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OPEN FILTERS AND CONGRUENCE RELATIONS ON SELF-DISTRIBUTIVE WEAK HEYTING ALGEBRAS

Abstract

In this paper, we study (open) filters and deductive systems of self-distributive weak Heyting algebras (SDWH-algebras) and obtain some results which determine the relationship between them. We show that the variety of SDWH-algebras is not weakly regular and every open filter is the kernel of at least one congruence relation. Finally, we characterize those SDWH-algebras which are weakly regular by using some properties involving principal congruence relations.

Keywords: SDWH-algebra, open filter, deductive system, congruence kernel, weakly regular.

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1. Introduction

Celani and Jansana introduced the concept of weak Heyting algebras in 2005 ([4]). A WH-algebra is a bounded distributive lattice with a binary operation → satisfying the properties of the strict implication in the modal logic K. These algebras are a generalization of Heyting algebras. Alizadeh and Joharizadeh ([1]) presented an algorithm to construct and count all

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nonisomorphic finite WH-algebras. San Martín ([14]) studied the compatible operations in some subvarieties of the variety of WH-algebras. He studied Principal congruences in WH-algebras in [15] and characterized the congruences in weak implicative semi-lattices in [16]. The notion of multipliers in weak Heyting algebras was defined in [10] and the relations between multipliers, closure operators, and homomorphisms in weak Heyting algebras were obtained.
Some of the known subvarieties of the variety WH-algebras are SRL, B, RWH and TWH. In 1976, years before the definition of WH-algebras, "the subresiduated lattices were defined and studied in a different way by George Epstein and Alfred Horn [9]. In the mentioned paper the authors proved that the Lindenbaum-Tarski algebra of the calculus R4 is a subresiduated lattice. They also introduced several subvarieties of SRL and counterpart logic.
Another subvariety of WH-algebras is the variety of basic algebras, first studied by Mohammad Ardeshir and Wim Rutenberg in 1998 ([2]). The counterpart logic of this variety, also called Basic logic, was first introduced by Albert Visser in 1981 ([17]) and then by Wim Ruitenberg in 1992 ([14]). As mentioned in [4], variety RWH corresponds to the logic defined by the class of reflexive Kripke models, and the variety TWH corresponds to the logic defined by the class of transitive Kripke models.
These five varieties (WH, SRL, B, RWH and TWH) are Archimedean varieties with congruence extension properties (CEP), but they are not locally finite either.
A self-distributive operation is distributive over itself. They have an important role in mathematics because of their connection with many fields such as knot theory, algebraic combinatorics, quantum groups ([7]), quandles ([11]) and Hilbert algebra ([8]). Also, self-distributive operations provide solutions of the Yang–Baxter equation.

Recently, we introduced self-distributive WH-algebras and obtained some of their properties. SDWH-algebras of orders 3 and orders 4 were characterized. Finally, we obtained the relation between SDWH-algebras and known subvarieties of WH-algebras, like TWH-algebras, RWH-algebras, SRL-algebras and Basic algebras ([13]). The relations between these subvarieties of WH-algebra are depicted in Figure 1.

Birkhoff studied the relation between congruence relations and ideals of lattices in [3]. He proposed in:

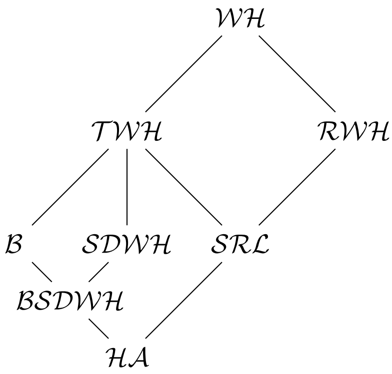

Figure 1. The order of $\mathcal{WH}$ subvarieties

Problem 73. Find necessary and sufficient conditions, in order that the correspondence between the congruence relations and ideals of a lattice be one-to-one.

Historically ideal theory for lattices was developed by Hashimoto ([12]). He established that there is a one-to-one correspondence between ideals and congruence relations of a lattice L under which the ideal corresponding to a congruence relation is a whole congruence class under it if and only if L is a generalized Boolean algebra. An algebra with a constant 1 is weakly regular if every two congruence relations coincide whenever they have the same congruence class containing 1 ([7]). An interesting problem is to find weakly regular algebras in varieties that are not varieties of weakly regular algebras (see [5]).

In this paper, we study the (generated) open filters of SDWH-algebras and prove that the lattice of open filters is a complete Heyting algebras such that the compact elements are principal open filters. Then the notion of deductive systems of an SDWH-algebra is introduced and the relations between deductive systems, open filters, and filters of SDWH-algebras are obtained. It is shown that every open filter is a kernel of at least one congruence relation on an SDWH-algebra. Moreover, the variety of SDWH-algebras is not weakly regular. We use the concepts of deductive systems and open filters to define two congruence relations on every SDWH-algebra and obtain the relation between them. Finally, we obtain the necessary and sufficient conditions for which an SDWH-algebra is weakly regular.

2. Preliminaries

In this section, we recall the basic definitions and some properties of weak Heyting-algebras which we will need in the next sections.

Definition 2.1 ([4]). An algebra $\mathcal{H} = (H, \wedge, \vee, \rightarrow, 0, 1)$ of type $(2, 2, 2, 0, 0)$ is called a weak Heyting algebra (or WH-algebra) if $(H, \vee, \wedge, 0, 1)$ is a bounded distributive lattice and the following conditions hold for all $x, y, z \in H$:

(WH1) $(x \rightarrow y) \wedge (x \rightarrow z) = x \rightarrow (y \wedge z)$,

(WH2) $(x \rightarrow z) \wedge (y \rightarrow z) = (x \vee y) \rightarrow z$,

(WH3) $(x \rightarrow y) \wedge (y \rightarrow z) \leq x \rightarrow z$,

(WH4) $x \rightarrow x = 1$.

The following proposition provides some properties of WH-algebras.

Proposition 2.2. ([1, 4]) Let $\mathcal{H}$ be a WH-algebra. Then the following hold for all $x, y, z \in H$:

(W1) if $x \leq y$, then $y \rightarrow z \leq x \rightarrow z$ and $z \rightarrow x \leq z \rightarrow y$,

(W2) if $x \leq y$, then $x \rightarrow y = 1$,

(W3) if $x \leq y \leq z$, then $z \rightarrow x = (z \rightarrow y) \wedge (y \rightarrow x)$,

(W4) $x \rightarrow y = x \rightarrow (x \wedge y)$,

(W5) $(x \rightarrow y) \rightarrow (y \rightarrow z) \leq (x \rightarrow y) \rightarrow (x \rightarrow z)$.

Definition 2.3 ([4, 13]). Let $\mathcal{H}$ be a WH-algebra.

(1) $\mathcal{H}$ is a Basic algebra iff satisfies the inequality $x \leq 1 \rightarrow x$ (I),

(2) $\mathcal{H}$ is a RWH-algebra iff satisfies the inequality $x \wedge (x \rightarrow y) \leq y$ (R),

(3) $\mathcal{H}$ is a TWH-algebra iff satisfies the inequality $x \rightarrow y \leq z \rightarrow (x \rightarrow y)$ (T),

(4) $\mathcal{H}$ is a subresiduated lattice, or sr-lattice iff satisfies the inequalities (T) and (R),

(5) $\mathcal{H}$ is an SDWH-algebra iff satisfies $x \to (y \to z) = (x \to y) \to (x \to z)$ (SD).

PROPOSITION 2.4 ([13]). Let $\mathcal{H}$ be a WH-algebra.

(1) $\mathcal{H}$ is a Heyting algebra if and only if $x = 1 \to x$, for all $x \in H$,

(2) $\mathcal{H}$ is an SDWH-algebra if and only if $x \to (y \to z) = y \to (x \to z)$, for all $x, y, z \in H$.

PROPOSITION 2.5 ([13]). Let $\mathcal{H}$ be an SDWH-algebra. Then the following hold, for all $x, y, z \in H$,

(1) $x \to (y \to x) = 1$,

(2) $x \to (x \to y) = 1 \to (x \to y) = x \to (1 \to y)$,

(3) $x \to (y \to (x \wedge y)) = 1$,

(4) $y \to z \leq x \to (y \to z)$,

(5) $x \to y \leq (z \to x) \to (z \to y)$,

(6) $x \to y \leq (y \to z) \to (x \to z)$.

