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FREE SPECTRA OF EQUIVALENTIAL ALGEBRAS WITH CONJUNCTION ON DENSE ELEMENTS

Abstract

We construct free algebras in the variety generated by the equivalential algebra with conjunction on dense elements and compute the formula for the free spectrum of this variety. Moreover, we describe the decomposition of free algebras into directly indecomposable factors.

Keywords: Fregean varieties, equivalential algebras, dense elements, free algebra, free spectra.

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1. Introduction

The equivalential algebra with conjunction on dense elements was introduced in [5]. This algebra turned out to be one of the four polynomially nonequivalent three-element algebras, that generates a congruence permutable Fregean variety. The other algebras are as follows. The first one is the three-element equivalential algebra (without any additional operation). It is very well known, also when it comes to the construction of the n-generated free algebras, as well as the cardinality of these algebras for small n and for some subvarieties (see [2], [9], [6], [7]). The other one is the three-element Brouwerian semilattice. This algebra also has been very well

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researched ([3]). The third of these algebras is the three-element equivalential algebra with conjunction on the regular elements. It was studied in [4]. The mentioned work contains the description of this algebra, its most important properties, the representation theorem, the construction of the free algebra and the free spectrum.

Whereas, when it comes to the equivalential algebra with conjunction on dense elements, we proved in [5] the representation theorem and we give a sketch of the construction the finitely generated free algebras. The aim of this paper is to extend the results of [5] by providing the formula for the free spectrum (Section 4). In this way we complete the full description (with accuracy to the polynomially equivalence) of the free algebras in congruence permutable Fregean varieties generated by three-element algebras.

The second aim of this article is to describe the directly indecomposability of the free algebras in the variety generated by the equivalential algebra with conjunction on dense elements (Theorem 3.6).

2. Equivalential algebras with conjunction on the dense elements

Preliminary facts can be found in [5], but for the convenience of the reader we recall some basic information.

Definition 2.1. **An equivalential algebra with conjunction on the dense elements** is an algebra $\mathbf{D} := (\{0, *, 1\}, \cdot, d, 1)$ of type $(2, 2, 0)$, where $(\{0, *, 1\}, \cdot, 1)$ is an equivalential algebra and d is a binary commutative operation presented in the table below (on the right):

$\cdot$	1	$*$	0
1	1	$*$	0
$*$	$*$	1	0
0	0	0	1

d	1	$*$	0
1	1	$*$	1
$*$	$*$	$*$	$*$
0	1	$*$	1

The interpretation of the name is given in [5, Definition 4.1]. We denote by $\mathcal{V}(\mathbf{D})$ the variety generated by $\mathbf{D}$.

A crucial role in the construction of the finitely generated free algebras is played by the subdirectly irreducible algebras in $\mathcal{V}(\mathbf{D})$.

PROPOSITION 2.2. [5, Proposition 4.6] There are only three (up to isomorphism) nontrivial subdirectly irreducible algebras in $\mathcal{V}(\mathbf{D})$: $\mathbf{D}, \mathbf{2}, \mathbf{2}^{\wedge}$, where:

$$\mathbf{2} := \{\{0,1\}, \cdot, d, 1\}, \text{ where } d \equiv 1,$$

$$\mathbf{2}^{\wedge} := \{\{*,1\}, \cdot, d, 1\}, \text{ where } d(x,y) := x \wedge y.$$

Whatsmore, $\mathbf{2}$ and $\mathbf{2}^{\wedge}$ are subalgebras of $\mathbf{D}$.

Let $\mathbf{A} \in \mathcal{V}(\mathbf{D})$. We denote by $\mathrm{Cm}(\mathbf{A})$ the set of all completely meet-irreducible congruences on $\mathbf{A}$.

We define an order $\leq$ on $\mathrm{Cm}(\mathbf{A})$ as follows:

$$\varphi \leq \psi \text{ iff } \varphi \subseteq \psi, \text{ for } \varphi, \psi \in \mathrm{Cm}(\mathbf{A}).$$

We use the following notation:

$$\overline{L} := \{\mu \in \mathrm{Cm}(\mathbf{A}) : \mathbf{A}/\mu \cong \mathbf{2}\},$$

$$\underline{L} := \{\mu \in \mathrm{Cm}(\mathbf{A}) : \mathbf{A}/\mu \cong \mathbf{D}\},$$

$$P := \{\mu \in \mathrm{Cm}(\mathbf{A}) : \mathbf{A}/\mu \cong \mathbf{2}^{\wedge}\},$$

$$L := \overline{L} \cup \underline{L}.$$

In our case it turns out that

$$\varphi \leq \psi \text{ iff } (\varphi \in \underline{L}, \psi \in \overline{L}, \varphi < \psi) \text{ or } \varphi = \psi. \tag{O1}$$

Moreover, if $\varphi < \psi$, then $\psi = \varphi^{+}$.

Let $\varphi, \psi \in \mathrm{Cm}(\mathbf{A})$. We introduce an equivalence relation on $\mathrm{Cm}(\mathbf{A})$ as follows (see [1, p. 51]):

$$\varphi \sim \psi \text{ iff the intervals } [\varphi, \varphi^{+}] \text{ and } [\psi, \psi^{+}] \text{ are projective.}$$

DEFINITION 2.3. Let $\mathbf{A} \in \mathcal{V}(\mathbf{D})$. The structure $\mathbf{Cm}(\mathbf{A}) := (\mathrm{Cm}(\mathbf{A}), \leq, \sim)$ is called a frame of $\mathbf{A}$.

From [5, Proposition 5.4, Theorem 5.5] we get that the equivalence classes of the relation $\sim$ on $\mathrm{Cm}(\mathbf{A})$ take the following form:

1. $\overline{L} \in \mathrm{Cm}(\mathbf{A})/_{\sim}$,
2. $\mu/_{\sim} = \{\mu\}$ for all $\mu \in \underline{L} \cup P$.

Moreover, $(\overline{L} \cup \{1_{\mathbf{A}}\}, \bullet, 1_{\mathbf{A}})$ forms a Boolean group, where $\mu_1 \bullet \mu_2 := (\mu_1 \div \mu_2)'$ for $\mu_1, \mu_2 \in \overline{L}$ ($\div$ denotes the symmetric difference and $'$ denotes the complement of a set).

