives '$\Box$' ('It is necessary that') and '$\Diamond$' ('It is possible that'), and brackets. Let For be the set of all modal propositional formulas.

Often, modal logics are examined in a set $\mathrm{For}_\Box$ of formulas built in a standard way without using the possibility sign '$\Diamond$'. This sign is just an abbreviation for '$\neg\Box\neg$'. Of course, $\mathrm{For}_{\Box} \subsetneq \mathrm{For}$. In both case, we put $\Box\Phi := \{\Box\varphi : \varphi \in \Phi\}$ for any subset Φ of formulas.

Moreover, let $\mathrm{For}_{\mathrm{cl}}$ be the set of all classical propositional formulas built without modal connectives.

PL-tautologies. Let $\mathrm{Taut}_{\mathrm{cl}}$ be the set of all tautologies from $\mathrm{For}_{\mathrm{cl}}$ and **PL** be the set of all their instances from For, which we will call *PL-tautologies*. Following [1], we say that a formula is *propositionally atomic* iff it is either atomic in the ordinary sense (i.e., it belongs to At) or modal (i.e., it has the form $\ulcorner\Box\varphi\urcorner$ or $\ulcorner\Diamond\varphi\urcorner$). Let PAt be the set of all propositionally atomic formulas. Moreover, let $\mathsf{Val}^{\mathsf{cl}}$ be the set of all valuations $V\colon \mathrm{For} \to \{0,1\}$ which preserve classical conditions for Boolean connectives. Of course, $V \in \mathsf{Val}^{\mathsf{cl}}$ iff for some assignment $v\colon \mathrm{PAt} \to \{0,1\}$, V is the unique extension of v by classical truth conditions for Boolean connectives. It is obvious:

LEMMA 1.1. *For any $\varphi \in \mathrm{For}$: $\varphi \in \mathbf{PL}$ iff $V(\varphi) = 1$ for any $V \in \mathsf{Val}^{\mathsf{cl}}$.*

A subset Ψ of For is *PL-consistent* iff that there is a $V \in \mathsf{Val}^{\mathsf{cl}}$ such that $V[\Psi] = \{1\}$. Moreover, for $\varphi \in \mathrm{For}$, we put $\Psi \models_{\mathbf{PL}} \varphi$ iff the set $\Psi \cup \{\neg\varphi\}$ is not PL-consistent. We have: $\Psi \models_{\mathbf{PL}} \varphi$ iff either $\varphi \in \mathbf{PL}$ or there are $n > 0$, $\psi_1, \dots, \psi_n \in \Psi$ such that $\ulcorner(\psi_1 \wedge \cdots \wedge \psi_n) \supset \varphi\urcorner \in \mathbf{PL}$. So $\emptyset \models_{\mathbf{PL}} \varphi$ iff $\varphi \in \mathbf{PL}$.

Modal logics. Following [1, p. 46], we say that a subset $\boldsymbol{L}$ of For is a *modal logic* iff $\boldsymbol{L}$ is closed under uniform substitution and the following rule for all $\Psi \subseteq \mathrm{For}$ and $\varphi \in \mathrm{For}$:[2]

(RPL) if $\Psi \subseteq \boldsymbol{L}$ and $\Psi \models_{\mathbf{PL}} \varphi$, then $\varphi \in \boldsymbol{L}$.

So $\boldsymbol{L}$ is a modal logic iff $\boldsymbol{L}$ includes $\mathrm{Taut}_{\mathrm{cl}}$ and is closed under substitution and *detachment*, i.e., for all $\varphi, \psi \in \mathrm{For}$:

(det) if $\ulcorner\varphi \supset \psi\urcorner \in \boldsymbol{L}$ and $\varphi \in \boldsymbol{L}$, then $\psi \in \boldsymbol{L}$.

All members of $\boldsymbol{L}$ are called its *theses*. We say that $\boldsymbol{L}$ is *consistent* iff $\boldsymbol{L} \neq \mathrm{For}$.

The set **PL** is the smallest modal logic. So all modal logic include **PL**.

We say that φ is *deducible* from a subset Ψ in $\boldsymbol{L}$ (written: $\Psi \vdash_{\boldsymbol{L}} \varphi$) iff either $\varphi \in \boldsymbol{L}$ or there are $n > 0$, $\psi_1, \dots, \psi_n \in \Psi$ such that $\ulcorner(\psi_1 \wedge \cdots \wedge \psi_n) \supset \varphi\urcorner \in \boldsymbol{L}$. Notice that:

[2] In [1], Chellas considers *systems of modal logic*, which do not have to be closed under uniform substitution.

- if $\Psi \models_{\mathbf{PL}} \varphi$ then $\Psi \vdash_{\boldsymbol{L}} \varphi$.
- $\varphi \in \boldsymbol{L}$ iff $\emptyset \vdash_{\boldsymbol{L}} \varphi$ iff $\boldsymbol{L} \vdash_{\boldsymbol{L}} \varphi$.

Moreover, we say that formulas φ and ψ are ***L**-equivalent* iff both $\varphi \vdash_{\boldsymbol{L}} \psi$ and $\psi \vdash_{\boldsymbol{L}} \varphi$, i.e. $\ulcorner \varphi \equiv \psi \urcorner \in \boldsymbol{L}$.

For any $\Phi \subseteq$ For, let $\boldsymbol{L}[\Phi]$ be the smallest modal logic including $\boldsymbol{L} \cup \Phi$.

1.2. Normal modal logics

Definition. A modal logic $\boldsymbol{L}$ is *normal* iff $\boldsymbol{L}$ contains the formulas:

$$\Diamond\varphi \equiv \neg\Box\neg\varphi \tag{df$\Diamond$}$$

$$\Box(p \supset q) \supset (\Box p \supset \Box q) \tag{K}$$

and is closed under the *rule of necessitation*, i.e., for any $\varphi \in$ For:

(nec) if $\varphi \in \boldsymbol{L}$ then $\ulcorner \Box\varphi \urcorner \in \boldsymbol{L}$.

Any normal logic $\boldsymbol{L}$ includes $\Box\mathbf{PL}$ and is closed under the following rules for all $\Psi \subseteq$ For and $\varphi, \psi, \chi \in$ For:

(rk) if $\Psi \vdash_{\boldsymbol{L}} \varphi$ then $\Box\Psi \vdash_{\boldsymbol{L}} \Box\varphi$;
($\mathrm{rk_d}$) if $\Psi, \psi \vdash_{\boldsymbol{L}} \varphi$ then $\Box\Psi \cup \{\Diamond\psi\} \vdash_{\boldsymbol{L}} \Diamond\varphi$;
(cgr) if $\ulcorner \varphi \equiv \psi \urcorner \in \boldsymbol{L}$, then $\ulcorner \Box\varphi \equiv \Box\psi \urcorner \in \boldsymbol{L}$;
(rep) $\ulcorner \varphi \equiv \psi \urcorner \in \boldsymbol{L}$, then $\ulcorner \chi \equiv \chi[{}^{\varphi}/\!/_{\psi}] \urcorner \in \boldsymbol{L}$.

where $\chi[{}^{\varphi}/\!/_{\psi}]$ is any formula that results from χ by replacing zero or more occurrences of φ, in χ, by ψ. Hence $\boldsymbol{L}$ is also closed under *replacement of tautological equivalents* iff for all $\chi, \varphi, \psi \in$ For we have:

(rte) if $\ulcorner \varphi \equiv \psi \urcorner \in \mathbf{PL}$, then $\ulcorner \chi \equiv \chi[{}^{\varphi}/\!/_{\psi}] \urcorner \in \boldsymbol{L}$.

So the following formulas are theses of any normal logic:

$$\Box p \equiv \neg\Diamond\neg p \tag{df$\Box$}$$

$$\Box(p \supset q) \supset (\Diamond p \supset \Diamond q) \tag{$\mathrm{K_d}$}$$

$$\Box(p \wedge q) \equiv (\Box p \wedge \Box q) \tag{R}$$

$$\Diamond(p \vee q) \equiv (\Diamond p \vee \Diamond q) \tag{$\mathrm{R_d}$}$$

$$\Diamond(p \supset q) \equiv (\Box p \supset \Diamond q) \tag{$\mathrm{R'_d}$}$$

A modal logic is normal iff it is closed under (cgr) and contains (K) and $\Box\top$ (for some $\top \in \mathrm{Taut_{cl}}$).

Remark 1.2. 1. If we consider a given normal logic in the set $\mathrm{For}_{\Box}$, then ($\mathtt{df}\Diamond$) is unnecessary because it is just a shortcut on one side of the PL-tautology '$\neg\Box\neg p \equiv \neg\Box\neg p$'. Moreover, ($\mathtt{df}\Box$) is a shortcut of the thesis '$\Box p \equiv \neg\neg\Box\neg\neg p$'.

2. For normal logics, it does not matter whether we examine them in For or their versions in $\mathrm{For}_{\Box}$. Namely, assume that for any formula φ of For, the formula $\varphi^{\Box}$ from $\mathrm{For}_{\Box}$ is its copy created by replacing each occurrence of '$\Diamond$' with '$\neg\Box\neg$'. Then φ is a thesis of a normal logic $\boldsymbol{L}$ iff $\varphi^{\Box}$ is its thesis in the $\mathrm{For}_{\Box}$-version denoted by $\boldsymbol{L}_{\Box}$. Moreover, $\boldsymbol{L}_{\Box} = \boldsymbol{L} \cap \mathrm{For}_{\Box}$ and $\boldsymbol{L}_{\Box} \subsetneq \boldsymbol{L}$. □

Selected normal logics. The smallest normal logic is denoted by **K**. Other known normal logics are build using (D), (T), (4), (B), (5) and the following:

$$\Diamond\top \supset (\mathtt{T}), \tag{$\mathtt{T^q}$}$$

where $\top$ is an arbitrary tautology of propositional classical logic.[3] Using the names of the above formulas, to simplify the naming of normal logics, we write $\mathbf{KX}_1 \ldots \mathbf{X}_n$ to denote the smallest normal logic containing formulas ($\mathtt{X}_1$), ..., ($\mathtt{X}_n$). We put $\mathbf{S5} := \mathbf{KT5}$ and $\mathbf{S4} := \mathbf{KT4}$. Since $\ulcorner\Diamond\top\urcorner \in \mathbf{KD}$, we have $\mathbf{KT} = \mathbf{KDT^q}$. Moreover, $\mathbf{KT^q} \subsetneq \mathbf{K4T^q} \subsetneq \mathbf{KB4} = \mathbf{KB5} = \mathbf{K5T^q} \subsetneq \mathbf{S5}$, $\mathbf{KBT^q} \subseteq \mathbf{KB4}$, $\mathbf{S5} = \mathbf{KTB4} = \mathbf{KDB4} = \mathbf{KDB5} = \mathbf{KD5T^q}$, $\mathbf{KD} \subsetneq \mathbf{KT} \subsetneq \mathbf{S4} \subsetneq \mathbf{S5}$, $\mathbf{KT^q} \subsetneq \mathbf{KT} \subsetneq \mathbf{KTB}$, $\mathbf{KB} \subsetneq \mathbf{KTB}$, $\mathbf{K4} \subsetneq \mathbf{K45} \subsetneq \mathbf{KB4}$ and $\mathbf{K5} \subsetneq \mathbf{K45} \subsetneq \mathbf{KD45} \subsetneq \mathbf{S5}$.

