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SEQUENT SYSTEMS FOR CONSEQUENCE RELATIONS OF CYCLIC LINEAR LOGICS

Abstract

Linear Logic is a versatile framework with diverse applications in computer science and mathematics. One intriguing fragment of Linear Logic is Multiplicative-Additive Linear Logic (MALL), which forms the exponential-free component of the larger framework. Modifying MALL, researchers have explored weaker logics such as Noncommutative MALL (Bilinear Logic, BL) and Cyclic MALL (Cy-MALL) to investigate variations in commutativity. In this paper, we focus on Cyclic Nonassociative Bilinear Logic (CyNBL), a variant that combines noncommutativity and nonassociativity. We introduce a sequent system for CyNBL, which includes an auxiliary system for incorporating nonlogical axioms. Notably, we establish the cut elimination property for CyNBL. Moreover, we establish the strong conservativeness of CyNBL over Full Nonassociative Lambek Calculus (FNL) without additive constants. The paper highlights that all proofs are constructed using syntactic methods, ensuring their constructive nature. We provide insights into constructing cut-free proofs and establishing a logical relationship between CyNBL and FNL.

Keywords: linear logic, Lambek calculus, nonassociative logics, noncommutative logics, substructural logics, consequence relation, nonlogical axioms, conservativeness.

1. Introduction and preliminaries

Linear Logic (PLL), introduced by Girard [7], is a powerful framework widely applied in computer science and mathematics. It offers a rich set of

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tools for reasoning about resources and provides a foundation for various formal systems. One intriguing fragment of PLL is Multiplicative-Additive Linear Logic (MALL), which focuses on the exponential-free aspects of PLL. In MALL, we encounter four binary connectives: $\otimes$ (product; multiplicative conjuntion), $\parr$ (par; multiplicative disjunction), $\wedge$ (additive conjuntion) and $\vee$ (additive disjunction). Additionally, MALL includes one unary connective: $^{\sim}$ (linear negation), and four constants: 1, 0, $\perp$, and $\top$. It is worth noting that MALL exhibits associativity and commutativity, as defined by the algebraic interpretation of the $\parr$ connective, further enhancing its expressive capabilities.

Abrusci [1] investigates Noncommutative MALL, a variant of the logic where the $\otimes$ connective is not required to be commutative. This exploration of noncommutativity adds an intriguing dimension to MALL and offers new avenues for reasoning about resources and implications. In Noncommutative MALL, we encounter two negations, $^{\sim}$ and $^{-}$, which exhibit an interesting algebraic property: for all a, the following equivalences hold: $a^{\sim -} = a = a^{-\sim}$. This property highlights the interplay between the two negations and underscores the expressive power of Noncommutative MALL. It is worth noting that this variant is also known as Bilinear Logic (BL), as named by Lambek [8].

Yetter [12] introduces Cyclic MALL (CyMALL) as a compromise between MALL and BL. While CyMALL maintains the noncommutative nature of BL, it distinguishes itself by adopting only one negation that satisfies the double negation law. This unique choice of negation adds a distinct flavor to the reasoning capabilities of CyMALL. Additionally, CyMALL allows for the relaxation of associativity, further differentiating it from traditional Bilinear Logic. Nonassociative Bilinear Logic (NBL) is another intriguing logic that explores the implications of nonassociativity. In this paper, we specifically focus on Cyclic NBL (CyNBL), a variant of NBL that inherits the noncommutative property from CyMALL while also incorporating nonassociativity.

In this paper, we present the sequent system for CyNBL in Section 2. Additionally, we introduce an auxiliary sequent system which allows for the inclusion of nonlogical axioms by treating them as special cases of the cut rules. As a result, we obtain an equivalent system that incorporates a form of the cut elimination property. Specifically, the cut elimination property applies to the pure logic, while the cut rules are restricted to handling assumptions only.

The proof of cut elimination and the development of the sequent systems in this paper draw inspiration from prior research. Specifically, Buszkowski [5] establishes the cut elimination property for constant-free MLL, which corresponds to the multiplicative fragment of MALL. Furthermore, Płaczek [10] extends this method to prove cut elimination for NBL. Notably, in the author's doctoral dissertation [11], there are remarks regarding the potential approach for proving cut elimination in CyNBL. It is worth mentioning that none of these previous results involve assumptions, as they focus primarily on the cut elimination property within the given logical frameworks.

Lin [9] has previously explored sequent systems for specific extensions of NL with assumptions. In these systems, confined to intuitionistic sequents of the form $\Gamma \Rightarrow A$, the assumption $A \Rightarrow B$ is replaced by a specific instance of the cut rule: from $\Gamma[B] \Rightarrow C$ and $\Delta => A$, we derive $\Gamma[\Delta] => C$. In this study, we adapt this concept to systems employing classical sequents.

Building upon these foundations, our paper further explores the cut elimination property within the context of CyNBL, considering both the pure logic aspects and the incorporation of nonlogical axioms.

In the third section, we prove the strong conservativeness of CyNBL over Full Nonassociative Lambek Calculus (FNL). This result highlights the relationship between CyNBL and FNL, shedding light on the expressive power and logical properties of CyNBL within the context of nonassociative Lambek calculus. Abrusci [2] has previously demonstrated that CyMALL, the commutative variant of CyNBL, is not a conservative extension of Full Lambek Calculus (FL) when considering the inclusion of additive constants such as $\bot$ and $\top$. However, when the additive constants are omitted, CyMALL exhibits conservativeness. V. M. Abrusci's work presents a sequent that is provable in CyMALL but not in FL with additive constants. A similar example can be provided for the nonassociative version.

In this paper, we establish that CyNBL without additive constants serves as a strongly conservative extension of FNL without additive constants, highlighting the logical relationship between the two systems. Additionally, a similar result can be obtained for CyMALL, as discussed in Section 4. Notably, this outcome has been previously demonstrated in Płaczek's work [11] and may also be inferred from other algebraic findings.

The crucial contribution of this paper lies in the application of syntactic methods, ensuring that all proofs are constructive in nature. We present a systematic approach to construct cut-free proofs based on existing theorems in CyNBL. Furthermore, we demonstrate how to construct proofs in FNL based on the proofs available in CyNBL. By showing these constructive methods, we provide valuable insights into the practical aspects of reasoning within CyNBL and its relationship with FNL, establishing a foundation for future research and application.

As a consequence of the results of this paper we can tell more about complexity of these logics. CyNBL has undecidable consequence relation, since it is a strongly conservative extension of FNL; see [6]. Also CyMALL has undecidable consequence relation, because it is a strongly conservative extension of FL; see [3]. The finitary consequence relation of multiplicative part of CyNBL is decidable in *PTIME*; see [4].

The other consequence is that NBL is also a strongly conservative extension of FNL. The open problem in this matter remains the decidability (and complexity) of the finitary consequence relation for multiplicative fragment of NBL.

1.1. Algebras

We will briefly introduce certain algebras that serve as models for the logics examined in this paper.

Let $(P, \leq)$ be a poset and let $^\sim$ be a unary operation on P such that for all $a, b \in P$: (i) if $a \leq b$, then $b^\sim \leq a^\sim$; (ii) $a^{\sim\sim} = a$. Such an operation $^\sim$ is called a *De Morgan negation*.

Definition 1.1. Let $\mathbf{M} = (M, \otimes, \wedge, \vee, ^\sim, 1, \bot, \leq)$ be a structure such that $\otimes, \wedge, \vee$ are binary operations, $^\sim$ is a unary operation, 1 and $\bot$ are constants and $\leq$ is a partial order on M. We say that $\mathbf{M}$ is a *bounded CyNBL–algebra*, if the following conditions hold:

(i) $^\sim$ is a De Morgan negation;

(ii) $(M, \wedge, \vee, \leq)$ is a lattice;

(iii) $a \otimes b \leq c$ iff $b \otimes c^\sim \leq a^\sim$ iff $c^\sim \otimes a \leq b^\sim$ for all $a, b, c \in M$;

(iv) $1 \otimes a = a = a \otimes 1$ for all $a \in M$;

(v) $\bot \leq a$ for all $a \in M$.

The analogous structre without constant $\bot$ and (iv) is called an *unbounded CyNBL–algebra*. One defines $a \parr b = (b^\sim \otimes a^\sim)^\sim$ and $0 = 1^\sim$ and $\top = \bot^\sim$.

Bounded CyNBL–algebras serve as models of CyNBL, while unbounded CyNBL–algebras model CyNBL without additive constants.

Definition 1.2. Let $\mathbf{M} = (M, \otimes, \multimap, \multimapinv, \wedge, \vee, 1, \leq)$ be a structure such that $\otimes, \multimap, \multimapinv, \wedge, \vee$ are binary operations, $1 \in M$ and $\leq$ is a partial order on M. We say that $\mathbf{M}$ is an *FNL–algebra*, if the following conditions hold:

(i) $(M, \wedge, \vee, \leq)$ is a lattice;

(ii) $a \otimes b \leq c$ iff $a \leq c \multimapinv b$ iff $b \leq a \multimap c$ for all $a, b, c \in M$;

(iii) $1 \otimes a = a = a \otimes 1$ for all $a \in M$.

FNL–algebras serve as models of FNL. It is possible to extend FNL by introducing additive constants $\bot$ and $\top$, or solely $\bot$ (given that $\top$ can be defined), resulting in bounded FNL–algebras. However, it's important to note that our paper does not explore FNL with additive constants.

It can be proved that every CyNBL–algebra, whether bounded or unbounded, is also an FNL–algebra. We define $a \multimap b = a^{\sim} \parr b$ and $a \multimapinv b = a \parr b^{\sim}$. One checks that the condition (ii) holds.

2. Sequent systems

Let $\mathcal{V}$ be an arbitrary, countable set of variables. We define the set of atoms $\mathcal{V}'$ as follows: if $p \in \mathcal{V}$, then both p and $p^{\sim}$ are elements of $\mathcal{V}'$. Variables in this set are referred to as *positive* atoms, while their negations ($p^{\sim}$) are termed *negative* atoms. We construct the set of *CyNBL–formulas* from $\mathcal{V}'$ by the binary connectives: $\otimes$, $\parr$, $\wedge$ and $\vee$ and the constants 1, 0, $\top$ and $\bot$.

It's worth noting that we do not treat negation as a connective. The systems we introduce adhere to the negation normal form, meaning that negations only appear in the form of atoms.

