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# D-COMPLETE SINGLE AXIOMS FOR THE EQUIVALENTIAL CALCULUS WITH THE RULES $D$ AND $R$

**Abstract**

Ulrich [9] showed that most of the known axiomatisations of the classical equivalence calculus (EC) are D-incomplete, that is, they are not complete with the condensed detachment rule (D) as the primary rule of the proof procedure. He proved that the axiomatisation $EEpEqrErEqp, EEEpppp$ by Wajsberg [10] is D-complete and pointed out a number of D-complete single axioms, including one organic single axiom. In this paper we present new single axioms for EC with the condensed detachment and the reversed condensed detachment rules that form D-complete bases and are organic.

*Keywords*: equivalential calculus, D-complete, single axiom, condensed detachment.

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## 1. Introduction

The main goal of the paper is to present new single axioms for the equivalential calculus (EC) with two rules of the proof procedure: the condensed detachment (D) and the reversed condensed detachement (R). The axioms form with the rules many different D-complete bases for EC. The first part of the article introduces EC and the basic concepts used in the paper. Then the issue of single inorganic axioms for a certain variant of EC calculus is discussed. In the third part of the paper, 8 new organic axioms for EC, unknown so far, are pointed out.

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## 2. Equivalential calculus

The well-formed formulas (wff) of the classical equivalential calculus are the formulas built from a binary connective $E$ and denumerably many sentence letters $p, q, r, \dots$. Each sentence letter is a wff. If $\alpha$ and $\beta$ are wff, so is $E\alpha\beta$.

The classical equivalential calculus (EC) is the set of all formulas that are tautologies of the standard matrix for the equivalence (E) from the classical propositional calculus. The set is identical to the set of such formulas in which each sentence letter occurs an even number of times Leśniewski [5] was the first to point this out.

In the early days of the study, EC axiomatisations were sets of formulas and two standard rules of the proof procedure: modus ponens for equivalence and substitution. The first axiomatisation was proposed by Leśniewski [5]. The first single shortest axiom was found by Łukasiewicz [12]. Currently, many different axiomatisations of EC are known, and research on EC is focused on the area of finding the single shortest axiom depending on an established set of rules of the proof procedure.

Instead of the modus ponens for equivalence and substitution, the condensed detachment ($D$) rule was introduced, which combines detachment with the best possible substitution. A detailed presentation of the rule $D$ may be found in, for example, [1, 2, 4, 9]. Suppose that $s(\beta)$ is some substitution of formula $\beta$. The rule $D$ allows one to write $s(\beta)$ in the proof if you have formula $E\alpha\beta$ and formula $\gamma$ for which the formulas $s(\alpha)$ and $s(\gamma)$ are identical. Moreover, the substitution $s$ needs to satisfy the condition that it is always a most general unifier (cf. [8]) for the formulas $\alpha$ and $\gamma$ and the formula $s(\beta)$ has the smallest possible number of common sentence letters with $E\alpha\beta$. In short, the result of applying the rule $D$ to the formulas $E\alpha\beta$ and $\gamma$ should be such a formula $s(\beta)$ from which, using only the substitution rule, one can obtain the set of all possible formulas that can be obtained from these formulas using the modus ponens rule for equivalence and the substitution rule. The formula $s(\beta)$ is therefore the most general formula possible.

By analogy, one can define the reversed condensed detachment ($R$). A detailed presentation of the rule $R$ may be found in, for example, [6, 3]. The difference between these rules is that the rule $D$ allows you to detach

$s(\alpha)$ from the formula $s(E\alpha\beta)$ and the result is $s(\beta)$. In contrast, the rule $R$ allows you to detach $s(\beta)$ from the formula $s(E\alpha\beta)$ and the result is $s(\alpha)$. The other conditions are exactly the same as for the rule $D$.

EC built solely with the rule $D$ we will denote as EC+D. EC with the rule $R$ we will denote as EC+R. We will abbreviate EC with the two rules $D$ and $R$ as EC+DR.

We will say of EC that it is complete if and only if it contains all expressions satisfying the standard matrix for the equivalence connective. So EC is complete if and only if it contains all such formulas in which each sentence letter occurs an even number of times. We will say that a theory is D-complete, if an axiomatisation based on the rule $D$ or the rule $R$, as the only rules allowed in the proof, forms a complete theory. We say that a calculus is D-incomplete, if it is based on the rule $D$ or the rule $R$ and there exists at least one formula satisfying the standard matrix for the equivalence functor that cannot be proved in the theory.

