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ABOUT LOGICALLY PROBABLE SENTENCES

Abstract

The starting point of this paper is the empirically determined ability to reason in natural language by employing probable sentences. A sentence is understood to be logically probable if its schema, expressed as a formula in the language of classical propositional calculus, takes the logical value of truth for the majority of Boolean valuations, i.e., as a logically probable formula. Then, the formal system **P** is developed to encode the set of these logically probable formulas. Based on natural semantics, a strong completeness theorem for **P** is proved. Alternative notions of consequence for logically probable sentences are also considered.

Keywords: probable sentences, majority, logically probable formula, Boolean valuation.

0. Intuitive motivation

Natural language reasoning can occasionally lead from true premises to false conclusions, which is incorrect from the standpoint of classical logic. Most of the time, the formulas of the classical propositional calculus (PC) that correlate to such erroneous inferences are not particularly interesting from a logical point of view. Consider the inference: "If it is raining, the roadway will be wet. Therefore (the conclusion): If it is not raining, then the roadway will not be wet." Similarly: "If it is raining, the roadway will be wet. Therefore (conclusion): If the roadway is wet, then it was raining." People routinely employ similar reasoning in their daily lives, even though the results of these inferences are not logically certain or do not follow logically from the premises. One non-trivial explanation for why

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this occurs is because, in the majority of real-world situations, the roadway simply will not be wet if it has not rained. In other words, determining if it has rained recently usually suffices to determine how wet the road is. We will not investigate whether the water accidentally leaked from somewhere because of a malfunction of a passing truck transporting mineral water or because a Zeppelin flying nearby dropped a massive water-filled balloon. Extreme situations, i.e., those that do not follow the ordinary course of things, are disregarded in our predictions. We are only interested in what is normal or typical, or what happens in most everyday situations. The precise sense in which the conclusions of the above two inferences follow from the premises will be given later in the paper.

We will provide one additional, perhaps distant analogy, which could be helpful in the intuitive grounding of the research undertaken in the next paragraphs. Consider a random device that is being utilized in a particular manner for a specific purpose. Typically, such a device will continue to function effectively until it wears out or malfunctions. As long as it is operated in accordance with the instruction manual, the device will function fairly effectively. In other words, the device will work if the manufacturer's requirements are satisfied, but it will not function well if the manufacturer's conditions are not met. The previous sentence contains two conditional assertions that one will undoubtedly run into in everyday life. Both are only probable, and we believe that is why we should try to find the rules for employing probable statements.

In order to summarize the overall issue, the key problem is how logical rules govern sentences that are merely probable, since it is known beforehand that they do not generally hold true yet – at the same time – are true in a limited number of or in *most* cases.

The observations outlined above and similar facts lead us to interest in reasoning that involves sentences based on patterns (formulas) that are true for most Boolean valuations.

1. Introduction

Let us take a typical propositional language based on an alphabet that comprises: (a) a countable set of propositional variables $V = \{p_1, p_2, p_3, \ldots\}$; (b) connectives: $\neg, \rightarrow, \vee, \wedge, \equiv$, respectively called negation, implication, disjunction, conjunction and equivalence; (c) the) and (brackets, i.e., respectively, the closing bracket and the opening bracket. For the sake of convenience, we shall represent the variables with the symbols: $p, q, r, s, t, \ldots$. The meaning of the connectives is characterized by the so-called truth tables for classical logic. The set of all well-formed formulas that are based on the aforementioned alphabet is denoted by the symbol $Form_{PC}$. The symbol $Form_{\to}$ denotes a proper subset of the set $Form_{PC}$ and contains formulas built only with the use of the variables, the sign of implication, and the brackets. The set of all subsets of the set X is denoted by the symbol 2^X, and the set of all finite subsets of the set X is denoted by the symbol $FinX$. The derivability relation for the language of PC will be denoted by $\vdash_{PC}$, and the corresponding consequence operation will be denoted by – C_{PC}. If X is a set, the symbol $|X|$ denotes the cardinality of X. If $A \in Form_{PC}$, the set of propositional variables in the formula A is denoted by the symbol $Var(A)$, and the symbol $|Var(A)|$ denotes the cardinality of this set, e.g., $Var((p \to q) \vee r)) = \{p, q, r\}$ and $|Var((p \to q) \vee r))| = |\{p, q, r\}| = 3$.

From the definition of the set of all Boolean valuations, which we denote by the symbol Val, we know that it has a power of continuum. Each element of the set Val is an extension of the *valuation of propositional variables* $v : V \to \{0, 1\}$. We will use the same symbol v for valuations of propositional variables and valuations of formulae, as this should not cause any confusion and is convenient.

Definition 1.1. Let $A \in Form_{PC}$. For every $v, v' \in Val :\ vR_Av'$ iff $v(p) = v'(p)$, for any $p \in Var(A)$.

The equivalence classes of the relation R_A will be denoted by $[v]_{R_A}$. Each valuation $v' \in [v]_{R_A}$ can be uniquely assigned the restriction of the valuation v' to the propositional variables occuring in A, denoted further by $v' \lceil Var(A)$. We have:

Fact 1.2. Let $A \in Form_{PC}$. For every valuation $v' \in [v]_{R_A}$ holds:

$$v' \lceil Var(A) = v \lceil Var(A).$$

Proof: For every $p \in Var(A)$, $(v' \lceil Var(A))(p) = (v \lceil Var(A))(p)$, which gives $v' \lceil Var(A) = v \lceil Var(A)$. □

We will call each such restriction $v \lceil Var(A)$: *significantly different Boolean valuation of the formula A or significantly different Boolean valuation* for short, when it is clear what formula is involved. For the established formula $A \in Form_{PC}$, there is a mutually one-to-one correspondence between equivalence classes and valuation restrictions.

This gives the following:

FACT 1.3. For any formula A, if $|Var(A)| = n$, then $|\{v\lceil Var(A) : v \in Val\}| = 2^n$ (the number of *significantly different Boolean valuations* is 2^n).

DEFINITION 1.4. A formula $A \in Form_{PC}$ is called a tautology (or a PC tautology) iff $v(A) = 1$ for every $v \in Val$.

The set of all tautologies of Classical Propositional Calculus (CPC) will be further denoted by $TAUT_{PC}$, with or without the index PC.

The obvious fact holds:

FACT 1.5. If A is a PC tautology, then $(v\lceil Var(A))(A) = 1$, for every valuation v.

2. Logically probable formulas

DEFINITION 2.1 (Logical probability function). We will call the function $m : Form_{PC} \to [0, 1]$ into a closed interval of real (rational) numbers **the logical probability function** if for any $A \in Form_{PC}$, $m(A) = |\{v\lceil Var(A) : v(A) = 1\}|/2^{|Var(A)|}$.

DEFINITION 2.2.[1] A formula $A \in Form_{PC}$ will be called a **logically probable formula** iff $m(A) > 1/2$.

DEFINITION 2.3. A set $X \subset Form_{PC}$ is called contradictory iff $A \in X$ and $\neg A \in X$ for some $A \in Form_{PC}$. A set of formulas of PC is called non-contradictory if it is not contradictory.

For the propositional language $Form_{PC}$ (recall the formulas inside the set $Form_{PC}$ can use the negation sign) and a consequence operation C defined for that language, the notion of a *contradictory* set of formulas is not equivalent to the notion of an *inconsistent* set of formulas because a

[1] According to the meaning, a 'probable sentence' is one whose probability is greater than 1/2, in the range of real numbers from 0 to 1. This sense of sentence probability was considered by [2, p. 7]. I owe the Reviewer a significant simplification of these two basic definitions.

contradictory set must simultaneously contain a formula and its negation, while an inconsistent set need not.

Recall that a set X is (*simply*) *consistent* (under C) iff there is no formula A such that both $A \in C(X)$ and $\neg A \in C(X)$.

A set X is absolutely consistent (under C) iff $C(X) \neq Form_{PC}$.[2]

In general, if $X \subset Form_{PC}$ is *simply consistent*, then X is *absolute consistent.*

Using Definition 2.3, we can see that: if $C(X)$ is non-contradictory, then X is consistent; but if X is contradictory, then X is inconsistent because $X \subset C(X)$.

We cannot say much about the consistency of the whole set P because we do not have a relevant operation of the consequence for the set P defined, and that goes beyond our present work. However, there is a sense in which the set P is provably consistent, namely by virtue of Lemma 2.4, under the *idle consequence* Id ($Id(X) = X$, for any set X); cf. [8, p. 38]: $Id(P) = P$ and P is non-contradictory.

Our further considerations (from Section 3 onwards) will concern the set P but they will be restricted to purely implicational language. We will then revisit the issue of consistency.

Lemma 2.4. *The set P is non-contradictory.*

Proof: Assume that P is contradictory, which means that some A and $\neg A$ are members of the set P. This means that $1/2 < m(A)$ and $1/2 < m(\neg A)$, but $m(\neg A) = (1 - m(A))$, hence $m(A) < 1/2$, which gives a contradiction. □

Lemma 2.5. *The following statements hold: (a) P is decidable, i.e., there exists an algorithm to determine in a finite number of steps whether any formula $A \in Form_{PC}$ belongs to the set P or not. (b) P is not closed under substitution. (c) P is not closed under modus ponens. (d) $TAUT \subset P$. (e) it exists such $A \in P$, that $A \notin P$ and $\neg A \notin P$. (f) the formulas built only from propositional variables, brackets and the disjunction connective belong to P. (g) P is inconsistent in propositional logic. (h) $P \neq Form_{PC}$. (i) P is not closed with respect to the rule with the schema $(A \to B), (B \to C)//(A \to C)$.*

[2] Such an understanding of consistency is also suitable for languages without the negation sign.