We define the metalanguage negation $^{\sim}$:

$$\begin{array}{ll} (p)^\sim = p^\sim & (p^\sim)^\sim = p \\ 1^\sim = 0 & 0^\sim = 1 \\ \top^\sim = \bot & \bot^\sim = \top \\ (A \otimes B)^\sim = B^\sim \parr A^\sim & (A \parr B)^\sim = B^\sim \otimes A^\sim \\ (A \wedge B)^\sim = A^\sim \vee B^\sim & (A \vee B)^\sim = A^\sim \wedge B^\sim \end{array}$$

One notices $A^{\sim\sim} = A$ for all formulas A.

We define *CyNBL–bunches*. A CyNBL–bunch is an element of the free unital groupoid generated by the set of all CyNBL–formulas. The neutral element of this unital groupoid is referred to as an empty bunch and is denoted by ϵ. A *CyNBL–sequent* is defined as any nonempty bunch, and we represent bunches using capital Greek letters.

An *anonymous variable* is a unique formula represented as $\_$, serving as a placeholder for substitution. It's important to note that if a bunch contains multiple anonymous variables, they are considered distinct, even if they share the same symbol. A *CyNBL–context* is a bunch with an anonymous variable. Contexts are denoted by $\Gamma[\_]$, and when we perform the substitution of Δ in place of $\_$, we represent it as $\Gamma[\Delta]$.

The axioms of CyNBL are:

$$(\text{a-id}) \ \frac{}{p, p^\sim} \qquad\qquad (\text{a-}0) \ \frac{}{0}$$

$$(\text{a-}\bot) \ \frac{}{\Gamma[\bot]}$$

The introduction rules (rules introducing connectives and constants) are:

$$(\text{r-}\otimes) \ \frac{\Gamma[(A, B)]}{\Gamma[A \otimes B]} \qquad\qquad (\text{r-}1) \ \frac{\Gamma[\Delta]}{\Gamma[(1, \Delta)]} \quad \frac{\Gamma[\Delta]}{\Gamma[(\Delta, 1)]}$$

$$(\text{r-}\parr 1) \ \frac{\Gamma[B] \qquad \Delta, A}{\Gamma[(\Delta, A \parr B)]} \qquad\qquad (\text{r-}\parr 2) \ \frac{\Gamma[A] \qquad B, \Delta}{\Gamma[(A \parr B, \Delta)]}$$

$$(\text{r-}\wedge) \ \frac{\Gamma[A]}{\Gamma[A \wedge B]} \quad \frac{\Gamma[A]}{\Gamma[B \wedge A]} \qquad\qquad (\text{r-}\vee) \ \frac{\Gamma[A] \qquad \Gamma[B]}{\Gamma[A \vee B]}$$

In (r-1) we assume $\Delta \neq \epsilon$.

The structural rules and the cut rule are:

$$\text{(r-shift)}\frac{(\Gamma,\Delta),\Theta}{\Gamma,(\Delta,\Theta)} \qquad\qquad \text{(r-cyc)}\frac{\Gamma,\Delta}{\Delta,\Gamma}$$

$$\text{(r-cut)}\ \frac{\Gamma[A] \qquad \Delta, A^{\sim}}{\Gamma[\Delta]}$$

The rules (r-shift) and (r-cyc) are reversible. For (r-cyc) it is obvious. To obtain reversed (r-shift) we apply consequtively (r-cyc), (r-shift), (r-cyc), (r-shift) and again (r-cyc). The reversibility of these rules is an important fact we use later. For the simplicity of proofs, we do not assume this fact in the definition of the system.

The models of CyNBL are bounded CyNBL–algebras. A valuation is a homomorphism μ of a free algebra of CyNBL–formulas to a bounded CyNBL–algebra extended by the following properties: $\mu((\Gamma,\Delta)) = \mu(\Gamma) \otimes \mu(\Delta)$ and $\mu(\epsilon) = 1$. A sequent Γ is satisfied by a valuation μ, if $\mu(\Gamma) \leq 0$.

CyNBL is strongly complete with respect to bounded CyNBL–algebras. The (r-shift) rule express the condition (iii) from definition 1.1. The rule (r-cyc) express the fact, that we have a De Morgan negation. One proves strong completeness in a usual way, using Lindenbaum–Tarski algebras.

2.1. Auxilary system

Let Φ be a set of sequents of the form $C, D^{\sim}$. We define the system $\mathbf{S}_\Phi$. The system $\mathbf{S}_\Phi$ has all axioms and introduction rules of CyNBL. We add the following rules and axioms:

$$\text{(a-id2)}\ \frac{}{p^{\sim}, p}$$

$$\text{(r-}\parr\text{3)}\ \frac{A,\Gamma \qquad B,\Delta}{A \parr B, (\Delta,\Gamma)} \qquad\qquad \text{(r-}\parr\text{4)}\ \frac{\Gamma, A \qquad \Delta, B}{(\Delta,\Gamma), A \parr B}$$

In (r-⅋3) and (r-⅋4) we assume Γ, Δ are nonempty; otherwise they are special cases of (r-⅋2) and (r-⅋1).

For every $(C, D^{\sim}) \in \Phi$ we add the *assumption rules*:

$$\text{(r-assm1)}\frac{D,\Gamma \qquad \Delta, C^{\sim}}{\Delta,\Gamma} \qquad \text{(r-assm2)}\frac{D,\Gamma \qquad \Delta, C^{\sim}}{\Gamma,\Delta}$$

$$\text{(r-assm3)}\frac{D,\Gamma_1 \qquad (\Gamma_2,\Gamma_3), C^{\sim}}{\Gamma_2,(\Gamma_3,\Gamma_1)} \qquad \text{(r-assm4)}\frac{D,(\Gamma_1,\Gamma_2) \quad \Gamma_3, C^{\sim}}{\Gamma_1,(\Gamma_2,\Gamma_3)}$$

$$\text{(r-assm5)}\frac{D,\Gamma_1 \qquad (\Gamma_2,\Gamma_3), C^{\sim}}{(\Gamma_3,\Gamma_1),\Gamma_2} \qquad \text{(r-assm6)}\frac{D,(\Gamma_1,\Gamma_2) \quad \Gamma_3, C^{\sim}}{(\Gamma_2,\Gamma_3),\Gamma_1}$$

$$\text{(r-assm7)}\frac{D,\Gamma_1 \qquad (\Gamma_2,\Gamma_3), C^{\sim}}{\Gamma_3,(\Gamma_1,\Gamma_2)} \qquad \text{(r-assm8)}\frac{D,(\Gamma_1,\Gamma_2) \quad \Gamma_3, C^{\sim}}{\Gamma_2,(\Gamma_3,\Gamma_1)}$$

$$\text{(r-assm9)}\frac{D,\Gamma_1 \qquad (\Gamma_2,\Gamma_3), C^{\sim}}{(\Gamma_1,\Gamma_2),\Gamma_3} \qquad \text{(r-assm10)}\frac{D,(\Gamma_1,\Gamma_2) \quad \Gamma_3, C^{\sim}}{(\Gamma_3,\Gamma_1),\Gamma_2}$$

We assume none of $\Gamma_1, \Gamma_2, \Gamma_3$ is empty. From now on we denote $\vdash_{\mathbf{S}_\Phi} \Gamma$ the provability of Γ in $\mathbf{S}_\Phi$.

We define inductively a function f:

$$f(A) = (A), \text{ for all CyNBL–formulas } A$$
$$f((\Gamma,\Delta)) = f(\Gamma) \otimes f(\Delta)$$
$$f(\epsilon) = 1$$

One proves that $\vdash_{\mathbf{S}_\Phi} \Gamma$ iff $\vdash_{\mathbf{S}_\Phi} f(\Gamma)$. Let Γ be a CyNBL–sequent. We represent Γ in the form $C, D^{\sim}$. If $\Gamma = (\Gamma_1, \Gamma_2)$, then $C = f(\Gamma_1)$ and $D = f(\Gamma_2)^{\sim}$. If $\Gamma = A$, then $C = A, D = 0$. Hence, every CyNBL–sequent may be represented by the sequent of the form $C, D^{\sim}$.

We define the relation $\Gamma \sim \Delta$, which holds for the CyNBL–bunches Γ and Δ, if Δ can be derived from Γ by finitely many applications of (r-cyc) and (r-shift).

Since both (r-cyc) and (r-shift) are reversible, this relation is an equivalence relation. The following lemma is a modification of lemma from Buszkowski [4] or Płaczek [11].

Lemma 2.1. *Let* $\Gamma[\_]$ *be an CyNBL–context. Then, there exists the unique CyNBL–bunch* Δ *such that* $\Gamma[\_] \sim (\Delta, \_)$.

Proof: We provide an algorithm which reduces $\Gamma[\_]$ to some sequent $(\Delta, \_)$. The reduction rules are as follows:

$$\text{(R1)} \qquad (\Psi[\_], \Phi) \to (\Phi, \Psi[\_])$$

$$\text{(R2)} \qquad (\Phi, (\Psi, \Xi[\_])) \to ((\Phi, \Psi), \Xi[\_])$$

$$\text{(R3)} \qquad (\Phi, (\Psi[\_], \Xi)) \to ((\Xi, \Phi), \Psi[\_])$$

(R1) is an application of (r-cyc), (R2) is an application of (r-cyc), (r-shift), (r-cyc), (r-shift), consequtively (i.e. reversed (r-shift) and (R3) is an application of (r-cyc), (r-shift) and (r-cyc), consequtively. The algorithm is deterministic and hence, after finitely many steps, terminates and yields $(\Delta, \_)$.

The rest of the proof is similar to Buszkowski [4] and Płaczek [11]. □

Corollary 2.2. Let $\Gamma[\_] \sim (\Delta, \_)$ and let Θ be a substructure of $\Gamma[\_]$, which does not contain this occurence of $\_$ (but it can contain occurences of other anonymous variables). Then, the reduction preserves Θ.

As a consequence, the relation $\sim$ is closed under substitution.

Proposition 2.3. Let $\Gamma \sim \Delta$. Then Γ is provable in $\mathbf{S}_\Phi$ iff Δ is provable.

Proof: We use the outer induction on the number of (r-shift) and (r-cyc) used to obtain Δ from Γ and the inner induction on the proof of Γ.

1° Assume Δ arises from Γ by one application of (r-cyc) or (r-shift); we denote: $\Gamma \sim_1 \Delta$. We run the inner induction. Let Γ be an axiom. Then Δ is an axiom, too.

Now we assume Γ is the conclusion of a rule.