Simplified Kripke-style semantics for K45, KB4, KD45 and S5. Following [4], for logics **K45**, **KB4** (= **KB5**) and **KD45** – instead of relational Kripke models – we can use *simplified models* of the form $\langle W, A, V\rangle$, where W is a non-empty set of worlds, $A \subseteq W$ (A is a set of *common alternatives* to all worlds from W), and V is a *valuation* as a function $V\colon \mathrm{For} \times W \to \{0,1\}$ which for any $x \in W$ gives $V(\cdot, x) \in \mathsf{Val}^{\mathsf{cl}}$ and, moreover, for any $\varphi \in \mathrm{For}$ we have:

($V_{\Box}$) $V(\Box\varphi, x) = 1$ iff for each $y \in A$ we have $V(\varphi, y) = 1$;
($V_{\Diamond}$) $V(\Diamond\varphi, x) = 1$ iff for some $y \in A$ we have $V(\varphi, y) = 1$.

[3] The name '$\mathtt{T^q}$' is an abbreviation for 'quasi-$\mathtt{T}$', because ($\mathtt{T}$) and ($\mathtt{T^q}$) are valid in all reflexive and quasi-reflexive Kripke frames, respectively. In a given quasi-reflexive Kripke frame, an *accessibility* relation R on a set W of worlds satisfies (see [1, p. 92, Exercise 3.51]): $\forall_{x\in W}(\exists_{y\in W}\, x\,R\,y \Rightarrow x\,R\,x)$.

We say that a simplified model $\langle W, A, V\rangle$ is *universal* (resp. *empty*, *non-empty*) iff $A = W$ (resp. $A = \emptyset$, $A \neq \emptyset$). Of course, a universal model $\langle W, W, V\rangle$ can be simplified to $\langle W, V\rangle$.[4] Commonly, such universal models are applied to **S5**.

We say that a formula φ is *true* in a model $\langle W, A, V\rangle$ iff $V(\varphi, x) = 1$ for each $x \in W$. We say that a formula is *valid* in a class $\boldsymbol{M}$ of models iff it is true in all models from $\boldsymbol{M}$. A class $\boldsymbol{M}$ *determines* a given logic if its theses are all those and only those formulas valid in $\boldsymbol{M}$.

The following fact is known:

THEOREM 1.3 ([1]). **S5** *is determined by the class of all universal models.*

Moreover, we have (see [4, Theorem 1.1]):

THEOREM 1.4. 1. **K45** *is determined by the class of all simplified models.*

2. **KB4** *is determined by the class of empty or universal models.*
3. **KD45** *is determined by the class of non-empty simplified models.*

1.3. T-normal modal logics

Definition. Following [5], a modal logic is *t-normal* iff it includes the set $\Box\mathrm{Taut}_{\mathrm{cl}}$ and contains (df◇), (K). Every t-normal logic also includes □**PL** and contains (df□), ($\mathrm{K_d}$), (R), ($\mathrm{R_d}$), ($\mathrm{R'_d}$). All normal logics are t-normal.[5] Every modal logic that extends a given t-normal logic is also t-normal.

Let $\boldsymbol{L}$ be a t-normal logic. Using □**PL**, (K), ($\mathrm{K_d}$), (R), ($\mathrm{R_d}$), we obtain:

(pk) if $\Psi \models_{\mathbf{PL}} \varphi$ then $\Box\Psi \vdash_{\boldsymbol{L}} \Box\varphi$;

($\mathrm{pk_d}$) if $\Psi, \psi \models_{\mathbf{PL}} \varphi$ then $\Box\Psi \cup \{\Diamond\psi\} \vdash_{\boldsymbol{L}} \Diamond\varphi$.

(pe) if $\ulcorner\varphi \equiv \psi\urcorner \in \mathbf{PL}$ then $\ulcorner\Box\varphi \equiv \Box\psi\urcorner \in \boldsymbol{L}$;

($\mathrm{pe_d}$) if $\ulcorner\varphi \equiv \psi\urcorner \in \mathbf{PL}$ then $\ulcorner\Diamond\varphi \equiv \Diamond\psi\urcorner \in \boldsymbol{L}$.

Remark 1.5. As for normal logics, if we consider a given t-normal logic in $\mathrm{For}_{\Box}$, then (df◇) is unnecessary (see Remark 1.2(1)). Also (df□) is a shortcut of the thesis '$\Box p \equiv \neg\neg\Box\neg\neg p$'. But, there may be some confusion regarding the two approaches to t-normal logics. We will show differences between both approaches in Section 2.4. □

[4] A universal model $\langle W, A\rangle$ also corresponds to the following relational model $\langle W, W \times W, V\rangle$ with the universal relation $R = W \times W$ *accessibility of worlds.*

[5] The term 't-normal' means that the rule of necessity from normal logics is limited to PL-tautologies, i.e., we have only $\Box\mathbf{PL} \subseteq \boldsymbol{L}$ instead of the rule (nec).

Very weak t-normal logics. If a t-normal logic is not closed under (rte), it will be called *very weak t-normal* (briefly: *vwt-normal*). In this paper, we will deal with such logics.[6]

2. The first four t-normal logics

2.1. Definitions and basic properties

Following [8], we denote the smallest t-normal logic by $\mathbf{S0.5}^{\circ}$. Following [3], by $\mathbf{S0.5}$, we denote the smallest t-normal logic containing (T). We have $\mathbf{S0.5} = \mathbf{S0.5}^{\circ}[\mathtt{T}]$; so the sign '$\mathbf{S0.5}^{\circ}$' means: $\mathbf{S0.5}$ without (T).

Notice that ($\mathtt{T}^{\mathtt{q}}$) is $\mathbf{S0.5}^{\circ}$-equivalent to each of the following formulas:

$$\Box p \supset (p \vee \Box q)$$
$$\Diamond q \supset (\Box p \supset p)$$
$$(\mathtt{D}) \supset (\mathtt{T})$$

Formulas (D) and $\ulcorner \Diamond \top \urcorner$ are $\mathbf{S0.5}^{\circ}$-equivalent. They and ($\mathtt{T_d}$) belong to $\mathbf{S0.5}$. We have $\mathbf{S0.5}^{\circ}[\mathtt{D}] \subsetneq \mathbf{S0.5}$, $\mathbf{S0.5}^{\circ}[\mathtt{T}^{\mathtt{q}}] \subsetneq \mathbf{S0.5}$ and $\mathbf{S0.5} = \mathbf{S0.5}^{\circ}[\mathtt{D}, \mathtt{T}^{\mathtt{q}}]$.

Remark 2.1. Lemmon [3] and Routley [8] investigated $\mathbf{S0.5}$ and $\mathbf{S0.5}^{\circ}$, respectively, in the set $\mathrm{For}_{\Box}$ (see Remark 1.5). The such version of $\mathbf{S0.5}$ was also presented in [2]. Moreover, the versions of $\mathbf{S0.5}^{\circ}$, $\mathbf{S0.5}^{\circ}[\mathtt{D}]$, $\mathbf{S0.5}^{\circ}[\mathtt{T}^{\mathtt{q}}]$ and $\mathbf{S0.5}$ in $\mathrm{For}_{\Box}$ was studied in [5, 6, 7]. □

2.2. Kripke-style semantics for $\mathbf{S0.5}^{\circ}$ and S0.5. Soundness and completeness

Let w be any object and A be any set. A *t-normal Kripke-style model* (briefly: *tn-model*) is any triple $\langle w, A, V\rangle$ such that V is a *valuation* as a function $V\colon \mathrm{For} \times (\{w\} \cup A) \to \{0, 1\}$ which for any $x \in A \cup \{w\}$ gives $V(\cdot, x) \in \mathsf{Val}^{\mathsf{cl}}$ and for any $\varphi \in \mathrm{For}$ we have:

($V^w_\Box$) $V(\Box\varphi, w) = 1$ iff for each $x \in A$ we have $V(\varphi, x) = 1$;
($V^w_\Diamond$) $V(\Diamond\varphi, w) = 1$ iff for some $x \in A$ we have $V(\varphi, x) = 1$.

We say that w is a *distinguished world*, A is a set of *alternative worlds to* w and $\langle w, A, V\rangle$ based on w and A. Moreover, we say that a tn-model is *self-associate* (resp. *empty*, *non-empty*) iff $w \in A$ (resp. $A = \emptyset$, $A \neq \emptyset$).

[6] In [5, 6, 7] various kinds of t-normal logics closed under (rte) were studied.

We say that a formula φ is *true* (resp. *false*) in a tn-model $\langle w, A, V\rangle$ iff $V(\varphi, w) = 1$ (resp. $V(\varphi, w) = 0$). We say that a formula is *valid* in a class $\boldsymbol{M}$ of tn-models (or $\boldsymbol{M}$-*valid*) iff it is true in all models from $\boldsymbol{M}$.

The following lemma shows how tn-models can be constructed:

Lemma 2.2. *Let w be an object, A be a set, $v_w \colon \mathrm{At} \to \{0,1\}$ and $V_x \in \mathsf{Val}^{\mathsf{cl}}$ for each $x \in A\setminus\{w\}$. Then there is the unique $V \colon \mathrm{For} \times (A \cup \{w\}) \to \{0,1\}$ such that $\langle w, A, V\rangle$ is a tn-model.*

Proof: For any $\alpha \in \mathrm{At}$ we put $V(\alpha, w) := v_w(\alpha)$ and for any $\varphi \in \mathrm{PAt}$ and $x \in A \setminus \{w\}$ we put $V(\varphi, x) := V_x(\varphi)$. Using truth conditions for Boolean connectives and ($V^w_\Box$), ($V^w_\Diamond$), we uniquely extend V. □

The following facts are also obvious:

Fact 2.3.