Now, we recall that every finite algebra from $\mathcal{V}(\mathbf{D})$ can be naturally decomposed as the direct product of two algebras:

Proposition 2.4. Let $\mathbf{A} \in \mathcal{V}(\mathbf{D})$ be finite. Then:

$$\mathbf{A} \cong \mathbf{A}/_{\bigwedge L} \times \mathbf{A}/_{\bigwedge P}.$$

To construct the free algebras in $\mathcal{V}(\mathbf{D})$ we need the notion of the hereditary sets [5, Definition 6.1] and the representation theorem [5, Theorem 6.2].

Definition 2.5. Let $\mathbf{A} \in \mathcal{V}(\mathbf{D})$ and $Z \subseteq \mathrm{Cm}(\mathbf{A})$. A set Z is **hereditary** if:

1. $Z = Z\uparrow$,
2. $\overline{L} \subseteq Z$ or $((\overline{L} \cap Z) \cup \{\mathbf{1_A}\}, \bullet)$ is a hyperplane in $(\overline{L} \cup \{\mathbf{1_A}\}, \bullet)$.

We will denote by $\mathcal{H}(\mathbf{A})$ the set of all hereditary subsets of $\mathrm{Cm}(\mathbf{A})$.

Theorem 2.6. *Let $\mathbf{A} \in \mathcal{V}(\mathbf{D})$ and let $\mathbf{A}$ be finite. Then the map $M : A \ni a \to M(a) := \{\mu \in \mathrm{Cm}(\mathbf{A}) : a \in 1/\mu\}$ is the isomorphism between $\mathbf{A}$ and $(\mathcal{H}(\mathbf{A}), \leftrightarrow, \mathsf{d}, \mathit{1})$, where*

$$Z \leftrightarrow Y := ((Z \div Y)\downarrow)'$$

$$\mathsf{d}(Z, Y) := [Z \cup ((Z\downarrow)' \cap L)] \cap [Y \cup ((Y\downarrow)' \cap L)],$$

$$\mathit{1} := \mathrm{Cm}(A),$$

for $Z, Y \in \mathcal{H}(\mathbf{A})$.

Using the above theorem we can build up elements of algebra $\mathbf{A}$ in $\mathcal{V}(\mathbf{D})$ from the set $\mathrm{Cm}(\mathbf{A})$ with the order and the partial Boolean operation, i.e. from the structure $(\mathrm{Cm}(\mathbf{A}), \leq, \bullet)$.

3. Free algebras

Let $n \in \mathbb{N}$ and let X be an n-element set of free generators of $\mathbf{F_D}(n)$, where $D = \{0, *, 1\}$ is ordered by $0 < * < 1$. In [5] (Section 7) we give only the sketch of the construction of $\mathbf{F_D}(n)$, which was based on the observation that we can identify any element of $\mathrm{Cm}(\mathbf{F_D}(n))$ with a certain map, which sends free generators in some subdirectly irreducible algebra in $\mathcal{V}(\mathbf{D})$. Now, we will give a more detailed description of this construction, however, based on a slightly different approach, using the fact that the only subdirectly irreducible algebras in $\mathcal{V}(\mathbf{D})$ are $\mathbf{D}$ and its subalgebras $\mathbf{2}^\wedge$ and $\mathbf{2}$ given by sets $\{*, 1\}$ and $\{0, 1\}$ (2.2). Recall that

$$\underline{L} = \{\mu \in \mathrm{Cm}(\mathbf{F_D}(n)) : \mathbf{F_D}(n)/\mu \cong \mathbf{D}\},$$

$$\overline{L} = \{\mu \in \mathrm{Cm}(\mathbf{F_D}(n)) : \mathbf{F_D}(n)/\mu \cong \mathbf{2}\},$$

$$L = \underline{L} \cup \overline{L},$$

$$P = \{\mu \in \mathrm{Cm}(\mathbf{F_D}(n)) : \mathbf{F_D}(n)/\mu \cong \mathbf{2}^\wedge\}.$$

We denote by e the map $e : \{0, *, 1\} \to \{0, 1\}$ given by $e(0) = 0$ and $e(*) = e(1) = 1$. Clearly, such defined e is a homomorphism of $\mathbf{D}$ onto $\mathbf{2}$.

Put now $S(n) := \big\{f : X \to D : f^{-1}(\{0, *\}) \neq \emptyset\big\}$. As $\mathbf{F_D}(n)$ is the free algebra, every $f \in S(n)$ can be uniquely extended to a homomorphism $\overline{f}$ from $\mathbf{F_D}(n)$ to $\mathbf{D}$ with $\mathrm{Im}\,\overline{f}$ equal to one of three algebras: $\mathbf{D}$, $\mathbf{2}^\wedge$, or $\mathbf{2}$. Thus $\ker \overline{f} \in \mathrm{Cm}(\mathbf{F_D}(n))$. In $S(n)$ we introduce an order relation $\preceq$ and a partial binary operation $\cdot$ in the following way. For $f, g \in S(n)$ we put $f \preceq g$ if and only if $f = g$ or $g = e \circ f$, and if $* \notin \mathrm{Im}\, f \cup \mathrm{Im}\, g$ we define $(f \cdot g)(x) := 1$ if $f(x) = g(x)$ and $(f \cdot g)(x) := 0$ if $f(x) \neq g(x)$ for $x \in X$.

The following theorem allows us to identify the structures $(S(n), \preceq, \cdot)$ and $(\mathrm{Cm}(\mathbf{F_D}(n)), \leq, \bullet)$, where '$\bullet$' is the partial Boolean operation on $\overline{L}$.

THEOREM 3.1. *The map* $\varphi : S(n) \ni f \to \ker \overline{f} \in$ *is an isomorphism of the structures* $(S(n), \preceq, \cdot)$ *and* $(\mathrm{Cm}(\mathbf{F_D}(n)), \leq, \bullet)$.