1.1° We consider (r-$\otimes$). We have $\Gamma = \Theta[A \otimes B]$.
Let $\Theta[\_] \sim_1 \Delta'[\_]$ and $\Delta = \Delta'[A \otimes B]$. The premise of (r-$\otimes$) is $\Theta[(A, B)]$. By the inner induction hypothesis and corollary 2.2, $\vdash_{\mathbf{S}_\Phi} \Delta'[(A, B)]$, so we apply (r-$\otimes$) and obtain $\vdash_{\mathbf{S}_\Phi} \Delta'[A \otimes B]$.

1.2° The cases for (r-$\vee$), (r-$\wedge$) and (r-1) are similar to (r-$\otimes$).

1.3° We consider (r-$\parr$1). We have

$$\frac{\Theta[B] \qquad \Xi, A}{\Theta[(\Xi, A \parr B)]}$$

and $\Gamma = \Theta[(\Xi, A \parr B)]$.

Assume Δ arises from Γ by an application of (r-cyc). We consider cases: (1) $\Theta[B] = B$, (2) $\Theta[B] \neq B$.

In the first case $\Gamma = (\Xi, A \parr B)$ and $\Delta = (A \parr B, \Xi)$. By the inner induction hypothesis, $\vdash_{\mathbf{S}_\Phi} A, \Xi$. So we apply (r-$\parr$2) to A, Ξ and B and obtain $\vdash_{\mathbf{S}_\Phi} A \parr B, \Xi$.

In the second case, we apply (r-cyc) to the premise $\Theta[B]$ and obtain $\Theta'[B]$. By the inner induction hypothesis, $\vdash_{\mathbf{S}_\Phi} \Theta'[B]$. We use (r-$\parr$1) with the premises $\Theta'[B]$ and Ξ, A.

Assume Δ arises from Γ by an application of (r-shift). We consider cases.

1.3°(i) We assume $\Theta[B] = B$. The derivation is as follows:

$$\frac{B \qquad (\Xi_1, \Xi_2), A}{(\Xi_1, \Xi_2), A \parr B}$$

Then $\Gamma = \big((\Xi_1, \Xi_2), A \parr B\big)$ and $\Delta = \big(\Xi_1, (\Xi_2, A \parr B)\big)$. By the inner induction hypothesis, $\vdash_{\mathbf{S}_\Phi} \Xi_1, (\Xi_2, A)$. We apply (r-$\parr$2).

1.3°(ii) We assume $\Theta[B] \neq B$. If $\Theta[B]$ consists of two formulas, then (r-shift) is not applicable to the conclusion. Otherwise we apply (r-shift) the first premise and use the same rule.

1.4° We consider (r-$\parr$2). We have

$$\frac{\Theta[A] \qquad B, \Xi}{\Theta[(A \parr B, \Xi)]}$$

The case when Δ arises by one application of (r-cyc) from Γ is similar to the previous one. The more interesting case is Δ arising by one application of (r-shift). The only possible case is when $\Theta[A] = \Theta'[A], \Psi$; otherwise (r-shift) is not applicable to the conclusion. In such a case, we apply (r-shift) to the first premise and then we use the same rule.

1.5° We consider (r-$\parr$3). We have:

$$\frac{A, \Theta \qquad B, \Xi}{A \parr B, (\Theta, \Xi)}$$

Δ must arise by an application of (r-cyc). Then $\Delta = ((\Theta, \Xi), A \parr B)$. We apply (r-cyc) to the premises. By the inner induction hypothesis, $\vdash_{\mathbf{S}_\Phi} \Theta, A$ and $\vdash_{\mathbf{S}_\Phi} \Xi, B$. We apply (r-$\parr$4) with these premises.

1.6° The case for (r-$\parr$4) is analogous to the previous cases.

1.7° We consider the assumption rules. Assume Δ arises by an application of (r-cyc). We apply other rule (as described in the table below) with the same premises:

original rule	new rule	original rule	new rule
(r-assm1)	(r-assm2)	(r-assm2)	(r-assm1)
(r-assm3)	(r-assm5)	(r-assm5)	(r-assm3)
(r-assm4)	(r-assm6)	(r-assm6)	(r-assm4)
(r-assm7)	(r-assm9)	(r-assm9)	(r-assm7)
(r-assm8)	(r-assm10)	(r-assm10)	(r-assm8)

Analogously, if Δ arises by (r-shift), we apply the table below:

original rule	new rule	original rule	new rule
(r-assm1)	(r-assm3)	(r-assm6)	(r-assm8)
(r-assm2)	(r-assm4)	(r-assm9)	(r-assm2)
(r-assm5)	(r-assm7)	(r-assm10)	(r-assm1)

2° Assume Δ arises from Γ by $n + 1$ applications of (r-cyc) or (r-shift). Then there exists Γ' such that Γ' arises from Γ by n applications and

Δ arises from Γ' by one application. By the outer induction hypothesis, $\vdash_{\mathbf{S}_\Phi} \Gamma'$. We have again only one application, so we proceed as above. $\square$

Remark 2.4. The transformation provided in the proof above does not change the length of the proof.

COROLLARY 2.5. The rules (r-shift) and (r-cyc) are admissible in $\mathbf{S}_\Phi$.

2.2. Cut elimination

In this section we prove that the cut rule (r-cut) is admissible in $\mathbf{S}_\Phi$ for every Φ. As a consequence, we obtain the cut–elimination property for CyNBL, since $\mathbf{S}_\emptyset$ is equivalent to CyNBL (they have the same theorems). The cut elimination for CyNBL can be proved simpler, since it is a property for pure logic (i.e. without assumptions), but our proof shows us not only cut elimination for pure logic, but also something like partial cut elimination for logics with nonlogical axioms (assumptions). This result will be useful later.

LEMMA 2.6. *Let* $(C, D^\sim) \in \Phi$.

(1) *if* $\vdash_{\mathbf{S}_\Phi} D, \Gamma$ *and* $\vdash_{\mathbf{S}_\Phi} \Delta[C^\sim]$, *then* $\vdash_{\mathbf{S}_\Phi} \Delta[\Gamma]$,

(2) *if* $\vdash_{\mathbf{S}_\Phi} \Gamma[D]$ *and* $\vdash_{\mathbf{S}_\Phi} \Delta, C^\sim$, *then* $\vdash_{\mathbf{S}_\Phi} \Gamma[\Delta]$

PROOF: We consider (1). We assume $\vdash_{\mathbf{S}_\Phi} D, \Gamma$ and $\vdash_{\mathbf{S}_\Phi} \Delta[C^\sim]$. Then, by proposition 2.3, $\vdash_{\mathbf{S}_\Phi} \Delta', C^\sim$ for some Δ' such that $\Delta[\_] \sim (\Delta', \_)$. Then, $\Delta[\Gamma] \sim \Delta', \Gamma$. We apply (r-assm1) to D, Γ and $\Delta', C^\sim$ and obtain $\vdash_{\mathbf{S}_\Phi} \Delta'$, Γ. By proposition 2.3, $\vdash_{\mathbf{S}_\Phi} \Delta[\Gamma]$.

We consider (2). We assume $\vdash_{\mathbf{S}_\Phi} \Gamma[D]$ and $\vdash_{\mathbf{S}_\Phi} \Delta, C^\sim$. Let Γ' be such that $\Gamma[\_] \sim (\Gamma', \_)$. Then $\vdash_{\mathbf{S}_\Phi} D, \Gamma'$ by proposition 2.3, since $(D, \Gamma') \sim (\Gamma', D)$. We apply (r-assm2) and obtain $\vdash_{\mathbf{S}_\Phi} \Gamma', \Delta$. Hence, $\vdash_{\mathbf{S}_\Phi} \Gamma[\Delta]$, by proposition 2.3. $\square$

THEOREM 2.7. *The rule (r-cut) is admissible in* $\mathbf{S}_\Phi$, *i.e. if* $\vdash_{\mathbf{S}_\Phi} \Gamma[A]$ *and* $\vdash_{\mathbf{S}_\Phi} \Delta, A^\sim$, *then* $\vdash_{\mathbf{S}_\Phi} \Gamma[\Delta]$.

PROOF: We assume $\vdash_{\mathbf{S}_\Phi} \Theta[C]$ and $\vdash_{\mathbf{S}_\Phi} \Psi, C^\sim$. We show $\vdash_{\mathbf{S}_\Phi} \Theta[\Psi]$.

The proof proceeds by the outer induction on the number of connectives in C, the intermediate induction on the length of the proof of $\Theta[C]$ and the inner induction on the length of the proof of $\Psi, C^\sim$.

We run the outer induction.

1° $C = p$. Then $C^\sim = p^\sim$. We run the intermediate induction.

1.1° Let $\Theta[C]$ be an axiom (a-id) or (a-id2). We have two possibilities: $p, p^\sim$ and $p^\sim, p$. We run the inner induction.

If $\Psi, C^\sim$ is an axiom, then $\Psi = p = C$ or $\Psi = \Xi[\perp]$ and $\Theta[\Psi]$ is an instance of (a-$\perp$). Now let $\Psi, C^\sim$ be the conclusion of an introduction rule. $C^\sim$ cannot be the active formula of any rule. We apply the inner induction hypothesis to the premise(s) with $C^\sim$ and use the same rule.

We consider the following special case:

$$\frac{A \qquad B, C^\sim}{A \parr B, C^\sim},$$

with $\Psi = A \parr B$. This may be obtained by (r-$\parr$1) or (r-$\parr$2). We apply the inner induction hypothesis to the premise $B, C^\sim$ and use (r-$\parr$1).

Now let $\Psi, C^\sim$ be the conclusion of an assumption rule. We have the following possibilities:

$$(1)\ \frac{F, \Psi \qquad C^\sim, E^\sim}{\Psi, C^\sim} \qquad (2)\ \frac{F, C^\sim \qquad \Psi, E^\sim}{\Psi, C^\sim}$$

$$(3)\ \frac{F, \Psi_2 \quad (C^\sim, \Psi_1), E^\sim}{\Psi, C^\sim} \qquad (4)\ \frac{F, (C^\sim, \Psi_1) \quad \Psi_2, E^\sim}{\Psi, C^\sim}$$

$$(5)\ \frac{F, \Psi_1 \quad (\Psi_2, C^\sim), E^\sim}{\Psi, C^\sim} \qquad (6)\ \frac{F, (\Psi_2, C^\sim) \quad \Psi_1, E^\sim}{\Psi, C^\sim}$$

where $(\Psi_1, \Psi_2) = \Psi$.