PROOF: (a) Use the method of truth tables. (b) $(p \to p) \to q \notin P$ while $p \to q \in P$. (c) $(p \vee q) \in P$ and $(p \vee q) \to q \in P$. (d) if $A \in TAUT$, then $m(A) = 1$. (e) $m(p) = m(\neg p) = 1/2$. (f) $m(p \vee q) = 3/4$. (g) $q \in C_{PC}(P)$ and $\neg q \in C_{PC}(P)$ with *modus ponens* and *substitution*. (h) $m(p) = 1/2$. (i) $((p \to (q \to p)) \to (p \to q))$, $((p \to q) \to r) \in P$, but $((p \to (q \to p)) \to r) \notin P$. □

LEMMA 2.6. *Let $A, B \in Form_{PC}$, the variable $p \notin Var(A)$, $|Var(A)| = n$ and $Var(B) = Var(A) \cup \{p\}$, then the following hold:*

1. $|Var(B)| = (n + 1)$;
2. $R_B \subset R_A$;
3. $[v']_{R_A} = [v'']_{R_B} \cup [v''']_{R_B}$; *where* $v''(p) = 1$ *and* $v'''(p) = 0$.

PROOF: Ad. 1. Case 1 is obvious.

Ad. 2. Suppose $< v, v' > \in R_B$, i.e., for any variable $q \in Var(B)$, $v(q) = v'(q)$. The set $Var(B)$ is a superset of $Var(A)$ ($Var(B) \supset Var(A)$), for each variable $r \in Var(A)$, $v(r) = v'(r)$, which results in vR_Av'.

Ad. 3. For the proof, take the pair $< v, v' > \in R_A$. Then, $v(q) = v'(q)$ for every $q \in Var(A)$. Any valuations v'', v''' that belong to R_B take the same logical value for the variables belonging to the set $Var(A)$ as the valuations v and v'. However, the only difference between the valuations v'' and v''' is the value they assign to the variable r, i.e., $v''(r) = 0$ and $v'''(r) = 1$, or reversely. The variable has a value of 0 in one of these valuations and a value of 1 in the other, yet both valuations fall under the class $[v']_{R_A}$. □

To explain it in another way and perhaps more intuitively, let us observe that significantly different valuations of the formula A i.e. each $v\lceil Var(A)$, for $v \in Val$, can be represented as finite sequences of 0s and 1s. If x represents such a string of length n, then the strings x0 and x1 represent strings of length $n + 1$. Up to each finite height (level) n, the full binary tree contains all such zero-one sequences with n-elements.[3]

We know from the previous lemma that if we add a new propositional variable to formula A by means of any of the binary connectives, the number of equivalence classes of the new formula will double. On the other

[3] A binary tree consisting of a root alone has a height of 0.

hand, if by equating two different variables we reduce the number of variables in formula A (for example, if $p, q \in Var(A)$ and $|Var(A)| = n$), and we substitute p for q in each place, we get formula B. Then, of course, $s = |Var(B)| = |Var(A)| - 1 = (n - 1)$ and the number of equivalence classes will decrease from $2n$ to 2^{n-1}, that is, it will be halved. If, on the other hand, we increase the number of variables appearing in formula A by combining it with the binary connective $* \in \{\vee, \wedge, \rightarrow, \equiv\}$ with any formula B and obtain $(A * B)$, then as long as $|Var(A) \cap Var(B)| = r$, the number of equivalence classes $R_{(A*B)}$ will be 2^{n+s-r}. This is also the number of all significantly different Boolean valuations of the formula $(A * B)$ i.e. $|\{v\lceil Var(A * B) \colon v \in Val\}| = 2^{n+s-r}$.

Let us pay attention to the following important lemma with a somewhat complex formulation:

LEMMA 2.7. *Let $A, B, (A * B) \in Form_{PC}$, where $* \in \{\rightarrow, \wedge, \vee, \equiv\}$, $A \in P$, $Var(A) = n$, $Var(B) = s$, $|Var(A) \cap Var(B)| = r$, $m(A) = k/2^n$, and let us denote with t the number of those valuations of the subformula A for which it takes the value 1 in the set of all significantly different Boolean valuations of the formula $(A * B)$ i.e. $|\{v\lceil(A * B) \colon v(A) = 1\}| = t$, then $k/2^n = t/2^{n+s-r}$.*

PROOF: Suppose that the number of all Boolean valuations of the subformula A in $(A * B)$ for which it takes the value 1 is t, i.e., $|\{v\lceil Var(A * B) : v(A) = 1\}| = t$. We know that $2^{n+s-r} = (2^n \cdot 2^{s-r})$, therefore $(2^{n+s-r}/2^n) = 2^{s-r}$. From here we can see that the valuations of the subformula A have been repeated 2^{s-r} times without change in the set of all valuations of the formula $(A * B)$, which means $t = (k \cdot 2^{s-r})$. Now $t/2^{n+s-r} = t/(2^n \cdot 2^{s-r}) = (k \cdot 2^{s-r})/(2^n \cdot 2^{s-r}) = k/2^n$. $\square$

The preceding lemmas should perhaps clarify the understanding of the following lemmas and their proofs.

LEMMA 2.8. *The set P is closed with respect to each of the following rules of conjunction elimination: $(A \wedge B)//A$; and $(A \wedge B)//B$.*

PROOF: Suppose that the formula $(A \wedge B)$ belongs to the set P. Hence, the majority of rows in the last column of its truth table contain 1. The truth table of this formula has 1 in some row of the last column iff the truth tables for each of formulas A and B have 1 in that row. $\square$

Lemma 2.9. *The set P is closed with respect to each of the following rules of disjunction introduction*: $A//(A \vee B)$ *and* $B//(A \vee B)$.

Proof: If the truth table for formula A has 1 in the majority of rows in the last column, then the last column of the truth table for the formula $(A \vee B)$ contains 1 in at least the same rows as formula A; the same holds for formula B. □

Lemma 2.10. *The set P is closed with respect to the rule given by* $A//(B \rightarrow A)$.

Proof: Let A be a member of the set P. The truth table for formula A contains 1 in most of the rows in the last column. There will also be 1 in the same rows of the truth table for $(B \rightarrow A)$ since an implication takes the value 1 if its successor takes the value 1[4]. □

Lemma 2.11. *The subset D of the set P (i.e., $D \subset P$) of formulas which contain just one propositional variable is a proper subset of the set $TAUT$ $(D \subset TAUT)$.*

Proof: The truth table for any formula $A \in D$ contains only two valuations of the single variable. There is only one majority for a two-element set, which is both elements of the set or all of them. □

Lemma 2.12. *For the set of countertautologies of PC, i.e., the set* $CTAUT := \{A : \neg A \in TAUT\}$, $(CTAUT \cap P) = \emptyset$ *holds.*

Proof: The last column of the truth table for the countertautologies contains only 0s. □

Theorem 2.13. *The set P is closed with respect to a weakened form of the rule of detachment of the scheme: if $(A \rightarrow B) \in P$ and $A \in TAUT$, then $B \in P$.*

Proof: Suppose that $(A \rightarrow B) \in P$ and $A \in TAUT$. If the formula B were not an element of the set P, then at least half of the rows in the last column of the truth table for formula $(A \rightarrow B)$ would contain 0, since every row in the last column of the truth table for the formula A would

[4] This is just a sketch of the proof as the exact proof requires longer presentation, but this should be clear enough.

contain 1, and then the whole implication would not be a member of the set P, which contradicts the assumption. □

THEOREM 2.14. *The set P is closed with respect to a weakened form of the detachment rule of the scheme: if $(A \to B) \in TAUT$ and $A \in P$, then $B \in P$.*

PROOF: As in the proof of Theorem 2.13. □

LEMMA 2.15. *The set P is not closed with respect to a rule of the scheme: $A, B//(A \wedge B)$.*

PROOF: The formulas $A = (p \to q)$ and $B = (q \to p)$ both belong to the set P, but $((p \to q) \wedge (q \to p)) \notin P$. □

3. A system P of logically probable formulas in an implicational language

Now, we will focus on the set $Form_{\to}$ of well-formed formulas built using only propositional variables, brackets, and the implication sign; we shall limit our consideration to the implicational part of PC, unless we explicitly indicate otherwise or it is clear from the context. Strictly speaking, we will consider the set $P_{\to} = (Form_{\to} \cap P)$; however, for the sake of convenience, we will continue to use the P symbol as long as this does not lead to confusion. We shall now define the syntactic consequence operation and the corresponding derivability relationship $\vdash_{\mathbf{P}}$. As is already known, the set of PC tautologies in a language with a single connective of implication can be axiomatized into the following system T:

$(T1)$ $((A \to B) \to ((B \to C) \to (A \to C)))$ (hypothetical syllogism);

$(T2)$ $(A \to (B \to A))$ (simplification);

$(T3)$ $(((A \to B) \to A) \to A)$ (Peirce's law);

(MP) $A, (A \to B)//B$ (rule of detachment).

The set $TAUT_{\to} := TAUT_{PC} \cap Form_{\to}$ is axiomatizable by means of rule schemes T in the sense that all classical PC tautologies in our language can be derived using formulas falling under the $(T1)$–$(T3)$ schemes and (MP), i.e., $C_T(\emptyset) = TAUT_{\to}$, where C_T is a consequence determined by T. In addition to these formulas, which are derivable in T, we still have *strictly probable formulas* in the set P which are true for most but not for all valuations (cf. Definition 3.1). We already know that the set of such formulas is not closed with respect to the rule of detachment or the substitution. The following question then arises:

[The Key Question] Is the entire set $P_{\rightarrow}$, and in particular the set of strictly probable formulas (see Definition 3.1 below), axiomatizable i.e., defining an effective set of axioms, being a proper subset of the set $P_{\rightarrow}$, when closed under the finite set of effective rules, gives the whole set $P_{\rightarrow}$?

Definition 3.1. We will call formula A a *strictly logically probable formula* when $A \in P$ and $A \notin C_T(\emptyset) = TAUT_{\rightarrow}$. We will denote the set of all such formulas by the symbol P'.