The converse formula is a formula in which every subformula of the form $E\alpha\beta$ is replaced with $E\beta\alpha$. E.g. the converse of $Epq$ is $Eqp$, the converse of $EEpqr$ is $ErEqp$. A formula is an organic formula if and only if no proper subformula of this formula is a theorem. Otherwise, we say that the formula is inorganic. E.g. the formula $EEpqEqp$ is organic, but the formula $EEppq$ is inorganic, since its frament $Epp$ is a theorem of EC. The set of all theorems of EC is identical to the set of such formulas in which each sentence letter occurs an even number of times. We say that a formula is two-property if and only if each sentence letter occuring occurs two times in the formula. So every formula of EC can be derived from some two-property formula by the substitution rule. It is interesting to note that if a formula is two-property, then using the rule $D$ or the rule $R$, only formulas with the same property can be derived from it.

We currently know fourteen single shortest (11-character long) axioms for EC+D, the last one found in 2003 [11]. In addition, fourteen corresponding converse formulas [7], which are axioms for EC+R. Furthermore, eleven (11-character long) axioms are known for EC+DR [3]. All these 39 formulas are single shortest axioms for EC+DR and are D-incomplete bases for EC+DR. We use the names of these axioms as in [3].

Ulrich [9] has shown that the axioms $\{EEpEqrErEqp, EEEpppp\}$ [10] form a D-complete base for EC+D. It is easy to show that the converse axioms constitute a D-complete base for EC+R.

THEOREM 2.1. *Formulas*

$$EEEpqrEErqp, \tag{2.1}$$

$$EpEpEpp \tag{2.2}$$

*constitute a D-complete base for EC+R.*

PROOF: These formulas are converse axioms of Wajsberg's axioms $\{EEpEqrErEqp, EEEpppp\}$. Since Wajsberg's axioms are D-complete with the rule $D$, their converses are D-complete with the rule $R$. □

The calculus EC+DR was investigated by Hodgson [3]. Each of the known classical single shortest axioms for this calculus is D-incomplete. In the following section, we will show the axiomatizations that are D-complete.

## 3. Inorganic single axioms for EC+DR

We will point out some general facts about the EC+DR calculus and discuss the single inorganic axioms for this calculus.

LEMMA 3.1. *If the axiomatisation is a D-complete base for EC+D or EC+R, it is a D-complete base for EC+DR.*

PROOF: The proof is immediate, it is sufficient to note that EC+DR is formed by adding one of the rules of the proof procedure ($D$ or $R$). Thus, EC+DR is formed from EC+D or EC+R by expanding the set of original rules of the proof procedure by the rule $D$ or the rule $R$, respectively. Monotonicity ensures that if EC+D or EC+R is D-complete, then EC+DR is also D-complete. □

Ulrich [9] proved that any formula of the scheme $EsEsEsEsA$, where $A$ is any single D-incomplete axiom for EC+D and does not contain the variable $s$, is a D-complete base for EC+D.

THEOREM 3.2. *Let $A$ be any single D-incomplete axiom for EC+R, such that $s$ does not occur in $A$. Then $EEEEAssss$ is a D-complete base for EC+R.*

PROOF: We conduct a 4-fold detachment using the rule $R$, which results in a single axiom $A$. From this axiom we derive the expression $EEEpqrEErqp$ (2.1), which is a converse of Wajsberg's first axiom, on the same basis we derive the expression $EEzEyExwEEEEAzyxw$. These derivations are possible because the formulas are two-property. From the second formula we detach $EEEEAssss$ using the rule $R$, as a result we get $EpEpEpp$ (2.2), which is the converse of Wajsberg's second axiom. □

The proof is analogous to the one in [9]. The axioms with the schemes $EsEsEsEsA$ and $EEEEAssss$ are each other's converses.