(1) By proposition 2.3 we have $\vdash_{\mathbf{S}_\Phi} E^\sim, C^\sim$ and the length of the proof of this sequent is the same as the length of the proof of $C^\sim, E^\sim$. We apply the inner induction hypothesis to $E^\sim, C^\sim$ and $\Theta[C]$ and obtain $\Theta[E^\sim]$. By lemma 2.6, $\vdash_{\mathbf{S}_\Phi} \Theta[\Psi]$.

(2) We apply the inner induction hypothesis to $F, C^\sim$ and $\Theta[C]$ and obtain $\Theta[F]$. Then, by lemma 2.6, $\vdash_{\mathbf{S}_\Phi} \Theta[\Psi]$.

(3) By proposition 2.3, $\vdash_{\mathbf{S}_\Phi} (\Psi_1, E^\sim), C^\sim$ and it has the proof of the same length as $(C^\sim, \Psi), E^\sim$. We apply the inner induction hypothesis to $(\Psi_1, E^\sim), C^\sim$ and $\Theta[C]$ and obtain $\vdash_{\mathbf{S}_\Phi} \Theta[(\Psi_1, E^\sim)]$. By lemma 2.6 we obtain $\vdash_{\mathbf{S}_\Phi} \Theta[\Psi]$.

(4)–(6) are similar to (1)–(3).

1.2° Let $\Theta[C]$ be an axiom (a-$\perp$). Then $\Theta[C] = \Theta'[\perp][C]$ and $\Theta'[\perp][\Psi]$ is another instance of (a-$\perp$).

1.3° We assume that $\Theta[C]$ is not an axiom, hence it is obtained by a rule. C cannot be the active formula of any rule. Hence it occurs in at least one premise, so we apply the intermediate induction hypothesis to the premise(s) with C and use the same rule.

2° The case for $C = p^\sim$ is similar to the previous one.

3° $C = 0$. Then $C^\sim = 1$. We run the intermediate induction.

Let $\Theta[0]$ be an axiom (a-0), then $\Theta[C] = C = 0$ and $\Theta[\Psi] = \Psi$. We run the inner induction. If $\Psi, 1$ is an axiom, then $\Psi = \Xi[\perp]$ and $\Theta[\Psi]$ is an instance of (a-$\perp$). Let $\Psi, 1$ be obtained by a rule. If $C^\sim = 1$ is not the active formula of a rule, we proceed as for $C = p$. If 1 is the active formula, then the rule is (r-1) of the form:

$$\frac{\Psi}{\Psi, 1}.$$

The premise is $\Psi = \Theta[\Psi]$.

Let $\Theta[C]$ be an axiom (a-$\perp$). Then $\Theta[C] = \Theta'[\perp][C]$ and $\Theta'[\perp][\Psi]$ is another instance of (a-$\perp$).

Now let $\Theta[C]$ be the conclusion of a rule. $C = 0$ cannot be the active formula of any rule. We apply the intermediate induction hypothesis to the premise(s) with $C = 0$ and use the same rule.

4° $C = 1$. Then $C^\sim = 0$. We run the intermediate induction.

Let $\Theta[C]$ be an axiom. Then $\Theta[C] = \Theta'[\perp][C]$ and $\Theta'[\perp][\Psi]$ is another instance of (a-$\perp$).

We assume $\Theta[1]$ is obtained by a rule. If $C = 1$ is the active formula, then $\Theta[1]$ is obtained by (r-1) admitting $\Delta = \epsilon$ in $\Theta[\Delta]$ as the premise. We run the inner induction. If $\Psi, 0$ is an axiom, then $\Psi = \epsilon$ and

$\Theta[\Psi] = \Theta[\epsilon]$ or $\Psi = \Xi[\bot]$ and $\Theta[\Psi]$ is an instance of (a-$\bot$). Let $\Psi, 0$ be obtained by a rule. $C^{\sim} = 0$ cannot be the active formula of any rule, so we proceed as for $C = p$.

If $\Theta[1]$ is obtained by a rule and $C = 1$ is not the active formula, then we proceed as above.

5° $C = \bot$. Then $C^{\sim} = \top$. We run the intermediate induction. Let $\Theta[C]$ be an axiom, we run the inner induction. We assume $\Psi, C^{\sim}$ is an axiom. Then $\Psi = \Xi[\bot]$ and $\Theta[\Psi]$ is another instance of (a-$\bot$). We assume $\Psi, C^{\sim}$ is the conclusion of a rule. Since $\top$ cannot be the active formula, then we apply the inner induction hypothesis to the premise(s) with $\top$ and use te same rule.

We assume $\Theta[C]$ is the conclusion of a rule. Since $\bot$ cannot be the active formula, then we proceed as above.

6° $C = \top$. Then $C^{\sim} = \bot$. We run the intermediate induction. Let $\Theta[C]$ be an axiom, then $\Theta[C] = \Theta'[\bot][\top]$ and $\Theta'[\bot][\Psi]$ is another instance of (a-$\bot$). We assume $\Theta[C]$ is the conclusion of a rule. Since $\top$ cannot be the active formula, then we proceed as above.

7° C is not an atomic formula. We run the intermediate induction.

Since C is not atomic, $\Theta[C]$ cannot be an instance of axiom (a-id), (a-id2) or (a-0). Let $\Theta[C]$ be an axiom (a-$\bot$). Then $\Theta[C] = \Theta'[\bot][C]$ and $\Theta'[\bot][\Psi]$ is another instance of (a-$\bot$). Let $\Theta[C]$ be the conclusion of a rule. If C is not the active formula, we apply the intermediate induction hypothesis to the premise(s) with C and use the same rule. We assume that C is the active formula.

7.1° $C = A \otimes B$. So $C^{\sim} = B^{\sim} \parr A^{\sim}$ and $\Theta[C]$ arises by (r-$\otimes$):

$$\frac{\Theta[(A, B)]}{\Theta[A \otimes B]}$$

We run the inner induction. Let $\Psi, C^{\sim}$ be an axiom, then $\Psi = \Xi[\bot]$ and $\Theta[\Psi]$ is another instance of (a-$\bot$). We assume $\Psi, C^{\sim}$ is the conclusion of a rule.

In the cases when $C^{\sim}$ does not occur in the active bunch, we apply the inner induction hypothesis to $\Theta[C]$ and the premise(s) with $C^{\sim}$, and use the same rule.

For example:

$$\frac{\Gamma[(D,E)], C^{\sim}}{\Gamma[D \otimes E], C^{\sim}}$$

changes into:

$$\frac{\Theta[\Gamma[(D,E)]]}{\Theta[\Gamma[D \otimes E]]},$$

where $\Psi = \Gamma[D \otimes E]$.

We consider cases when $C^{\sim}$ occurs in the active bunch, but is not the active formula.

$$\frac{D \qquad E, C^{\sim}}{D \parr E, C^{\sim}} \qquad \frac{D, C^{\sim} \qquad E}{D \parr E, C^{\sim}}$$

We apply the inner induction hypothesis to the premise with $C^{\sim}$ and use (r-$\parr$1).

Let $C^{\sim}$ be the active formula:

$$\frac{\Psi, A^{\sim} \qquad B^{\sim}}{\Psi, C^{\sim}} \qquad \frac{\Psi, B^{\sim} \qquad A^{\sim}}{\Psi, C^{\sim}} \qquad \frac{\Psi_2, B^{\sim} \quad \Psi_1, A^{\sim}}{(\Psi_1, \Psi_2), C^{\sim}}$$

The first case is obtained by (r-$\parr$1). We apply the outer induction hypothesis to $\Theta[(A,B)]$ and $\Psi, A^{\sim}$ and then to $\Theta[(\Psi, B)]$ and $B^{\sim}$, obtaining $\Theta[\Psi]$. The second one is obtained by (r-$\parr$1) or (r-$\parr$2). We proceed as above: we apply twice the outer induction hypothesis to both premises. The third case is obtained by (r-$\parr$4), where $\Psi = (\Psi_1, \Psi_2)$. We apply the outer induction hypothesis twice, obtaining $\Theta[(\Psi_1, \Psi_2)] = \Theta[\Psi]$.

7.2° $C = A \parr B$, then $C^{\sim} = B^{\sim} \otimes A^{\sim}$. We have to consider four cases, one for each (r-$\parr$i).

(1) $\dfrac{\Gamma[B] \qquad \Delta, A}{\Gamma[(\Delta, A \parr B)]}$

We run the inner induction. Let $\Psi, C^{\sim}$ be an axiom, then $\Psi = \Xi[\bot]$ and $\Theta[\Psi]$ is another instance of (a-$\bot$). We assume $\Psi, C^{\sim}$ is the conclusion of a rule. We skip cases when $C^{\sim}$ is not the active formula of a rule (in these cases we proceed as above). We consider (r-$\otimes$) as the only possibility:

$$\frac{\Psi, (B^\sim, A^\sim)}{\Psi, C^\sim}$$

We apply proposition 2.3 to $\Psi, (B^\sim, A^\sim)$, then we apply the outer induction hypothesis to Δ, A and $(\Psi, B^\sim), A^\sim$ and obtain: $\Delta, (\Psi, B^\sim)$. By proposition 2.3 and the outer induction hypothesis applied to $\Theta[B]$ and $(\Delta, \Psi), B^\sim$ we obtain $\Gamma[(\Delta, \Psi)] = \Theta[\Psi]$.

(2) $\dfrac{\Gamma[A] \qquad B, \Delta}{\Gamma[(A \parr B, \Delta)]}$

We run the inner induction and consider the same instance as above. We apply proposition 2.3 to $\Psi, (B^\sim, A^\sim)$, obtaining $(\Psi, B^\sim), A^\sim$. By proposition 2.3 we get $A^\sim, (\Psi, B^\sim)$. We use proposition 2.3 and apply the outer induction hypothesis to $(A^\sim, \Psi), B^\sim$ and B, Δ. We obtain $(A^\sim, \Psi), \Delta$ and apply proposition 2.3 and proposition 2.3. We use the outer induction hypothesis with $(\Psi, \Delta), A^\sim$ and $\Gamma[A]$, obtaining $\Gamma[(\Psi, \Delta)] = \Theta[\Psi]$.