PROOF: (1) φ is onto. Let $\mu \in \mathrm{Cm}(\mathbf{F_D}(n))$. Then $\mathbf{F_D}(n)/\mu$ is isomorphic to $\mathbf{K} \in \{\mathbf{D}, \mathbf{2}, \mathbf{2}^\wedge\}$. In all three cases we denote the isomorphism by ι. Put $\pi_\mu(t) = t/\mu$ for $t \in F_{\mathbf{D}}(n)$. Then $\iota \circ \pi_\mu : \mathbf{F_D}(n) \to \mathbf{K}$ is a surjective homomorphism. Hence $\iota \circ \pi_\mu|_X \in S(n)$ and $\varphi\left(\iota \circ \pi_\mu|_X\right) = \ker(\iota \circ \pi_\mu) = \ker(\pi_\mu) = \mu$, as desired.

(2) φ is one-to-one. Suppose, on the contrary, that $f, g \in S(n)$, $f \neq g$ and $\ker \overline{f} = \ker \overline{g}$. There is no loss of generality in assuming that there exists $x \in X$ such that $f(x) < g(x)$. Clearly, $f(x) = 0$ and $g(x) = *$. Then from $\operatorname{Im} \overline{f} \cong \mathbf{F_D}(n)/\ker \overline{f} = \mathbf{F_D}(n)/\ker \overline{g} \cong \operatorname{Im} \overline{g}$ we deduce that $\operatorname{Im} \overline{f} = \operatorname{Im} \overline{g}$. As $0 \in \operatorname{Im} \overline{f}$ and $* \in \operatorname{Im} \overline{g}$, we have $\mathbf{D} = \operatorname{Im} \overline{f} = \operatorname{Im} \overline{g}$. In consequence, there exists $y \in X$ such that $f(y) = *$, and hence $\overline{f}(yxx) = 1 = \overline{f}(1)$, and so $1 = \overline{g}(yxx) = g(y) * * = g(y) = f(y)$, a contradiction.

(3) φ nad φ^{-1} are monotone. Let $f, g \in S(n)$. If $f \prec g$, then $g = e \circ f$. Since e is a homomorphism, we get $\overline{g} = e \circ \overline{f}$, and so $\ker \overline{f} \leq \ker \overline{g}$. Conversely, assume that $\ker \overline{f} < \ker \overline{g}$. From (O1) we have $\operatorname{Im} \overline{g} \cong \mathbf{F_D}(n)/\ker \overline{g} \cong \mathbf{2}$ and $\operatorname{Im} \overline{f} \cong \mathbf{F_D}(n)/\ker \overline{f} \cong \mathbf{D}$, and, in consequence, $\operatorname{Im} \overline{f} = \mathbf{D}$. Then there exists $x \in X$ such that $f(x) = *$, and so $e(f(x)) = 1$. Hence $\ker \overline{f} < \ker(\overline{e \circ f}) \in \operatorname{Cm}(\mathbf{F_D}(n))$. Thus, using (O1) we obtain $\ker \overline{g} = \ker(\overline{e \circ f})$, which implies $g = e \circ f$, as required.

(4) φ preserves the partial operations. Let $f, g \in S(n)$, $f \neq g$, and $* \notin \operatorname{Im} f \cup \operatorname{Im} g$. Then $\mathbf{F_D}(n)/\ker \overline{f} \cong \operatorname{Im} \overline{f} = \mathbf{2}$ and $\mathbf{F_D}(n)/\ker \overline{g} \cong \operatorname{Im} \overline{g} = \mathbf{2}$. Moreover, $\ker \overline{f} \bullet \ker \overline{g} = (\ker \overline{f} \div \ker \overline{g})' \in \operatorname{Cm}(\mathbf{F_D}(n))$ and $\operatorname{Cm}(\mathbf{F_D}(n))/(\ker \overline{f} \bullet \ker \overline{g}) \cong \mathbf{2}$. Put $h := \varphi^{-1}(\ker \overline{f} \bullet \ker \overline{g})$. For $x \in X$ we have $h(x) = 1$ iff $(x, 1) \in \ker \overline{f} \bullet \ker \overline{g}$ iff $(f(x) = 1$ and $g(x) = 1)$ or $(f(x) = 0$ and $g(x) = 0)$ iff $(f \cdot g)(x) = 1$. Thus $\ker \overline{f} \bullet \ker \overline{g} = \varphi(f \cdot g)$, which completes the proof. $\square$

From the above theorem we get the following corollaries.

COROLLARY 3.2. $|\operatorname{Cm}(\mathbf{F_D}(n))| = 3^n - 1$.

COROLLARY 3.3.

1. $\overline{L} = \{\ker \overline{f} : f \in S(n) \text{ and } * \notin \operatorname{Im} f\}$;

   $\underline{L} = \{\ker \overline{f} : f \in S(n) \text{ and } \{0, *\} \subseteq \operatorname{Im} f\}$;

   $P = \{\ker \overline{f} : f \in S(n) \text{ and } 0 \notin \operatorname{Im} f\}$.

Moreover, for $f, g \in S(n)$ we have

2. $\ker \overline{f} \leq \ker \overline{g}$ if and only if $\{0, *\} \subseteq \operatorname{Im} f$, $* \notin \operatorname{Im} g$ and $f^{-1}(\{*, 1\}) = g^{-1}(\{1\})$;
3. the Boolean operation on $\overline{L} \cup \{\mathbf{1}_{\mathbf{F_D}(n)}\}$ is defined by $(1, x) \in \ker \overline{f} \bullet \ker \overline{g}$ if and only if $f(x) = g(x)$ for $x \in X$.

Theorem 3.1 allows us to identify elements from $\mathrm{Cm}(\mathbf{F_D}(n))$ with maps f from X to D. In the diagram, we will label these maps by the set of generators belonging to the kernel of f.

Observe that the construction of the frame $\mathrm{Cm}(\mathbf{F_D}(n))$ is similar to the construction of the frame of the equivalential algebras with conjunction on the regular elements, described in [4]. The number of elements of the frame is the same in both cases, but the equivalence classes of relation $\sim$ are different.