The set P' is closed under the following version of the non-standard rule, called the *Successor Rule* (RN):

Lemma 3.2. *If* $A \in P', m(B) = 1/2$*, and* $((A \rightarrow B) \rightarrow B) \notin TAUT_{\rightarrow}$*, then* $(A \rightarrow B) \in P'$*.*

Proof: Following the assumption of this lemma, if $((A \rightarrow B) \rightarrow B) \notin TAUT_{\rightarrow}$, then for a certain valuation $v, v(A) = v(B) = 0$ and $v(A \rightarrow B) = 1$. Since $m(B) = 1/2$, i.e., $B \notin P$, then in the worst case exactly half of the truth table for formula $(A \rightarrow B)$ will contain zeros, and there will be $m(A \rightarrow B) = 1/2$. That is, half of the last column of the truth table for the whole formula will then contain zeros in those rows where formula B takes the value zero. The valuation v gives us the guarantee that $|\{v\lceil Var(A \rightarrow B) \colon v(A \rightarrow B) = 1\}| > |\{v\lceil Var(A \rightarrow B) \colon v(A \rightarrow B) = 0\}|$ will occur: the whole implication will have at least one valuation v (which assigns the whole formula the value) more than the number of valuations assigning the value 0 to the implication. $\square$

The above objections are exemplified by the following formulas: $(p \rightarrow q) \rightarrow ((r \rightarrow r) \rightarrow s)$ (satisfies the assumptions of the lemma and belongs to P') and $(p \rightarrow q) \rightarrow ((p \rightarrow p) \rightarrow p)$ (does not satisfy the assumptions of the lemma and does not belong to the set P').

Lemma 3.3. *If* $A \in P'$ *and the variable* $p \notin Var(A)$*, then* $(A \rightarrow p) \in P'$*.*

Proof: Directly from Lemma 3.2. for $B = p$. $\square$

At the same time we have, in a sense, a dual to Lemma 3.2.:

Lemma 3.4. *If $A \in P$, then $(B \to A) \in P$.*

Proof: The proof is straightforward. □

Lemma 3.5. *Let $A \in Form_{\to}$ and $|Var(A)| = n$. Then $2^{n-1} \leq |\{v\lceil Var(A)\colon v(A) = 1\}|$ i.e. $m(A) \geq 1/2$.*

Proof: Since Lemma 3.5 is very general, to demonstrate its validity we shall use structural induction by the number of instances of connectives in formula A. Let us assume the assumptions of the lemma. Base step: A is a single variable p. Hence, there is only one Boolean valuation for which p takes the value 1, and $2^0 = 1 \leq 1$. If A is a simple implication $(p \to q)$, then the cardinality of the set of Boolean valuations for which this implication takes the Boolean value of truth is obviously 3 and is greater than $2^1 = 2$. Inductive step: suppose that the lemma holds for formulas B and C, and we want to prove that it holds for $A = (B \to C)$. Suppose $|Var(B) \cap Var(C)| = k$. We then have to consider cases where $k = 0$, i.e., $Var(B) \cap Var(C) = \emptyset$, and where $k > 0$, i.e., $Var(B) \cap Var(C) \neq \emptyset$. In the first case, assuming that $|Var(B)| = n$ and $|Var(C)| = m$, the truth table for formula $(B \to C)$ has 2^{n+m} rows. The column under formula C will contain 1 in half or more of the rows. The last column of such a table will contain 1 in at least the same rows since an implication with a true successor takes the logical value of truth. We will consider the second case, where formulas B and C share at least one propositional variable, i.e., $k > 0$. In this case, formula $(B \to C)$ will have $(n + m - k)$ different propositional variables, and its truth table will have 2^{n+m-k} rows. The number of valuations of the output formula is decreased by 2^k times because some valuations of the $n + m$ variables are discarded as a result of the equivocation of the shared variables. Let us take formula C as a starting point for consideration and assume that it has 2^m significantly different valuations, while formula B has 2^{n-k} such valuations. In this case, the final truth table T_A for the whole formula $A = (B \to C)$ will have 2^{n+m-k} valuations and will have, as a fragment, a 2^{n-k}-fold repetition of the table for C with 2^m rows. In each such fragment for C, at least half of the rows will contain 1 (based on the inductive assumption), therefore the same rows under formula A will also contain 1 since implication A with the true successor is true. Since this is the case in every occurrence of the fragment for C, so it is the same in the whole T_A table for A. □

Lemma 3.6. *Let* $Form_2 = \{A : A \in Form_{\to}$ *and* $|Var(A)| = 2\}$. *Then* $Form_2 = (X \cup Y) \cup Z$, *where:* $X = \{A : m(A) = 1\}$, $Y = \{A : m(A) = 1/2\}$ *and* $Z = \{A : m(A) = 3/4\}$.

Proof: This follows from Lemma 3.5. because $2^{2-1} = 2^1 = 2$ for $n = 2$. The number of all significantly different Boolean valuations of A is equal to 4, so the numbers k for which $2 \leq k \leq 4$ are only $2, 3$ and 4. □

Some known theorems on classical propositional calculus can be applied to our considerations on the set of exclusively implicational formulas. The compactness theorem is one of these.

Lemma 3.7. *The set* P *is closed with respect to a weakened form of the detachment rule of the scheme:* $A \in P$ *and* $(A \to B) \in P$ *and* $m(A \wedge B) > 1/2$, *then* $B \in P$.

Proof: Let $A \in P$ and $(A \to B) \in P$ and $m(A \wedge B) > 1/2$. Then it is straightforward that $m(B) > 1/2$. □

Lemma 3.8. *Every subset of the set of formulas* $X \subset Form_{\to}$ *is satisfiable.*

Proof: Straightforward from Lemma 3.5. □

Definition 3.9. A rule of the form $A//e(A)$, where e is an endomorphism, will be called a *restricted substitution rule* (RSu).

To avoid going into the technical details, it suffices to say that $e(A)$ is the result of substituting only propositional variables for propositional variables in the formula A, with the following caveats:

- a propositional variable has been substituted for a propositional variable;
- one and the same variable is substituted for a particular variable in all places where it occurs;
- the cardinality of the set $Var(A)$, cannot change as a result of the substitution i.e., $|Var(A)| = |Var(e(A))|$.

Let $h : V \to V$ be a permutation of the set V onto V. The set of all such permutations will be denoted by $Perm := \{h : h$ is a permutation of the set $V\}$. Each such permutation can be extended uniquely to the substitution e, which maps $Form_{PC}$ to $Form_{PC}$ and acts as a substitution in our system[5]. Strictly speaking, $e(A)$ is the value which is taken by a substitution e applied to the formula A. We will use the letter h, possibly with a subscript to denote arbitrary permutation of the set V, and we will use e for the substitution of the entire set $Form_{PC}$[6].

LEMMA 3.10. *The set P is closed under the rule (RSu) of the scheme: $A//e(A)$, for any substitution e.*

PROOF: Suppose $A \in P'$, since the case is obvious for $A \in TAUT_{\rightarrow}$. Therefore, $m(A) > 1/2$. It is clear that we can always find such endomorphisms e_1, e_2 that $A//e_1(A)$ and $e_1(A)//e_2(e_1(A)) = e(A)$ when $Var(A) \cap Var(e_1(A)) = \emptyset$ and $Var(e_1(A)) \cap Var(e_2(e_1(A))) = \emptyset$. For a better understanding of what happens when performing substitutions such as e_1 or e_2, let us imagine a truth table for the formula A. The substitution consists solely of respectively replacing, in the row describing the table, some sentence variables with others, as a result of which we get a truth table for $e_1(A)$, and similarly for $e_2(e_1(A))$. The places in the table where the logical values of the subformulas occur remain unchanged; in particular, the number $m(A)$ remains unchanged. Hence, $m(e(A)) > 1/2$. □

a. Axiom schemas and inference rules for the system P

We are now in a position to answer the Key Question posed above about the axiomatic system.

DEFINITION 3.11. The system **P** is defined by the axioms arising from the following schemes:

$(T1)$ $((A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C)))$;

$(T2)$ $(A \rightarrow (B \rightarrow A))$;

$(T3)$ $(((A \rightarrow B) \rightarrow A) \rightarrow A)$;

$(T4)$ formula $(p \rightarrow q)$;[7]

[5] This designation of e has its own tradition; cf. [8, pp. 18–22].

[6] Such substitutions, also called automorphisms, form a group and satisfies the conditions of composition, associativity, identity and inverse. To denote substitution in our sense we will use the signs: e, (possible with a subscript) and α, β.

[7] The variables p and q are distinct. By pure coincidence, a system with our axioms, specially $(T4)$, is mentioned in the paper [4, p. 193].

and the following inference rules:

(RO) if $\vdash_{\mathbf{P}} A$ and $\vdash_{\mathbf{P}} (A \to B)$ and $m(A \wedge B) > 1/2$, then $\vdash_{\mathbf{P}} B$;

(RN) if $\vdash_{\mathbf{P}} A$ and $m((A \to B) \to B) \neq 1$, then $\vdash_{\mathbf{P}} (A \to B)$;

(RSu) if $\vdash_{\mathbf{P}} A$, then $\vdash_{\mathbf{P}} e(A)$, for any substitution e.

Note. The Reviewer of the paper has requested to explain why in the rule RO above is allowed to use the semantical information in the form $m(A \wedge B) > 1/2$. Firstly, it should be noted that the system presented differs from the usual semantic presentation of classical logic by the 'quantifier' binding the set of logical valuations of the formula. Usually metalogic allows only two types of the quantifiers in its semantic description: 'for each' and 'it exists'. Our description additionally allows 'for most'. Secondly, the properties of such systems are not well recognized and studied, and our quantifier can be introduced into the metalogic of various non-classical systems. Third, the questionable condition can be thrown out beyond the formulation of the rule itself to the form:

(RO) if $\vdash_{\mathbf{P}} A$ and $\vdash_{\mathbf{P}} (A \to B)$, then $\vdash_{\mathbf{P}} B$; provided that $m(A \wedge B) > 1/2$.