1. The rules (RPL) and (det) preserve the truth in each tn-model.
2. All instances of formulas (K) and (df$\Diamond$), and all formulas of $\mathbf{PL} \cup \Box\mathbf{PL}$ are valid in the class of all tn-models.

Fact 2.4. Let w be any object and A be any set. Then:

1. For any tn-model $\mathfrak{M}$ based on w and A: (D) is true in $\mathfrak{M}$ iff $A \neq \emptyset$.
2. (T) are true in all tn-models based on w and A iff $w \in A$.
3. ($\mathtt{T}^{\mathtt{q}}$) are true in all tn-models based on w and A iff either $A = \emptyset$ or $w \in A$.

Theorem 2.5 (Soundness).

1. *All theses of* $\mathbf{S0.5}^{\circ}$ *are valid in the class of all tn-models.*
2. *All theses of* $\mathbf{S0.5}^{\circ}[\mathtt{D}]$ *are valid in the class of all non-empty tn-models.*
3. *All theses of* $\mathbf{S0.5}^{\circ}[\mathtt{T}^{\mathtt{q}}]$ *are valid in the class of all tn-models which are empty or self-associate.*
4. *All theses of* $\mathbf{S0.5}$ *are valid in the class of all self-associate tn-models.*

Given the above theorem, we can assume that the classes of models mentioned in the following items are suitable for the logics $\mathbf{S0.5}^{\circ}$, $\mathbf{S0.5}^{\circ}[\mathtt{D}]$, $\mathbf{S0.5}^{\circ}[\mathtt{T}^{\mathtt{q}}]$ and $\mathbf{S0.5}$, respectively. We denote this classes by $\mathsf{M}_{\mathbf{S0.5}^{\circ}}$, $\mathsf{M}_{\mathbf{S0.5}^{\circ}[\mathtt{D}]}$, $\mathsf{M}_{\mathbf{S0.5}^{\circ}[\mathtt{T}^{\mathtt{q}}]}$ and $\mathsf{M}_{\mathbf{S0.5}}$. For all models of these classes we can assume that for all worlds from $A \setminus \{w\}$, all modal propositionally atomic formulas have arbitrary values.

Finally, Theorem A.7 in Appendix give the completeness of the logics **S0.5**°, **S0.5**°[D], **S0.5**°[T^q] and **S0.5**.

THEOREM 2.6 (Completeness). *All formulas valid in the class* $\mathbf{M}_{S0.5^\circ}$ (*resp.* $\mathbf{M}_{S0.5^\circ[D]}$, $\mathbf{M}_{S0.5^\circ[T^q]}$, $\mathbf{M}_{S0.5}$) *are theses of* **S0.5**° (*resp.* **S0.5**°[D], **S0.5**°[T^q], **S0.5**).

2.3. Some conclusions

By Fact 2.4 and Theorem 2.5, we get:

FACT 2.7.

1. (T^q), (D) and any formula of the form $\ulcorner\Diamond\varphi\urcorner$ do not belong to **S0.5**°.
2. (D) and any formula of the form $\ulcorner\Diamond\varphi\urcorner$ do not belong to **S0.5**°[T^q].
3. (T^q) does not belong to **S0.5**°[D].
4. (T) belong neither to **S0.5**°[T^q] nor **S0.5**°[D].
5. **S0.5**° $\subsetneq$ **S0.5**°[D] $\subsetneq$ **S0.5** and **S0.5**° $\subsetneq$ **S0.5**°[T^q] $\subsetneq$ **S0.5**.

FACT 2.8. The following implications are not theses of **S0.5**:

$$\Box\Box p \supset \Box\Box\neg\neg p \qquad\qquad \Box\Box\neg\neg p \supset \Box\Box p$$
$$\Box\Diamond p \supset \Box\neg\Box\neg p \qquad\qquad \Box\neg\Box\neg p \supset \Box\Diamond p$$
$$\Box\Box p \supset \Box\neg\Diamond\neg p \qquad\qquad \Box\neg\Diamond\neg p \supset \Box\Box p$$

So **S0.5**°, **S0.5**°[D], **S0.5**°[T^q] and **S0.5** are not closed under (rte).[7]

PROOF: It is easy to point out suitable self-associate tn-models in which the above formulas are false. Hence, by Theorem 2.5(4) and Fact 2.7, **S0.5**°, **S0.5**°[D], **S0.5**°[T^q] and **S0.5** are not closed under (rte). □

The theorems below concern modal propositionally atomic formulas.[8]

THEOREM 2.9. *For any* $\boldsymbol{L} \in \{\mathbf{S0.5}^\circ, \mathbf{S0.5}^\circ[T^q], \mathbf{S0.5}^\circ[D], \mathbf{S0.5}\}$ *and* $\varphi \in$ For:

$$\ulcorner\Box\varphi\urcorner \in \boldsymbol{L} \quad \textit{iff} \quad \varphi \in \mathbf{PL}.$$

PROOF: Firstly, $\Box$**PL** $\subsetneq$ **S0.5**°[T^q] $\subsetneq$ **S0.5** and $\Box$**PL** $\subsetneq$ **S0.5**°[D] $\subsetneq$ **S0.5**. Secondly, let $\varphi \notin \mathbf{PL}$, $w \neq a$, $A := \{w, a\}$. Then, by Lemma 1.1, for some $V_a \in \mathsf{Val}^{\mathsf{cl}}$ we have that $V_a(\varphi) = 0$. By Lemma 2.2, for V_a and any

[7] In [5, 6, 7], t-normal logics closed under (rte) in versions built-in $\mathrm{For}_\Box$ are examined.

[8] [7, Facts 3.8 and 3.9] provides these theorems in versions for logics built-in $\mathrm{For}_\Box$.

assignment $v_w\colon \mathrm{At} \to \{0,1\}$ there is a self-associate tn-model $\langle w, A, V\rangle$ such that $V(\Box\varphi, w) = 0$. Hence $\ulcorner\Box\varphi\urcorner \notin \mathbf{S0.5}$, by Theorem 2.5(4). Moreover, we use Fact 2.7. □

THEOREM 2.10. *For any* $\boldsymbol{L} \in \{\mathbf{S0.5}^\circ, \mathbf{S0.5}^\circ[\mathtt{D}]\}$, $\Psi \subseteq \mathrm{For}$ *and* $\varphi \in \mathrm{For}$:

$$\Box\Psi \vdash_{\boldsymbol{L}} \Box\varphi \quad \textit{iff} \quad \Psi \models_{\mathbf{PL}} \varphi.$$

PROOF: Firstly, by (pk), if $\Psi \models_{\mathbf{PL}} \varphi$, then $\Box\Psi \vdash_{\mathbf{S0.5}^\circ} \Box\varphi$ and it entails $\Box\Psi \vdash_{\mathbf{S0.5}^\circ[\mathtt{D}]} \Box\varphi$. Secondly, suppose that $\Psi \not\models_{\mathbf{PL}} \varphi$ and $w \neq a$. Then, by Lemma 1.1, for some $V_a \in \mathsf{Val}^{\mathsf{cl}}$ we have $V_a[\Psi] = \{1\}$ and $V_a(\varphi) = 0$. By Lemma 2.2, for V_a and any $v_w\colon \mathrm{At} \to \{0,1\}$ there is a non-empty tn-model $\langle w, \{a\}, V\rangle$ such that $V[\Box\Psi] = \{1\}$ and $V(\Box\varphi, w) = 0$. Hence $\Box\Psi \nvdash_{\mathbf{S0.5}^\circ[\mathtt{D}]} \Box\varphi$, by Theorem 2.5. □

Remark 2.11. For $\mathbf{S0.5}^\circ[\mathtt{T^q}]$ and $\mathbf{S0.5}$, the "⇒"-part of Theorem 2.10 does not hold. Indeed, '$\Box\Box p \supset \Box p$' belong to $\mathbf{S0.5}^\circ[\mathtt{T^q}]$ ($\subsetneq \mathbf{S0.5}$). Therefore, $\Box\Box p \vdash_{\mathbf{S0.5}^\circ[\mathtt{T^q}]} \Box p$ and $\Box\Box p \vdash_{\mathbf{S0.5}} \Box p$, but $\Box\Box p \not\models_{\mathbf{PL}} \Box p$. □

2.4. Similarities and differences between the two approaches

Versions of t-normal logic built-in the set $\mathrm{For}_\Box$ include $\mathrm{Taut_{cl}}$ and $\Box\mathrm{Taut_{cl}}$, contain (K) and are closed under (det) and uniform substitutions. All such versions include $\mathbf{PL}_\Box$ ($:= \mathbf{PL} \cap \mathrm{For}_\Box$) and $\Box\mathbf{PL}_\Box$. We use the sign '$\Diamond$' as an abbreviation for '$\neg\Box\neg$'. As theses of such versions of t-normal logics, we obtain these formulas whose shortcuts are (df$\Diamond$), (df$\Box$), ($\mathtt{K_d}$), ($\mathtt{R_d}$), ($\mathtt{R'_d}$) (see Remark 1.2(1)).

Let us denote by $\mathbf{S0.5}^\circ_\Box$ the smallest t-normal logic built-in $\mathrm{For}_\Box$. Moreover, let $\mathbf{S0.5}_\Box$ be the smallest t-normal logic built-in $\mathrm{For}_\Box$ containing (T) (see Remark 2.1). The formulas for which ($\mathtt{T_d}$), (D), $\ulcorner\Diamond\top\urcorner$ and all $\mathbf{S0.5}^\circ$-equivalents to ($\mathtt{T^q}$) are shortcuts belong to $\mathbf{S0.5}_\Box$.

Let $\mathbf{S0.5}^\circ_\Box[\mathtt{T^q}]$ be the smallest t-normal logic built-in $\mathrm{For}_\Box$ containing ($\mathtt{T^q}$). As theses of $\mathbf{S0.5}^\circ_\Box[\mathtt{T^q}]$, we obtain these formulas whose shortcuts are $\mathbf{S0.5}^\circ$-equivalents to ($\mathtt{T^q}$). Moreover, let $\mathbf{S0.5}^\circ_\Box[\mathtt{D}]$ be the smallest t-normal logic built-in $\mathrm{For}_\Box$ containing '$\Box p \supset \neg\Box\neg p$', whose shortcut is (D).