This construction proceeds as follows:

1. Each $\mu \in \mathrm{Cm}(\mathbf{F_D}(n))$ is labelled by the set of indices $\{i : x_i \in X \cap (1/\mu)\} \subseteq \{1, \ldots, n\}$.
2. $\overline{L}$ has $2^n - 1$ elements labelled by all proper subsets of $\{1, \ldots, n\}$ and these elements form only one equivalence class.
3. P has $2^n - 1$ elements labelled by all proper subsets of $\{1, \ldots, n\}$, but in this case each element forms a one-element equivalence class.
4. If $\mu \in \overline{L}$ is labelled by $S \subsetneq \{1, \ldots, n\}$, so below μ (i. e. in $\underline{L}$) there are elements labelled by all proper subsets of S.
5. Each $\mu \in \underline{L}$ forms a one-element equivalence class.

We will also use the following designations in the figures:

1. Each dot denotes an element of the frame.
2. Straight lines denote a partial ordering directed upwards.
3. The equivalence class with more than one element is marked with an ellipse.
4. Each dot that does not lie in an ellipse denotes a one-element equivalence class.

3.1. $\mathbf{F_D}(2)$

$\mathrm{Cm}(\mathbf{F_D}(2))$ has 8 elements (Fig. 1): So, there are 9 hereditary sets on the left side and 8 hereditary sets on the right side. Finally, $|\mathbf{F_D}(2)| = 9{\cdot}8 = 72$.

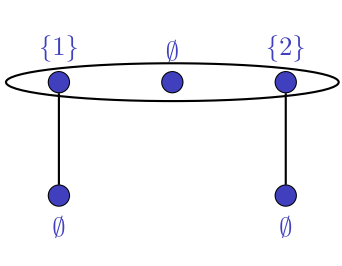

Figure 1. $\mathrm{Cm}(\mathbf{F_D}(2))$

3.2. $\mathbf{F_D}(3)$

$\mathrm{Cm}(\mathbf{F_D}(3))$ has 26 elements (Fig. 2): On the left side there are 4536 hereditary sets, and on the right side there are 128 hereditary sets. Consequently, $|\mathbf{F_D}(3)| = 4536 \cdot 128 = 580608$.

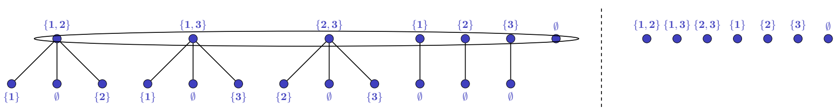

Figure 2. $\mathrm{Cm}(\mathbf{F_D}(3))$

3.3. Direct indecomposability of $\mathbf{F_D}(n)$

Let us start from the following observation.

Proposition 3.4. Let $g \equiv 0 \in S(n)$. Then $\ker \overline{g}$ is the only minimal element of $(\mathrm{Cm}(\mathbf{F_D}(n)), \leq)$ lying in $\overline{L}$.

Proof: Assume to the contrary that $f \in S(n)$ and $\ker \overline{f} < \ker \overline{g}$. Then $* \in \mathrm{Im}\, f$ and $g^{-1}(\{1\}) = f^{-1}(\{*, 1\}) \neq \emptyset$, a contradiction.

To prove the uniqueness take $h \in S(n)$ such that $* \notin \mathrm{Im}\, h$, and $h \neq g$. Take $x_0 \in X$ such that $h(x_0) = 1$. Define $f_0 : X \to D$ given by $f_0(x) = h(x)$ for $x \neq x_0$ and $f_0(x) = *$ for $x = x_0$. Since $0 \in \mathrm{Im}\, h$, we get $\{0,*\} \subseteq \operatorname{Im} f_0$ and $f_0^{-1}(\{*,1\}) = h^{-1}(\{1\})$. From Corollary 3.3 we obtain $\ker \overline{f_0} < \ker \overline{h}$. □

Example 3.5.

1. $|X| = 1$. Then $\mathrm{Cm}(\mathbf{F_D}(1)) = L \cup P$, where $L := \{\ker \overline{f} : f(x_1) = 0\}$ and $P := \{\ker \overline{f} : f(x_1) = *\}$. From Proposition 2.4 we get $\mathbf{F_D}(1) \cong \mathbf{2} \times \mathbf{2}^{\wedge}$.
2. $|X| = 2$. From Fig. 1 we see that $\underline{L} = \{\mu_1, \mu_2\}$, $\mu_1 \leq \ker \overline{f}$, where $f(x_1) = 1$, $f(x_2) = 0$, and $\mu_2 \leq \ker \overline{g}$, where $g(x_1) = 0$, $g(x_2) = 1$. Then $\mu_1 \wedge \mu_2 \leq \ker \overline{f} \wedge \ker \overline{g} \leq \ker \overline{f} \bullet \ker \overline{g} = \ker \overline{h}$, where $h(x_1) = 0$, $h(x_2) = 0$. Hence $\bigwedge L = \mu_1 \wedge \mu_2$. Moreover $\ker \overline{f} < \mu_1 \vee \mu_2$, and so $\mu_1 \vee \mu_2 = \mathbf{1}_{\mathbf{F_D}(2)}$. Thus $\mathbf{F_D}(2)/\bigwedge L \cong \mathbf{F_D}(2)/\mu_1 \times \mathbf{F_D}(2)/\mu_2 \cong \mathbf{D} \times \mathbf{D}$, and finally, from [Proposition 2.4] $\mathbf{F_D}(2) \cong \mathbf{D}^2 \times (\mathbf{2}^{\wedge})^3$.

Unfortunately, for $n \geq 3$, the situation is not so easy, since $\mathbf{F_D}(n)/\bigwedge L$ is not directly decomposable.