And fourth, some conditions of a semantic nature are excluded because they are too general and trivialize the Key Question of axiomatizability, as in the example: $\vdash_{\mathbf{P}} A$, provided that $m(A) > 1/2$.

Lemma 3.12. *The rules $RO1$ and $RO2$ are derivable in* $\mathbf{P}$*; the scheme of $RO1$ is: if $A \in TAUT_{\to}$ and $\vdash_{\mathbf{P}} (A \to B)$, then $\vdash_{\mathbf{P}} B$, and the scheme of $RO2$ is: $(A \to B) \in TAUT_{\to}$, then $\vdash_{\mathbf{P}} B$.*

Proof: We will give the proof for $RO1$ only because the proof for $RO2$ is similar. Let us suppose that $A \in TAUT_{\to}$ and $\vdash_{\mathbf{P}} (A \to B)$, then $m(A \wedge (A \to B)) = m(A \wedge B) > 1/2$, so $m(B) > 1/2$, and $\vdash_{\mathbf{P}} B$. □

Lemma 3.13. *In the system* $\mathbf{P}$*, there is a derivable rule of the scheme: $A//((A \to B) \to B)$.*

Proof: Suppose that: $\vdash_{\mathbf{P}} A$. The formula $(A \to ((A \to B) \to B))$ is a tautology of PC. Hence, by virtue of the (RO) rule, we have: $((A \to B) \to B)$. □

This rule is important because it corresponds in our system to the rule described in Lemma 3.2.

In order to clarify the restrictions imposed on the (RN) rule, let us note that the formulas $((p \to q) \to p)$ and $((p \to p) \to q)$ are not elements of the set P, but the formula $((q \to p) \to p)$ is. However, since we cannot use the RN rule for the derivation of this formula, we derive it slightly differently. Here is the Hilbert-style proof in the system **P**, which is characterized in Definition 3.11:

1. $(p \to q)$ from axiom $(T4)$;
2. $(q \to r)$ from $(T4)$ by rule (RSu);
3. $((q \to r) \to p)$ by virtue of Lemma 3.3;
4. $((q \to r) \to p) \to ((q \to p) \to p)$ the PC-tautology;
5. $((q \to p) \to p)$ by virtue of Theorem 2.14. from 4. and 3.

Recall the following properties that a consequence operator $C : 2^{FormPC} \to 2^{FormPC}$ might satisfy for any set $X, Y \subset Form_{PC}$:

(i) (Reflexivity) $X \subset C(X)$;

(ii) (Monotonicity) $X \subset Y \Rightarrow C(X) \subset C(Y)$;

(iii) (Idempotency) $CC(X) \subset C(X)$;

(iv) (Structurality) $eC(X) \subset C(eX)$ for each endomorphism e;

(v) (Finite) $C(X) = \cup\{C(Y) : Y \subset X \wedge Y \in FinX\}$.

If C satisfies the conditions (i)–(iv), it is called *structural*; if it satisfies conditions (i)–(iii), (v) it is called *finitistic*; and if it satisfies all the above conditions, it is called *standard*. If we restrict the number of endomorphisms[8] to the class of all automorphisms of the set $Form_{PC}$, we obtain a more detailed notion of the structural consequence operation. When $C_{\mathbf{P}}$ is the consequence that corresponds to our system **P**, then the following occurs:

[8]In general, an endomorphism of $Form_{PC}$ need not be a function from $Form_{PC}$ onto $Form_{PC}$. In our case, it is always so because automorphisms are the unique extensions of permutations of the set V.

Lemma 3.14. *The consequence operation* $C_{\mathbf{P}}$ *satisfies conditions (i.)-(v.), except that condition (iv.) holds not for each endomorphism but for any substitution* e *that is a unique extension of the certain permutation* $h \in Perm$.

Proof: The proof of condition (iv) requires special attention. Suppose that $A \in \alpha C_{\mathbf{P}}(X)$. This means that such B exists that $B \in C_{\mathbf{P}}(X)$ and $A = \alpha B$. Hence, there also exists a proof of αB based on the set αX. □

Definition 3.15. A non-atomic formula A of the PC language is called quasi-Horn if the following conditions are met: a) A is in canonical conjunctive normal form ($CCNF$); b) each literal clause (disjunction) contains **at least** one positive literal.

The original *Horn clause* is supposed to contain at most one positive literal, while our *quasi-Horn* formulas are supposed to contain at least one positive literal.

Theorem 3.16 (weak completeness theorem). *The consequence operation* $C_{\mathbf{P}}$ *that corresponds to the system* $\mathbf{P}$ *has the property:* $C_{\mathbf{P}}(\emptyset) = P$.

Proof: ($\Rightarrow$) Let us first prove the implications from left to right. Suppose that the formula $A \in C_{\mathbf{P}}(\emptyset)$; thus, it has a proof based on a set of axioms of the system $\mathbf{P}$ of the form $< D_1, D_2, D_3, \ldots, D_n >$, where each $D_i (0 < i \leq n)$ is either an axiom of the system or has been obtained from prior expressions of this sequence by means of any of the rules of the system. We will give a sketch of a well-known inductive proof, the essence of which consists in showing that the property being proved is preserved by the rules of the system. When the formula is an axiom, the matter is evident because each axiom belongs to P. Also, for the rules (RO) and (RN), the proof is straightforward by virtue of the corresponding theorems: Theorem 2.13, Theorem 2.14., and Lemma 3.2. Consequently, we will concentrate on the case of the restricted substitution rule (RSu). Therefore, let us assume that $A \in P'$, i.e., A, is a strictly probable formula. Let us take any bijection $\alpha : V \to V$. Let $Var(A) = \{p_1, \ldots, p_k\}$, hence $Var(e(A)) = \{e(p_1), \ldots, e(p_k)\}$. According to the definition, $v(A) = 1$ for the majority of the 2^k valuations. If $v_j (0 < j \leq 2^k)$ is such a valuation that $v_j(A) = 1$, then for each such valuation v_j the variables will take corresponding logical values. Let us define the new valuation v_m in the form $v_j(p_i) = v_m(h(p_i))$, hence we have $v_j(A) = v_m(e(A))$, and so on for every j.

($\Leftarrow$) We now want to prove the converse implication. Let us therefore assume that $A \in P$ and that $Var(A) = \{p_1, \ldots, p_k\}$. If A is a tautology, then the case is obvious. Let be $A \in P'$, then $m(A) > 1/2$. By virtue of the relevant metatheorems, there is a formula of canonical conjunctive normal form for A, which we will denote by A_{CCNF}. Such a formula is a conjunction of the clauses $A_i (1 \leq i < 2^{k-1})$, of disjunctive form whose members, called literals, are single propositional variables or their negations occurring in the formula A. Let us note that if A is a purely implicational formula, then A_{CCNF} is *quasi-Horn*. It can easily be seen by virtue of Lemma 2.12. that A_{CCNF} is a conjunction of at most 2^{k-1} conjuncts. But the disjunction in the form $(\neg p_1 \vee \neg p_2 \vee \ldots \vee \neg p_k)$ is excluded as a such conjunct because, for the valuation v, if $v(p_1) = v(p_2) = \ldots = v(p_k) = 1$, then $v(A) = 1$. So, each conjunct (elementary disjunction) of the formula A_{CCNF} contains at least one positive literal, and the number of all conjuncts is less than 2^{k-1}. It is a fact that $(A_{CCNF} \equiv A) \in TAUT_{PC}$, and this is provable in PC in the functionally complete language, which, by virtue of the law of exportation of the scheme $(B \wedge C \to D) \equiv (B \to (C \to D))$ and the rule of equivalence elimination, we can transform to the form $(A_1 \to (A_2 \to (\ldots (A_{((2^{k-1})-1)} \to A) \ldots) \in TAUT_{\to}$, that is $\vdash_{\mathbf{P}} (A_1 \to (A_2 \to (\ldots (A_{((2^{k-1})-1)} \to A) \ldots)$. By applying the (RO) rule 2^{k-1} times and detaching in each step the subsequent formula A_i, we get $\vdash_{\mathbf{P}} A$. Applying the RO rule is in any case permissible since each formula A_i and formula A satisfy the constraint imposed on its application. Note also that every clause A_i is quasi-Horn, therefore part of such a formula has always one of the following formulas or an equivalent formula: $\neg p \vee q$; $p \vee \neg q$; $p \vee q$, where $p, q \in X$.[9] The corresponding formulas in the form of an implication $p \to q$, $q \to p$, and $((p \to q) \to q)$ are equivalent to each of the preceding formulas and are derivable in $\mathbf{P}$. If the A_{CCNF} clause is more complex, then by the rule in Lemma 3.13. we can include further disjunctive members of each clause of formula A. This completes the proof. □

Lemma 3.17. *If* $A \vdash_{\mathbf{P}} e(A)$, *then* $\vdash_{\mathbf{P}} (A \to e(A))$.

Proof: Suppose that $|Var(A)| = m$ and $|Var(e(A))| = n$. If $A \in TAUT_{\to}$, then $e(A)$ is also a tautology. So, suppose that $A \notin TAUT$ and $A \vdash_{\mathbf{P}} e(A)$.

[9] Case: $\neg p \vee \neg q$, is excluded as not being a quasi-Horn.