THEOREM 3.3. *Let $A$ be any single D-incomplete axiom for EC+DR, such that $s$ does not occur in $A$. Then $EsEsEsEsA$ and $EEEEAssss$ is a D-complete base for EC+DR.*

PROOF: For all single axioms $A$ D-incomplete for EC+D and EC+R the theorem is true by Lemma 3.1. For the single D-incomplete axioms $A$ of EC+DR, it can be shown that by 4-fold detachment via the rule $D$ or the rule $R$, one can always derive $A$ from $EsEsEsEsA$ or $EEEEAssss$. Since $A$ is a single axiom of EC+DR, it is possible to derive (2.1), $EEzEyExwEEEEAzyxw$ and $EEEEEEEAzyxwEzEyExw$ from it, and by the latter two axiom (2.2) can be derived from the corresponding axiom $A$ by means of an appropriate rule. □

Since we know 39 single 11-character D-incomplete axioms for EC+DR, by virtue of Theorem 3.3 above, 78 single axioms of EC+DR can be identified that constitute D-complete bases.

THEOREM 3.4. *Let $A$ be any single D-incomplete axiom for EC+DR, such that $s$ does not occur in $A$. Then formulas:*

$$EsEEEAsss \tag{3.1}$$

$$EsEsEEAss \tag{3.2}$$

$$EsEsEsEAs \tag{3.3}$$

*are D-complete bases for EC+DR, each separately as a single axiom.*

Proof: As there are two rules available to us, $D$ or $R$, we can apply them as required. By fourfold detachment we always obtain the axiom $A$. Then from $A$ and the given axiom it will always be possible to derive Wajsbegr's axioms (2.1), (2.2), analogous to the proof of Theorem 3.3. □

As a result, we have 117 new axioms. In total, we can generate 195 axioms with these techniques. All these axioms are inorganic. All these axioms are 19 characters long. Whether there is a shorter-than-19-character single D-complete inorganic axiom for EC+D, EC+R, EC+DR remains an open question.

## 4. Organic single axioms EC+DR

We discuss some organic D-complete axioms for EC+DR calculus. The axioms (4.2), (4.3), (4.4), (4.5), (4.7), (4.8), (4.9), (4.10) were previously unknown.

Ulrich [9] has shown that the formula

$$EEpqEEqrEsEsEsEsEpr \tag{4.1}$$

is a D-complete base for EC+D. The converse of this formula,

$$EEEEEErpssssErqEqp, \tag{4.2}$$

constitutes the D-complete base for EC+R. Both of these expressions are organic and constitute, by virtue of Lemma 3.1, each separately, a D-complete bases for EC+DR.

In the proofs of Theorems 4.3 and 4.4 we use the standard notation for the rules $D$ or the rule $R$. E.g. the description D1.2 means that the rule $D$ was applied to line 1 and line 2, which in this case were the minor and major premises for the rule $D$. The description D1.1 means that rule D was applied to line 1, which in this case was the minor and major premises for the rule $D$. Similarly, the description DD1.1.1 means the application of the rule $D$ to line 1 and to a certain formula D1.1, which is formed from the application of the rule $D$ to line 1.

THEOREM 4.1. *Formula*

$$EEEpqrEsEsEsEsEErpq \tag{4.3}$$

*is a single organic axiom of EC+DR, which forms a D-complete base.*

PROOF:

$$\begin{aligned}
&1.\ EEEpqrEsEsEsEsEErpq\\
D1.1{=}&2.!\ EtEtEtEtEEEsEsEsEsEErpqEpqr\\
DDDD2.1.1.1.1{=}&3.\ EEEsEsEsEsEErpqEpqr\\
R3.1{=}&4.\ EEwEwEwEwEEEEEpqrEsEsEsEsEErpqtuEtu\\
R4.1{=}&5.\ EwEwEwEwEEEEEtuvExExExExEEvtuEEpqr\\
&EsEsEsEsEErpq\\
DDDD5.1.1.1.1{=}&6.\ EEEEEtuvExExExExEEvtuEEpqr\\
&EsEsEsEsEErpq\\
R6.1{=}&7.\ EEEEstuExExExExEEustEErEErpqEpq\\
D7.1{=}&8.\ EErEErpqEpq\\
D1.8{=}&9.\ EsEsEsEsEEEpqrEErpq\\
DDDD9.1.1.1.1{=}&10.EEEpqrEErpq
\end{aligned}$$

Formula $EEEpqrEErpq$ (TN) is a single D-incomplete axiom for EC+R, so any two-property formula can be derived from it, including these two;

$$11.EEpEqrErEqp$$
$$12.EEwExEyEzEEEpqrEErpqEEEwxyz$$

Formula 11. is one of the Wajsberg D-complete axioms for EC+D. The second axiom can be derived in one step.