Theorem 3.6. *$\mathbf{F_D}(n)/\bigwedge L$ is directly indecomposable for $n \geq 3$.*

Proof: Let $n \geq 3$. For contradiction assume that $\mathbf{F_D}(n)/\bigwedge L$ is directly decomposable. Then, since $\mathrm{Con}(\mathbf{F_D}(n)/\bigwedge L) = \{\varphi/\bigwedge L : \varphi \in \mathrm{Con}\,\mathbf{F_D}(n)$ and $\bigwedge L \leq \varphi\}$, we can find $\alpha_1, \alpha_2 \in \mathrm{Con}\,\mathbf{F_D}(n)$ such that $\bigwedge L < \alpha_i$ for $i = 1,2$, $\bigwedge L = \alpha_1 \wedge \alpha_2$ and $\alpha_1 \vee \alpha_2 = \mathbf{1}_{\mathbf{F_D}(n)}$. For $i = 1,2$ define $M(\alpha_i) := \{\mu \in \mathrm{Cm}(\mathbf{F_D}(n)) : \alpha_i \leq \mu\}$. First, we show that $M(\alpha_i) \cap L \neq \emptyset$. For this purpose, we deduce from properties of α_i $(i = 1,2)$ that $\alpha_i \neq \mathbf{1}_{\mathbf{F_D}(n)}$. Thus $M(\alpha_i) \neq \emptyset$ and if $M(\alpha_i) \cap L = \emptyset$, then we would find $\mu \in M(\alpha_i)$ such that $\mu \in P$, and so $\bigwedge L < \mu$. From [5, Proposition 5.4] we know that $|\mu/\sim| = 1$, and this means, by [1, Lemma 22], that there exists $\gamma \in L$ such that $\gamma \leq \mu$, which contradicts (O1). Thus, we get $M(\alpha_i) \cap L \neq \emptyset$, and, consequently, $M(\alpha_i) \cap \overline{L} \neq \emptyset$.

Moreover, using [8, Theorem 4.4.(1)], we deduce that $M(\alpha_i) \cap (\overline{L} \cup \{\mathbf{1}_{\mathbf{F_D}(n)}\})$ $(i = 1,2)$ is a Boolean subgroup of $(\overline{L} \cup \{\mathbf{1}_{\mathbf{F_D}(n)}\}, \bullet)$. Boolean groups can be treated as vector spaces over $\mathbb{Z}_2$. We show that $\overline{L} \cup \{\mathbf{1}_{\mathbf{F_D}(n)}\}$ can be split as the direct sum of its vector subspaces $M(\alpha_i) \cap (\overline{L} \cup \{\mathbf{1}_{\mathbf{F_D}(n)}\})$ $(i = 1,2)$. Firstly, we observe that $M(\alpha_1) \cap M(\alpha_2) = \emptyset$, since otherwise there exists $\mu \in M(\alpha_1) \cap M(\alpha_2)$, and so $\mathbf{1}_{\mathbf{F_D}(n)} = \alpha_1 \vee \alpha_2 \leq \mu$, a contradiction. Take now $\mu \in \overline{L}$ such that $\mu \notin M(\alpha_1) \cup M(\alpha_2)$. Then, applying

[5, Proposition 5.5] and [1, Lemma 22] again, by $\alpha_1 \wedge \alpha_2 \leq \mu$ we deduce that there exist $\mu_1, \mu_2 \in \overline{L}$ such that $\alpha_1 \leq \mu_1$, $\alpha_2 \leq \mu_2$ and $\mu_1 \wedge \mu_2 \leq \mu$. Now from [8, Lemma 3.10] we get that $\mu = \mu_1 \bullet \mu_2$, $\mu_1 \in M(\alpha_1)$, and $\mu_2 \in M(\alpha_2)$. Hence $M(\alpha_i) \cap (\overline{L} \cup \{\mathbf{1}_{\mathbf{F_D}(n)}\})$ $(i = 1, 2)$ form a direct sum equal $\overline{L} \cup \{\mathbf{1}_{\mathbf{F_D}(n)}\}$ of dimension n. Hence $\left|\{\mu \in \overline{L} : \mu \notin M(\alpha_1) \cup M(\alpha_2)\}\right| > 2$ for $n \geq 3$. We know from Proposition 3.4 that in $\overline{L}$ there is a unique minimal element of $(\mathrm{Cm}(\mathbf{F_D}(n)), \leq)$. Take $\mu \in \overline{L}$ such that $\alpha_1 \nleq \mu$, $\alpha_2 \nleq \mu$, and μ is not minimal in $(\mathrm{Cm}(\mathbf{F_D}(n)), \leq)$. Then there is $\gamma \in \underline{L}$ such that $\gamma < \mu$. As $\alpha_1 \wedge \alpha_2 \leq \gamma$ and, by [5, Proposition 5.4], $|\gamma/_\sim| = 1$, we obtain, using [1, Lemma 22] again, $\alpha_1 \leq \gamma$ or $\alpha_2 \leq \gamma$. Thus $\alpha_1 \leq \mu$ or $\alpha_2 \leq \mu$, a contradiction. □

4. Free spectrum

In this section we compute the cardinality of the free algebras in $\mathbf{F_D}(n)$, which is a laborious task. However, it is finally possible to find the explicit formula on the free spectrum.

From the definition of L and P, property O1 and Definition 2.5 it follows that:

Proposition 4.1.

$$|\mathbf{F_D}(n)| = |\mathcal{H}(L)| \cdot |\mathcal{H}(P)|,$$

where $\mathcal{H}(L) := \{Z \cap L : Z \in \mathcal{H}(\mathbf{F_D}(n))\}$ and $\mathcal{H}(P) := \{Z \cap P : Z \in \mathcal{H}(\mathbf{F_D}(n))\}$.

We first compute the right factor of this product.

Proposition 4.2. Let $P = \{\mu \in \mathrm{Cm}(\mathbf{F_D}(n)) : \mathbf{F_D}(n)/\mu \cong \mathbf{2}^\wedge\}$. Then:

$$|\mathcal{H}(P)| = 2^{2^n - 1}.$$

Proof: The set P contains $2^n - 1$ elements and every subset of the P is a hereditary set. Therefore, the number of the hereditary sets is equal to 2^{2^n-1}. □

Next, we compute the left factor. For this, we will use the following lemma. This fact has been used in the proof of [6, Theorem 10]. However, it was given without proof. We will denote by $P(n)$ the family of all subsets of the set $\{1, \ldots, n\}$.