If we show that $m(A \to e(A)) > 1/2$, then by virtue of the weak completeness theorem $\vdash_{\mathbf{P}} (A \to e(A))$. We know that $m(A) = m(e(A)) \geq 1/2$. The table for the whole formula $(A \to e(A))$, when m(A)=m(e(A))=1/2, will have ones in the rows where its predecessor, formula A, has zeros, and this will be exactly half of all the values of the whole implication. In addition to this, the whole formula will have ones in the rows in which formula A and $e(A)$ take the value 1. Such rows certainly exist, hence the last column of the table for the whole formula will be more than half full of ones. We perform analogous reasoning for the case when $m(A) = m(e(A)) > 1/2$. So, $\vdash_{\mathbf{P}} (A \to e(A))$ by virtue of Theorem 3.16. □

To prove the weak deduction theorem, we need to weaken the **P** system to a system, which we will tentatively denote by the symbol $\mathbf{P}_{-}$. In this new system, we will abandon the RSu rule, but we will close the axioms of the system $\mathbf{P}_{-}$ to any bijections and their endomorphisms, that is, to our substitution.

THEOREM 3.18 (weak deduction theorem). *If* $\{A\} \vdash_{\mathbf{P}_{-}} B$, *then* $\vdash_{\mathbf{P}_{-}} (A \to B)$.

PROOF: We will base our reasoning on the principle of ordinal induction, which is equivalent to normal induction. We will prove precisely the following: $\forall k(\forall i(i < k \to W(i)) \to W(k))$. Then, we will consider a general sentence of the form: $\forall n W(n)$. In our case, the formula $W(n)$ will have the following meaning: "a proof of $B(< D_1, \ldots, D_n = B >)$ based on A, having length n, can be transformed into a certain proof of $(A \to B)$ based on the set $\emptyset$". The theorem holds in the case where A is a tautology, so we assume that $A \notin TAUT_{\to}$. If $k = 1$, then it must be shown that there is $(\forall i(i < 1 \to W(i)) \to W(1)$. Since the antecedent is true, it is equivalent to the sentence $W(1)$, i.e., a proof of length 1 has the property W. Thus, two cases must be considered: (i) $B = A$; (ii) B is an axiom. If $B = A$, then $(A \to B) = (A \to A)$, and this is the theorem of the system $\mathbf{P}_{-}$. When B is an axiom of $\mathbf{P}_{-}$, and if we take an axiom of the form $(B \to (A \to B))$, then we can detach B (using RO), and we have $(A \to B)$. Now, if we suppose that the theorem of the system $\mathbf{P}_{-}$ holds for $\forall i(i < k \to W(i))$, then we need to show that it holds for a proof of length k, i.e., $W(k)$. Our system has three inference rules, so we need to examine each of the three cases. If formula D_k is an axiom of the system or is identical to formula A, then we repeat the reasoning for $k = 1$. For rule RO, suppose that there exist such indices $i, j < k$ that $D_i = (D_j \to D_k) = (D_j \to B)$ and

$m(D_j \wedge B) > 1/2$. By virtue of the induction assumption, both $A \to D_i$ and $A \to D_j$ are theorems of the system. Let us now take a tautology of the form $(A \to (D_j \to D_k)) \to ((A \to D_j) \to (A \to D_k))$. We can detach $(A \to (D_j \to D_k))$ from it because it is a theorem of the system; next, from the formula $((A \to D_j) \to (A \to D_k))$, which is also a theorem of the system, we can detach $(A \to D_j)$ because $m(D_i \wedge D_k) > 1/2$, and also $(A \to (D_i \wedge D_k)) > 1/2$. Let us now consider the RN rule. Let us assume that $D_k = (D_j \to C)$ for some formula $D_j = B$ (which has a proof shorter than k), and let us assume that $((B \to C) \to C) \notin TAUT_{\to}$. By virtue of the induction assumption, we have a proof of $(A \to D_j) = (A \to B)$. Now, we need a proof for $(A \to (B \to C))$, where $A \notin TAUT_{\to}$. By the RN rule, we have $\vdash_{\mathbf{P}\_} ((A \to B) \to C)$, because $\vdash_{\mathbf{P}\_} (A \to B)$ and $(((A \to B) \to C) \to C) \notin TAUT_{\to}$, i.e., there exists such a valuation v that $v(C) = v(B) = 0$ and $v(A) = 1$. From the tautology $((A \to B) \to C) \to (A \to (B \to C))$ and its predecessor, we also have the successor $\vdash_{\mathbf{P}\_} (A \to (B \to C))$. $\square$

The converse of the above theorem does not hold for **P**.

Theorem 3.19. *It is not true that if $\vdash_{\mathbf{P}} (A \to B)$, then $\{A\} \vdash_{\mathbf{P}} B$.*

Proof: The theorem $\vdash_{\mathbf{P}} ((q \to p) \to p)$, whose proof is provided above, serves as the counterexample to the implication from Theorem 3.20. The thesis of the system **P** is also $\vdash_{\mathbf{P}} (q \to p)$; we obtain this thesis from the axiom $(T4)(p \to q)$ by applying RSu. On the other hand, because the rule of detachment is not a rule of **P**, the single variable is not a **P**-theorem by virtue of Theorem 3.16. and the proof of Lemma 2.5. $\square$

Lemma 3.20. *The set P is absolutely consistent under the consequence $C_{\mathbf{P}}$.*

Proof: Directly from Theorem 3.16. and that, for example, $((p \to p) \to p) \notin P$. $\square$

4. Three propositions for the definition of the entailment relation

Definition 4.1 (entailment_1). The formula A follows from a set of formulas X (symbolically: $X \models_1 A$) iff for every $B \in X$, if $B \in P$, then $A \in P$ (or else: if $X \subset P$, then $A \in P$).

Definition 4.2 (entailment_2). Formula A follows from a finite set of formulas X (symbolically: $X \models_2 A$) iff $(\wedge X \to A) \in P$, where $\wedge X$ is a generalized conjunction of the elements of the set X, i.e., $\wedge X := (A_1 \wedge \ldots \wedge A_n)$.

Definition 4.3 (entailment_3). The formula A follows from a finite set of formulas X (symbolically: $X \models_3 A$) iff for most Boolean valuations v of the formulas which belong to X – if these valuations have been assigned the value 1 – the value 1 has also been assigned to the formula A.

5. Some remarks on the family of all majorities

We should bear in mind that the proper object of the present work is some notion of **majority**, which here we have fortunately managed to *insert* into the consideration of the classical propositional calculus. So, the general work on the notion of majority is still to be done. In this section, we will assume that the family $\pi(X)$ of the subsets of the set X is a majority in the set of all valuations of the set X. In this section, we will try to give suggestions for applying some typical algebraic concepts to the family $\pi(X)$. We do so because the findings of this section might, for someone, form the basis of possible further investigations.

Definition 5.1. Let X be any finite set of propositional variables such that $|X| = n$, and let V_X denote the set of all valuations of variables of the set X of the form $v : X \to \{0, 1\}$. Then, by the symbol $\pi(X)$ we denote a subset of the set 2^{V_X} of the form $\pi(X) = \{Y : Y \subset V_X \wedge 2^{n-1} < |Y|\}$.

Each element of the set V_X can be naturally assigned to the formulas of the language of the classical propositional calculus, but this is only possible for a functionally complete language. In the case of a functionally incomplete language, there are valuations of variables to which the formulas of such a language do not correspond. On the other hand, a certain set V_X can be assigned to every formula A, where $X = Var(A)$. Among formulas with two different propositional variables in a purely implicational language, there are infinitely many tautologies, e.g., $(p \to (q \to p))$, $((p \to p) \to (q \to q))$, $((p \to (p \to q)) \to (p \to q))$, and many others. The same is true for other formulas from the set P, especially P'. As already mentioned, when a language is not functionally complete, such as a purely implicational language, then the set $\pi(X)$ may have less cardinality since there may be no formulas in a functionally incomplete language that define certain Boolean valuations. For example, in the case of formulas of the $Form_{\rightarrow}$ language that are built with only two propositional variables, p and q, we have only four elements of the set $V_{\{p,q\}}$, and five elements of the set $\pi(\{p,q\})$, respectively:

- tautologies, i.e., formulas that are true for all valuations;
- $v(p) = 1$ and $v(q) = 1$, or $v(p) = 1$ and $v(q) = 0$, or $v(p) = 0$ and $v(q) = 1$, and the corresponding formula is $(p \rightarrow q) \rightarrow q$;
- $v(p) = 1$ and $v(q) = 1$, or $v(p) = 1$ and $v(q) = 0$, or $v(p) = 0$ and $v(q) = 0$, and the corresponding formula is $(q \rightarrow p)$;
- $v(p) = 1$ and $v(q) = 1$, or $v(p) = 0$ and $v(q) = 0$, or $v(p) = 0$ and $v(q) = 1$, and the corresponding formula is $(p \rightarrow q)$.

LEMMA 5.2. *For the valuations $v(p) = 1$ and $v(q) = 0$, or $v(p) = 0$ and $v(q) = 0$, or $v(p) = 0$ and $v(q) = 1$, there is no purely implicational formula that defines them.*

PROOF: If there were such a purely implicational formula with two variables $A(p,q)$ that were true only for these cases, then $p/q := A(p,q)$ could serve as the definition of a Sheffer stroke by the implication alone. This would make it possible to define all binary connectives of the classical logic solely by implication, which is not possible. $\square$

The set $\pi(X)$ from Definition 5.1. is particularly interesting because of its cardinality. For example, $|\pi(X)| = 5$ for $|X| = 2$; while $|\pi(X)| = 94$ when $|X| = 3$. The general formula for determining the cardinality of this set looks like this: $|\pi(X)| = \binom{2^n}{2^{(n-1)}+1} + \ldots + \binom{2^n}{2^{(n-1)}+2^{(n-1)}}$, when $|X| = n$. Note that this function can be composed with the Var function and extended to the set of all $Form_{PC} : \pi'(A) := \pi(Var(A))$. For the sake of emphasis, let us observe that for a language with implication alone there are only four sets of valuations that the implication formulas correspond to.