(3) $\dfrac{A, \Gamma \qquad B, \Delta}{A \parr B, (\Delta, \Gamma)}$

We run the inner induction and consider the same instance as above. We apply proposition 2.3 to $\Psi, (B^\sim, A^\sim)$ and obtain $(\Psi, B^\sim), A^\sim$. We apply proposition 2.3 and get $A^\sim, (\Psi, B^\sim)$. We use proposition 2.3 and apply the outer induction hypothesis to $(A^\sim, \Psi), B^\sim$ and B, Δ. We have $(A^\sim, \Psi), \Delta$. We apply proposition 2.3 and proposition 2.3. We use the outer induction hypothesis to $(\Psi, \Delta), A^\sim$ and A, Γ, obtaining $(\Psi, \Delta), \Gamma$. We use proposition 2.3.

(4) $\dfrac{\Gamma, A \qquad \Delta, B}{(\Delta, \Gamma), A \parr B}$

We run the inner induction and consider the same instance as above. We apply proposition 2.3 to $\Psi, (B^\sim, A^\sim)$, obtaining $(\Psi, B^\sim), A^\sim$. We apply the outer induction hypothesis to $(\Psi, B^\sim), A^\sim$ and Γ, A. We get $\Gamma, (\Psi, B^\sim)$. We use proposition 2.3 and apply the outer induction hypothesis to $(\Gamma, \Psi), B^\sim$ and Δ, B. We obtain $\Delta, (\Gamma, \Psi)$ and use proposition 2.3.

7.3° $C = A \wedge B$. So $C^\sim = A^\sim \vee B^\sim$. We have the following instances:

$$\frac{\Theta[A]}{\Theta[C]} \qquad \frac{\Theta[B]}{\Theta[C]}$$

We run the inner induction. Let $\Psi, C^{\sim}$ be an axiom, then $\Psi = \Xi[\bot]$ and $\Theta[\Psi]$ is another instance of (a-$\bot$). We assume $\Psi, C^{\sim}$ is the conclusion of a rule. We skip the cases with $C^{\sim}$ not being the active formula. We have only one possibility:

$$\frac{\Psi, A^{\sim} \qquad \Psi, B^{\sim}}{\Psi, C^{\sim}}$$

We apply the outer induction hypothesis to $\Theta[A]$ and $\Psi, A^{\sim}$ or to $\Theta[B]$ and $\Psi, B^{\sim}$, depending on the proof of $\Theta[C]$. In both cases we obtain $\Theta[\Psi]$.

7.4° $C = A \vee B$. So $C^{\sim} = A^{\sim} \wedge B^{\sim}$. We have the following case:

$$\frac{\Theta[A] \qquad \Theta[B]}{\Theta[C]}$$

We run the inner induction. Let $\Psi, C^{\sim}$ be an axiom, then $\Psi = \Xi[\bot]$ and $\Theta[\Psi]$ is another instance of (a-$\bot$). We assume $\Psi, C^{\sim}$ is the conclusion of a rule. We consider only the cases with $C^{\sim}$ as the active formula:

$$\frac{\Psi, A^{\sim}}{\Psi, C^{\sim}} \qquad \frac{\Psi, B^{\sim}}{\Psi, C^{\sim}}$$

In the first case we apply the outer induction hypothesis to $\Theta[A]$ and $\Psi, A^{\sim}$ and in the second case to $\Theta[B]$ and $\Psi, B^{\sim}$. □

Lemma 2.8. *Let A be any CyNBL–formula. Then $A, A^{\sim}$ and $A^{\sim}, A$ are provable in $\mathbf{S}_{\Phi}$.*

Proof: The proof proceeds by the induction on the complexity of the formula A. Let $A = p$. Then $(A, A^{\sim}) = (p, p^{\sim})$ and $(A^{\sim}, A) = (p^{\sim}, p)$ are axioms. Analogously for $A = p^{\sim}$.

Let $A = 1$, then $(A, A^{\sim}) = (1, 0)$. We have $\vdash 0$. We apply (r-1) and obtain $\vdash 1, 0$. Analogously, $\vdash (0, 1)$. Similarly for $A = 0$.

Now let $A = A_1 \otimes A_2$. Then $(A, A^{\sim}) = (A_1 \otimes A_2, A_2^{\sim} \parr A_1^{\sim})$ and $(A^{\sim}, A) = (A_2^{\sim} \parr A_1^{\sim}, A_1 \otimes A_2)$. By the induction hypothesis: $\vdash A_1, A_1^{\sim}$, $\vdash A_1^{\sim}, A_1$, $\vdash A_2, A_2^{\sim}$ and $\vdash A_2^{\sim}, A_2$. We have the following derivations:

$$\dfrac{\dfrac{\dfrac{\dots}{A_1, A_1^{\sim}} \qquad \dfrac{\dots}{A_2, A_2^{\sim}}}{(A_1, A_2), A_2^{\sim} \parr A_1^{\sim}}}{A_1 \otimes A_2, A_2^{\sim} \parr A_1^{\sim}} \text{(r-}\parr\text{4)} \qquad \dfrac{\dfrac{\dfrac{\dots}{A_1^{\sim}, A_1} \qquad \dfrac{\dots}{A_2^{\sim}, A_2}}{A_2^{\sim} \parr A_1^{\sim}, (A_1, A_2)}}{A_2^{\sim} \parr A_1^{\sim}, A_1 \otimes A_2} \text{(r-}\parr\text{3)}$$

Let $A = A_1 \parr A_2$. Then $(A, A^{\sim}) = (A_1 \parr A_2, A_2^{\sim} \otimes A_1^{\sim})$ and $(A^{\sim}, A) = (A_2^{\sim} \otimes A_1^{\sim}, A_1 \parr A_2)$. The proof is analogous to the case for $\otimes$, but we use in the first case (r-$\parr$3) instead of (r-$\parr$4) and (r-$\parr$4) instead of (r-$\parr$3).

Let $A = A_1 \wedge A_2$. Then $(A, A^{\sim}) = (A_1 \wedge A_2, A_1^{\sim} \vee A_2^{\sim})$. By the induction hypothesis: $\vdash A_1, A_1^{\sim}$ and $\vdash A_2, A_2^{\sim}$. We use (r-$\wedge$) and obtain $\vdash A_1 \wedge A_2, A_1^{\sim}$ and $\vdash A_1 \wedge A_2, A_2^{\sim}$. We apply (r-$\vee$) and obtain $\vdash A_1 \wedge A_2, A_1^{\sim} \vee A_2^{\sim}$. The second part is proved in the similar way.

The case $A = A_1 \vee A_2$ is similar to the previous one. □

Proposition 2.9 (Phi=SPhi). $\Phi \vdash \Gamma$ iff $\vdash_{\mathbf{S}_\Phi} \Gamma$.

Proof: Let $\Phi \vdash \Gamma$. We show $\vdash_{\mathbf{S}_\Phi} \Gamma$. All rules of CyNBL are admissible in $\mathbf{S}_\Phi$. We show that every sequent $(C, D^{\sim}) \in \Phi$ is provable in $\mathbf{S}_\Phi$. By lemma 2.8, $\vdash D, D^{\sim}$ and $\vdash C, C^{\sim}$. We apply (r-assm1) and obtain $\vdash_{\mathbf{S}_\Phi} C, D^{\sim}$. Hence, $\vdash_{\mathbf{S}_\Phi} \Gamma$.

Now we assume $\vdash_{\mathbf{S}_\Phi} \Gamma$. We show $\Phi \vdash \Gamma$. We take the proof of Γ in $\mathbf{S}_\Phi$ and replace all applications of the assumption rules as follows:

$$\dfrac{\dfrac{\dots}{D, \Gamma} \qquad \dfrac{\dots}{\Delta, C^{\sim}}}{\Delta, \Gamma} \quad \rightarrow \quad \dfrac{\dfrac{\dfrac{\dots}{D, \Gamma} \qquad C, D^{\sim}}{C, \Gamma} \qquad \dfrac{\dots}{\Delta, C^{\sim}}}{\Delta, \Gamma} \text{(r-cut)}$$

$$\dfrac{\dfrac{\dots}{D, \Gamma} \qquad \dfrac{\dots}{\Delta, C^{\sim}}}{\Gamma, \Delta} \quad \rightarrow \quad \dfrac{\dfrac{\dfrac{\dfrac{\dots}{D, \Gamma} \qquad C, D^{\sim}}{C, \Gamma} \qquad \dfrac{\dots}{\Delta, C^{\sim}}}{\Delta, \Gamma} \text{(r-cut)}}{\Gamma, \Delta} \text{(r-cut)}$$

$$\dfrac{\dfrac{\dots}{D, \Gamma_1} \qquad \dfrac{\dots}{(\Gamma_2, \Gamma_3), C^{\sim}}}{\Gamma_2, (\Gamma_3, \Gamma_1)} \quad \rightarrow \quad \dfrac{\dfrac{\dfrac{\dfrac{\dots}{D, \Gamma_1} \qquad C, D^{\sim}}{C, \Gamma_1} \qquad \dfrac{\dots}{(\Gamma_2, \Gamma_3), C^{\sim}}}{(\Gamma_2, \Gamma_3), \Gamma_1} \text{(r-cut)}}{\Gamma_2, (\Gamma_3, \Gamma_1)} \text{(r-cut)}$$

And analogously with other rules. The rules (r-$\parr$3) and (r-$\parr$4) are admissible in CyNBL. □

Provability in pure CyNBL is equivalent to provability in $\mathbf{S}_{\emptyset}$. Hence, CyNBL admits the cut elimination.

Let T be an arbitrary set of CyNBL–formulas. By a *T–sequent* we mean a CyNBL–sequent containing only formulas from T. By a *T–proof* we mean a proof consisting of only T–sequents.

Proposition 2.10 (subformula property). Let Φ be a set of sequents of the form $C, D^{\sim}$. Let Γ be a CyNBL–sequent and let T be a subformula–closed set such that every formula in Γ occurs in T and for every $(C, D^{\sim}) \in \Phi$ we have $C, D, C^{\sim}, D^{\sim} \in T$. Then, Γ is provable in $\mathbf{S}_{\Phi}$ iff it has a T–proof in $\mathbf{S}_{\Phi}$.

3. Strong conservativeness

In this section we define Full Nonassociative Lambek Calculus (FNL) without $\bot$. We know, that CyNBL is not a conservative extension of FNL with $\bot$. We prove that CyNBL is strongly conservative extension of FNL. This result may be easily proved for CyNBL without additive constants, using the subformula property.