LEMMA 4.3. *[6, p. 1352] The map*

$$S : P(n) \ni A \to S(A) := \{C \in P(n) : |A \setminus C| \ \text{ is even}\}$$

gives a one-to-one correspondence between $P(n)$ *and* $\{H \subseteq P(n) : (H, \bullet)$ *is a hyperplane* $(P(n), \bullet)$ *or* $H = P(n)\}$, *where* $(P(n), \bullet)$ *is a Boolean group with the operation* $\bullet$ *defined as follows:* $B \bullet C := (B \div C)'$ *for* $B, C \in P(n)$.

PROOF: We give only the main ideas of the proof (we will skip the tedious details). First, we note that $(S(A), \bullet)$ is a subgroup $(P(n), \bullet)$ for $A \in P(n)$. This is because $\{1, \ldots, n\} \in S(A)$ is a neutral element of$(P(n), \bullet)$ and it is easy to check (by considering parity) that $C_1 \bullet C_2 \in S(A)$ for $C_1, C_2 \in S(A)$.

In the same manner we can see that if D_1, $D_2 \notin S(A)$, so $D_1 \bullet D_2 \in S(A)$. Therefore, if $A \neq \emptyset$, so $S(A)$ is a maximal subgroup of $(P(n), \bullet)$ (if $A = \emptyset$, so it is obvious that $S(A) = P(n)$).

It remains to prove that S is bijective. Since the sets $\{H \subseteq P(n) :$ H jest is a hyperplane or $H = P(n)\}$ and $P(n)$ have the same cardinality (equal to 2^n), thus it is sufficient to prove that S is injective. Let $A, B \in P(n)$ such that $A \neq B$. We give the proof only for the case $|A|$—is odd, $|B|$—is odd; the other cases are left to the reader. Thus there exists $x \in P(n)$, such that $x \in A$ and $x \notin B$ (or, conversely). Then $|A \setminus \{x\}|$ is even, so $\{x\} \in S(A)$ and $|B \setminus \{x\}|$ is odd. In consequence, $\{x\} \notin S(B)$. Therefore $S(A) \neq S(B)$. □

Next, we use [6, Theorem 10.1], which we adapted to our case.

THEOREM 4.4. *We have*

$$|\mathcal{H}(L)| = \sum_{k=0}^{n} \binom{n}{k} \prod_{m=1}^{n-1} (l(m))^{\alpha_k(n,n-m)} \tag{4.1}$$

where

$$\alpha_k(n,j) := \sum_{p} \binom{n-k}{j-2p} \binom{k}{2p} \tag{4.2}$$

for $p \in \mathbb{N}$, *such that* $\max(0, k+j-n) \leq 2p \leq \min(k,j)$ *for* $k, j \in \mathbb{N}$, $0 \leq k \leq n$ *and* $1 \leq j \leq n$, *where*

$$l(m) := 2^{2^m - 1}.$$

Proof: Let $A \in P(n)$. Therefore, from Lemma 4.3 it follows that $(S(A), \bullet)$ is hyperplane in $(P(n), \bullet)$ or $(S(A), \bullet)=(P(n), \bullet)$. Write $\mathcal{H}(A) := \{Z \in \mathcal{H}(L) : Z \cap \overline{L} = S(A)\}$. Then $\mathcal{H}(L) = \bigcup\{\mathcal{H}(A) : A \in P(n)\}$. Similarly to [6, p. 1352], every $\mathcal{H}(A)$ can be identified with the Cartesian product of the family subsets of C, such that $C \in S(A) \setminus \{1, \dots, n\}$. However, in our case the number of such subsets is $2^{2^m - 1}$, where $m = |C|$.

From this we deduce that:

$$|\mathcal{H}(L)| = \sum_{A \in P(n)} |\mathcal{H}(A)| =$$

$$\sum_{A \in P(n)} \prod_{m=0}^{n-1} l(m)^{|\{C \in S(A) : |C| = m\}|}.$$

Next, note that if $|A| = |B|$, so:

$$|\{C \in S(A) : |C| = m\}| = |\{C \in S(B) : |C| = m\}|,$$

for $A, B \in P(n)$. Therefore:

$$|\mathcal{H}(L)| = \sum_{k=0}^{n} \binom{n}{k} \prod_{m=1}^{n-1} l(m)^{\beta_k(n,m)},$$

where

$$\beta_k(n, m) := |\{C \in S(A) : |C| = m\}|,$$

for $A \in P(n)$, such that $|A| = k$.

Now, we calculate $\beta_k(n, m)$. If $C \in S(A)$ and $|C| = m$, so $|A \setminus C| = 2p$, for $p \in \mathbb{N}$. Since $|A| = k$, so $|A \cap C| = k - 2p$. Thus $|C \setminus A| = m - (k - 2p)$. Consequently, $|\{C \in S(A) : |C| = m\}| = \binom{k}{2p} \cdot \binom{n-k}{m-(k-2p)}$. Finally we get that

$$\beta_k(n,m) = \sum_p \binom{n-k}{m-(k-2p)}\binom{k}{2p}$$
$$= \sum_p \binom{n-k}{(n-k)-(m-(k-2p))}\binom{k}{2p}$$
$$= \sum_p \binom{n-k}{n-m-2p}\binom{k}{2p},$$

where $p \in \mathbb{N}$, such that $\max(0, k-m) \le 2p \le \min(k, n-m)$, for $k, m \in \mathbb{N}$, $0 \le k \le n$ and $0 \le m \le n-1$. Taking $\alpha_k(n,j) := \beta_k(n, n-j)$ (then $j = n-m$) we get (4.1). □

To get the explicit formula, we use the following Lemmas.

LEMMA 4.5 ([6, Proposition 11]). *The functions α_k ($k \in \mathbb{N}$) fulfill:*

1. *The recurrence equation*

$$\alpha_k(n+1, j) = \alpha_k(n,j) + \alpha_k(n, j-1),$$

for $n \ge k$ and $1 \le j \le n$.