Let $\boldsymbol{L} \in \{\mathbf{S0.5}^\circ, \mathbf{S0.5}^\circ[\mathtt{D}], \mathbf{S0.5}^\circ[\mathtt{T^q}], \mathbf{S0.5}\}$. For $\boldsymbol{L}_\Box$ we use tn-models, which we define in the same way as tn-models for $\boldsymbol{L}$ with the only difference that the set For is replaced by $\mathrm{For}_\Box$, and we only use ($V^w_\Box$). We have

(∗) All formulas from $\mathrm{For}_\Box$ true in all tn-models for $\boldsymbol{L}$ are also true in all tn-models for $\boldsymbol{L}_\Box$.

In [5, Theorem 4.8], an appropriate version of the completeness theorem for $\boldsymbol{L}_{\Box}$ is given.[9] We can prove:

THEOREM 2.12. *$\boldsymbol{L}_{\Box} = \boldsymbol{L} \cap \mathrm{For}_{\Box}$. So $\boldsymbol{L}_{\Box} \subsetneq \boldsymbol{L}$.*

PROOF: It is obvious that $\boldsymbol{L}_{\Box} \subseteq \boldsymbol{L} \cap \mathrm{For}_{\Box}$. Suppose that $\varphi \in \boldsymbol{L} \cap \mathrm{For}_{\Box}$. We take any tn-models for $\boldsymbol{L}_{\Box}$. By ($*$) and Theorem 2.5, φ is true in this model. From the completeness theorem for $\boldsymbol{L}_{\Box}$, we obtain that $\varphi \in \boldsymbol{L}_{\Box}$. $\Box$

From Theorems 2.9, 2.10 and 2.12 we obtain:

COROLLARY 2.13 ([5]). *For all $\varphi \in \mathrm{For}_{\Box}$ and $\Psi \subseteq \mathrm{For}_{\Box}$:*

1. *For $\boldsymbol{L} \in \{\mathbf{S0.5}^{\circ}, \mathbf{S0.5}^{\circ}[\mathtt{D}], \mathbf{S0.5}^{\circ}[\mathtt{T^q}], \mathbf{S0.5}\}$, $\ulcorner\Box\varphi\urcorner \in \boldsymbol{L}_{\Box}$ iff $\varphi \in \mathbf{PL}_{\Box}$.*
2. *For $\boldsymbol{L} \in \{\mathbf{S0.5}^{\circ}, \mathbf{S0.5}^{\circ}[\mathtt{D}]\}$, $\Box\Psi \vdash_{\boldsymbol{L}_{\Box}} \Box\varphi$ iff $\Psi \models_{\mathbf{PL}_{\Box}} \varphi$.*

As we mentioned in Remark 1.5, there may be some confusion regarding the two approaches to these logics. The difference between them is visible from Fact 2.8. Namely, the implications below are not theses of $\mathbf{S0.5}$ but even are theses even of $\mathbf{S0.5}^{\circ}_{\Box}$:

$$\Box\Diamond p \supset \Box\neg\Box\neg p \qquad\qquad \Box\neg\Box\neg p \supset \Box\Diamond p$$

Indeed, in $\mathbf{S0.5}^{\circ}_{\Box}$ the above implications are just shortcuts on one side of the PL-tautology '$\Box\neg\Box\neg p \supset \Box\neg\Box\neg p$'. Hence also, '$\Box(\Box\Diamond p \equiv \Box\neg\Box\neg p)$' belongs to $\mathbf{S0.5}^{\circ}_{\Box}$. However, it does not contradict Corollary 2.13 because, in $\mathbf{S0.5}^{\circ}_{\Box}$, these three forms are just abbreviations of suitable formulas from $\mathbf{PL}_{\Box}$ and $\Box\mathbf{PL}_{\Box}$, respectively.

Finally, note that the following implications are also not theses of $\mathbf{S0.5}_{\Box}$:

$$\Box\Box p \supset \Box\neg\Diamond\neg p \qquad\qquad \Box\neg\Diamond\neg p \supset \Box\Box p$$

Indeed, for $\mathbf{S0.5}_{\Box}$, these formulas are just abbreviations of the following:

$$\Box\Box p \supset \Box\neg\neg\Box\neg\neg p \qquad\qquad \Box\neg\neg\Box\neg\neg p \supset \Box\Box p$$

which are $\mathbf{S0.5}^{\circ}_{\Box}$-equivalent to '$\Box\Box p \supset \Box\Box\neg\neg p$' and '$\Box\Box\neg\neg p \supset \Box\Box p$', respectively. Fact 2.8 and Theorem 2.12 say that the last formulas are not theses of $\mathbf{S0.5}^{\circ}_{\Box}$.

[9] Its proof is an appropriate version of the proof of Theorem A.7.

3. Other t-normal logics with additional axioms

3.1. Additional axioms

Theorem 2.5(4) shows that none of formulas ($\mathtt{D}^{\mathtt{m}}$), ($\mathtt{D}^{\mathtt{m}}_{\mathtt{d}}$), ($\mathtt{T}^{\mathtt{m}}$), ($\mathtt{T}^{\mathtt{m}}_{\mathtt{d}}$), ($\mathtt{4}$), ($\mathtt{4}_{\mathtt{d}}$), ($\mathtt{B}$), ($\mathtt{B}_{\mathtt{d}}$), ($\mathtt{5}$), ($\mathtt{5}_{\mathtt{d}}$) belongs to **S0.5**. The formulas listed here are additional axioms with which we will extend $\mathbf{S0.5}^{\circ}$ and **S0.5**. It is evident that:

- ($\mathtt{D}^{\mathtt{m}}$) $\in \mathbf{S0.5}[\mathtt{T}^{\mathtt{m}}]$, ($\mathtt{T}^{\mathtt{m}}$) $\in \mathbf{S0.5}[\mathtt{B}_{\mathtt{d}}]$ and ($\mathtt{B}$) $\in \mathbf{S0.5}^{\circ}[\mathtt{5}]$;
- ($\mathtt{D}^{\mathtt{m}}_{\mathtt{d}}$) $\in \mathbf{S0.5}[\mathtt{T}^{\mathtt{m}}_{\mathtt{d}}]$, ($\mathtt{T}^{\mathtt{m}}_{\mathtt{d}}$) $\in \mathbf{S0.5}[\mathtt{B}]$ and ($\mathtt{B}_{\mathtt{d}}$) $\in \mathbf{S0.5}[\mathtt{5}_{\mathtt{d}}]$;
- ($\mathtt{T}^{\mathtt{m}}$) $\in \mathbf{S0.5}^{\circ}[\mathtt{D}^{\mathtt{m}},\mathtt{4}_{\mathtt{d}}]$ and ($\mathtt{T}^{\mathtt{m}}_{\mathtt{d}}$) $\in \mathbf{S0.5}^{\circ}[\mathtt{4},\mathtt{D}^{\mathtt{m}}_{\mathtt{d}}]$;
- ($\mathtt{T}^{\mathtt{m}}$) $\in \mathbf{S0.5}^{\circ}[\mathtt{5}_{\mathtt{d}},\mathtt{D}]$ and ($\mathtt{T}^{\mathtt{m}}_{\mathtt{d}}$) $\in \mathbf{S0.5}^{\circ}[\mathtt{D},\mathtt{5}]$;
- ($\mathtt{D}^{\mathtt{m}}$), ($\mathtt{D}^{\mathtt{m}}_{\mathtt{d}}$) $\in \mathbf{S0.5}^{\circ}[\mathtt{D},\mathtt{5}_{\mathtt{d}},\mathtt{T}^{\mathtt{m}}_{\mathtt{d}}]$ and ($\mathtt{D}^{\mathtt{m}}$), ($\mathtt{D}^{\mathtt{m}}_{\mathtt{d}}$) $\in \mathbf{S0.5}^{\circ}[\mathtt{D},\mathtt{5},\mathtt{T}^{\mathtt{m}}]$;
- ($\mathtt{D}^{\mathtt{m}}$), ($\mathtt{D}^{\mathtt{m}}_{\mathtt{d}}$) $\in \mathbf{S0.5}^{\circ}[\mathtt{D},\mathtt{5},\mathtt{5}_{\mathtt{d}}]$.

Further, we will show that there are no other dependencies between additional axioms.

We are interested in such t-normal logics, which have a given additional axiom and its dual form. To simplify naming of logics, we will write $\mathbf{S0.5}^{\circ}.\mathbf{X}_1 \ldots \mathbf{X}_n$ to denote the smallest t-normal logic containing formulas ($\mathtt{X}_1$), ..., ($\mathtt{X}_n$) and their dual forms. Moreover, the notation $\mathbf{S0.5}.\mathbf{X}_1 \ldots \mathbf{X}_n$ will indicate the suitable smallest extension of **S0.5**. For example:

- $\mathbf{S0.5}^{\circ}.\mathbf{4T}^{\mathrm{m}} \subseteq \mathbf{S0.5}^{\circ}.\mathbf{4D}^{\mathrm{m}}$ and $\mathbf{S0.5.4D}^{\mathrm{m}} = \mathbf{S0.5.4T}^{\mathrm{m}}$.

Further, we will show that the following combinations of additional axioms give normal logics (see Theorem 4.1):

(†) $\mathbf{S0.5}^{\circ}.\mathbf{45} = \mathbf{K45}$, $\mathbf{S0.5}^{\circ}.\mathbf{D45} = \mathbf{KD45}$ and $\mathbf{S0.5}^{\circ}.\mathbf{B45} = \mathbf{KB4}$ (= **KB5**);

(‡) $\mathbf{S0.5.45} = \mathbf{S5}$.

Moreover, we will show that the remaining combinations of additional axioms give vwt-normal logics (see Fact 3.7).

The following fact and results obtained in Section 2.3 will show differences between the logics thus obtained and the logics $\mathbf{S0.5}^{\circ}$ and **S0.5**.

Fact 3.1. For all $\varphi, \psi \in \mathrm{For}$:

1. (a) If $\ulcorner \varphi \equiv \psi \urcorner \in \mathbf{PL}$ then $\ulcorner \Box\Box\varphi \equiv \Box\Box\psi \urcorner \in \mathbf{S0.5}[\mathtt{4}]$.
   (b) $\ulcorner \Diamond\varphi \equiv \Diamond\neg\Box\neg\varphi \urcorner \in \mathbf{S0.5}[\mathtt{4}]$.
2. (a) If $\ulcorner \varphi \equiv \psi \urcorner \in \mathbf{PL}$ then $\ulcorner \Diamond\Diamond\varphi \equiv \Diamond\Diamond\psi \urcorner \in \mathbf{S0.5}[\mathtt{4}_{\mathtt{d}}]$.
   (b) $\ulcorner \Box\varphi \equiv \Box\neg\Diamond\neg\varphi \urcorner \in \mathbf{S0.5}[\mathtt{4}_{\mathtt{d}}]$.