DEFINITION 5.3. For a finite set of variables X, the family $\pi(X)$ forms a certain algebra $< \pi(X), \cup, ' >$, which satisfies the following conditions:

i. $X \in \pi(X)$;

ii. If $Y \subset Z$ and $Y \in \pi(X)$, then $Z \in \pi(X)$;

iii. If $Y \in \pi(X)$, then $Y' \notin \pi(X)$.

We will now apply the definition of a filter to the family $\pi(X)$.

Definition 5.4. A family of sets $F \subset \pi(X)$ is called a filter in the family $\pi(X)$ if the following conditions are satisfied:

I. $X \in F$;

II. $Y \subset Z$ and $Y \in F$, then $Z \in F$;

III. If $Z \in F$ and $Y \in F$, then $Z \cap Y \in F$;

IV. $\emptyset \notin F$.

As can be seen, the family F being the filter significantly reduces the cardinality of the family $\pi(X)$. For example, let us take the already considered case of $|X| = 2$ and $|\pi(X)| = 5$. We then have $V_{\{p,q\}} = \{11, 10, 01, 00\}$ and $\pi(\{p, q\}) = \{X_1 = \{11, 10, 01, 00\}$, $X_2 = \{11, 10, 01\}$, $X_3 = \{11, 10, 00\}$, $X_4 = \{11, 00, 01\}$, $X_5 = \{10, 01, 00\}\}$. There is only one filter F in this family and it is non-proper: $F = \{X_1\}$.

Based on the presented facts, we shall determine how we will understand the semantic model of our system **P**. Since the traditional semantic model that we are familiar with suffices for our purposes, we have so far been able to understand it intuitively.

Definition 5.5. A structure $M_P =<< \{1, 0\}; f_{\rightarrow} >; D = \{1\} >$ is called a **normal model** for **P**.[10]

Functions, which are valuations, assign logical values of truth (1) or false (0) to all propositional variables and formulas.

We have the following typology of formulas:

A. Formula A is *satisfiable* iff there is a valuation for which A takes the logical truth value;

B. Formula A is *logically probable* iff it takes the logical truth value for 'most' Boolean valuations of A i.e. for most restrictions $v\lceil Var(A)$ of Boolean valuations of A.

C. Formula A is a *tautology* iff it takes the logical value of true for all Boolean valuations.

[10] Of course, usually there are more functions: $f_{\neg}, f_{\vee}, f_{\wedge}, f_{\equiv}$.

6. The strong completeness of the system P

Let us now consider Definition 4.1. as the fundamental definition of the entailment. This definition is similar to Tarski's definition of the entailment, which preserves truthfulness in the sense that a proposition A does not follow from the set of true propositions X when A itself is not true. A formula A does not follow from the set of logically probable formulas X when it is not itself logically probable, as stated in Definition 4.1. Additionally, unlike other definitions that only permit finite sets X, this definition admits any cardinality of the set X. It should be noted that, from the perspective of natural language, the requirement that the set of premises is finite is not very unreasonable. Let us see, therefore, whether the consequence that results from Definition 4.1. has the properties of a consequence in the Tarskian (classical) sense. Since the so-understood consequence is a relation, $\models_1 \subset 2^{Form} \times Form$, we need to determine the properties of this relation. We will show that it has the properties of Tarski's consequence, namely reflexivity, cut and monotonicity [5, p. 5].

Lemma 6.1.

a. *If $A \in X$, then $X \models_1 A$; (reflexivity).*

b. *If $X \models_1 B$, for any $B \in Y$ and $X \cup Y \models_1 C$, then $X \models_1 C$; (cut).*[11]

c. *If $X \models_1 A$ and $X \subset Y$, then $Y \models_1 A$; (monotonicity).*

Proof: Ad a. Suppose that $A \in X$ and $X \subset P$ holds, then, of course, by the definition of inclusion we have $A \in P$.

Ad b. Suppose $X \models_1 B$, for every $B \in Y$ and $X \cup Y \models_1 C$ and $X \subset P$. From these presumptions, we aim to demonstrate that $X \models_1 C$. We therefore have for every $B \in Y$, $B \in P$, that is $Y \subset P$. If $X \subset P$ and $Y \subset P$, then $(X \cup Y) \subset P$. And from this, we straightforwardly obtain $C \in P$.

Ad c. Suppose that $X \models_1 A$ and $Y \subset P$. Hence, we have $X \subset P$, and by the first assumption we have $A \in P$. □

[11] Makinson [5] calls cut also a cumulative transitivity and characterizes it in the terms of the consequence operator as follows: if $X \subset Y \subset C(X)$, then $C(Y) \subset C(X)$. Cut, given the above definition and in the presence of monotonicity, is equivalent to the idempotence condition $C(C(X)) \subset C(X)$.

It seems natural to ask about the completeness of the system **P**, that is, more precisely, whether every true formula of the language is simultaneously provable in this system. Such a theorem has the following strong version: if $X \models_1 A$, then $X \vdash_{\mathbf{P}} A$. We will attempt to provide a proof of this important theorem later in this paper. On the other hand, the converse implication if $X \vdash_{\mathbf{P}} A$, then $X \models_1 A$ is called the *soundness theorem* for the system P, and we shall attempt to prove it first. The proof is similar to the left-to-right implication of Theorem 3.16.

Theorem 6.2 (strong soundness of the system **P**). *For any $X \subset Form_{\rightarrow}$ and $A \in Form_{\rightarrow}$, if $X \vdash_{\mathbf{P}} A$, then $X \models_1 A$.*

Proof: Suppose that $X \vdash_{\mathbf{P}} A$ holds for given X and A, which we will write in an equivalent way: $A \in C_{\mathbf{P}}(X)$. Thus, A has a proof that is based on the set of formulas X and the set of axioms of the system **P** of the form $d =< D_1, D_2, D_3, \ldots, D_n = A >$, where each $D_i (1 \leq i \leq n)$ is either a member of the set X, an axiom of the system, or has been obtained from the prior expressions of this sequence using any of the four rules of the system. The sequence that is a proof of a formula A based on the set of formulas X will only contain a finite subset Y of the elements of the set X, and for this set we have $Y \vdash_{\mathbf{P}} A$. This is due to the finite length of the proof of a formula A. We will give a sketch of the well-known proof of the inductive hypothesis $W(n)$, which states that if d is a proof of A based on the finite set $Y \subset X$, then $Y \models_1 A$, which we shall demonstrate for any n. To do so, it suffices to prove $\forall n(\forall k(k < n \rightarrow W(k)) \rightarrow W(n))$. Let us assume that the antecedent of the implication holds for any $k < n$ and $W(k)$. The formula $D_n = A$ in the proof d must appear as a result of any of the following steps. When the formula A is an axiom or an element of the set Y, then, of course, $Y \models_1 A$. Also, for rules (RO) and (RN), the case is evident by virtue of the corresponding theorems: Theorem 2.13., Theorem 2.14. and Lemma 3.2. Therefore, only the proof for the case of the restricted substitution rule (RSu) is needed. Let us therefore assume that $A \in P'$, i.e., A, is a strictly probable formula. Let us take any bijection $h : V \rightarrow V$. Let $Var(A) = \{p_1, \ldots, p_m\}$. Hence, $Var(e(A)) = \{h(p_1), \ldots, h(p_m)\}$. By definition of the 2^m valuations of formula A, for most of them $v(A) = 1$. Let $v_j(0 < j < 2^m)$ be such that $v_j(A) = 1$, then for each valuation v_j of the propositional variables of formula A, this formula will take the corresponding logical values. Let us define $v_j(p_i) = v_j(h(p_i))$, hence $v_j(A) = v_j(e(A))$ for every j. That is, if $X \models_1 A$, then $\{A\} \models_1 e(A)$, and finally $X \models_1 e(A)$. $\square$

We now proceed to the proof of completeness theorem: this is the most important theorem for the system **P**; the proof is akin to the proof of the right-to-left implication of Theorem 3.16. But first we draw the corollary, the proof of which is based on Theorem 6.2 and Lemma 3.20:

Corollary 6.3. The system **P** is absolutely consistent, i.e., $C_{\mathbf{P}}(\emptyset) \neq Form_{\rightarrow}$.

Theorem 6.4 (strong completeness theorem for **P**). *For any $X \subset Form_{\rightarrow}$ and $A \in Form_{\rightarrow}$: if $X \models_1 A$, then $X \vdash_{\mathbf{P}} A$.*

Proof: We prove this theorem using some variant of Lindenbaum's lemma for our language that has no negation. Suppose then that $X \models_1 A$, i.e., that if $X \subset P$, then $A \in P$. For an indirect proof, suppose that $A \notin C_{\mathbf{P}}(X)$, hence the set X is consistent in the Post sense. Using Lindenbaum's lemma, we can extend the set X to a maximal and consistent set X^* for a purely implicational language. $X^* = \cup_{m \in N} X_m$, where $X_0 = X$; $X_{m+1} = X_m \cup \{w_m\}$, if $X_m \cup \{W_m\}$ is consistent; and $X_{m+1} = X_m$, if $X_m \cup \{W_m\}$ is inconsistent. It is known that all formulas of the set $Form_{\rightarrow}$ can be put on an infinite list: $w_0, w_1, w_2, \ldots$ We must now prove the auxiliary lemmas concerning the set X^*. $\square$

Lemma 6.5. *Let a formula $A \in Form_{\rightarrow}$ be such that $Var(A) = \{p_1, p_2, \ldots, p_n\}$: if for any Boolean valuation v and for any $n \geq i > 0$, $v(p_i) = 1$, then $v(A) = 1$.*

Proof: Induction by the complexity degree of a formula A. If $A = p$ and $v(p) = 1$, $v(A) = 1$. Suppose $A = (B \rightarrow C)$ and let formulas B, C satisfy the assumptions of the theorem, i.e., $v(B) = v(C) = 1$, then $v(A) = v(B \rightarrow C) = v(B) \rightarrow v(C) = 1 \rightarrow 1 = 1$. $\square$