2. *The boundary conditions:*

$$\alpha_k(k,j) = \begin{cases} \binom{k}{j}, & j - \textit{even} \\ 0, & j - \textit{odd} \end{cases} \quad : 0 \le j \le k.$$

3. $$\alpha_k(n,n) = \begin{cases} 1, & k - \textit{even} \\ 0, & k - \textit{odd} \end{cases} \quad : n \ge k.$$

4. $$\alpha_k(n,0) = 1 : n \ge k.$$

LEMMA 4.6. *[6, Lemma 12] Let $n, k \in \mathbb{N}$ and $n \ge k$. Let us consider the generating functions for the coefficients $\alpha_k(n,j)$ ($j = 0, \ldots, n$) given by*

$$t_{n,k}(z) := \sum_{j=0}^{n} \alpha_k(n,j) z^j.$$

Then:

$$t_{k,k}(z) = \sum_{\substack{j=0 \\ j-parzyste}}^{k} \binom{k}{j} z^j \tag{4.3}$$

where

$$t_{n,k}(z) = (z+1)^{n-k} t_{k,k}(z). \tag{4.4}$$

Next, we prove the following Lemma.

LEMMA 4.7. *Let $k, j \in \mathbb{N}$. Then:*

1)

$$\sum_{\substack{j=0 \\ j-even}}^{k} \binom{k}{j} 2^{k-j} = \frac{3^k+1}{2},$$

2)

$$\sum_{\substack{j=0 \\ j-even}}^{k} \binom{k}{j} = 2^{k-1}.$$

PROOF: Ad. 1. Let:

$$a := \sum_{\substack{j=0 \\ j-even}}^{k} \binom{k}{j} 2^{k-j} \text{ and } b := \sum_{\substack{j=0 \\ j-odd}}^{k} \binom{k}{j} 2^{k-j}.$$

Then $a + b = \sum_{j=0}^{k} \binom{k}{j} 2^{k-j} \cdot 1^j = (2+1)^k = 3^k$.
In turn

$$a-b = \sum_{\substack{j=0 \\ j-even}}^{k} \binom{k}{j} 2^{k-j} - \sum_{\substack{j=0 \\ j-odd}}^{k} \binom{k}{j} 2^{k-j} = \sum_{j=0}^{k} \binom{k}{j} 2^{k-j} \cdot (-1)^j = (2-1)^k = 1.$$

From the system of equations: $a + b = 3^k$ and $a - b = 1$ we get $a = \frac{3^k+1}{2}$.

Ad. 2. Let:

$$a := \sum_{\substack{j=0 \\ j-even}}^{k} \binom{k}{j} \text{ and } b := \sum_{\substack{j=0 \\ j-odd}}^{k} \binom{k}{j}.$$

Hence $a + b = \sum_{j=0}^{k} \binom{k}{j} = 2^k$,
and

$$a - b = \sum_{\substack{j=0 \\ j-even}}^{k} \binom{k}{j} - \sum_{\substack{j=0 \\ j-odd}}^{k} \binom{k}{j}$$

$$= \sum_{j=0}^{k} \binom{k}{j} 1^{k-j} \cdot (-1)^j = (1-1)^k = 0.$$

From the system of equations: $a + b = 2^k$ and $a - b = 0$ it follows that $a = 2^{k-1}$. □

Now, we prove the following result:

COROLLARY 4.8.

$$t_{n,k}(1) = \begin{cases} 2^n & k = 0, \\ 2^{n-1} & k \neq 0. \end{cases} \tag{4.5}$$

PROOF: From (4.4) and then from (4.3), we have for $k = 0$:

$$t_{n,0}(1) = 2^n t_{0,0}(1) = 2^n \sum_{\substack{j=0 \\ j-even}}^{0} \binom{0}{j} 1^j = 2^n.$$

Now, let $k \neq 0$:

$$t_{n,k}(1) = 2^{n-k} t_{k,k}(1) = 2^{n-k} \sum_{\substack{j=0 \\ j-even}}^{k} \binom{k}{j} 1^j.$$

Lemma 4.7 shows that

$$\sum_{\substack{j=0 \\ j-even}}^{k} \binom{k}{j} = 2^{k-1}.$$

Hence

$$t_{n,k}(1) = 2^{n-k} \cdot 2^{k-1} = 2^{n-k+k-1} = 2^{n-1}.$$ □

Finally we prove the following theorem.

THEOREM 4.9.

$$|\mathcal{H}(L)| = 2^{3^n - 2^{n+1} + 1} + (2^{-2^{n-1} - 2^n + 1}) \sum_{k=1}^{n} \binom{n}{k} 2^{\frac{3^{n-k}(3^k+1)}{2}}.$$

PROOF: From (4.1) it follows that:

$$|\mathcal{H}(L)| = \sum_{k=0}^{n} \binom{n}{k} \prod_{m=1}^{n-1} (2^{2^m} - 1)^{\alpha_k(n, n-m)}. \tag{4.6}$$

Replacing $n - m$ by j, we get:

$$|\mathcal{H}(L)| = \sum_{k=0}^{n} \binom{n}{k} 2^{\sum_{j=1}^{n-1} (2^{n-j} - 1)\alpha_k(n,j)}. \tag{4.7}$$

We calculate separately the above exponent.
Let $W_k := \sum_{j=1}^{n-1} (2^{n-j} - 1)\alpha_k(n, j)$. Then:

$$\begin{aligned} W_k = \sum_{j=1}^{n-1} (2^{n-j} - 1)\alpha_k(n, j) &= \sum_{j=1}^{n-1} 2^{n-j}\alpha_k(n, j) - \sum_{j=1}^{n-1} \alpha_k(n, j) \\ &= \sum_{j=0}^{n} 2^{n-j}\alpha_k(n, j) - 2^{n-0}\alpha_k(n, 0) - 2^{n-n}\alpha_k(n, n) \\ &\quad - (\sum_{j=0}^{n} \alpha_k(n, j) - \alpha_k(n, 0) - \alpha_k(n, n)) \\ &= \sum_{j=0}^{n} 2^{n-j}\alpha_k(n, j) - 2^n\alpha_k(n, 0) - \alpha_k(n, n) \\ &\quad - \sum_{j=0}^{n} \alpha_k(n, j) + \alpha_k(n, 0) + \alpha_k(n, n). \end{aligned}$$

Simplifying and applying Lemma 4.5(4), we get:

$$W_k = \sum_{j=0}^{n} 2^{n-j}\alpha_k(n, j) - \sum_{j=0}^{n} \alpha_k(n, j) - 2^n + 1. \tag{4.8}$$