DEFINITION 2.6 ([4]). Let $\mathcal{L}$ be a lattice. A non-empty subset F of L is called a filter of $\mathcal{L}$, if it is satisfies the following conditions, for all $x, y \in L$

(F1) If $x, y \in F$, then $x \wedge y \in F$,

(F2) If $x \in F$ and $x \leq y$, then $y \in F$.

A filter F of a WH-algebra $\mathcal{H}$ is called an open filter of $\mathcal{H}$, if it is satisfies the following condition, for all $x \in H$.

(OF) If $x \in F$, then $1 \to x \in F$.

We denote by $OF(\mathcal{H})$ the set of all open filters of $\mathcal{H}$.

PROPOSITION 2.7 ([3]). Let $(L, \wedge, \vee, 0, 1)$ be a bounded distributive lattice. If $\langle a \rangle$ is the filter generated by element $a \in L$, we have

(1) $\langle a \rangle = \{x \in L |\ a \leq x\ \}$,

(2) $a \leq b$, then $\langle b \rangle \subseteq \langle a \rangle$,

(3) $\langle a\rangle \vee \langle b\rangle = \langle a \wedge b\rangle$,

(4) $\langle a\rangle \cap \langle b\rangle = \langle a \vee b\rangle$.

PROPOSITION 2.8 ([13, 14]). Let $\mathcal{H}$ be an SDWH-algebra. Given an integer $n \geq 1$, we define inductively

$$\Box^0(x) = x, \quad \Box^1(x) = 1 \to x, \quad \Box^n(x) = 1 \to (\Box^{n-1}(x)),$$
$$x \to^0 y = y, \quad x \to^n y = x \to (x \to^{n-1} y).$$

Then the following hold for all $x, y, z \in H$,

(N1) $x \to^{n+1} y = \Box^n(x \to y)$,

(N2) $\Box^n(x \wedge y) = \Box^n(x) \wedge \Box^n(y)$,

(N3) $n \leq m$ implies $\Box^n(x) \leq \Box^m(x)$,

(N4) $\Box^n(x \to (y \to z)) = \Box^{n+1}(x \to y) \to \Box^{n+1}(x \to z)$.

Let $\mathcal{H}$ be WH-algebra and $a, b \in H$. By $\Phi(a, b)$, we denote the principal congruence relation of $\mathcal{H}$ generated by (a, b), i.e., the smallest congruence relation that contains (a, b).

PROPOSITION 2.9 ([16]). Let $\mathcal{H}$ be WH-algebra. The binary term is defined

$$t_n(a, b) = (a \leftrightarrow b) \wedge \Box(a \leftrightarrow b) \wedge \cdots \wedge \Box^n(a \leftrightarrow b),$$

where $a \leftrightarrow b = (a \to b) \wedge (b \to a)$. Then $(x, y) \in \Phi(a, b)$ if and only if there exists $n \in \mathbb{N}$ satisfying:

(C1) $x \wedge a \wedge b \wedge t_n(a, b) = y \wedge a \wedge b \wedge t_n(a, b)$,

(C2) $(x \vee a \vee b) \wedge t_n(a, b) = (y \vee a \vee b) \wedge t_n(a, b)$,

(C3) $t_n(a, b) \leq x \leftrightarrow y$.

DEFINITION 2.10 ([6]). An algebra $\mathcal{A}$ with a constant 1 is called weakly regular iff for each congruence relations θ, ϕ on $\mathcal{A}$, we have $\theta = \phi$ whenever $[1]_\theta = [1]_\phi$.
A variety V is weakly regular if every $\mathcal{A} \in V$ has this property.

3. Open filters and deductive systems

In this section, we study the structure of open filters and deductive systems of SDWH-algebras.

Let S be a non-empty subset of a WH-algebra $\mathcal{H}$. The smallest open filter of $\mathcal{H}$ containing S, (i.e. $\cap\{F \in OF(\mathcal{H}) | S \subseteq F\}$), is called the open filter generated by S and it will be denoted by $\langle S\rangle_O$. If $S = \{a\}$, we write $\langle a\rangle_O$ instead of $\langle\ \{a\}\ \rangle_O$ and it is called principal open filter.

Proposition 3.1. Let $\mathcal{H}$ be an SDWH-algebra. Then the following statements are hold, for all $a, b \in H$:

(1) $\langle a\rangle_O = \{x \in H | \Box(a) \wedge a \leq x\} = \langle \Box(a) \wedge a\rangle$,

(2) if $a \leq 1 \to a$, then $\langle a\rangle_O = \langle a\rangle$,

(3) $a \leq b$ implies $\langle b\rangle_O \subseteq \langle a\rangle_O$,

(4) $\langle a\rangle_O \vee \langle b\rangle_O = \langle a \wedge b\rangle_O$,

(5) $\langle a \vee b\rangle_O \subseteq \langle a\rangle_O \cap \langle b\rangle_O = \langle(\Box(a) \wedge a) \vee (\Box(b) \wedge b)\rangle_O$,

(6) if $\langle a\rangle_O = \langle b\rangle_O$, then $x \to a = x \to b$ for all $x \in H$,

(7) $\langle a \to b\rangle_O = \langle a \to b\rangle$.

Proof: (1) By Proposition 2.7 part (1), we have $\langle a \wedge (1 \to a)\rangle = \{x \in H | a \wedge (1 \to a) \leq x\}$. Thus $F = \{x \in H | a \wedge (1 \to a) \leq x\}$ is a filter. We will prove that F is open. Let $x \in F$. Since $1 \to a \leq 1 \to (1 \to a)$ by Proposition 2.5 part (4), then

$a \wedge (1 \to a) \leq (1 \to a) \wedge (1 \to (1 \to a)) = 1 \to (a \wedge (1 \to a)) \leq 1 \to x$ by (WH1) and (W1). Then $1 \to x \in F$. Hence F is open filter containing a. But $\langle a\rangle_O$ is the smallest open filter containing a, therefore $\langle a\rangle_O \subseteq F$. On the other hand, since $a, 1 \to a \in \langle a\rangle_O$, then $a \wedge (1 \to a) \in \langle a\rangle_O$ by (F1). For any $x \in F$, we get $x \in \langle a\rangle_O$ by (F2). Hence $F \subseteq \langle a\rangle_O$.
(2) It follows from part (1).
(3) Since $a \leq b$, then $a \wedge (1 \to a) \leq b \wedge (1 \to b)$ by (W1). Using Proposition 2.7 part (2), we get $\langle b \wedge (1 \to b)\rangle \subseteq \langle a \wedge (1 \to a)\rangle$. Hence $\langle b\rangle_O \subseteq \langle a\rangle_O$ by part (1).
(4) Using part (1), (WH1) and Proposition 2.7 part (3), we have

$$\begin{aligned}\langle a \wedge b\rangle_O =& \langle a \wedge b \wedge (1 \to (a \wedge b))\rangle = \langle a \wedge (1 \to a) \wedge b \wedge (1 \to b)\rangle \\ =& \langle a \wedge (1 \to a)\rangle \vee \langle b \wedge (1 \to b)\rangle = \langle a\rangle_O \vee \langle b\rangle_O.\end{aligned}$$

(5) Using part (1) and then Proposition 2.7 part (4), we have
$\langle a\rangle_O \cap \langle b\rangle_O = \langle \Box(a)\wedge a\rangle \cap \langle \Box(b)\wedge b\rangle = \langle (\Box(a)\wedge a)\vee(\Box(b)\wedge b)\rangle$.