Lemma 6.6. *For any $A \in Form_{\rightarrow} : A \in P$ iff $A \in X^*$.*

Proof: Suppose that $A \in P$. By virtue of Theorem 3.16, $\vdash_{\mathbf{P}} A$, and from the monotonicity of the consequence operation (Lemma 6.1(c)), we obtain $X^* \vdash_{\mathbf{P}} A$, hence $A \in X^*$. If $A \notin X^*$ and the set X^* is maximal, then $X^* \cup \{A\}$ is inconsistent. Conversely, suppose that $A \in X^*$, hence $X^* \vdash_{\mathbf{P}} A$. For the purpose of an indirect proof, we will assume that

$A \notin P$. By virtue of Lemma 3.5, in exactly half of the Boolean valuations the formula A takes the Boolean value 1, and it takes the Boolean value 0 for the other half. Then, for such an A, $\{A\} \vdash_{\mathbf{P}} p$, where p is some variable belonging to $Var(A)$.[12] By further applying the rule (RSu) $\{p\} \vdash_{\mathbf{P}} q$, we get $\{A\} \vdash_{\mathbf{P}} q$ for any variable q. This, in turn, leads us to assert that X^* is inconsistent, which contradicts the assumption. □

One final step still needs to be proven:

Lemma 6.7. *If $A \in Form_{\rightarrow}$ and $m(A) = 1/2$, then $\{A\} \vdash_{\mathbf{P}} p$ for some variable $p \in Var(A)$.*

Proof: Assume that the assumptions of the lemma hold and let $|Var(A)| = n$. Thus, exactly 2^{n-1} possible Boolean valuations of formula A take the value 1, and the other half of the valuations obviously take the value 0. If the formula $A = p$, the lemma obviously holds. Suppose, then, that A is a compound formula and has the form $A = (A_1 \rightarrow B_1)$ for some A_1 and $B_1 \in Form_{\rightarrow}$. Then, $v(A) = 0$ iff $v(A_1) = 1$ and $v(B_1) = 0$. Assuming that $B_1 = (A_2 \rightarrow B_2)$, then $v(B_1) = 0$ iff $v(A_2 \rightarrow B_2) = 0$ iff $v(A_2) = 1$ and $v(B_2) = 0$. Thus, $v(A) = 0$ iff $v(A_1) = 1$ and $v(B_1) = 0$; then $v(A_2) = 1$ and $v(B_2) = 0$. Following this pattern, after $k > 1$ steps we arrive at $B_{k-1} = (A_k \rightarrow p)$, where $p \in Var(A)$; iff $(v(A_1) = 1) \wedge (v(A_2) = 1) \wedge \ldots \wedge (v(A_k) = 1) \wedge (v(B_k) = v(p) = 0)$. From transitivity, we have if $v(A) = 0$, then $v(p) = 0$, for some variable $p \in Var(A)$. By contraposition for this variable, if $v(p) = 1$, then $v(A) = 1$. For the proof of the converse implication, the key issue is whether it would be possible that $v(p) = 0$, while $v(A) = 1$ for some valuation v. This case is ruled out since the table for the formula A has 2^n rows, half of which contain 0 and half of which contain 1 in the last column. In the column under the variable p, also half of the cells contain 0 and half contain 1. Consequently, if $v(A) = 0$, then $v(p) = 0$, so 0 occurs at least in those rows of the column under the variable p where the valuation of formula A equals 0, and exactly half of the valuations equal 0. According to Definition 2.1, none of the other rows in the column under the variable p can contain 0 because there are 2^{n-1} rows containing 1. Let us now construct a disjunctive normal form of the formula A, that is A_{APN}. This formula is a disjunction of 2^{n-1} conjunctions of literals with n members. In each such conjunction, there is as its

[12] The proof of this claim is given in Lemma 6.6 below.

member a variable p. We can move this variable to the front of the conjunction by applying the law of distributivity of the disjunction over the conjunction to A_{APN}. This allows us to derive p within PC: $A_{APN} \vdash_{PC} p$. Also, in PC it holds that $A_{APN} \equiv A$ and $A_{APN} \vdash_{PC} A$. By virtue of the extensionality rule, we have $A \vdash_{PC} p$. From the deduction theorem for PC, we have $\vdash_{PC} (A \to p)$; and from the definition of the system $\mathbf{P}$, we have $\vdash_P (A \to p)$. Hence from A, by rule $(RO1)$, we will obtain p, and finally we will also obtain $p \vdash_P q$, for any variable q, by rule (RSu). $\square$

LEMMA 6.8. *The set $X^* \cup \{A\}$ is inconsistent when $A \in Form_{\to}$ and exactly half of its Boolean valuations take the value* 1 *(as in Lemma 6.6).*

PROOF: The set $\{A\}$ is inconsistent, as follows from Lemma 6.6. So, by virtue of monotonicity, also $X^* \cup \{A\}$ is inconsistent.

This completes the proof of Lemmas 6.6 and 6.7, and also the proof of Theorem 6.4 (the strong completeness theorem). $\square$

As an illustration, let us observe that within the formulas of the PC language written with three different variables (p, q, r), there are 70 formulas in the disjunctive normal form that have exactly half of the rows occupied by 1s in the last column of their respective truth tables. Only three of these formulas have equivalents that are simply implicational formulas. Such formulas are characterized by the fact that applying the Quine-McCluskey method – also known as Karnaugh's method – to minimize Boolean functions yields a single variable. The formulas in disjunctive normal form used in the example are:

- $(p \wedge q \wedge r) \vee (p \wedge \neg q \wedge r) \vee (\neg p \wedge q \wedge r) \vee (\neg p \wedge \neg q \wedge r)$;
- $(p \wedge q \wedge r) \vee (p \wedge \neg q \wedge r) \vee (p \wedge q \wedge \neg r) \vee (p \wedge \neg q \wedge \neg r)$;
- $(p \wedge q \wedge r) \vee (\neg p \wedge q \wedge r) \vee (p \wedge q \wedge \neg r) \vee (\neg p \wedge q \wedge \neg r)$.

When understood as sets, the consequence relation $\models_1$ and the relation $\models_{PC}$ are distinct since they intersect. This is because $\{q \to p\} \models_1 (p \to q)$, yet $\{q \to p\} \models_{PC} (p \to q)$ does not hold since, for valuations $v(p) = 1$ and $v(q) = 0$, $v(q \to p) = 1$, while $v(p \to q) = 0$. Both relations hold when premises and conclusions are tautologies. On the other hand, $\{(p \to q) \to r, p \to q\} \models_{PC} r$, but $\{(p \to q) \to r, p \to q\} \models_1 r$ does not hold because the premises are elements of the set P, while the single variable r is not.

A study of the consequence relation $\models_1$ reveals that it has some unexpected properties, such as $\{q \to p\} \models_1 p \to q$ or – even more contentious – $\{q \to p\} \models_1 r \to s$. Due to the proven completeness theorem, $\{q \to p\} \vdash_P r \to s$ holds because the derivation is allowed by the substitution rule (RSu). This observation can serve as the starting point for consideration of a system without this rule, which can be a challenging issue.

7. Further considerations regarding the entailment relation

Since this paragraph concerns only purely implicational language, we need to adapt Definition 4.2, which is of the logical consequence in the second sense we have given, i.e., $\models_2$.

LEMMA 7.1 (entailment_2′). *The formula A follows from a finite set of formulas $X = \{A_1, A_2, \ldots, A_n\}$ (i.e., symbolically, $\{A_1, A_2, \ldots, A_n\} \models_2 A$) iff $(A_1 \to (A_2 \to (\ldots(A_n \to A))\ldots) \in P$.*

PROOF: This transformation is made possible by the equivalence that is the theorem of the PC: $(A \wedge B) \to C) \equiv (A \to (B \to C))$. □

We will now demonstrate how the above understanding of the consequence relation does not meet the classical properties of the consequence.

LEMMA 7.2. *The relation $\models_2$ satisfies the conditions a. and c., but it does not satisfy the condition b.:*

a. *If $A \in X$, then $X \models_2 A$; (reflexivity).*

b. *It is not true that if for all $B \in Y$, $X \models_2 B$, and $X \cup Y \models_2 C$, then $X \models_2 C$; (cut).*

c. *If $X \models_2 A$ and $X \subset Y$, then $Y \models_2 A$; (monotonicity).*

PROOF: Ad a. Suppose that there exists $A \in X = \{A_1, A_2, \ldots, A_n\}$. Therefore, $A = A_i$, for some $0 < i < n+1$. Hence, $\{A_1, A_2, \ldots, A_n\} = \{A_1, A_2, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n, A_i\} \models_2 A$, and since $(A_1 \to (A_2 \to (\ldots (A_{i-1} \to (A_{i+1} \to (\ldots(A_n \to (A_i \to A)))\ldots)$ is a tautology; therefore, by virtue of Definition 2.2, $\{A_1, A_2, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n, A_i\} \models_2 A$ because $A_i = A$.

Ad b. Suppose that for every $B \in Y : X \models_2 B$, and $X \cup Y \models_2 C$, where $Y = \{(q \to p)\}$; $B = (q \to p)$; $X = \{(p \to q)\}$; $C = p$. We have $(q \to p) \in Y$; $\{(p \to q)\} \models_2 (q \to p)$, because $((p \to q) \to (q \to p)) \in P$; $\{(p \to q)\} \cup \{(q \to p)\} = \{(p \to q), (q \to p)\} \models_2 p$, because $(((p \to q) \wedge (q \to p)) \to p) \in P$; but $X = \{(p \to q)\} \models_2 p$ does not hold because $((p \to q) \to p) \notin P$.