We now compute the first sum in (4.8). We denote it by S_k. Then

$$S_k = \sum_{j=0}^{n} 2^{n-j}\alpha_k(n,j) = \sum_{j=0}^{n} 2^n \cdot 2^{-j}\alpha_k(n,j) = 2^n \sum_{j=0}^{n} 2^{-j}\alpha_k(n,j).$$

On account of Lemma 4.6, we have:

$$S_k = 2^n \cdot t_{n,k}(\frac{1}{2}) = 2^n \cdot (\frac{1}{2}+1)^{n-k} \cdot t_{k,k}(\frac{1}{2}) = 2^n \frac{3^{n-k}}{2^{n-k}} \sum_{\substack{j=0 \\ j-even}}^{k} \binom{k}{j} 2^{-j} =$$

$$3^{n-k} \cdot 2^k \sum_{\substack{j=0 \\ j-even}}^{k} \binom{k}{j} 2^{-j} = 3^{n-k} \sum_{\substack{j=0 \\ j-even}}^{k} \binom{k}{j} 2^{k-j}.$$

From Lemma 4.7 we conclude that

$$S_k = 3^{n-k}\frac{3^k+1}{2}.$$

It follows from Lemma 4.6 that the second sum in (4.8) is equal to $t_{n,k}(1)$. Hence:

$$W_k = 3^{n-k}\frac{3^k+1}{2} - t_{n,k}(1) - 2^n + 1.$$

Applying Corollary 4.8 and Lemma 4.6 we deduce that:

$$W_0 = 3^n - 2^{n+1} + 1,$$

In turn for $k \neq 0$ we get:

$$W_k = 3^{n-k}\frac{3^k+1}{2} - 2^{n-1} - 2^n + 1.$$

We can now return to (4.7). We get:

$$|\mathcal{H}(L)| = 2^{3^n - 2^{n+1}+1} + \sum_{k=1}^{n}\binom{n}{k} 2^{\frac{3^{n-k}(3^k+1)}{2} - 2^{n-1} - 2^n + 1} =$$

$$2^{3^n - 2^{n+1}+1} + (2^{-2^{n-1}-2^n+1}) \sum_{k=1}^{n}\binom{n}{k} 2^{\frac{3^{n-k}(3^k+1)}{2}}.$$

□

We can now formulate our main result.

Theorem 4.10. *Let $n \in \mathbb{N}$. Then*

$$|\mathbf{F_D}(n)| = 2^{3^n - 2^n} + \sum_{k=1}^{n} \binom{n}{k} 2^{\frac{3^n + 3^{n-k}}{2} - 2^{n-1}}.$$

Proof: Combining Theorem 4.9 with Proposition 4.2 we deduce that

$$|\mathbf{F_D}(n)| = 2^{2^n - 1}(2^{3^n - 2^{n+1} + 1} + (2^{-2^{n-1} - 2^n + 1}) \sum_{k=1}^{n} \binom{n}{k} 2^{\frac{3^{n-k}(3^k + 1)}{2}}) =$$

$$2^{3^n - 2^{n+1} + 1 + 2^n - 1} + (2^{-2^{n-1} - 2^n + 1 + 2^n - 1}) \sum_{k=1}^{n} \binom{n}{k} 2^{\frac{3^{n-k}(3^k + 1)}{2}} =$$

$$2^{3^n - 2^n} + 2^{-2^{n-1}} \sum_{k=1}^{n} \binom{n}{k} 2^{\frac{3^{n-k}(3^k + 1)}{2}} = 2^{3^n - 2^n} + \sum_{k=1}^{n} \binom{n}{k} 2^{\frac{3^n + 3^{n-k}}{2} - 2^{n-1}}.$$

□

Corollary 4.11. Let $n \in \mathbb{N}$. Then $|\mathbf{F_D}(n)|$ is asymptotically equal to $2^{3^n - 2^n}$.

Proof: According to the above theorem, it is sufficient to show that:

$$\frac{\sum_{k=1}^{n} \binom{n}{k} 2^{\frac{3^n + 3^{n-k}}{2} - 2^{n-1}}}{2^{3^n - 2^n}} \xrightarrow{n \to +\infty} 0.$$

First observe that:

$$0 \leq \frac{\sum_{k=1}^{n} \binom{n}{k} 2^{\frac{3^n + 3^{n-k}}{2} - 2^{n-1}}}{2^{3^n - 2^n}} \leq \frac{2^n \cdot 2^{\frac{3^n + 3^{n-1}}{2} - 2^{n-1}}}{2^{3^n - 2^n}} \leq \frac{2^{n + \frac{3^n + 3^{n-1}}{2} - 2^{n-1}}}{2^{3^n - 2^n}}.$$

We next show that: $\frac{2^{n + \frac{3^n + 3^{n-1}}{2} - 2^{n-1}}}{2^{3^n - 2^n}} \xrightarrow{n \to +\infty} 0$.

Since $n + \frac{3^n + 3^{n-1}}{2} - 2^{n-1} = n + \frac{4 \cdot 3^{n-1}}{2} - 2^{n-1} = n + 2 \cdot 3^{n-1} - 2^{n-1}$, it follows that:

$$\begin{aligned}
\frac{2^{n + \frac{3^n + 3^{n-1}}{2} - 2^{n-1}}}{2^{3^n - 2^n}} &= \frac{2^{n + 2 \cdot 3^{n-1} - 2^{n-1}}}{2^{3^n - 2^n}} \\
&= 2^{n + 2 \cdot 3^{n-1} - 2^{n-1} - 3^n + 2^n} \\
&= 2^{n - 3^{n-1} + 2^{n-1}}.
\end{aligned}$$

Now that $n - 3^{n-1} + 2^{n-1} = 3^{n-1}(\frac{n}{3^{n-1}} - 1 + (\frac{2}{3})^{n-1}) \xrightarrow{n\to+\infty} -\infty$, we have

$$\frac{2^{n+\frac{3^n+3^{n-1}}{2}-2^{n-1}}}{2^{3^n-2^n}} \xrightarrow{n\to+\infty} 0,$$

and the proof is complete. □

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