Put $u := (\Box(a)\wedge a)\vee(\Box(b)\wedge b) = (a\wedge(1\to a))\vee(b\wedge(1\to b))$. We will show that $u \leq 1\to u$. By (W1), (WH1) and Proposition 2.5 part (4), we obtain

$$\begin{aligned}1\to u &= 1\to[(a\wedge(1\to a))\vee(b\wedge(1\to b))]\\ &\geq [1\to(a\wedge(1\to a))]\vee[1\to(b\wedge(1\to b))]\\ &= [(1\to a)\wedge(1\to(1\to a))]\vee[(1\to b)\wedge(1\to(1\to b))]\\ &\geq (1\to a)\vee(1\to b)\\ &\geq (a\wedge(1\to a))\vee(b\wedge(1\to b)) = u.\end{aligned}$$

So $\langle u\rangle_O = \langle u\rangle$ by part (2). Hence $\langle a\rangle_O\cap\langle b\rangle_O = \langle(\Box(a)\wedge a)\vee(\Box(b)\wedge b))\rangle_O$. Also, $\langle a\vee b\rangle_O \subseteq \langle a\rangle_O\cap\langle b\rangle_O$ by part (3).
(6) Let $\langle a\rangle_O = \langle b\rangle_O$. Then $\langle a\wedge(1\to a)\rangle = \langle b\wedge(1\to b)\rangle$. We get $a\wedge(1\to a) = b\wedge(1\to b)$. So $x\to(a\wedge(1\to a)) = x\to(b\wedge(1\to b))$. Using Proposition 2.5 parts (2), (4) and (WH1) we get $x\to a = x\to b$.
(7) Using part (1) and then Proposition 2.5 part (4), we get
$\langle a\to b\rangle_O = \{x\in H|(a\to b)\wedge(1\to(a\to b))\leq x\} = \{x\in H|a\to b\leq x\} = \langle a\to b\rangle$. $\square$

In an SDWH-algebra $\langle a\rangle_O\cap\langle b\rangle_O, \langle a\vee b\rangle_O$ may not be equal in general. See the following example:

Example 3.2. Let $H=\{0,a,b,1\}$ where $0<a,b<1$ such that a,b are not comparable. Consider the following binary operation:

$\to$	0	a	b	1
0	1	1	1	1
a	0	1	0	1
b	1	1	1	1
1	0	1	0	1

It is easy to see that $\mathcal{H}=(H,\vee,\wedge,\to,0,1)$ is an SDWH-algebra and
$\langle a\rangle_O = \{c\in H|x\geq a\wedge(1\to a)\} = \{x\in H|x\geq a\} = \{1,a\}$,
$\langle b\rangle_O = \{c\in H|x\geq b\wedge(1\to b)\} = \{x|\in H|x\geq 0\} = \{1,b,a,0\}$.
Then $\langle a\rangle_O\cap\langle b\rangle_O = \{1,a\}$, but $\langle a\vee b\rangle_O = \langle 1\rangle_O = \{1\}$. Therefore $\langle a\vee b\rangle_O \subsetneq \langle a\rangle_O\cap\langle b\rangle_O$.

LEMMA 3.3. *Let F be an open filter of an SDWH-algebra $\mathcal{H}$ and $y \in F$. Then $x \to y \in F$ for all $x \in H$.*

PROOF: Let $x \in H$ be arbitrary. We have $1 \to y \in F$ by (OF). Since $1 \to y \leq x \to y$ by (W1), then $x \to y \in F$ by (F2). □

The next proposition gives a concrete description of the open filter generated by a subset of an SDWH-algebra.

PROPOSITION 3.4. Let $\{F_i\}_{i \in I}$ be a family of open filters of an SDWH-algebra $\mathcal{H}$, $S \subseteq H$ and $a \in H \backslash S$. Then

(1) $\langle S \rangle_O = \{x \in H | \ s_1 \wedge \cdots \wedge s_n \wedge \Box(s'_1) \wedge \cdots \wedge \Box(s'_m) \leq x$ for some $m, n \in N, s_1, \ldots, s_n, s'_1, \ldots, s'_m \in S\} = \langle S \cap \Box(\langle S \rangle) \rangle$,

(2) $\langle S \cup \{a\} \rangle_O = \{x \in H | a \wedge s_1 \wedge \cdots \wedge s_n \wedge \Box(a) \wedge \Box(s'_1) \wedge \cdots \wedge \Box(s'_m) \leq x$ for some $m, n \in N, s_1, \ldots, s_n, s'_1, \ldots, s'_m \in S\}$,

(3) $\langle \cup_{i \in I} F_i \rangle_O = \{x \in H | f_{i_1} \wedge f_{i_2} \wedge ... \wedge f_{i_m} \leq x$ for some $j = 1, ..., m$ and $f_{i_j} \in F_{i_j}\}$.

PROOF: (1) We denote by F the set from the right part of equality from announce (above). It is easy to prove that F is a filter containing S. We will show that F is open. Let $x \in F$. Then there exist $m, n \in N$ and $s_1, \ldots, s_n, s'_1, \ldots, s'_m \in S$ such that $s_1 \wedge \cdots \wedge s_n \wedge \Box(s'_1) \wedge \cdots \wedge \Box(s'_m) \leq x$. Since $1 \to (1 \to s'_i) \geq 1 \to s'_i$ by Proposition 2.5 part (4), then

$$\begin{aligned} 1 \to x &\geq (1 \to s_1) \wedge \cdots \wedge (1 \to s_n) \wedge (1 \to (1 \to s'_1)) \wedge \cdots \wedge (1 \to (1 \to s'_m)) \\ &\geq (1 \to s_1) \wedge \cdots \wedge (1 \to s_n) \wedge (1 \to s'_1) \wedge \cdots \wedge (1 \to s'_m). \end{aligned}$$

by (WH1). Hence $1 \to x \in F$ by (F2). But $\langle S \rangle_O$ is smallest open filter containing S, therefore $\langle S \rangle_O \subseteq F$.
Now, we have $s_i, 1 \to s'_i \in \langle S \rangle_O$. Thus $s_1 \wedge \cdots \wedge s_n \wedge (1 \to s'_1) \wedge \cdots \wedge (1 \to s'_n) \in \langle S \rangle_O$ by (F1). So for any $x \in F$, we have $x \in \langle S \rangle_O$. Hence $F \subseteq \langle S \rangle_O$.
(2) and (3) are a direct consequence of (1). □

PROPOSITION 3.5. Let $\mathcal{H}$ be an SDWH-algebra. Then $(OF(\mathcal{H}), \wedge, \vee, \{1\}, H)$ is a complete distributive lattice.

PROOF: Suppose that $\{F_i\}_{i \in I}$ is a family of open filters of $\mathcal{H}$. It is easy to check that the infimum of of this family is $\wedge_{i \in I} F_i = \cap_{i \in I} F_i$ and the supermum is $\vee_{i \in I} F_i = \langle \cap_{i \in I} F_i \rangle_O$ Therefore $(OF(\mathcal{H}), \wedge, \vee, \{1\}, H)$ is a complete

lattice.
We will show that for every open filter F and every family $\{F_i\}_{i\in I}$ of open filters, $F \wedge (\vee_{i\in I} F_i) = \vee_{i\in I}(F \wedge F_i)$. Clearly, $\vee_{i\in I}(F \wedge F_i) \subseteq F \wedge (\vee_{i\in I} F_i)$. Conversely, suppose that $x \in F \wedge (\vee_{i\in I} F_i)$. Then $x \in F$ and $x \geqslant f_{i_1} \wedge f_{i_2} \wedge \ldots \wedge f_{i_m}$ for some $j = 1, \ldots, m$ and $f_{i_j} \in F_{i_j}$. Since $(H, \vee, \wedge, 0, 1)$ is a distributive lattice, then $x = x \vee (f_{i_1} \wedge f_{i_2} \wedge \ldots \wedge f_{i_m}) \geqslant (x \vee f_{i_2}) \wedge \ldots \wedge (x \vee f_{i_m})$. We have $x \vee f_{i_j} \in F \cap F_{i_j}$, for every $1 \leq j \leq m$. So $x \in \vee_{i\in I}(F \wedge F_i)$ by Proposition 3.4 part (3). Hence $F \wedge (\vee_{i\in I} F_i) \subseteq \vee_{i\in I}(F \wedge F_i)$. □

Proposition 3.6. Let F_1, F_2 be open filters of an SDWH-algebra $\mathcal{H}$. Put $F_1 \to F_2 := \{x \in H | \langle x\rangle_O \cap F_1 \subseteq F_2\}$. Then $F_1 \to F_2 = \{x \in H | (x \wedge \Box(x)) \vee y \in F_2, \text{ for all } y \in F_1\} \in OF(\mathcal{H})$.