3. (a) $\ulcorner\Box\Box\varphi \equiv \Box\neg\Diamond\neg\varphi\urcorner$ and (b) $\ulcorner\Diamond\Diamond\varphi \equiv \Diamond\neg\Box\neg\varphi\urcorner$ belong to **S0.5.4**.

4. (a) If $\ulcorner\varphi \equiv \psi\urcorner \in \mathbf{PL}$ then $\ulcorner\Box\Diamond\varphi \equiv \Box\Diamond\psi\urcorner \in \mathbf{S0.5}[5] \cap \mathbf{S0.5}^\circ[\mathtt{D},4_\mathtt{d},5]$.
(b) $\ulcorner\Box\varphi \equiv \Diamond\neg\Diamond\neg\varphi\urcorner \in \mathbf{S0.5}[5] \cap \mathbf{S0.5}^\circ[\mathtt{D},4_\mathtt{d},5]$.

5. (a) If $\ulcorner\varphi \equiv \psi\urcorner \in \mathbf{PL}$ then $\ulcorner\Diamond\Box\varphi \equiv \Diamond\Box\psi\urcorner \in \mathbf{S0.5}[5_\mathtt{d}] \cap \mathbf{S0.5}^\circ[\mathtt{D},4,5_\mathtt{d}]$.
(b) $\ulcorner\Diamond\varphi \equiv \Box\neg\Box\neg\varphi\urcorner \in \mathbf{S0.5}[5_\mathtt{d}] \cap \mathbf{S0.5}^\circ[\mathtt{D},4,5_\mathtt{d}]$.

6. (a) $\ulcorner\Box\Diamond\varphi \equiv \Box\neg\Box\neg\varphi\urcorner$ and (b) $\ulcorner\Diamond\Box\varphi \equiv \Diamond\neg\Diamond\neg\varphi\urcorner$ belong to **S0.5.5** and $\mathbf{S0.5}^\circ\mathbf{.D45}$.

7. (a) $\Box(\mathtt{T_d}) \in \mathbf{S0.5}^\circ[5]$, (b) $\Box(\mathtt{T}) \in \mathbf{S0.5}^\circ[5_\mathtt{d}]$ and (c) $\Box(\mathtt{D}) \in \mathbf{S0.5}^\circ\mathbf{.5}$.

Proof: *Ad* 1. (a) By (T), (4), (pe); (b) By (T), (4), (pe), (df$\Diamond$).

Ad 2. (a) By ($\mathtt{T_d}$), ($4_\mathtt{d}$), ($\mathrm{pe_d}$). (b) By ($\mathtt{T_d}$), ($4_\mathtt{d}$), ($\mathrm{pe_d}$), (df$\Box$).

Ad 3. (a) By (4), (T) and item 2(b). (b) By ($4_\mathtt{d}$), ($\mathtt{T_d}$) and item 1(b).

Ad 4. (a) From (T), (5), ($\mathrm{pe_d}$) we have: $\Box\Diamond\varphi \equiv \Diamond\varphi \equiv \Diamond\psi \equiv \Box\Diamond\psi$. Moreover, by (D), ($4_\mathtt{d}$), ($\mathrm{pe_d}$), (5): $\Box\Diamond\varphi \supset \Diamond\Diamond\varphi \supset \Diamond\varphi \equiv \Diamond\psi \supset \Box\Diamond\psi$. (b) From (T), (5), (pe) we have: $\Diamond\neg\varphi \equiv \Box\Diamond\neg\varphi \equiv \Box\neg\neg\Diamond\neg\varphi$. Hence and (df$\Diamond$), (df$\Box$) we have: $\Box\varphi \equiv \Diamond\neg\Diamond\neg\varphi$. Moreover, by (D), ($4_\mathtt{d}$), (5) we have: $\Box\Diamond\neg\varphi \supset \Diamond\Diamond\neg\varphi \supset \Diamond\neg\varphi \supset \Box\Diamond\neg\varphi$. So we use (df$\Diamond$), (df$\Box$) and (pe).

Ad 5. (a) By ($\mathtt{T_d}$), ($5_\mathtt{d}$) and ($\mathrm{pe_d}$) we have: $\Diamond\Box\varphi \equiv \Box\varphi \equiv \Box\psi \equiv \Diamond\Box\psi$. Moreover, by ($5_\mathtt{d}$), (pe), (4), (D): $\Diamond\Box\varphi \supset \Box\varphi \supset \Box\psi \equiv \Box\Box\psi \supset \Diamond\Box\psi$. (b) From ($\mathtt{T_d}$), ($5_\mathtt{d}$) and (pe): $\Box\neg\varphi \equiv \Diamond\Box\neg\varphi \equiv \Diamond\neg\neg\Box\neg\varphi$. Hence and (df$\Diamond$), (df$\Box$) we have: $\Diamond\varphi \equiv \Box\neg\Box\neg\varphi$. Moreover, by ($5_\mathtt{d}$), (4), (D) we have: $\Diamond\Box\neg\varphi \supset \Box\neg\varphi \supset \Box\Box\neg\varphi \supset \Diamond\Box\neg\varphi$. So we use (df$\Diamond$), (df$\Box$) and (pe).

Ad 6. (a) By (T), (5): $\Box\Diamond\varphi \equiv \Diamond\varphi$. Moreover, by (D), ($4_\mathtt{d}$), (5), we have: $\Box\Diamond\varphi \supset \Diamond\Diamond\varphi \supset \Diamond\varphi \supset \Box\Diamond\varphi$. So in both cases we use item 5(b). (b) By ($\mathtt{T_d}$), ($5_\mathtt{d}$): $\Diamond\Box\varphi \equiv \Box\varphi$. Moreover, by ($5_\mathtt{d}$), (4), (D), we have: $\Diamond\Box\varphi \supset \Box\varphi \supset \Box\Box\varphi \supset \Diamond\Box\varphi$. So in both cases we use item 4(b).

Ad 7. (a) By (pk), $\Box\Diamond p \vdash_{\boldsymbol{L}} \Box(p \supset \Diamond p)$ and $\Box\neg p \vdash_{\boldsymbol{L}} \Box(p \supset \Diamond p)$ for any t-normal logic $\boldsymbol{L}$. Moreover, $\Diamond p \vdash_{\mathbf{S0.5}^\circ[5]} \Box\Diamond p$ and $\neg\Diamond p \vdash_{\mathbf{S0.5}^\circ} \Box\neg p$. Thus, '$\Box(p \supset \Diamond p)$' $\in \mathbf{S0.5}^\circ[5]$.

(b) By (pk), $\Box p \vdash_{\boldsymbol{L}} \Box(\Box p \supset p)$ and $\Box\neg\Box p \vdash_{\boldsymbol{L}} \Box(\Box p \supset p)$ for any t-normal logic $\boldsymbol{L}$. Moreover, $\neg\Box p \vdash_{\mathbf{S0.5}^\circ[5_\mathtt{d}]} \Box\neg\Box p$. Therefore, '$\Box(\Box p \supset p)$' belongs to $\mathbf{S0.5}^\circ[5_\mathtt{d}]$.

(c) By (pk), $\Box(\mathtt{T}), \Box(\mathtt{T_d}) \vdash_{\boldsymbol{L}} \Box(\mathtt{D})$ for any t-normal logic $\boldsymbol{L}$. So we use (a) and (b). □

3.2. Kripke-style semantics for additional axioms. Soundness

To use tn-models for additional axioms, we must assume an appropriate condition for a given axiom. In a tn-model $\langle w, A, V\rangle$, every one of these conditions will apply to any formula φ:

$$\exists_{x\in A}\, V(\Box\varphi, x) = 1 \implies \exists_{y\in A}\, V(\Diamond\varphi, y) = 1, \qquad (\mathrm{cD}^{\mathtt{m}}\varphi)$$

$$\forall_{x\in A}\, V(\Box\varphi, x) = 1 \implies \forall_{y\in A}\, V(\Diamond\varphi, y) = 1, \qquad (\mathrm{cD}^{\mathtt{m}}_{\mathtt{d}}\varphi)$$

$$\exists_{x\in A}\, V(\Box\varphi, x) = 1 \implies \exists_{y\in A}\, V(\varphi, y) = 1, \qquad (\mathrm{cT}^{\mathtt{m}}\varphi)$$

$$\forall_{x\in A}\, V(\varphi, x) = 1 \implies \forall_{y\in A}\, V(\Diamond\varphi, y) = 1, \qquad (\mathrm{cT}^{\mathtt{m}}_{\mathtt{d}}\varphi)$$

$$V(\varphi, w) = 1 \implies \forall_{y\in A}\, V(\Diamond\varphi, y) = 1, \qquad (\mathrm{cB}\varphi)$$

$$V(\varphi, w) = 0 \implies \forall_{y\in A}\, V(\Box\varphi, y) = 0, \qquad (\mathrm{cB}_{\mathtt{d}}\varphi)$$

$$\forall_{x\in A}\big(\exists_{y\in A}\, V(\varphi, y) = 1 \implies V(\Diamond\varphi, x) = 1\big), \qquad (\mathrm{c}5\varphi)$$

$$\forall_{x\in A}\big(V(\Box\varphi, x) = 1 \implies \forall_{y\in A}\, V(\varphi, x) = 1\big), \qquad (\mathrm{c}5_{\mathtt{d}}\varphi)$$

$$\forall_{x\in A}\big(\forall_{y\in A}\, V(\varphi, y) = 1 \implies V(\Box\varphi, x) = 1\big), \qquad (\mathrm{c}4\varphi)$$

$$\forall_{x\in A}\big(V(\Diamond\varphi, x) = 1 \implies \exists_{y\in A}\, V(\varphi, y) = 1\big). \qquad (\mathrm{c}4_{\mathtt{d}}\varphi)$$

Moreover, for (T), (D) and ($\mathtt{T}^{\mathtt{q}}$) we use the conditions '$w \in A$', '$A \neq \emptyset$' and 'either $w \in A$ or $A = \emptyset$', respectively.