Let $\mathcal{V}$ be an arbitrary, countable set of variables. We construct the set of *FNL*–formulas from $\mathcal{V}$ by the binary connectives $\otimes, \multimap, \multimapinv, \wedge, \vee$ and the constant 1.

An *FNL–bunch* is an element of free unital groupoid generated by the set of all FNL–formulas. The neutral element of this unital groupoid is called an empty bunch and denoted ϵ. We define an *FNL–context* analogously as a CyNBL–context. An *FNL–sequent* is a pair $\Gamma \Rightarrow A$, where Γ is an FNL–bunch and A is an FNL–formula.

The axioms and the rules of FNL are as follows:

$$(\mathrm{id})\ \frac{}{A \Rightarrow A} \qquad (\mathrm{cut})\ \frac{\Gamma \Rightarrow A \qquad \Delta[A] \Rightarrow C}{\Delta[\Gamma] \Rightarrow C}$$

$$(\otimes \Rightarrow)\ \frac{\Gamma[(A, B)] \Rightarrow C}{\Gamma[A \otimes B] \Rightarrow C} \qquad (\Rightarrow \otimes)\ \frac{\Gamma \Rightarrow A \qquad \Delta \Rightarrow B}{\Gamma, \Delta \Rightarrow A \otimes B}$$

$$(\multimap\Rightarrow)\ \frac{\Gamma[B] \Rightarrow C \qquad \Theta \Rightarrow A}{\Gamma[(\Theta, A \multimap B)] \Rightarrow C} \qquad (\Rightarrow\multimap)\ \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \multimap B}$$

$$(\multimapinv\Rightarrow)\ \frac{\Gamma[A] \Rightarrow C \qquad \Theta \Rightarrow B}{\Gamma[(A \multimapinv B, \Theta)] \Rightarrow C} \qquad (\Rightarrow\multimapinv)\ \frac{\Gamma, B \Rightarrow A}{\Gamma \Rightarrow A \multimapinv B}$$

$$(1\Rightarrow)\ \frac{\Gamma[\Delta]\Rightarrow C}{\Gamma[(1,\Delta)]\Rightarrow C}\quad \frac{\Gamma[\Delta]\Rightarrow C}{\Gamma[(\Delta,1)]\Rightarrow C}\quad (\Rightarrow 1)\ \frac{}{\epsilon\Rightarrow 1}$$

$$(\vee\Rightarrow)\ \frac{\Gamma[A]\Rightarrow C\qquad \Gamma[B]\Rightarrow C}{\Gamma[A\vee B]\Rightarrow C}\qquad (\Rightarrow\vee)\ \frac{\Gamma\Rightarrow A_i}{\Gamma\Rightarrow A_1\vee A_2}(i=1,2)$$

$$(\wedge\Rightarrow)\ \frac{\Gamma[A]\Rightarrow C}{\Gamma[A\wedge B]\Rightarrow C}\quad \frac{\Gamma[B]\Rightarrow C}{\Gamma[A\wedge B]\Rightarrow C}\quad (\Rightarrow\wedge)\ \frac{\Gamma\Rightarrow A\qquad \Gamma\Rightarrow B}{\Gamma\Rightarrow A\wedge B}$$

FNL is strongly complete with respect to FNL–algebras. One proves that fact in a standard way, using Lindenbaum–Tarski algebras. Since every CyNBL–algebra is an FNL–algebra, then CyNBL is an extension of FNL.

Definition 3.1. We define two sets of CyNBL–formulas: $\mathcal{F}_1$ and $\mathcal{F}_2$. The former is the set of FNL–forumlas translated into CyNBL and the latter is the set of negated translated FNL–formulas.

(i) For every $p\in\mathcal{V}$, $p\in\mathcal{F}_1$ and $p^\sim\in\mathcal{F}_2$.

(ii) $1\in\mathcal{F}_1$ and $0\in\mathcal{F}_2$.

(iii) If $A,B\in\mathcal{F}_1$, then $A\otimes B, A\wedge B, A\vee B\in\mathcal{F}_1$.

(iv) If $A,B\in\mathcal{F}_2$, then $A\parr B, A\wedge B, A\vee B\in\mathcal{F}_2$.

(v) If $A\in\mathcal{F}_1$ and $B\in\mathcal{F}_2$, then $A\parr B, B\parr A\in\mathcal{F}_1$ and $A\otimes B, B\otimes A\in\mathcal{F}_2$.

(vi) No other formula belongs to $\mathcal{F}_1$ nor $\mathcal{F}_2$.

Notice that for every $A\in\mathcal{F}_1$ its metalanguage negation $A^\sim\in\mathcal{F}_2$ and conversely, if $A\in\mathcal{F}_2$, then $A^\sim\in\mathcal{F}_1$. Moreover, $\mathcal{F}_1\cap\mathcal{F}_2=\emptyset$. We define $\mathcal{F}=\mathcal{F}_1\cup\mathcal{F}_2$. We see that $\mathcal{F}$ is not the set of all CyNBL–formulas; e.g. $p\vee p^\sim\notin\mathcal{F}$.

CyNBL is an extension of FNL. We translate $A\multimap B$ into $A^\sim\parr B$ and $A\multimapinv B$ into $A\parr B^\sim$. We translate the FNL–sequent $\Gamma\Rightarrow A$ to the CyNBL–sequent $\Gamma, A^\sim$. One notices that every FNL–bunch is a CyNBL–bunch (if we replace $\multimap$ and $\multimapinv$ with $\parr$). Also, every translated FNL–sequent is an $\mathcal{F}$–sequent.

Lemma 3.2. *Let Φ be a set of sequents of the form $C, D^{\sim}$, where C, D are FNL–formulas. If an $\mathcal{F}$–sequent contains some formulas $A, B \in \mathcal{F}_2$, then this sequent is unprovable in $\mathbf{S}_\Phi$.*

Proof: Since $\mathcal{F}$ is closed under subformulas and contains all formulas $C, D, C^{\sim}$ and $D^{\sim}$ for every sequent in Φ, then every provable (in $\mathbf{S}_\Phi$) $\mathcal{F}$–sequent has an $\mathcal{F}$–proof, by the subformula property.

We observe that none of the axioms of $\mathbf{S}_\Phi$ has more than one formula from $\mathcal{F}_2$. We show, that if the premises of a rule are $\mathcal{F}$–sequents with at most one formula from $\mathcal{F}_2$, then the conclusion also has at most one $\mathcal{F}_2$–formula or is not an $\mathcal{F}$–sequent.

We consider (r-$\otimes$). The premise is of the form $\Gamma[(A, B)]$. The conclusion is of the form $\Gamma[A \otimes B]$. We consider two cases: (1) A or B belongs to $\mathcal{F}_2$, (2) Neither A, B belongs to $\mathcal{F}_2$. In the first case, $A \otimes B \in \mathcal{F}_2$, by definition 3.1(v). There is no other $\mathcal{F}_2$–formula, since in premise there is only one. In the second case, if $A, B \in \mathcal{F}_1$, then $A \otimes B \in \mathcal{F}_1$, by definition 3.1(iii). Hence, it is impossible to be more $\mathcal{F}_2$–formulas in the conclusion than in the premise.

We consider (r-1). Since $1 \in \mathcal{F}_1$, then there cannot be two negated formulas in the conclusion.

We consider (r-$\wedge$). Let the premise be $\Gamma[A]$. If $A \in \mathcal{F}_1$, then $A \wedge B \in \mathcal{F}_1$ or $A \wedge B \notin \mathcal{F}$. If $A \in \mathcal{F}_2$, then $A \wedge B \in \mathcal{F}_2$ or $A \wedge B \notin \mathcal{F}$.

We consider (r-$\vee$). Let the premises be $\Gamma[A]$ and $\Gamma[B]$. Then the conclusion is $\Gamma[A \vee B]$. The formula $A \vee B$ belongs to $\mathcal{F}$ iff both $A, B \in \mathcal{F}_1$ or both $A, B \in \mathcal{F}_2$.

We consider (r-$\parr$1). Let the premises be $\Gamma[B]$ and Δ, A. If $A, B \in \mathcal{F}_1$, then $A \parr B \notin \mathcal{F}$ and the conclusion is not an $\mathcal{F}$–sequent. If $A, B \in \mathcal{F}_2$, then there are only $\mathcal{F}_1$–formulas in $\Gamma[\_]$ and Δ. If one of A, B is in $\mathcal{F}_1$ and the other in $\mathcal{F}_2$, then $A \parr B \in \mathcal{F}_1$, by definition 3.1(v). So if in the conclusion were two $\mathcal{F}_2$–formulas, then one of the premises would also have two $\mathcal{F}_2$–formulas, which is impossible by assumption.

Cases for (r-$\parr$2), (r-$\parr$3), (r-$\parr$4) are similar.

We consider the assumption rules. All of them have premises of the form D, Γ and $\Delta, C^{\sim}$. Since C, D are FNL–formulas, then $D \in \mathcal{F}_1$ and $C^{\sim} \in \mathcal{F}_2$. Thus, Γ contains at most one formula from $\mathcal{F}_2$ and Δ does not contain any formula of $\mathcal{F}_2$. For every assumption rule the conclusion is built from the formulas of Γ and Δ, so the conclusion can have at most one $\mathcal{F}_2$–formula.

Now, assume that some $\mathcal{F}$–sequent with two formulas from $\mathcal{F}_2$ is provable in $\mathbf{S}_\Phi$. Since it is provable, it has an $\mathcal{F}$–proof. But none of the axioms and none of the conclusions of the rules has two formulas form $\mathcal{F}_2$, unless they are not $\mathcal{F}$–sequents. Hence, there is no $\mathcal{F}$–proof and the sequent is unprovable. □

One notices that the lemma above does not hold if we admit $\perp$ and $\top$ in FNL. In such a case, $\mathcal{F}_1 \cap \mathcal{F}_2 \neq \emptyset$. We see, that $\perp, (A, B)$ is an axiom for all formulas A, B, even those belonging to $\mathcal{F}_2$.

THEOREM 3.3. *Let $\Gamma[A^\sim]$ be an $\mathcal{F}$–sequent where $A \in \mathcal{F}_1$ and $\Gamma[A^\sim] \sim \Delta, A^\sim$ for some Δ. Then $\vdash_{\mathbf{S}_\Phi} \Gamma[A^\sim]$, if and only if $\Phi \vdash_{FNL} \Delta \Rightarrow A$.*

PROOF: The if-part immediately follows from **??** and the fact that CyNBL is an extension of FNL. We prove only the if-part.