Proof: Put $F := \{x \in H | (x \wedge \Box(x)) \vee y \in F_2, \text{ for all } y \in F_1\}$. We will prove that $F_1 \to F_2 = F$. Suppose that $x \in F_1 \to F_2$. Then $\langle x\rangle_O \cap F_1 \subseteq F_2$. Let $y \in F_1$ be arbitrary. We get that $(x \wedge \Box(x)) \vee y \in \langle x\rangle_O \cap F_1$. So $(x \wedge \Box(x)) \vee y \in F_2$. Therefore $x \in F$. Hence $F_1 \to F_2 \subseteq F$.
Conversely, suppose that $x \in F$ and $y \in \langle x\rangle_O \cap F_1$. Then $(x \wedge \Box(x)) \leq y$ and $y \in F_1$. We get that $y = (x \wedge \Box(x)) \vee y \in F_2$. Thus $x \in F_1 \to F_2$. Hence $F \subseteq F_1 \to F_2$.
Now, we will prove that $F_1 \to F_2$ is an open filter. Since $(1 \wedge \Box(1)) \vee y = 1 \in F_2$ for all $y \in F_1$, then $1 \in F_1 \to F_2$ and $F_1 \to F_2$ is a non-empty subset of H. Let $x, y \in H$ such that $x \leq y$ and $x \in F_1 \to F_2$. So $\langle x\rangle_O \cap F_1 \subseteq F_2$ and $\langle y\rangle_O \subseteq \langle x\rangle_O$ by Proposition 3.1 part (3). Then $\langle y\rangle_O \cap F_1 \subseteq \langle x\rangle_O \cap F_1 \subseteq F_2$. Hence $y \in F_1 \to F_2$.
Let $x, y \in H$ such that $x, y \in F_1 \to F_2$. Then $\langle x\rangle_O \cap F_1 \subseteq F_2$ and $\langle y\rangle_O \cap F_1 \subseteq F_2$. Using Proposition 3.1 part (4) and Proposition 3.4, we have $\langle x \wedge y\rangle_O \cap F_1 = (\langle x\rangle_O \vee \langle y\rangle_O) \cap F_1 \subseteq F_2$. Therefore $x \wedge y \in F_1 \to F_2$. Hence $F_1 \to F_2$ is a filter.
Let $x \in F_1 \to F_2$. Then $(x \wedge \Box(x)) \vee y \in F_2$, for all $y \in F_1$. Since $\Box(x) \leq \Box^2(x)$ by (N3), then $(x \wedge \Box(x)) \vee y \subseteq \Box(x) \vee y = (\Box(x) \wedge \Box^2(x)) \vee y$. So $(\Box(x) \wedge \Box^2(x)) \vee y \in F_2$. Hence $F_1 \to F_2$ is open. □

In the next proposition, we will prove that $OF(\mathcal{H})$ forms a complete Heyting algebra with respect to inclusion.

Proposition 3.7. Let $\mathcal{H}$ be an SDWH-algebra. Define binary operations $\wedge$, $\vee$ and $\to$ on $OF(\mathcal{H})$ as follows: for all $F_1, F_2 \in OF(\mathcal{H})$, $F_1 \wedge F_2 = F_1 \cap F_2$,

$F_1 \vee F_2 = \langle F_1 \cup F_2 \rangle_O$, $F_1 \to F_2 = \{x \in H | \langle x \rangle_O \cap F_1 \subseteq F_2\}$. Then $(OF(\mathcal{H}), \wedge, \vee, \to, \{1\}, H)$ is a complete Heyting algebra.

PROOF: By Proposition 3.5, $(OF(\mathcal{H}), \wedge, \vee, \{1\}, H)$ is a complete lattice. Next, we will prove that $F_1 \wedge F_2 \subseteq F_3$ if and only if $F_1 \subseteq F_2 \to F_3$. Suppose $F_1 \wedge F_2 \subseteq F_3$ and $x \in F_1$. Then $\langle x \rangle_O \subseteq F_1$, hence $\langle x \rangle_O \wedge F_2 \subseteq F_1 \wedge F_2 \subseteq F_3$. Thus $x \in F_2 \to F_3$.
Conversely, suppose that $F_1 \subseteq F_2 \to F_3$ and $x \in F_1 \wedge F_2$. Then $x \in F_1$. So $x \in F_2 \to F_3$. We get $\langle x \rangle_O \wedge F_2 \subseteq F_3$. Then $x \in F_3$. Hence $F_1 \wedge F_2 \subseteq F_3$. □

PROPOSITION 3.8. Let F be an open filter of an SDWH-algebra $\mathcal{H}$. Then F is a compact element of $(OF(\mathcal{H}), \wedge, \vee, \to, \{1\}, H)$ if and only if F is a principal open filter of $\mathcal{H}$.

PROOF: Suppose that F is a compact element of $(OF(\mathcal{H}), \wedge, \vee, \to, \{1\}, H)$. Since $F = \vee_{x \in F} \langle x \rangle_O$, then there exist $x_1, x_2, ..., x_n \in F$ such that $F = \langle x_1 \rangle_O \vee \langle x_2 \rangle_O \vee ... \vee \langle x_n \rangle_O$. Using Proposition 3.1 part (4), we have $F = \langle x_1 \wedge x_2 \wedge ... \wedge x_n \rangle_O$. Hence F is a principal open filter of $\mathcal{H}$.
Conversely, let F be a principal open filter of of $\mathcal{H}$. Then there exists $x \in F$ such that$F = \langle x \rangle_O$. Suppose that $\{F_i\}_{i \in I}$ is a family of open filters of $\mathcal{H}$ such that $F \subseteq \vee_{i \in I} F_i$. Then $x \in \langle \cup_{i \in I} F_i \rangle_O$. Then there exist $i_j \in I$, $f_{i_j} \in F_{i_j}$ $(j = 1, ..., m)$ such that $f_{i_1} \wedge f_{i_2} \wedge ... \wedge f_{i_m} \leq x$ by Proposition 3.4 part (3). So $x \in \langle F_{i_1} \cup F_{i_2} \cup ... \cup F_{i_m} \rangle_O$. Hence $F = \langle x \rangle_O \subseteq F_{i_1} \vee F_{i_2} \vee ... \vee F_{i_m}$. □

We define deductive system of an SDWH algebra in a usual way:

DEFINITION 3.9. A subset D is called a deductive system of an SDWH algebra $\mathcal{H}$ if it is satisfies the following conditions, for all $x, y \in H$:

(D1) $1 \in D$,

(D2) $x, x \to y \in D$ imply $y \in D$.

The set of all deductive system of $\mathcal{H}$ is denoted by $Ds(\mathcal{H})$.

PROPOSITION 3.10. Let $\mathcal{H}$ be an SDWH algebra. Then $Ds(\mathcal{H}) \subseteq OF(\mathcal{H}) \subseteq F(\mathcal{H})$.

PROOF: Let $D \in Ds(\mathcal{H})$. We will show D is an open filter.
(F1) Let $x \in D$, $y \in H$ and $x \leq y$. Then $x \to y = 1 \in D$ by (W2). So $y \in D$ by (D2).

(F2) Let $x, y \in D$. By Proposition 2.5 part (3), we have $x \to (y \to (x \wedge y)) = 1 \in D$. Then $y \to x \wedge y \in D$. Hence $x \wedge y \in D$ by (D2).
(OF3) Let $x \in D$. We have $x \to (1 \to x) = 1 \in D$ by Proposition 2.5 part (1). Thus $1 \to x \in D$ by (D2). Therefore $D \in OF(\mathcal{H})$.
It is clear that every open filter is a filter of $\mathcal{H}$. □

In the following example, we will see that every open filter may not be a deductive system of an SDWH-algebra and there exists a filter that is not an open filter.

Example 3.11. Let $H = \{0, a, b, 1\}$ with $0 < a, b < 1$, such that a, b are not comparable. Consider the following binary operation:

$\to$	0	a	b	1
0	1	1	1	1
a	1	1	1	1
b	b	b	1	1
1	b	b	1	1

It is easy to see that $\mathcal{H} = (H, \vee, \wedge, \to, 0, 1)$ is an SDWH-algebra and
$F(\mathcal{H}) = \{\{1\}, \{1, b\}, \{1, a\}, H\}$,
$OF(\mathcal{H}) = \{\{1\}, \{1, b\}, H\}$,
$Ds(\mathcal{H}) = \{H\}$.
So $Ds(\mathcal{H}) \subsetneqq OF(\mathcal{H}) \subsetneqq F(\mathcal{H})$.