Remark 3.2. (i) In all self-associate tn-models: ($\mathrm{cT}^{\mathtt{m}}\varphi$) entails ($\mathrm{cD}^{\mathtt{m}}\varphi$); ($\mathrm{cT}^{\mathtt{m}}_{\mathtt{d}}\varphi$) entails ($\mathrm{cD}^{\mathtt{m}}_{\mathtt{d}}\varphi$); ($\mathrm{cB}_{\mathtt{d}}\varphi$) entails ($\mathrm{cT}^{\mathtt{m}}\varphi$); ($\mathrm{cB}\varphi$) entails ($\mathrm{cT}^{\mathtt{m}}_{\mathtt{d}}\varphi$); ($\mathrm{c}5\varphi$) entails ($\mathrm{cB}\varphi$); ($\mathrm{c}5_{\mathtt{d}}\varphi$) entails ($\mathrm{cB}_{\mathtt{d}}\varphi$).

(ii) Apart from the above, no other dependencies exist between the given conditions. □

The following lemma is easy to prove:

Lemma 3.3. *Let χ is an additional axiom, $\varphi \in$ For and $\mathfrak{M}$ be a tn-model. We put $\chi^{\varphi} := \chi[p/\varphi]$. Then:*

$$\chi^{\varphi} \textit{ is true in } \mathfrak{M} \textit{ iff } \varphi \textit{ satisfies the condition } (\mathrm{c}\chi\varphi) \textit{ in } \mathfrak{M}.$$

Let Φ be a non-empty set of formulas which contains some or all of the formulas used as additional axioms (including (T), (D) and ($\mathtt{T}^{\mathtt{q}}$)). Then we will call $\mathbf{S0.5}^{\circ}[\Phi]$-*model* all those and only those tn-models in which conditions for all instances of the formulas in Φ are satisfied.

Theorem 3.4 (Soundness). *All theses of* $\mathbf{S0.5}^{\circ}[\Phi]$ *are valid in the class of all* $\mathbf{S0.5}^{\circ}[\Phi]$*-models.*

We will further use the following lemma:

Lemma 3.5. *Let* $\{(4), (4_{\mathtt{d}}), (5), (5_{\mathtt{d}})\} \subseteq \Phi$*,* $\langle w, A, V\rangle$ *be an* $\mathbf{S0.5}^{\circ}[\Phi]$*-model and* $W := \{w\} \cup A$*. Then:*

1. $\langle W, A, V\rangle$ *is a simplified Kripke-style model.*
2. *If also* $(\mathtt{B}) \in \Phi$ *then* $\langle W, A, V\rangle$ *is an empty or universal Kripke model.*
3. *If also* $(\mathtt{D}) \in \Phi$ *then* $\langle W, A, V\rangle$ *is a non-empty simplified model.*
4. *If also* $(\mathtt{T}) \in \Phi$ *then* $\langle W, V\rangle$ *is a universal Kripke model.*

Proof: *Ad* 1. Let $\varphi \in \mathrm{For}$. By $(V^{w}_{\Box})$, $(\mathrm{c4}\varphi)$ and $(\mathrm{c5}_{\mathtt{d}}\varphi)$, for any $x \in W$: $V(\Box\varphi, x) = 1$ iff $V(\varphi, y) = 1$ for each $y \in A$. By $(V^{w}_{\Diamond})$, $(\mathrm{c4}_{\mathtt{d}}\varphi)$ and $(5_{\mathtt{d}})$, for any $x \in W$: $V(\Diamond\varphi, x) = 1$ iff $V(\varphi, y) = 1$ for some $y \in A$. Thus, $\langle W, A, V\rangle$ satisfies conditions $(V_{\Box})$ and $(V_{\Diamond})$ from p. 516.

Ad 2. By item 1, $\langle W, A, V\rangle$ satisfies $(V_{\Box})$ and $(V_{\Diamond})$. Assume that $A \neq \emptyset$.

For $(V_{\Box})$ with $A = W$: Let $\varphi \in \mathrm{For}$. By $(\mathrm{cB}_{\mathtt{d}}\varphi)$, we have:

(i) for any $x \in A$: if $V(\Box\varphi, x) = 1$ then $V(\varphi, w) = 1$).

Moreover, assume that $V(\varphi, w) = 0$. Then, by $(\mathrm{cB}_{\mathtt{d}}\varphi)$, $V(\Box\varphi, x) = 0$ for each $x \in A$. So $V(\Box\varphi, x_0) = 0$ for some $x_0 \in A$ because $A \neq \emptyset$. Hence $V(\Box\varphi, w) = 0$, by $(V^{w}_{\Box})$. So we obtain:

(ii) if $V(\Box\varphi, w) = 1$ then $V(\varphi, w) = 1$.

Thus, using (i), (ii), $(V^{w}_{\Box})$ and $(V_{\Box})$, we obtain:

$(V_{\Box})$ for any $x \in W$: $V(\Box\varphi, x) = 1$ iff $\forall_{y \in W}\, V(\varphi, y) = 1$.

For $(V_{\Diamond})$ with $A = W$: Let $\varphi \in \mathrm{For}$. By $(\mathrm{cB}\varphi)$, we have:

(i′) for any $x \in A$: $V(\varphi, w) = 1 \Rightarrow V(\Diamond\varphi, x) = 1$.

Moreover, using (ii) and $(\mathtt{df}\Diamond)$ for $\neg\varphi$, we obtain:

(ii′) if $V(\varphi, w) = 1$ then $V(\Diamond\varphi, w) = 1$.

Thus, using (i′), (ii′), $(V^{w}_{\Diamond})$ and $(V_{\Diamond})$, we obtain:

$(V_{\Box})$ for any $x \in W$: $V(\Diamond\varphi, x) = 1$ iff $\exists_{y \in W}\, V(\varphi, y) = 1$.

Ad 3. $A \neq \emptyset$, by Fact 2.4(1).

Ad 4. Suppose that $(\mathtt{T}) \in \boldsymbol{L}$. Then $(\mathtt{D})$, $(\mathtt{B})$ and $(\mathtt{B}_{\mathtt{d}})$ belong to $\boldsymbol{L}$. Hence, by item 3, $A \neq \emptyset$. So $\langle W, V\rangle$ is a universal Kripke model, by item 2. □

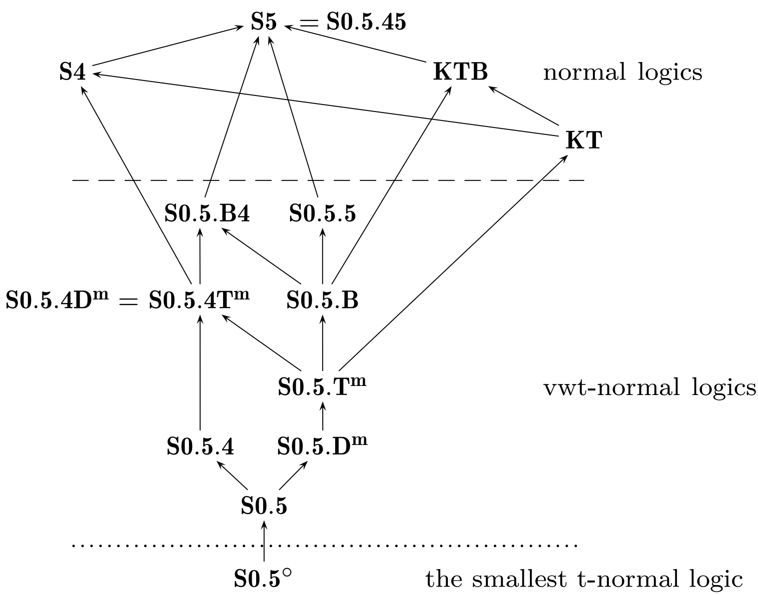

Figure 1. The dependencies between the considered extensions of **S0.5**

3.3. Some conclusions

By constructing appropriate countermodels, by Theorem 3.4, we have the following facts (cf. Remark 3.2(ii)):

FACT 3.6.

1. ($\mathtt{D^m}$) $\notin$ **S0.5**[$\mathtt{D^m_d}$] and ($\mathtt{D^m_d}$) $\notin$ **S0.5**[$\mathtt{D^m}$].
2. ($\mathtt{T^m}$)($\mathtt{T^m_d}$) $\notin$ **S0.5.D**^{m}.
3. Neither ($\mathtt{B}$) nor ($\mathtt{B_d}$) belongs to neither **S0.5.T**^{m} nor **S0.5.4**.
4. ($\mathtt{4}$) $\notin$ **S0.5**[$\mathtt{4_d}$] and ($\mathtt{4_d}$) $\notin$ **S0.5**[$\mathtt{4}$].
5. ($\mathtt{B}$) $\notin$ **S0.5**[$\mathtt{B_d}$] and ($\mathtt{B_d}$) $\notin$ **S0.5**[$\mathtt{B}$].
6. ($\mathtt{5}$) $\notin$ **S0.5**[$\mathtt{5_d}$] and ($\mathtt{5_d}$) $\notin$ **S0.5**[$\mathtt{5}$].
7. ($\mathtt{4}$), ($\mathtt{4_d}$) $\notin$ **S0.5.5** and ($\mathtt{5}$), ($\mathtt{5_d}$) $\notin$ **S0.5.B4**.

The dependencies between the considered extensions of the logics **S0.5** and **S0.5**° are presented in Figures 1 and 2, respectively.

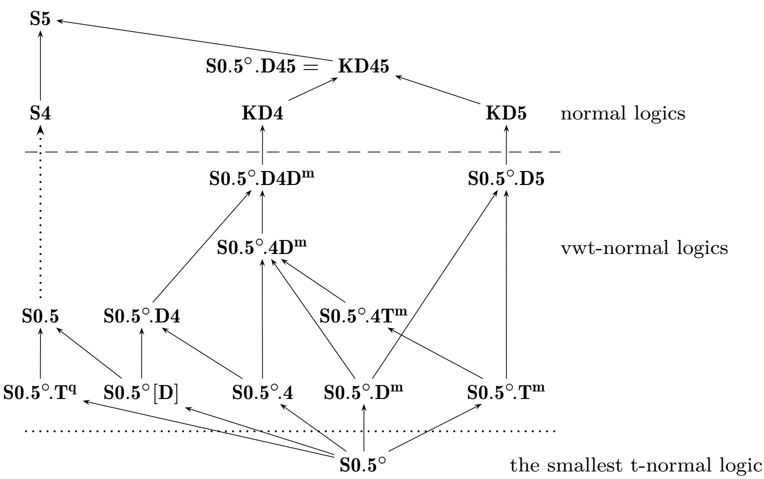

Figure 2. The dependencies between the considered extensions of **S0.5**$^\circ$

Also, by constructing appropriate two-element self-associate counter-models and using Theorem 3.4, we obtain the following fact, which shows that the logics **S0.5.B4** and **S0.5.5** (and all others included therein) are not closed under (rte) (cf. Fact 3.1).