Let $\Theta[C^\sim]$ be an $\mathcal{F}$–sequent provable in $\mathbf{S}_\Phi$ and $\Theta[\_] \sim (\Psi, \_)$. We show the construction of the proof in FNL. We proceed by the outer induction on the number of connectives of C and the inner induction on the proof of $\Theta[C^\sim]$. Notice that $\Theta[\_]$ has only $\mathcal{F}_1$–formulas.

1° Let $C = p$. We run the inner induction. If $\Theta[p^\sim]$ is an axiom, then $\Theta[p^\sim] = (p, p^\sim)$ or $\Theta[p^\sim] = (p^\sim, p)$. In both cases, $(\Psi, C^\sim) = (p, p^\sim)$ and $p \Rightarrow p$ is an axiom in FNL.

Now we assume $\Theta[p^\sim]$ is the conclusion of a rule. We observe $p^\sim$ cannot be the active formula of a rule.

We consider (r-$\otimes$). The premise is $\Gamma[(A, B)][p^\sim]$. Then, by proposition 2.3, $\vdash_{\mathbf{S}_\Phi} \Delta[(A, B)], p^\sim$ for $(\Psi, p^\sim) = (\Delta[A \otimes B], p^\sim)$. By the inner induction hypothesis, $\Delta[(A, B)] \Rightarrow p$ is provable in FNL from Φ. We apply ($\otimes \Rightarrow$) and obtain $\Delta[A \otimes B] \Rightarrow p$.

The cases for (r-$\vee$), (r-$\wedge$) and (r-1) are analogous.

We consider (r-$\parr$1). We recall that $A \parr B = A^\sim \multimap B = A \multimapinv B^\sim$. We consider the following cases:

$$\frac{\Gamma[B][p^\sim] \qquad \Delta, A^\sim}{\Gamma[(\Delta, A \multimap B)][p^\sim]} \qquad \frac{\Gamma[B^\sim] \qquad \Delta[p^\sim], A}{\Gamma[(\Delta[p^\sim], A \multimapinv B)]}$$

$$\frac{\Gamma[B^\sim][p^\sim] \qquad \Delta, A}{\Gamma[(\Delta, A \multimapinv B)][p^\sim]} \qquad \frac{\Gamma[B] \qquad \Delta[p^\sim], A^\sim}{\Gamma[(\Delta[p^\sim], A \multimap B)]}$$

In the first case we have $\Gamma[B][p^{\sim}] \sim \Gamma'[B], p^{\sim}$. By the inner induction hypothesis, $\Phi \vdash_{\mathrm{FNL}} \Gamma'[B] \Rightarrow p$ and $\Phi \vdash_{\mathrm{FNL}} \Delta \Rightarrow A$. We apply $(\multimap\Rightarrow)$ and obtain $\Phi \vdash_{\mathrm{FNL}} \Gamma'[(\Delta, A \multimap B)] \Rightarrow p$. Also, $(\Psi, p^{\sim}) = (\Gamma'[(\Delta, A \multimap B)], p^{\sim})$ by corollary 2.2.

In the second case we know that $\Gamma[B^{\sim}]$ reduces to $\Gamma', B^{\sim}$ and, by the inner induction hypothesis, $\Phi \vdash_{\mathrm{FNL}} \Gamma' \Rightarrow B$. Also, $\Delta[p^{\sim}], A$ reduces to $\Delta'[A], p^{\sim}$. So, $\Phi \vdash_{\mathrm{FNL}} \Delta'[A] \Rightarrow p$. We apply $(\multimapinv\Rightarrow)$ and obtain $\Phi \vdash_{\mathrm{FNL}} \Delta'[(A \multimapinv B, \Gamma')] \Rightarrow p$. One easily checks that $\Gamma[(\Delta[p^{\sim}], A \multimapinv B)] \sim (\Delta'[(A \multimapinv B, \Gamma')], p^{\sim})$.

The last two cases have premises with two negated FNL–formulas. By lemma 3.2, it is impossible. The cases for (r-$\parr$2), (r-$\parr$3) and (r-$\parr$4) are similar.

We consider (r-assm1). We have two possibilities.

$$(1)\ \frac{F, \Gamma[p^{\sim}] \qquad \Delta, E^{\sim}}{\Delta, \Gamma[p^{\sim}]},$$

where $(\Delta, \Gamma[p^{\sim}]) = \Theta[p^{\sim}]$ and $(\Delta, \Gamma[p^{\sim}]) \sim \Gamma'[\Delta]$ and $\Gamma'[\Delta] = \Psi$.

By the inner induction hypothesis, $\Phi \vdash_{\mathrm{FNL}} \Delta \Rightarrow E$ and $\Phi \vdash_{\mathrm{FNL}} \Gamma'[F] \Rightarrow p$. Since $E \Rightarrow F \in \Phi$, then $\Phi \vdash_{\mathrm{FNL}} E \Rightarrow F$. We apply (cut) to $\Gamma'[F] \Rightarrow p$ and $E \Rightarrow F$ and obtain $\Phi \vdash_{\mathrm{FNL}} \Gamma'[E] \Rightarrow p$. We apply (cut) to this and $\Delta \Rightarrow E$ and obtain $\Phi \vdash_{\mathrm{FNL}} \Gamma'[\Delta] \Rightarrow p$.

$$(2)\ \frac{D, \Gamma \qquad C^{\sim}, \Delta[p^{\sim}]}{\Delta[p^{\sim}], \Gamma},$$

where $(\Delta[p^{\sim}], \Gamma) = \Theta[p^{\sim}]$ and $\Delta'[\Gamma] = \Psi$. This case is impossible, since, by lemma 3.2, the second premise is unprovable.

The cases for other assumption rules are similar.

2° Let $C = 1$. We run the inner induction. $\Theta[0]$ may be an axiom (a-0). So $\Theta[\_] = \_$ and $\Psi = \epsilon$. Hence, $(\Psi, C^{\sim}) = 1^{\sim}$ and $\Phi \vdash_{\mathrm{FNL}} \epsilon \Rightarrow 1$ by $(\Rightarrow 1)$.

We assume $\Theta[0]$ is not an axiom. Then, 0 is not the active formula of any rule. We proceed as above.

3° Let $C = A \otimes B$. Then $C^{\sim} = B^{\sim} \parr A^{\sim}$. We run the inner induction. We know that $\Theta[C^{\sim}]$ cannot be an axiom. So it has to be the conclusion of a rule. If $C^{\sim}$ is not the active formula of a rule, we proceed as above. We assume $C^{\sim}$ is the active formula, so we have the following possibilities:

$$(1)\,\frac{\Gamma[A^{\sim}] \qquad \Delta, B^{\sim}}{\Gamma[(\Delta, B^{\sim} \parr A^{\sim})]} \qquad (2)\,\frac{\Gamma[B^{\sim}] \qquad A^{\sim}, \Delta}{\Gamma[(B^{\sim} \parr A^{\sim}, \Delta)]}$$
$$(3)\,\frac{B^{\sim}, \Gamma \qquad A^{\sim}, \Delta}{B^{\sim} \parr A^{\sim}, (\Delta, \Gamma)} \qquad (4)\,\frac{\Gamma, B^{\sim} \qquad \Delta, A^{\sim}}{(\Delta, \Gamma), B^{\sim} \parr A^{\sim}}$$

In (1) we have $\Gamma[\_] \sim (\Gamma', \_)$, so, $\Gamma[(\Delta, B^{\sim} \parr A^{\sim})] \sim ((\Gamma', \Delta), B^{\sim} \parr A^{\sim})$. By the inner induction hypothesis, $\Phi \vdash_{\mathrm{FNL}} \Gamma' \Rightarrow A$ and $\Phi \vdash_{\mathrm{FNL}} \Delta \Rightarrow B$. We apply $(\Rightarrow \otimes)$ and obtain $\Phi \vdash_{\mathrm{FNL}} \Gamma', \Delta \Rightarrow A \otimes B$. In (2), (3) and (4) we proceed similarly.

4° Let $C = A \multimap B$. Then $C = A^{\sim} \parr B$ and $C^{\sim} = B^{\sim} \otimes A$. We run the inner induction. $\Theta[C^{\sim}]$ cannot be an axiom. Hence it has to be the conclusion of a rule. If $C^{\sim}$ is not the active formula of a rule, we proceed as above. We assume $C^{\sim}$ is the active formula, so we have only one possibility:

$$\frac{\Gamma[(B^{\sim}, A)]}{\Gamma[B^{\sim} \otimes A]}$$

Then $\Gamma[\_] \sim (\Gamma', \_)$ and $\Gamma[(B^{\sim}, A)] \sim ((A, \Gamma'), B^{\sim})$. By the induction hypothesis, $\Phi \vdash_{\mathrm{FNL}} A, \Gamma' \Rightarrow B$. We apply $(\Rightarrow\multimap)$ and obtain $\Phi \vdash_{\mathrm{FNL}} \Gamma' \Rightarrow A \multimap B$.

The case when $C = A \multimapinv B$ is analogous.

5° Let $C = A \vee B$, then $C^{\sim} = A^{\sim} \wedge B^{\sim}$. We run the inner induction. $\Theta[C^{\sim}]$ cannot be an axiom. Hence it has to be the conclusion of a rule. If $C^{\sim}$ is not the active formula of a rule, we proceed as above. We assume $C^{\sim}$ is the active formula, so we have two possibilities:

$$\frac{\Theta[A^{\sim}]}{\Theta[A^{\sim} \wedge B^{\sim}]} \qquad \frac{\Theta[B^{\sim}]}{\Theta[A^{\sim} \wedge B^{\sim}]}$$

In both cases we apply the inner induction hypothesis to the premise and then we apply $(\Rightarrow \vee)$.

6° Let $C = A \wedge B$, then $C^{\sim} = A^{\sim} \vee B^{\sim}$. We run the inner induction. $\Theta[C^{\sim}]$ cannot be an axiom. Hence it has to be the conclusion of a rule. If $C^{\sim}$ is not the active formula of a rule, we proceed as above. We assume $C^{\sim}$ is the active formula, so we have one case:

$$\frac{\Theta[A^{\sim}] \qquad \Theta[B^{\sim}]}{\Theta[A^{\sim} \vee B^{\sim}]}$$

We apply the inner induction hypothesis to both premises and then we apply $(\Rightarrow \wedge)$. $\square$

Corollary 3.4. CyNBL is a strongly conservative extension of FNL.