THEOREM 3.12. *Let $\mathcal{H}$ be an SDWH-algebra. The following are equivalent:*

(1) $1 \to x \leq x$, *for all* $x \in H$,

(2) $OF(\mathcal{H}) = Ds(\mathcal{H})$.

PROOF: (1) $\Rightarrow$ (2) By Proposition 3.10, we have $Ds(\mathcal{H}) \subseteq OF(\mathcal{H})$. Let $F \in OF(\mathcal{H})$ and $x, x \to y \in F$. We will show that (D2) is true. By $y \leq x \vee y \leq 1$ and (W3) we have:

$$1 \to y = (1 \to (x \vee y)) \wedge ((x \vee y) \to y).$$

But $(x \vee y) \to y = x \to y \in F$ by (WH2). Since $1 \to x \leq 1 \to (x \vee y)$ and $1 \to x \in F$ by (F3), then $1 \to (x \vee y) \in F$ by (F2). So $1 \to y \in F$ by (F1). Thus $y \in F$ by assumption and (F2). Therefore $y \in Ds(\mathcal{H})$ and $OF(\mathcal{H}) = Ds(\mathcal{H})$.

(2) $\Rightarrow$ (1) Let $x \in H$. Then open filter $F_x := \langle 1 \to x \rangle_O$ is a deductive system by assumption. Obviously, $1, 1 \to x \in F_x$. So $x \in F_x$ by (D2). Hence $1 \to x \leq x$ by Proposition 3.1 part (7). $\square$

PROPOSITION 3.13. Let $\mathcal{H}$ be an SDWH-algebra. The following are equivalent:

(1) $x \leq 1 \to x$, for all $x \in H$,

(2) $F(\mathcal{H}) = OF(\mathcal{H})$.

PROOF: (1) $\Rightarrow$ (2) By Proposition 3.10, we have $OF(\mathcal{H}) \subseteq F(\mathcal{H})$. Let $F \in F(\mathcal{H})$. We will show that F is open. Let $x \in F$. By assumption, we have $x \leq 1 \to x$. Hence $1 \to x \in F$ that is, $F \in OF(\mathcal{H})$.

(2) $\Rightarrow$ (1) Let $x \in H$. Then the filter $F_x = \{y \in H | x \leq y\}$ is an open filter by assumption. Thus $1 \to x \in F_x$. So $x \leq 1 \to x$. $\square$

COROLLARY 3.14. An SDWH-algebra $\mathcal{H}$ is a basic algebra if and only if $F(\mathcal{H}) = OF(\mathcal{H})$.

COROLLARY 3.15. Let $\mathcal{H}$ be an SDWH-algebra. The following are equivalent:

(1) $x = 1 \to x$, for all $x \in H$,

(2) $\mathcal{H}$ is Heyting algebra,

(3) $F(\mathcal{H}) = OF(\mathcal{H}) = Ds(\mathcal{H})$,

(4) $F(\mathcal{H}) = Ds(\mathcal{H})$.

The smallest deductive system of an SDWH-algebra $\mathcal{H}$ containing S, (i.e. $\cap\{D \in Ds(\mathcal{H}) | S \subseteq D\}$), is called the deductive system generated by S and it will be denoted by $\langle S \rangle_D$ ($\langle a \rangle_D$ is called principal deductive system.)

PROPOSITION 3.16. Let $\mathcal{H}$ be an SDWH-algebra. If $a, b \in H$, then

(1) $$\begin{aligned} \langle a \rangle_D &= \{x \in H \mid \square^n (a \to x) = 1, \text{ for some } n \in \mathbb{N}\} \\ &= \{x \in H \mid a \to^n x = 1, \text{ for some } n \in \mathbb{N}\}, \end{aligned}$$

(2) $a \leq b$ implies $\langle b \rangle_D \subseteq \langle a \rangle_D$,

(3) $\langle a \vee b \rangle_D = \langle a \rangle_D \cap \langle b \rangle_D$,

(4) $\langle a \wedge b \rangle_D = \langle a \rangle_D \vee \langle b \rangle_D$.

Proof: (1) We will show $D = \{x \in H \mid \Box^n(a \to x) = 1,$ for some $n \in \mathbb{N}\}$ is a deductive system of $\mathcal{H}$. We have $a \to 1 = 1$, so $1 \in D$. Let $x, x \to y \in D$. Then there exist $m, n \in \mathbb{N}$ such that $\Box^n(a \to x) = 1$ and $\Box^m(a \to (x \to y)) = 1$. Then $\Box^{n+m}(a \to x) = \Box^m(1) = 1$ and $\Box^{m+n}(a \to (x \to y)) = \Box^n(1) = 1$ by (N3). So we have

$$\Box^{m+n+1}(a \to y) = 1 \to (\Box^{m+n}(a \to y)) = \Box^{m+n}(a \to x) \to \Box^{m+n}(a \to y) = \Box^{m+n}(a \to (x \to y)) = \Box^n(1) = 1.$$

by (N4). Thus $y \in D$. Hence $D \in Ds(\mathcal{H})$. Also, we have $\Box^1(a \to a) = 1$. Hence $a \in D$. Then there exists $n \in \mathbb{N} \cup \{0\}$ such that $\Box^n(a \to x) = 1 \in \langle a \rangle_D$. Since $1 \to \Box^{n-1}(a \to x) = 1 \in \langle a \rangle_D$ and $1 \in \langle a \rangle_D$, then $\Box^{n-1}(a \to x) = 1 \in \langle a \rangle_D$ by (DS2). By inductively, we obtain $a \to x \in \langle a \rangle_D$. But $a \in \langle a \rangle_D$. So $x \in \langle a \rangle_D$ by (DS2). Hence $D \subseteq \langle a \rangle_D$. Since $\langle a \rangle_D$ is the smallest deductive system containing a, we obtain $D = \langle a \rangle_D$. Using (N1), we have $a \to^n x = \Box^{n-1}(a \to x)$. So it is easy to prove that $\langle a \rangle_D = \{x \in H \mid a \to^n x = 1,$ for some $n \in \mathbb{N}\}$.
(2) Let $x \in \langle b \rangle_D$. Then there exists $n \in \mathbb{N}$ such that $\Box^n(b \to x) = 1$ by part (1). By assumption $a \leq b$. So $b \to x \leq a \to x$ by (W1). Using (N5), we obtain $\Box^n(b \to x) \leq \Box^n(a \to x)$. Therefore $\Box^n(a \to x) = 1$. So $x \in \langle a \rangle_D$ by part (1). Hence $\langle b \rangle_D \subseteq \langle a \rangle_D$.
(3) Let $x \in \langle a \rangle_D \cap \langle a \rangle_D$. Then there exist $n, m \in \mathbb{N}$ such that we have $\Box^n(a \to x) = 1$ and $\Box^m(b \to x) = 1$ by part (2). Put $p := max\{m, n\}$. By (N3), we obtain $\Box^p(a \to x) \geq \Box^n(a \to x) = 1$. Similarly $\Box^p(b \to x) = 1$. Using (WH3) and then (N2), we get $\Box^p((a \vee b) \to x) = \Box^p((a \to x) \wedge (b \to x)) = \Box^p(a \to x) \wedge \Box^p(b \to x) = 1$. Hence $x \in \langle a \vee b \rangle_D$. Therefore $\langle a \rangle_D \cap \langle b \rangle_D \subseteq \langle a \vee b \rangle_D$.
Conversely, we have $a, b \leq a \vee b$. By part (2), we obtain $\langle a \vee b \rangle_D \subseteq \langle a \rangle_D, \langle b \rangle_D$. Hence $\langle a \vee b \rangle_D \subseteq \langle a \rangle_D \cap \langle b \rangle_D$. □

4. Congruence relations on SDWH algebras

In this section, we study some properties that establish some connections among the congruence relations, the open filters, and the deductive systems of an SDWH-algebra $\mathcal{H}$.
We denote by $Con(\mathcal{H})$ the congruence lattice of an SDWH-algebra $\mathcal{H}$. As usual, for a $\theta \in Con(\mathcal{H})$ denote by $[1]_\theta$ its congruence class containing the element 1, so-called kernel of θ.