Fact 3.7. 1. The formulas '$\Box\Diamond p \supset \Box\Diamond\neg\neg p$', '$\Box\Diamond\neg\neg p \supset \Box\Diamond p$', '$\Diamond\Box p \supset \Diamond\Box\neg\neg p$' and '$\Diamond\Box\neg\neg p \supset \Diamond\Box p$' do not belong to **S0.5.B4**. So '$\Box(\Diamond p \supset \Diamond\neg\neg p)$' and '$\Box(\Diamond\neg\neg p \supset \Diamond p)$' too.

2. The formulas '$\Box\Box p \supset \Box\Box\neg\neg p$', '$\Box\Box\neg\neg p \supset \Box\Box p$', '$\Diamond\Diamond p \supset \Diamond\Diamond\neg\neg p$' and '$\Diamond\Diamond\neg\neg p \supset \Diamond\Diamond p$' do not belong to **S0.5.5**. So '$\Box(\Box p \supset \Box\neg\neg p)$' and '$\Box(\Box\neg\neg p \supset \Box p)$' too.

Moreover, we have (cf. Fact 3.1(3,5)):

Fact 3.8. Neither '$\Box(\Diamond p \supset \neg\Box\neg p)$', '$\Box(\neg\Box\neg p \supset \Diamond p)$', '$\Box(\Box p \supset \neg\Diamond\neg p)$' nor '$\Box(\neg\Diamond\neg p \supset \Box p)$' belongs to either **S0.5.5** or **S0.5.B4**.

Remark 3.9. Logics considered here can also be built in the set $\mathrm{For}_\Box$. Facts 3.8 and 3.8 show the differences between the two approaches. Moreover, we will show that for versions built in the set $\mathrm{For}_\Box$, we can omit abbreviations of (5), (B) and ($\mathtt{T}^{\mathtt{m}}_{\mathtt{d}}$).

Indeed, ($5_{\mathtt{d}}$) is an abbreviation of '$\neg\Box\neg\Box p \supset \Box p$'. From it, by **PL** and the substitution $p/\neg p$, we have '$\neg\Box\neg p \supset \Box\neg\Box\neg p$', an abbreviation of (5). Therefore, this last shortcut belongs to $\mathbf{S0.5}^{\circ}_{\Box}[5_{\mathtt{d}}]$.

($\mathtt{B}_{\mathtt{d}}$) is an abbreviation of '$\neg\Box\neg\Box p \supset p$'. From it, by **PL** and the substitution $p/\neg p$, we have '$p \supset \Box\neg\Box\neg p$', an abbreviation of (B). Therefore, this last shortcut belongs to $\mathbf{S0.5}^{\circ}_{\Box}[\mathtt{B}_{\mathtt{d}}]$.

($\mathtt{T}^{\mathtt{m}}$) is an abbreviation of '$\neg\Box\neg\Box p \supset \neg\Box\neg p$'. From it, by **PL** and the substitution $p/\neg p$, we have '$\Box\neg\neg p \supset \Box\neg\Box\neg p$'. Hence, by (pe), we have '$\Box p \supset \Box\neg\Box\neg p$, an abbreviation of ($\mathtt{T}^{\mathtt{m}}_{\mathtt{d}}$). Therefore, this last shortcut belongs to $\mathbf{S0.5}^{\circ}_{\Box}[\mathtt{T}^{\mathtt{m}}]$. □

3.4. Completeness

Let $\boldsymbol{L}$ be a t-normal logic and $\langle w_\Gamma, A_\Gamma, V_\Gamma\rangle$ be the canonical model for $\boldsymbol{L}$ and $\Gamma \in \mathrm{Max}_{\boldsymbol{L}}$ (see Appendix A.2).

Lemma 3.10. *Let χ be a formula from* (4), ($4_{\mathtt{d}}$), (B), ($\mathtt{B}_{\mathtt{d}}$), (5), ($5_{\mathtt{d}}$), ($\mathtt{D}^{\mathtt{m}}$), ($\mathtt{D}^{\mathtt{m}}_{\mathtt{d}}$), ($\mathtt{T}^{\mathtt{m}}$), ($\mathtt{T}^{\mathtt{m}}_{\mathtt{d}}$). *If $\boldsymbol{L}$ contains χ, then any formula φ satisfies condition* (c$\chi\varphi$) *in $\langle w_\Gamma, A_\Gamma, V_\Gamma\rangle$.*

Proof: For any $\varphi \in$ For, using the definition of canonical models and Lemmas A.1 and A.5, and conditions ($V^w_\Box$) and ($V^w_\Diamond$) for V_Γ, we obtain that φ satisfies condition (c$\chi\varphi$) in $\langle w_\Gamma, A_\Gamma, V_\Gamma\rangle$. □

Let Φ be a non-empty set of formulas which contains some or all of the formulas used as additional axioms (including (T), (D) and ($\mathtt{T}^{\mathtt{q}}$)). We put $\boldsymbol{L} := \mathbf{S0.5}^{\circ}[\Phi]$. Let $\mathsf{M}_{\boldsymbol{L}}$ be the class of all $\boldsymbol{L}$-models. From Lemmas A.5, A.6 and 3.10 we have:

Fact 3.11. All canonical models for $\boldsymbol{L}$ belong to $\mathsf{M}_{\boldsymbol{L}}$.

We can show that $\boldsymbol{L}$ is complete with respect to $\mathsf{M}_{\boldsymbol{L}}$.

Theorem 3.12. *All formulas valid in the class $\mathsf{M}_{\boldsymbol{L}}$ are theses of $\boldsymbol{L}$.*

Proof: Let φ be valid in $\mathsf{M}_{\boldsymbol{L}}$ and $\Gamma \in \mathrm{Max}_{\boldsymbol{L}}$. By Fact 3.11, the canonical model $\langle w_\Gamma, A_\Gamma, V_\Gamma\rangle$ for $\boldsymbol{L}$ and Γ belongs to $\mathsf{M}_{\boldsymbol{L}}$. So $V_\Gamma(\varphi, w_\Gamma) = 1$. Hence $\varphi \in \Gamma$. Therefore, φ belongs to all $\boldsymbol{L}$-maximal sets. Hence $\varphi \in \boldsymbol{L}$, by Lemma A.3(2). □

4. Determining K45, KB4, KD45 and S5 without using modal rules

Using Lemma 3.5 and Theorems 1.3, 1.4 and 3.12, we obtain (†) and (‡), i.e., **K45**, **KB4**, **KD45** and **S5** can be defining without using modal rules.

THEOREM 4.1. *(†) and (‡) hold.*

PROOF: It is obvious that $\mathbf{S0.5^{\circ}.45} \subseteq \mathbf{K45}$, $\mathbf{S0.5^{\circ}.B45} \subseteq \mathbf{KB4}$ (= **KB5**), $\mathbf{S0.5^{\circ}.D45} \subseteq \mathbf{KD45}$ and $\mathbf{S0.5.45} \subseteq \mathbf{S5}$. We will show that we also have the reverse inclusions.

For $\mathbf{S5} \subseteq \mathbf{S0.5.45}$: Suppose that $\varphi \in \mathbf{S5}$. We will prove that φ is valid in $\mathsf{M}_{\mathrm{S0.5.45}}$. Let $\langle w, A, V\rangle$ be any **S0.5.45**-model. Then, by Lemma 3.5, $\langle W, V\rangle$ is a universal Kripke model. So, by the assumption and Theorem 1.3, for any $x \in W$ we have $V(\varphi, x) = 1$. So also $V(\varphi, w) = 1$; i.e., φ is true in $\langle w, A, V\rangle$. Therefore, φ is valid in $\mathsf{M}_{\mathrm{S0.5.45}}$. Hence $\varphi \in \mathbf{S0.5.45}$, by Theorem 3.12.

Similarly, using Lemma 3.5 and Theorems 1.4 and 3.12, we obtain that $\mathbf{K45} \subseteq \mathbf{S0.5^{\circ}.45}$, $\mathbf{KB4} \subseteq \mathbf{S0.5^{\circ}.B45}$ and $\mathbf{KD45} \subseteq \mathbf{S0.5^{\circ}.D45}$. □

A. Completeness of S0.5°, S0.5°[D], S0.5°[T^q], S0.5

The results reported here are adapted for **S0.5°**, **S0.5°**[D], **S0.5°**[$\mathtt{T^q}$] and **S0.5** built-in For from those obtained in [5] (where these logics are analyzed in $\mathrm{For}_{\Box}$ and the broader class of t-regular logics is analyzed).

A.1. Notions and facts concerning maximal consistent sets

Let $\boldsymbol{L}$ be a consistent t-normal logic. A set Ψ is $\boldsymbol{L}$*-consistent* iff for some $\varphi \in \mathrm{For}$ we have $\Psi \nvdash_{\boldsymbol{L}} \varphi$; equivalently in the light of **PL**, iff $\Psi \nvdash_{\boldsymbol{L}} p \wedge \neg p$. Every $\boldsymbol{L}$-consistent set is **PL**-consistent.

We say that Γ is $\boldsymbol{L}$*-maximal* iff Γ is $\boldsymbol{L}$-consistent and Γ has only $\boldsymbol{L}$-inconsistent proper extensions. By changing $\boldsymbol{L}$ to **PL**, we will obtain the definition of **PL**-*maximal sets*. Let $\mathrm{Max}_{\boldsymbol{L}}$ and $\mathrm{Max}_{\mathrm{PL}}$ be the sets of all $\boldsymbol{L}$-maximal and **PL**-maximal sets, respectively.

We will use the following lemmas (which can be proven as in [1]).

LEMMA A.1. *Let $\Gamma \in \mathrm{Max}_{\boldsymbol{L}}$. Then $\boldsymbol{L} \subseteq \Gamma$ and for all $\varphi, \psi \in \mathrm{For}$:*

1. *$\Gamma \vdash_{\boldsymbol{L}} \varphi$ iff $\varphi \in \Gamma$.*

2. $\ulcorner\neg\varphi\urcorner \in \Gamma$ *iff* $\varphi \notin \Gamma$.
3. $\ulcorner\varphi \wedge \psi\urcorner \in \Gamma$ *iff both* $\varphi \in \Gamma$ *and* $\psi \in \Gamma$.
4. $\ulcorner\varphi \vee \psi\urcorner \in \Gamma$ *iff either* $\varphi \in \Gamma$ *or* $\psi \in \Gamma$.
5. $\ulcorner\varphi \supset \psi\urcorner \in \Gamma$ *iff either* $\varphi \notin \Gamma$ *or* $\psi \in \Gamma$.
6. $\ulcorner\varphi \equiv \psi\urcorner \in \Gamma$ *iff either* $\varphi, \psi \in \Gamma$ *or* $\varphi, \psi \notin \Gamma$.