$$D12.9 = 13.EEEpppp.$$ □

THEOREM 4.2. *Formula*

$$EEEEEEqEprssssErEqp \tag{4.4}$$

*is a single organic axiom for EC+DR, which forms a D-complete base.*

PROOF:

$$
\begin{aligned}
&1.\ EEEEEEqEprssssErEqp\\
R1.1=&2.\ EEEEErEEqpEEEEEqEprsssstttt\\
RRRR2.1.1.1.1=&3.\ ErEEqpEEEEEqEprssss\\
D3.1=&4.\ EEutEEEEEuEtEEEEEEqEprssss\\
&\quad ErEqpvvvv\\
D4.1=&5.\ EEEEEEEEEEqEprssssEErEqp\\
&\quad EEEEEEuEtvwwwwEvEutxxxx\\
RRRR5.1.1.1.1=&6.\ EEEEEEqEprssssEErEqp\\
&\quad EEEEEEuEtvwwwwEvEut\\
D6.1=&7.\ EEEqpEEqEprrEEEEEEtEsuwwwwEuEts\\
R7.1=&8.\ EEtsEEtEsuu\\
R1.8=&9.\ EEEEEEqEprErEqpssss\\
RRRR9.1.1.1.1=&10. EEqEprErEqp
\end{aligned}
$$

Formula $EEpEqrErEpq$ (WN) is a single D-incomplete axiom for EC+D, so any two-property formula can be derived from it, including these two:

$$
\begin{aligned}
&11. EEEpqrEErqp\\
&12. EEzEyExwEEEEEEqEprErEqpzyxw
\end{aligned}
$$

Formula in the row 11. is the converse of Wajsberg D-complete axiom (2.1) for EC+R. The converse (2.2) of the second axiom can be derived in one step.

$$D12.9 = 13. EpEpEpp. \qquad \square$$

THEOREM 4.3. *Formula*

$$EEEpqrEsEsEsEsEpEqr \tag{4.5}$$

*is a single organic axiom for EC+DR, which forms a D-complete base.*

PROOF:

$$1. EEEpqrEsEsEsEsEpEqr$$
$$R.1.1 = 2. EEEEpqrpEqr$$

Formula $EEEEpqrpEqr$ (OYJ) is a single D-complete axiom of EC+DR. Two formulas can be derived from it:

$$3. EEEpqrEErqp$$
$$4. EEEEpqrExEyEzEwEpEqrExEyEzw$$

Formula in the row 3. is the converse of Wajsberg's D-complete axiom (2.1). The second axiom (2.2) can be derived in one step.

$$D4.1 = 5. EpEpEpp. \qquad \square$$

Formula

$$EEEEEEEpqrssssEpEqr, \tag{4.6}$$

the reverse of (4.5), is a single D-complete axiom of EC+DR as well.

Applying an analogous proof technique to Theorem 4.3, it can be proved that the following formulas are single D-complete axioms for EC+DR:

$$EEpEqrEsEsEsEsErEpq, \tag{4.7}$$

$$EEEpqrEsEsEsEsEqErp \tag{4.8}$$

Axiom 4.7 allows the derivation of the axiom $EpEEqrEqErp$ (XIM), which is a single D-incomplete axiom of EC+DR. Axiom 4.8 allows for a derivation of the axiom $EEEpqEEqrpr$ (HXH), which is a single D-incomplete axiom of EC+R.

The corresponding reverses of these formulas are single D-complete axioms for EC+DR. The reverse of (4.7), a formula

$$EEEEEEEpqrssssEErpq, \tag{4.9}$$

allows for a derivation of the axiom $EEEEpqrEqrp$ (DXN), which is a single D-incomplete axiom of EC+DR. The reverse of (4.8), a formula

$$EEEEEEEpqrssssEqErp, \tag{4.10}$$

allows for a derivation of the axiom $EpEEqEprErq$ (XGF), which is a single D-incomplete axiom EC+D.

Axioms (4.2), (4.3), (4.4), (4.5) (4.7), (4.8), (4.9), (4.10) are organic. All these axioms are 19–characters long. Whether there is a shorter-than-19-character single D-complete axiom for EC+D, EC+R, EC+DR remains an open question.

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