Definition 4.1. Let F be an open filter of an SDWH-algebra $\mathcal{H}$. Define two binary relations Θ_F and Γ_F on H as follows:
$\Theta_F = \{(x,y) \in H \times H | x \wedge f \leq y \text{ and } y \wedge f \leq x \text{ for some} f \in F\}$,
$\Gamma_F = \{(x,y) \in H \times H | x \to y, y \to x \in F\}$.

Proposition 4.2. Let F be an open filter of an SDWH-algebra $\mathcal{H}$. Then Θ_F is the least congruence relation on H such that $[1]_{\Theta_F} = F$.

Proof: Clearly, Θ_F is reflexive and symmetric. In order to prove transivity, let $x, y, z \in H$ such that $(x,y), (y,z) \in \Theta_F$. We have $x \wedge f_1 \leq y$, $y \wedge f_1 \leq x$, $y \wedge f_2 \leq z$ and $z \wedge f_2 \leq y$ for some $f_1, f_2 \in F$. Then $x \wedge (f_1 \wedge f_2) \leq z$, $z \wedge (f_1 \wedge f_2) \leq x$ and $f_1 \wedge f_2 \in F$ by (F1). So $(x,z) \in \Theta_F$. Therefore Θ_F is an equivalence relation on H.
Now, we will prove that Θ_F compatible with $\wedge, \vee, \to$. Let $(x,y), (a,b) \in \Theta_F$, then $x \wedge f_1 \leq y$, $y \wedge f_1 \leq x$, $a \wedge f_2 \leq b$ and $b \wedge f_2 \leq a$, for some $f_1, f_2 \in F$. Put $f = f_1 \wedge f_2$. Then $f \in F$ by (F1). Thus $x \wedge a \wedge f \leq y \wedge b$, $y \wedge b \wedge f \leq x \wedge a$, $(x \vee a) \wedge f \leq (y \vee b)$ and $(y \vee b) \wedge f \leq (x \vee a)$ for $f \in F$. So $(x \wedge a, y \wedge b) \in \Theta_F$ and $(x \vee a, y \vee b) \in \Theta_F$. Therefore Θ_F compatible with $\wedge$ and $\vee$.
We will show that $(a \to x, a \to y) \in \Theta_F$, and $(a \to y, b \to y) \in \Theta_F$ which implies by transitivity of Θ_F that $(a \to x, b \to y) \in \Theta_F$.
Since $x \wedge f_1 \leq y$, then $(a \to x) \wedge (a \to f_1) = a \to (x \wedge f_1) \leq a \to y$ by (WH1) and (W1). Similarly, we have $(a \to y) \wedge (a \to f_1) \leq a \to x$. By Lemma 3.3, we have $a \to f_1 \in F$ because $f_1 \in F$ and F is an open filter. So $(a \to x, a \to y) \in \Theta_F$.

Since $f_2 \in F$ and F is open, then $1 \to f_2 \in F$. We have $1 \to f_2 \leq a \to f_2 = a \to (a \wedge f_2) \leq a \to b$, so $a \to b \in F$. Similarly, we obtain $b \to a \in F$. Thus $(a \to b) \wedge (b \to a) \in F$ by (F1). We have $(b \to y) \wedge [(a \to b) \wedge (b \to a)] \leq (a \to b) \wedge (b \to y) \leq a \to y$ and $(a \to y) \wedge [(a \to b) \wedge (b \to a)] \leq b \to y$. Hence $(a \to y, b \to y) \in F$. Therefore Θ_F is compatible with $\to$.
Suppose that $x \in [1]_{\Theta_F}$, so $(x,1) \in \Theta_F$. Then $x \wedge f \leq 1$ and $1 \wedge f \leq x$ for some $f \in F$. So $x \in F$ by (F2). Conversely, let $x \in F$, we have $x \wedge x \leq 1$ and $1 \wedge x \leq x$ for $x \in F$. Then $(x,1) \in \Theta_F$, so $x \in [1]_{\Theta_F}$. Therefore $[1]_{\Theta_F} = F$.
Suppose that $\theta \in Con(\mathcal{H})$ such that $[1]_\theta = F$. Let $(x,y) \in \Theta_F$. Then there exists $f \in F$ such that $x \wedge f \leq y$ and $y \wedge f \leq x$. We get $x \wedge f = y \wedge f$. Since $f \in F = [1]_\theta$, then $(f,1) \in \theta$. So $(x, x \wedge f) = (x \wedge 1, x \wedge f) \in \theta$ and $(y, y \wedge f) = (y \wedge 1, y \wedge f) \in \theta$. Thus $(x,y) \in \theta$ by transitivity. Hence $\Theta_F \subseteq \theta$. $\square$

Corollary 4.3. Let F be an open filter of an SDWH-algebra $\mathcal{H}$. Then F is a congruence kernel in $\mathcal{H}$ that is, $F = [1]_\theta$ for some $\theta \in Con(\mathcal{H})$,

Proposition 4.4. Let F be an open filter of an SDWH-algebra $\mathcal{H}$. Then Γ_F is congruence relation on $\mathcal{H}$ such that $F \subseteq [1]_{\Gamma_F}$.

Proof: Obviously, Γ_F is reflexive and symmetric. The transitivity follows from (WH3). So Γ_F is an equivalence relation on H.

Let $(x, y) \in \Gamma_F$, and $(a, b) \in \Gamma_F$. Then $x \to y, y \to x \in F$ and $a \to b, b \to a \in F$. By (W1) and (WH1), we obtain $x \to y \leq (x \wedge a) \to y = (x \wedge a) \to (y \wedge a)$ and $y \to x \leq (y \wedge a) \to x = (y \wedge a) \to (x \wedge a)$. Then $(x \wedge a) \to (y \wedge a) \in F$ and $(y \wedge a) \to (x \wedge a) \in F$ by (F2). So $(x \wedge a, y \wedge a) \in \Gamma_F$. Similarly, we can prove that $(a \wedge y, b \wedge y) \in \Gamma_F$. Then $(x \wedge a, y \wedge b) \in \Gamma_F$ by transitivity. Therefore Γ_F compatible with $\wedge$.
Using (W1) and (WH2), we get $x \to y \leq x \to (y \vee a) = (x \vee a) \to (y \vee a)$ and $y \to x \leq y \to (x \vee a) = (y \vee a) \to (x \vee a)$. Thus $(x \vee a, y \vee a) \in \Gamma_F$. Similarly, $(a \vee y, b \vee y) \in \Gamma_F$. Thus $(x \vee a, y \vee b) \in \Gamma_F$ Therefore Γ_F compatible with $\vee$.
By Proposition 2.5 part (5), we have $x \to y \leq (a \to x) \to (a \to y)$ and $y \to x \leq (a \to y) \to (a \to x)$. By (F2), we obtain $(a \to x, a \to y) \in \Gamma_F$. By Proposition 2.5 part (6), $b \to a \leq (a \to y) \to (b \to y)$ and $a \to b \leq (b \to y) \to (a \to y)$. Using (F2), we get $(a \to y, b \to y) \in \Gamma_F$. Hence $(a \to x, b \to y) \in \Gamma_F$. So Γ_F compatible with $\to$. Therefore Γ_F is a congruence relation on H. Let $x \in F$ be arbitrary. Since F is open, then $1 \to x \in F$. By (W2) and (F2), $x \to 1 \in F$. Thus $F \subseteq [1]_{\Gamma_F}$. □

In Proposition 4.4, F and $[1]_{\Gamma_F}$ may not be equal in general. See the following example:

Example 4.5. Consider the open filter $F = \{1, b\}$ of the SDWH-algebra $\mathcal{H}$ in Example 3.11. Then $F \subsetneq [1]_{\Gamma_F} = H$.

Corollary 4.6. Let D be a deductive system of an SDWH-algebra $\mathcal{H}$. Then Γ_D is the greatest congruence relation on H such that $D = [1]_{\Gamma_D}$.