Notice that from Lemma A.1(2) we obtain:

FACT A.2. Every $\boldsymbol{L}$-maximal set is **PL**-maximal.

LEMMA A.3. *For all* $\Psi \subseteq \mathrm{For}$ *and* $\varphi \in \mathrm{For}$:

1. $\Psi \vdash_{\boldsymbol{L}} \varphi$ *iff* $\varphi \in \Gamma$ *for each* $\Gamma \in \mathrm{Max}_{\boldsymbol{L}}$ *such that* $\Psi \subseteq \Gamma$.
2. $\varphi \in \boldsymbol{L}$ *iff* $\varphi \in \Gamma$ *for each* $\Gamma \in \mathrm{Max}_{\boldsymbol{L}}$.

LEMMA A.4. *For all* $\Gamma \in \mathrm{Max}_{\boldsymbol{L}}$ *and* $\varphi \in \mathrm{For}$ *the following conditions are equivalent*:

(a) $\ulcorner\Box\varphi\urcorner \in \Gamma$.
(b) $\Gamma \vdash_{\boldsymbol{L}} \Box\varphi$.
(c) $\{\psi : \ulcorner\Box\psi\urcorner \in \Gamma\} \vdash_{\mathbf{PL}} \varphi$.
(d) $\varphi \in \Delta$ *for each* $\Delta \in \mathrm{Max}_{\mathbf{PL}}$ *such that* $\{\, \psi : \ulcorner\Box\psi\urcorner \in \Gamma \,\} \subseteq \Delta$.

PROOF: "(a) $\Rightarrow$ (d)" It is trivial. "(d) $\Leftrightarrow$ (c)" By Lemma A.3(1).

"(c) $\Rightarrow$ (b)" Ether $\varphi \in \mathbf{PL}$ or for some $\psi_1, \ldots, \psi_n \in \{\psi : \ulcorner\Box\psi\urcorner \in \Gamma\}$, $n > 0$, we have $\ulcorner(\psi_1 \wedge \cdots \wedge \psi_n) \supset \varphi\urcorner \in \mathbf{PL}$. But the first case entails the second case. Hence $\ulcorner(\Box\psi_1 \wedge \cdots \wedge \Box\psi_n) \supset \Box\varphi\urcorner \in \boldsymbol{L}$, by (pk). But Γ contains each of $\ulcorner\Box\psi_1\urcorner, \ldots, \ulcorner\Box\psi_n\urcorner$ since $\Box\mathbf{PL} \subseteq \Gamma$. So $\Gamma \vdash_{\boldsymbol{L}} \Box\varphi$.

"(a) $\Leftrightarrow$ (b)" By Lemma A.1(1). □

A.2. Canonical models. Completeness

Let $\boldsymbol{L}$ be a t-normal logic and $\Gamma \in \mathrm{Max}_{\boldsymbol{L}}$. We say that $\langle w_\Gamma, A_\Gamma, V_\Gamma\rangle$ is the *canonical model for* $\boldsymbol{L}$ *and* Γ iff it satisfies the following conditions:

- $w_\Gamma := \Gamma$,
- $A_\Gamma := \{\, \Delta \in \mathrm{Max}_{\mathbf{PL}} : \forall_{\psi \in \mathrm{For}}(\ulcorner\Box\psi\urcorner \in \Gamma \Rightarrow \psi \in \Delta\,) \,\}$,
- $V_\Gamma \colon \mathrm{For} \times (\{w_\Gamma\} \cup A_\Gamma) \to \{0, 1\}$ is the valuation such that for all $\varphi \in \mathrm{For}$ and $\Delta \in \{w_\Gamma\} \cup A_\Gamma$

$$V_\Gamma(\varphi,\Delta) \coloneqq \begin{cases} 1 & \text{if } \varphi \in \Delta \\ 0 & \text{otherwise} \end{cases}$$

We need the following lemmas to prove the completeness of $\mathbf{S0.5}^\circ$, $\mathbf{S0.5}^\circ[\mathtt{D}]$, $\mathbf{S0.5}^\circ[\mathtt{T^q}]$ and $\mathbf{S0.5}$.

Lemma A.5. *$\langle w_\Gamma, A_\Gamma, V_\Gamma\rangle$ is a tn-model.*

Proof: Thanks to properties of maximal sets (see Lemma A.1), for every $\Delta \in \{w_\Gamma\} \cup A_\Gamma$ the assignment $V_\Gamma(\cdot,\Delta)$ belongs to $\mathsf{Val}^{\mathsf{cl}}$. Moreover, we prove that $V_\Gamma(\cdot, w_\Gamma)$ satisfies $(V^w_\Box)$ and $(V^w_\Diamond)$ for each $\varphi \in \mathrm{For}$.

Firstly, $V_\Gamma(\Box\varphi, w_\Gamma) = 1$ iff $\ulcorner\Box\varphi\urcorner \in \Gamma$ iff $\varphi \in \Delta$ for each $\Delta \in \mathrm{Max}_{\mathrm{PL}}$ such that $\{\psi \in \mathrm{For} : \ulcorner\Box\psi\urcorner \in \Gamma\} \subseteq \Delta$ (by Lemma A.4) iff $\varphi \in \Delta$ for each $\Delta \in A_\Gamma$ iff $V_\Gamma(\varphi,\Delta) = 1$ for each $\Delta \in A_\Gamma$.

Secondly, since $\boldsymbol{L} \subseteq \Gamma$, $\ulcorner\Diamond\varphi \equiv \neg\Box\neg\varphi\urcorner \in \Gamma$. Hence, by Lemma A.1, $V_\Gamma(\Diamond\varphi, w_\Gamma) = 1$ iff $\ulcorner\Diamond\varphi\urcorner \in \Gamma$ iff $\ulcorner\Box\neg\varphi\urcorner \notin \Gamma$ iff $V_\Gamma(\neg\varphi,\Delta) = 0$ for some $\Delta \in A_\Gamma$ iff $V_\Gamma(\varphi,\Delta) = 1$ for some $\Delta \in A_\Gamma$. □

Lemma A.6. 1. *If $\boldsymbol{L}$ contains $(\mathtt{T})$ then $w_\Gamma \in A_\Gamma$.*

2. *If $\boldsymbol{L}$ contains $(\mathtt{D})$ then $A_\Gamma \neq \emptyset$.*

3. *If $\boldsymbol{L}$ contains $(\mathtt{T^q})$ then either $A_\Gamma = \emptyset$ or $w_\Gamma \in A_\Gamma$.*

Proof: By Lemma A.1, $\boldsymbol{L} \subseteq \Gamma$. So in any specific case we have:

1. For any $\psi \in \mathrm{For}$, $\ulcorner\Box\psi \supset \psi\urcorner \in \Gamma$. So, if $\ulcorner\Box\psi\urcorner \in \Gamma$ then $\psi \in \Gamma$, by Lemma A.1(5). Hence $\Gamma \in A_\Gamma$. Moreover, $\Gamma \in \mathrm{Max}_{\mathrm{PL}}$, by Fact A.2.

2. For any $\tau \in \mathrm{Taut}_{\mathrm{cl}}$ we have $\ulcorner\Box\tau\urcorner$ and $\ulcorner\Box\tau \supset \Diamond\tau\urcorner$ belong to Γ. So, $\Diamond\tau \in \Gamma$, by Lemma A.1(5). Hence $V(\Diamond\tau, \Gamma) = 1$. So, by Lemma A.5, for some $\Delta \in A_\Gamma$ we have $V(\tau,\Delta) = 1$. Therefore, $A_\Gamma \neq \emptyset$.

3. For any $\psi \in \mathrm{For}$ we have $\ulcorner(\mathtt{D}) \supset (\Box\psi \supset \psi)\urcorner \in \Gamma$. Suppose that $A_\Gamma \neq \emptyset$. Then $(\mathtt{D}) \in \Gamma$, by Fact 2.4(1) and Lemma A.5. Thus, $\ulcorner\Box\psi \supset \psi\urcorner \in \Gamma$. Therefore, as in item 1, we can show that $\Gamma \in A_\Gamma$. □

For $\boldsymbol{L} \in \{\mathbf{S0.5}^\circ, \mathbf{S0.5}^\circ[\mathtt{D}], \mathbf{S0.5}^\circ[\mathtt{T^q}], \mathbf{S0.5}\}$. Let $\mathbf{M}_{\boldsymbol{L}}$ be the class of all $\boldsymbol{L}$-models. We can show that $\boldsymbol{L}$ is complete with respect to $\mathbf{M}_{\boldsymbol{L}}$.

Theorem A.7 (Completeness). *All formulas valid in $\mathbf{M}_{\boldsymbol{L}}$ are theses of $\boldsymbol{L}$.*

Proof: For $\mathbf{S0.5}^\circ$: Suppose that φ is valid in $\mathbf{M}_{\mathbf{S0.5}^\circ}$ and $\Gamma \in \mathrm{Max}_{\mathbf{S0.5}^\circ}$. By Lemma A.5, $\langle w_\Gamma, A_\Gamma, V_\Gamma\rangle$ belongs to $\mathbf{M}_{\mathbf{S0.5}^\circ}$. Thus, $V_\Gamma(\varphi, w_\Gamma) = 1$. Hence $\varphi \in \Gamma$. So, we have shown that φ belongs to all $\mathbf{S0.5}^\circ$-maximal sets. Hence $\varphi \in \mathbf{S0.5}^\circ$, by Lemma A.3(2).

For $\boldsymbol{L} \in \{\mathbf{S0.5}^\circ[\mathtt{D}], \mathbf{S0.5}^\circ[\mathtt{T^q}], \mathbf{S0.5}\}$: Same as above, taking $\boldsymbol{L}$ instead of $\mathbf{S0.5}^\circ$. By Lemmas A.5 and A.6, $\langle w_\Gamma, A_\Gamma, V_\Gamma \rangle$ belongs to $\mathsf{M}_{\boldsymbol{L}}$. □

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