4. Application to similar logics

The results of this paper may be adapted to other logics similar to CyNBL. In this section we provide sequent systems for these logics and some remarks about the results of this paper.

4.1. Cyclic Multiplicative–Additive Linear Logic

Cyclic Multiplicative-Additive Linear Logic (CyMALL) serves as the associative counterpart to CyNBL. Key distinctions in the sequent systems include the use of finite sequences of formulas rather than bunches and the absence of the (r-shift) rule.

CyMALL employs the same formulas as CyNBL, and metalanguage negation is defined in a similar manner. CyMALL–sequents consist of nonempty finite sequences of CyMALL–formulas. Notably, an empty sequence, denoted by ϵ, is not considered a sequent.

The axioms of CyMALL are:

(a-id) $\dfrac{}{p, p^{\sim}}$ (a-0) $\dfrac{}{0}$

(a-$\bot$) $\dfrac{}{\Gamma, \bot, \Delta}$

The introduction rules are:

$$\text{(r-}\otimes\text{)}\ \frac{\Gamma, A, B, \Delta}{\Gamma, A \otimes B, \Delta} \qquad \text{(r-1)}\ \frac{\Gamma, \Delta}{\Gamma, 1, \Delta}$$

$$\text{(r-}⅋1\text{)}\ \frac{\Gamma_1, B, \Gamma_2 \qquad \Delta, A}{\Gamma_1, \Delta, A ⅋ B, \Gamma_2} \qquad \text{(r-}⅋2\text{)}\ \frac{\Gamma_1, A, \Gamma_2 \qquad B, \Delta}{\Gamma_1, A ⅋ B, \Delta, \Gamma_2}$$

$$\text{(r-}\wedge\text{)}\ \frac{\Gamma, A, \Delta}{\Gamma, A \wedge B, \Delta} \quad \frac{\Gamma, A, \Delta}{\Gamma, B \wedge A, \Delta} \qquad \text{(r-}\vee\text{)}\ \frac{\Gamma, A, \Delta \qquad \Gamma, B, \Delta}{\Gamma, A \vee B, \Delta}$$

The structural rule and the cut rule are:

$$\text{(r-cyc)}\ \frac{\Gamma, \Delta}{\Delta, \Gamma} \qquad \text{(r-cut)}\ \frac{\Gamma_1, A, \Gamma_2 \qquad \Delta, A^{\sim}}{\Gamma_1, \Delta, \Gamma_2}$$

Let Φ be a set of sequents of the form $C, D^{\sim}$. We define the system $\mathbf{S}_{\Phi}$. The system $\mathbf{S}_{\Phi}$ has all axioms and introduction rules of CyMALL. We add the following axioms:

(a-id2) $p^{\sim}, p$

For every $(C, D^{\sim}) \in \Phi$ we add the assumption rules:

$$\text{(r-assm1)}\ \frac{D, \Gamma \qquad \Delta, C^{\sim}}{\Delta, \Gamma} \qquad \text{(r-assm2)}\ \frac{D, \Gamma \qquad \Delta, C^{\sim}}{\Gamma, \Delta}$$

As one may notice, the system for associative version is much less complex. The proofs are then also simpler. For example, the analogue for lemma 2.1 is the following:

LEMMA. *Let Γ_1, A, Γ_2 be an CyMALL–sequent. Then, there exists unique Γ such that $\Gamma_1, A, \Gamma_2 \sim \Gamma, A$.*

Two sequences of formulas are related by $\sim$ if one can be obtained from the other through finitely many applications of (r-cyc). This signifies that one sequence is a cyclic permutation of the other. It's worth noting that this can always be achieved in a single application of (r-cyc). Furthermore, the counterpart of (r-shift) in the associative system is a trivial rule.

We consider CyMALL as an extension of FL, i.e. associative version of FNL. The FL–formulas are the same as for FNL. Instead of bunches we use finite sequences of formulas. The axioms and rules of FL are as follows:

$$(\mathrm{id}) \quad A \Rightarrow A$$

$$(\mathrm{cut}) \ \frac{\Gamma \Rightarrow A \qquad \Delta_1, A, \Delta_2 \Rightarrow C}{\Delta_1, \Gamma, \Delta_2 \Rightarrow C}$$

$$(\otimes \Rightarrow) \ \frac{\Gamma, A, B, \Delta \Rightarrow C}{\Gamma, A \otimes B, \Delta \Rightarrow C} \qquad (\Rightarrow \otimes) \ \frac{\Gamma \Rightarrow A \qquad \Delta \Rightarrow B}{\Gamma, \Delta \Rightarrow A \otimes B}$$

$$(\multimap\Rightarrow) \ \frac{\Gamma, B, \Delta \Rightarrow C \qquad \Theta \Rightarrow A}{\Gamma, \Theta, A \multimap B, \Delta \Rightarrow C} \qquad (\Rightarrow\multimap) \ \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \multimap B}$$

$$(\multimapinv\Rightarrow) \ \frac{\Gamma, A, \Delta \Rightarrow C \qquad \Theta \Rightarrow B}{\Gamma, A \multimapinv B, \Theta, \Delta \Rightarrow C} \qquad (\Rightarrow\multimapinv) \ \frac{\Gamma, B \Rightarrow A}{\Gamma \Rightarrow A \multimapinv B}$$

$$(1 \Rightarrow) \ \frac{\Gamma, \Delta \Rightarrow C}{\Gamma, 1, \Delta \Rightarrow C} \qquad (\Rightarrow 1) \ \epsilon \Rightarrow 1$$

$$(\vee \Rightarrow) \ \frac{\Gamma, A, \Delta \Rightarrow C \qquad \Gamma, B, \Delta \Rightarrow C}{\Gamma, A \vee B, \Delta \Rightarrow C} \ (\Rightarrow \vee) \ \frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_1 \vee A_2}(i = 1, 2)$$

$$(\wedge \Rightarrow) \ \frac{\Gamma, A_i, \Delta \Rightarrow C}{\Gamma, A_1 \wedge A_2, \Delta \Rightarrow C}(i = 1, 2) \quad (\Rightarrow \wedge) \ \frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}$$

One may easily adjust the proofs of all the results in this paper, since the rules are similar.

4.2. Logics without multiplicative constants

We can explore variations of CyNBL and CyMALL by removing the multiplicative constants 1 and 0. These versions are denoted as CyNBL$^+$ and CyMALL$^+$. We eliminate all rules and axioms involving 1 and 0 and adjust the definitions of sequents.

For CyNBL$^+$, we define bunches as the element of the free groupoid generated by the set of all CyNBL$^+$–formulas. A CyNBL$^+$–sequent is every bunch which has at least two formulas.

CyNBL$^+$ is not an extension of FNL, but of FNL$^+$. FNL$^+$ is derived from FNL by removing constant 1, all axioms and rules associated with 1, and imposing a restriction on sequents to have a nonempty antecedent (known as the Lambek restriction).

Since CyMALL$^+$ is an associative variant of CyNBL$^+$, we define a CyMALL$^+$–sequent as a sequence of CyMALL$^+$–formulas, consising of at least two formulas. All other modifications are similar to those for CyNBL$^+$.

All the results proved in this paper remain true, since we do not use 1 in any important way.

References

[1] V. M. Abrusci, *Phase semantics and sequent calculus for pure noncommutative classical linear propositional logic*, **The Journal of Symbolic Logic**, vol. 56(4) (1991), pp. 1403–1451, DOI: https://doi.org/10.2307/2275485.

[2] V. M. Abrusci, *Classical Conservative Extensions of Lambek Calculus*, **Studia Logica**, vol. 71(3) (2002), pp. 277–314, DOI: https://doi.org/10.1023/A:1020560613199.

[3] W. Buszkowski, *Lambek calculus with nonlogical axioms*, CSLI Lecture Notes, [in:] C. Casadio, P. J. Scott, R. A. G. Seely (eds.), **Language and Grammar. Studies in Mathematical Linguistics and Natural Language**, CSLI Publications, Stanford, CA (2005), pp. 77–93.

[4] W. Buszkowski, *On classical nonassociative Lambek calculus*, [in:] M. Amblard, P. de Groote, S. Pogodalla, C. Retoré (eds.), **Logical Aspects of Computational Linguistics**, vol. 10054 of Lecture Notes in Computer Science, Springer, Berlin–Heidelberg (2016), pp. 68–84, DOI: https://doi.org/10.1007/978-3-662-53826-5_5.

[5] W. Buszkowski, *Involutive nonassociative Lambek calculus: Sequent systems and complexity*, **Bulletin of the Section of Logic**, vol. 46(1/2) (2017), DOI: https://doi.org/10.18778/0138-0680.46.1.2.07.

[6] K. Chvalovskỳ, *Undecidability of consequence relation in full non-associative Lambek calculus*, **The Journal of Symbolic Logic**, (2015), pp. 567–586, DOI: https://doi.org/10.1017/jsl.2014.39.

[7] J.-Y. Girard, *Linear logic*, **Theoretical Computer Science**, vol. 50(1) (1987), pp. 1–101, DOI: https://doi.org/10.1016/0304-3975(87)90045-4.

[8] J. Lambek, *Cut elimination for classical bilinear logic*, **Fundamenta Informaticae**, vol. 22(1–2) (1995), pp. 53–67, DOI: https://doi.org/10.3233/FI-1995-22123.

[9] Z. Lin, *Modal nonassociative Lambek calculus with assumptions: complexity and context-freeness*, [in:] **Language and Automata Theory and Applications: 4th International Conference, LATA 2010, Trier, Germany, May 24-28, 2010. Proceedings 4**, vol. 6031 of

Lecture Notes in Computer Science, Springer (2010), pp. 414–425, DOI: https://doi.org/10.1007/978-3-642-13089-2_35.

[10] P. Płaczek, *One-Sided Sequent Systems for Nonassociative Bilinear Logic: Cut Elimination and Complexity*, **Bulletin of the Section of Logic**, vol. 50(1) (2020), pp. 55–80, DOI: https://doi.org/10.18778/0138-0680.2020.25.

[11] P. Płaczek, **Extensions of Lambek calculi: Sequent systems, conservativeness and computational complexity**, Ph.D. thesis, Adam Mickiewicz University, Poznań (2021).

[12] D. N. Yetter, *Quantales and (Noncommutative) Linear Logic*, **The Journal of Symbolic Logic**, vol. 55(1) (1990), pp. 41–64, DOI: https://doi.org/10.2307/2274953.

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