Proof: By Proposition 3.10 and Proposition 4.4, Γ_D is congruence relation on H such that $D \subseteq [1]_{\Gamma_D}$. Now, suppose that $x \in [1]_{\Gamma_D}$. Then $1 = x \to 1 \in D$ and $1 \to x \in D$. By (D2), we get $x \in D$. Hence $[1]_{\Gamma_D} \subseteq D$. Suppose that $\theta \in Con(\mathcal{H})$ such that $D = [1]_{\Gamma_D}$. Let $(x, y) \in \theta$. Then

$(x \to y, y \to y) = (x \to y, 1)$. So $x \to y \in [1]_\theta = D$. Similarly $y \to x \in D$. Hence $(x, y) \in \Gamma_D$. Therefore $\theta \subseteq \Gamma_D$. □

PROPOSITION 4.7. Let F be an open filter of an SDWH-algebra $\mathcal{H}$. Then $\frac{\mathcal{H}}{\Gamma_F}$ is a Heyting algebra if and only if F is a deductive system of H.

PROOF: Suppose that F is a deductive system of H and $x \in H$ be arbitrary. We have $1 \to ((1 \to x) \to x) = (1 \to x) \to (1 \to x) = 1 \in F$. Since F is a deductive system and $1 \in F$, then $(1 \to x) \to x \in F$. Also, $x \to (1 \to x) = 1 \to (x \to x) = 1 \in F$. Thus $(x, 1 \to x) \in \Gamma_F$ that is, $[x]_{\Gamma_F} = [1]_{\Gamma_F} \to [x]_{\Gamma_F}$ for all $x \in H$. Hence $\frac{\mathcal{H}}{\Gamma_F}$ is a Heyting algebra.
Conversely, let $x, x \to y \in F$. Then we have $[x] = [x \to y] = [1]$. Since $\frac{\mathcal{H}}{\Gamma_F}$ is a Heyting algebra, then $[x \wedge y] = [x \wedge (x \to y)] = [1]$. We obtain $x \wedge y \in F$. Hence $y \in F$. □

PROPOSITION 4.8. Let θ be a congruence relation on an SDWH-algebra $\mathcal{H}$. Then

(1) $[1]_\theta \in OF(\mathcal{H})$,

(2) $\Theta_{[1]_\theta} \subseteq \theta \subseteq \Gamma_{[1]_\theta}$.

PROOF: (1) Let $x, y \in [1]_\theta$. Then $(x, 1) \in \theta$ and $(y, 1) \in \theta$. By compatibility of θ with $\wedge$, we have $(x \wedge y, 1) \in \theta$. Thus $x \wedge y \in [1]_\theta$.
Let $x \in [1]_\theta$ such that $x \leq y$. Then $(x, 1) \in \theta$ and $x \vee y = y$. So $(y, 1) = (x \vee y, 1 \vee y) \in \theta$. Hence $[1]_\theta$ is a filter. If $x \in [1]_\theta$, then $(x, 1) \in \theta$. So $(1 \to x, 1 \to 1) \in \theta$. Hence $[1]_\theta$ is an open filter.
(2) Let $(x, y) \in \theta_{[1]_\theta}$. Then there exists $f \in [1]_\theta$ such that $x \wedge f \leq y$ and $y \wedge f \leq x$. We obtain $x \wedge f = y \wedge f$ and $(f, 1) \in \theta$. So $(x \wedge f, x) \in \theta$ and $(y \wedge f, y) \in \theta$. Thus $(x, y) \in \theta$. Hence $\theta_{[1]_\theta} \subseteq \theta$.
Now, suppose that $(x, y) \in \theta$. Then $(x \to y, y \to y) = (x \to y, 1) \in \theta$ and $(y \to x, x \to x) = (y \to x, 1) \in \theta$. So $x \to y \in [1]_\theta$ and $y \to x \in [1]_\theta$. Hence $(x, y) \in \Gamma_{[1]_\theta}$. Therefore $\theta \subseteq \Gamma_{[1]_\theta}$ □

In an SDWH-algebra θ and $\Theta_{[1]_\theta}$ may not be equal in general. See the following example.

Example 4.9. Let $H = \{0, a, b, 1\}$ where $0 < a < b < 1$. Consider the following binary operation:

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	1	1	1	1
b	a	a	1	1
1	a	a	b	1

Obviously $\mathcal{H} = (H, \vee, \wedge, \rightarrow, 0, 1)$ is an SDWH-algebra. Consider the congruence relation $\theta = \{(1,1), (b,b), (1,b), (b,1), (a,a), (0,0), (a,0), (0,a)\}$ on H. Then $[1]_\theta = \{1, b\}$ is an open filter of $\mathcal{H}$ and

$$\Theta_{[1]_\theta} = \{(1,1), (b,b), (1,b), (b,1), (a,a), (0,0)\}.$$

Hence $\Theta_{[1]_\theta} \subsetneq \theta$.

For every RWH-algebra $\mathcal{H}$, there is an isomorphism between the lattice of open filters of $\mathcal{H}$ and the lattice congruence relation of $\mathcal{H}$ (see [4, 15]).

If $\mathcal{H}$ is an SDWH-algebra, then he natural map $\theta \mapsto [1]_\theta$ associated with $Con(\mathcal{H})$ and $OF(\mathcal{H})$ is well defined and onto, but not one-to-one in general as you can see in the following example. This example also shows that the open filters of SDWH-algebras can be kernels of more than one congruence relation. Hence the variety of SDWH-algebras is not weakly regular.

Example 4.10. Let $H = \{0, a, b, 1\}$ where $0 < a < b < 1$. Consider the following binary operation:

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	0	1	1	1
b	0	1	1	1
1	0	a	a	1

It is easy to see that $\mathcal{H} = (H, \vee, \wedge, \rightarrow, 0, 1)$ is an SDWH-algebra and for all $x \in H$ we have $1 \rightarrow x \leq x$. We have $OF(\mathcal{H}) = Ds(\mathcal{H}) = \{\{1\}, \{1, b, a\}, H\}$. But we have four congruence relations on H as follows:

$$\theta_1 = \Delta = \{(0,0), (a,a), (b,b), (1,1)\},\ [1]_{\theta_1} = F_1 = \{1\},$$
$$\theta_2 = \Delta \cup \{(a,b), (b,a)\},\ [1]_{\theta_2} = F_1 = \{1\},$$
$$\theta_3 = \Delta \cup \{(a,b), (b,a), (1,b), (b,1), (1,a), (a,1)\},\ [1]_{\theta_3} = F_2 = \{1, b, a\},$$
$$\theta_4 = \nabla = \{(x,y) | x, y \in H\},\ [1]_{\theta_4} = F_3 = H.$$

Remark. Let $\mathcal{H}$ be an SDWH-algebra. Then the natural map $\theta \mapsto [1]_\theta$ is an order isomorphism from $Con(\mathcal{H})$ to $OF(\mathcal{H})$ if and only if $\mathcal{H}$ is a weakly regular algebra.

In the following, we will obtain a characterization of weakly regular SDWH-algebras.

LEMMA 4.11. *Let $\mathcal{H}$ be an SDWH-algebra and $a, b \in H$. Then $(x, y) \in \Phi(a, b)$ if and only if*

(1) $x \wedge a \wedge b \wedge (a \leftrightarrow b) = y \wedge a \wedge b \wedge (a \leftrightarrow b)$,

(2) $(x \vee a \vee b) \wedge (a \leftrightarrow b) = (y \vee a \vee b) \wedge (a \leftrightarrow b)$,

(3) $a \leftrightarrow b \leq x \leftrightarrow y$.

PROOF: For all $n \in \mathbb{N}$, we have $a \to b \leq \Box^n(a \to b)$ and $b \to a \leq \Box^n(b \to a)$ by (N6). Then $a \leftrightarrow b \leq \Box^n(a \leftrightarrow b)$ by (N2). So $t_n(a, b) = a \leftrightarrow b$. The result follows from Proposition 2.9. $\Box$

Recall that a variety V has equationally definable principal congruences (EDPC) if there exists a finite family of quaternary terms $\{u_i, v_i\}_{i=1}^r$ such that for every algebra $\mathcal{A}$ in V and every principal congruence $\Phi(a, b)$ of $\mathcal{A}$, if and only if $u_i(a, b, c, d) = v_i(a, b, c, d)$ for each $i = 1, ..., r$ ([6]). For algebraizable logics, EDPC corresponds to the deduction-detachment theorem.

COROLLARY 4.12. The variety of SDWH-algebras has EDPC.