Ad c. Suppose $X \models_2 A$ and $X \subset Y$ and $|Y| = n$ for some n. If Y is not finite, then the consequence holds vacuously. By Definition 4.2., we have $(\wedge X \to A) \in P$ and $(\wedge Y \to \wedge X) \in TAUT$, and from the transitivity $(\wedge Y \to \wedge X) \to ((\wedge X \to A) \to (\wedge Y \to A))$. After commutating and detaching (using $(RO1)$), we get $(\wedge Y \to \wedge X) \to (\wedge Y \to A)$; then by detaching the antecedent which is a tautology (using $(RO2)$), we get $(\wedge Y \to A)$. This formula is a member of P. This, in turn, by virtue of Lemma 7.1, gives $Y \models_2 A$. $\square$

Let us now examine the properties of the third relation of consequence. Note that here we are dealing with richer language since there is at least a conjunction in addition to the implication. Nevertheless, we will try to remove it. As we know, the set P itself is not closed with respect to the rule of conjunction introduction (cf. Lemma 2.15). Let us note that the following holds:

Lemma 7.3. *Let $X \subset Form_{PC}$ and $|X| = n$. For any Boolean valuation $v : v(X) = 1$ iff $v(\wedge X) = 1$.*

Proof: Suppose $v(X) = 1$. This is so iff $v(A) = 1$ for any $A \in X$. Then, of course, $v(\wedge X) = 1$. Conversely, if $v(\wedge X) = 1$, then $v(A) = 1$, for every member A of the conjunction $\wedge X$. $\square$

Based on above consideration, we can reformulate the definition of the consequence $\models_3$ and use it for the following lemma:

Lemma 7.4. *Let $X \subset Form_{\to}$ be a finite set and $A \in Form_{\to}$. Then, $X \models_3 A$ iff every Boolean valuation v from the set of the majority of Boolean valuations satisfying the set X, i.e., $v(X) = 1$, also satisfies A, i.e., $v(A) = 1$.*

Proof: From Definition 4.3 and Lemma 7.3. $\square$

The idea behind this term is that we want every valuation which belongs to a majority of valuations and assigns a logical truth value to the conjunction of all premises to assign a logical truth value to the conclusion as well. This term is different from both $\models_1$ (even for finite sets of premises) and $\models_2$. For example, let us take the formulas: $(p \to q)$, $(q \to p) \in P$. Then, $((p \to q) \wedge (q \to p)) \to p \in P$, that is $\{(p \to q), (q \to p)\} \models_2 p$, but $\{(p \to q), (q \to p)\} \not\models_1 p$. To distinguish $\models_3$ from $\models_2$, note that $\{(p \to q)\} \models_2 (q \to p)$ because $((p \to q) \to (q \to p)) \in P$; however, $\{(p \to q)\} \models_3 (q \to p)$ does not hold because, for a valuation where $v(p) = 0$ and $v(q) = 1$, we have $v(p \to q) = 1$ and $v(q \to p) = 0$.

Lemma 7.5. *For any finite $X, Y \subset Form_{PC}$, and $A, B, C \in Form_{PC}$:*

A. *If $A \in X$, then $X \models_3 A$; (reflexivity)*

B. *If for all $B \in Y : X \models_3 B$, and $X \cup Y \models_3 C$, then $X \models_3 C$; (cut)*

C. *If $X \models_3 A$ and $X \subset Y$, then $Y \models_3 A$; (monotonicity).*

Proof: Ad A. Let $A \in X$, $|X| = n$. Suppose also that $\wedge X \in P$, i.e., it takes the value 1 for most Boolean valuations of this formula. For each such Boolean valuation v, if $v(\wedge X) = 1$, then $v(A) = 1$. For if $v(A) = 0$, then there would also be $v(X) = 0$. Thus $X \models_3 A$.

Ad B. Suppose that for every $B \in Y, X \models_3 B$, and $X \cup Y \models_3 C$. Therefore, any valuation v of the conjunction $\wedge X$ that assigns it the value 1 assigns the value 1 to any formula $B \wedge Y$, therefore any such valuation assigns the value 1 to the conjunction $\wedge Y$. Hence, if $v(\wedge X) = 1$, then $v(\wedge Y) = 1$. Thus, the set of such valuations that $v(X) = 1$ is contained in the set of valuations such that $v(Y) = 1$. With $X \cup Y \models_3 C$, by Definition 4.3 we can assert that each valuation (belonging to the majority of valuations) that assigns value 1 to the conjunction $\wedge(X \cup Y) = (\wedge X) \wedge (\wedge Y)$ assigns the same value to the formula C. Based on the corresponding PC tautology, we have $(\wedge X) \wedge (\wedge Y) \equiv (\wedge X) \wedge (\wedge X \to \wedge Y)$. So, there is an inclusion of the set of Boolean valuations v such that $v((\wedge X) \wedge (\wedge Y)) = 1$ in the set of valuations such that $v(\wedge X) = 1$. Hence, $X \models_3 C$.

Ad C. Suppose that $|X| = n$, $|Y| = m$, $X \models_3 A$, $X \subset Y$ and, for an indirect proof, for any $Y : Y \not\models_3 A$. The majority of valuations in the set of all Boolean valuations of the conjunction $\wedge X$ for which $v(\wedge X) = 1$, under the assumption $X \subset Y$, is the superset of the set of those Boolean valuations for which $v(\wedge Y) = 1$. Thus, $Y \models_3 A$, which is a contradiction. $\square$

Because of the finiteness condition, it is debatable whether Definition 4.3 of the consequence is adequate for the realm of logically probable formulations.

8. Conclusions

In our deliberations, the notion of majority plays an important role. There are other research fields for which this concept is also important. For example, there are studies focused on decision-making in the fields of social choice theory, political sciences and economics, whose authors have examined how "[...] individual preferences and interests can be combined into a collective decision." [3, p. 1]. These studies are synthetically described in [3], and the latest results are presented in [7]. This fact alone indicates some conceptual affinity between these studies and considerations presented in this paper. Moreover, the notion of majority in both my theory and the theory of decision making is analysed in the context of inferences and logic, although from different angles. In studies devoted to group decision-making, this notion refers to the majority in a certain group of subjects as a whole. In this case, the majority is a result of individual choices made by decision-makers, and the decision of the group as a whole (i.e., a collective judgement) is obtained through aggregation functions. In the case analysed in this paper, the majority is formed only by the logical valuations, and – unlike in the decision-making process – we do not need to involve any extralogical apparatus. In short, group decision-making theory uses means outside the arsenal of logic, while in our conception we remain within purely logical concepts and the standard language of PC. We cannot rule out combining our take on a majority within the framework of group choice theory.

Another related line of research can be found in the field of logical probability, which stems from Carnap and was developed by, for example, [1]. This probability is defined for first-order sentences as the fraction of the set of finite models for which a sentence is true in the set of all finite models of the sentence. However, this study focused on logical probability, unlike the study described in this paper; cf. [1].

The basic ideas of our paper were developed in the post-graduate thesis of Olszewski, later summarized in a paper [6]. As I have already mentioned, the real *powerbroker* of these considerations is the notion of *majority*. While in the presented paper *majority* appears in some tricky way, it seems that the main further considerations should focus on the abstract notion of majority. Here by way of example, we will give five proposals of the definitions of majority for some set U to show the richness of this concept. Below we define the families of subsets of the universe U, denoted by $\pi^k(U)$; the elements of these families are subsets which are the *majorities* in the set U; cf. [6]:

Def. A. Let U be any set, finite or infinite: $\pi^1(U) = \{Y \subset U\colon\ |Y| > |Y'|\}$.

Def. B. Let U be an infinite set of any cardinality: $\pi^2(U) = \{Y \subset U\colon |Y'| = n\}$.

Def. C. Let U be an infinite set of any cardinality: $\pi^3(X) = \{Y \subset U\colon |Y| > |Y'|\}$.

Def. D. Let U be a metric space with its metric $d : \pi^4(U) = \{Y \subset U\colon |Y| > |Y'| \text{ and } dY > dY'\}$.

Def. E. Let U be a topological space: $\pi^5(U) = \{Y \subset U\colon\ |Y| > |Y'| \text{ and if } Y \text{ a } \textit{dense} \text{ subset of } U\}$.

A quite natural direction for further research is to extend the main results, including the completeness theorems, to the whole language of propositional calculus. In a sense, the concept presented can be extended to the concept of logical probability (majority) of first order formulas for a finite universe. The concepts of the *probability of the propositional language formulas* and the *majority*, understood as a form of modality operator, seem equally promising.

References

[1] R. Fagin, *Probabilities on Finite Models*, **The Journal of Symbolic Logic**, vol. 41(1) (1976), pp. 50–58, DOI: https://doi.org/10.2307/2272945.

[2] D. Grossi, G. Pigozzi, **Symbolic Logic and Its Applications**, Longmans, Green, and Co., London (1906), URL: https://archive.org/details/symboliclogicits00macc/.

[3] D. Grossi, G. Pigozzi, **Judgment aggregation: A primer**, Synthesis Lectures on Artificial Intelligence and Machine Learning, Springer, Cham (2014), DOI: https://doi.org/10.1007/978-3-031-01568-7.

[4] H. Hiż, *Extendible Sentential Calculus*, **The Journal of Symbolic Logic**, vol. 24(3) (1959), pp. 193–202, DOI: https://doi.org/10.2307/2963776.

[5] D. Makinson, **Bridges from Classical to Nonmonotonic Logic**, College Publications, Rickmansworth (1906).

[6] A. Olszewski, *Kilka uwag na temat demokracji (A Few Remarks on Democracy)*, **Logos i Ethos**, vol. 1(7) (1999), pp. 116–122.

[7] R. Takemura, *Logic and Majority Voting*, **Journal of Philosophical Logic**, vol. 51 (2022), pp. 347–382, DOI: https://doi.org/10.1007/s10992-021-09631-7.

[8] R. Wójcicki, **Theory of Logical Calculi**, Springer, Dordrecht (1988), DOI: https://doi.org/10.1007/978-94-015-6942-2.

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