PROPOSITION 4.13. Let c be an element of an SDWH-algebra $\mathcal{H}$. Then

(1) $\Phi(1, c) = \Theta_{\langle c \rangle_O}$,

(2) $\Phi(1, c) = \Theta_{[1]_{\Phi(1,c)}}$.

PROOF: (1) Let $(x, y) \in \Phi(1, c)$. We have $x \wedge c \wedge (1 \to c) = y \wedge c \wedge (1 \to c)$ by Lemma 4.11 part (1) and (W2). Since $c \wedge (1 \to c) \in \langle c \rangle_O$, then we obtain $(x, y) \in \Theta_{\langle c \rangle_O}$. So $\Phi(1, c) \subseteq \Theta_{\langle c \rangle_O}$.
Conversely, suppose that $(x, y) \in \Theta_{\langle c \rangle_O}$. Then there exists $f \in \langle c \rangle_O$ such that $x \wedge f \leq y$ and $y \wedge f \leq x$. We get $c \wedge (1 \to c) \leq f$ and $x \wedge f = y \wedge f$. So

$$x \wedge c \wedge (1 \to c) = (x \wedge f) \wedge c \wedge (1 \to c) = (y \wedge f) \wedge c \wedge (1 \to c) = y \wedge c \wedge (1 \to c).$$

Hence $x \wedge c \wedge 1 \wedge (1 \leftrightarrow c) = y \wedge c \wedge 1 \wedge (1 \leftrightarrow c)$. It is obvious that $(x \vee c \vee 1) \wedge (1 \leftrightarrow c) = (y \vee c \vee 1) \wedge (1 \leftrightarrow c)$. We have $x \wedge f = y \wedge f \leq y$, so $x \to f = x \to (x \wedge f) \leq x \to y$ by (W4). Thus

$$1 \to c = 1 \to (c \wedge (1 \to c)) \leq 1 \to f \leq x \to f \leq x \to y.$$

Similarly, we can prove that $1 \to c \leq y \to x$. Thus $1 \leftrightarrow c \leq x \leftrightarrow y$. Hence $(x,y) \in \Phi(1,c)$ by Lemma 4.11.
(2) By Proposition 4.8 part (2), we have $\Theta_{[1]_{\Phi(1,c)}} \subseteq \Phi(1,c)$. Conversely, suppose that $(x,y) \in \Phi(1,c) = \Theta_{\langle c \rangle_O}$. Then there exists $f \in \langle c \rangle_O$ such that $c \wedge (1 \to c) \leq f$ and $x \wedge f = y \wedge f$. We have
$1 \wedge c \wedge 1 \wedge (1 \leftrightarrow c) = f \wedge c \wedge 1 \wedge (1 \leftrightarrow c)$,
$(1 \vee c \vee 1) \wedge (1 \leftrightarrow c) = (f \vee c \vee 1) \wedge (1 \leftrightarrow c)$,
$1 \leftrightarrow c \leq x \leftrightarrow f$.
Therefore $(1,f) \in \Phi(1,c)$ by Lemma 4.11. Thus $f \in [1]_{\Phi(1,c)}$. Hence $(x,y) \in \Theta_{[1]_{\Phi(1,c)}}$. □

THEOREM 4.14. *Let $\mathcal{H}$ be an SDWH-algebra. Then $\mathcal{H}$ is weakly regular if and only if $\Phi(a,b) = \Theta_{[1]_{\Phi(a,b)}}$, for all $a,b \in H$.*

PROOF: Suppose that $\mathcal{H}$ is weakly regular and a,b are two arbitrary elements of H. We have $\Phi(a,b) \in Con(\mathcal{H})$, so $F = [1]_{\Phi(a,b)} \in OF(\mathcal{H})$ by Proposition 4.8. Also, $\Theta_F \in Con(\mathcal{H})$ such that $[1]_{\Theta_F} = F$ by Proposition 4.2. Since $\mathcal{H}$ is weakly regular, then $\Theta_F = \Phi(a,b)$. Hence $\Phi(a,b) = \Theta_{[1]_{\Phi(a,b)}}$.

Conversely, let $\theta_1, \theta_2 \in Con(\mathcal{H})$ such that $[1]_{\theta_1} = [1]_{\theta_2}$. Suppose that $(x,y) \in \theta_1$, then $\Phi(x,y) \subseteq \theta_1$. We obtain $[1]_{\Phi(x,y)} \subseteq [1]_{\theta_1} = [1]_{\theta_2}$. It is easy to show that $\Theta_{[1]_{\Phi(x,y)}} \subseteq \Theta_{[1]_{\theta_2}}$. Using assumption and Proposition 4.8 part (2), we obtain $\Phi(x,y) \subseteq \theta_2$. Thus $(x,y) \in \theta_2$. Hence $\theta_1 \subseteq \theta_2$. Similarly, we can prove that $\theta_2 \subseteq \theta_1$. Therefore $\theta_1 = \theta_2$. □

COROLLARY 4.15. Let $\mathcal{H}$ be an SDWH-algebra. If for all $a,b \in H$, there exist $c \in H$ such that $\Phi(a,b) = \Phi(1,c)$, then $\mathcal{H}$ is weakly regular.

PROOF: It follows from Theorem 4.14 and Proposition 4.13. □

PROPOSITION 4.16. Let $\mathcal{H}$ be an SDWH-algebra such that $H = \{0,a,b,1\}$, $0 < a,b < 1$, a,b are not comparable. Then $\mathcal{H}$ is weakly regular.

PROOF: We will show that $\Phi(a,0) = \Phi(b,1)$, $\Phi(b,0) = \Phi(a,1)$ and $\Phi(a,b) = \Phi(1,0)$. We have

$a \wedge b \wedge 1 \wedge (1 \to b) = 0 \wedge b \wedge 1 \wedge (1 \to b)$,
$(a \vee b \vee 1) \wedge (1 \to b) = (0 \vee b \vee 1) \wedge (1 \to b)$,
$1 \wedge a \wedge 0 \wedge (a \to 0) = b \wedge a \wedge 0 \wedge (a \to 0)$,

$(1 \vee a \vee 0) \wedge (a \to b) = (1 \vee a \vee 0) \wedge (a \to 0).$
Also, we have $a \to 0 = 1 \to b$ by (W5). Then $a \leftrightarrow 0 = 1 \leftrightarrow b$. So $\Phi(a, 0) = \Phi(b, 1)$. Similarly, we can prove $\Phi(b, 0) = \Phi(a, 1)$. We have
$a \wedge 1 \wedge 0 \wedge (1 \to 0) = b \wedge 1 \wedge 0 \wedge (1 \to 0),$
$(a \vee 1 \vee 0) \wedge (1 \to 0) = (b \vee 1 \vee 0) \wedge (1 \to 0),$
$1 \wedge a \wedge b \wedge (a \leftrightarrow b) = 0 \wedge a \wedge b \wedge (a \leftrightarrow b),$
$(1 \vee a \vee b) \wedge (a \leftrightarrow b) = (0 \vee a \vee b) \wedge (a \leftrightarrow b).$
By (W5), we have $a \to b = 1 \to b$ and $b \to a = 1 \to a$. Then
$a \leftrightarrow b = (1 \to b) \wedge (1 \to a) = 1 \to (a \wedge b) = 1 \leftrightarrow 0.$
So $\Phi(a, b) = \Phi(1, 0)$. Hence $\mathcal{H}$ is weakly regular by Corollary 4.15. □

5. Conclusions and future works

In this paper, we have studied SDWH-algebras in the context of Birkhoff's Problem 73, that is we have studied whether or not SDHW-algebras are weakly regular. To do this, we have considered open filters and deductive systems in SDWH-algebras to show that in general, they are not weakly regular, and give necessary and sufficient conditions for an SDWH-algebra to be weakly regular by using principal congruence relations.
In the future, we will introduce and study a corresponding logic to SDWH-algebras and investigate some basic properties of this logic. But here are some open questions still about SDW-algebras to be studied. Is there any representation theorem for SDWH-algebras? Is the class of weakly regular SDWH-algebras a variety? With positive answer to this, we should know the relation between this proper subvariety of SDWH-algebras and the other subvarieties of WH- algebras such as the variety the varieties of RWH, TWH, SRL and B. It would be interesting to find a characterization of the WH-spaces that correspond to the algebras in the subvariety of SDWH-algebras.

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