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ON PRE-HILBERT AND POSITIVE IMPLICATIVE PRE-HILBERT ALGEBRAS

Abstract

In the paper, pre-Hilbert algebras are defined as a generalization of Hilbert algebras (namely, a Hilbert algebra is just a pre-Hilbert algebra satisfying the property of antisymmetry). Pre-Hilbert algebras have been inspired by Henkin's Positive Implicative Logic. Their properties and characterizations are investigated. Some important results and examples are given. Moreover, positive implicative pre-Hilbert algebras are introduced and studied, their connections with some algebras of logic are presented. The hierarchies existing between the classes of algebras considered here are shown.

Keywords: Hilbert algebra, pre-Hilbert algebra, BCK-algebra, BCC-algebra, BE-algebra, positive implicativity.

1. Introduction

L. Henkin [5] introduced the notion of "implicative model", as a model of positive implicative propositional calculus. In 1960, A. Monteiro [14] has given the name "Hilbert algebras" to the dual algebras of Henkin's implicative models. In 1966, K. Iséki [7] introduced a new notion called a BCK algebra. It is an algebraic formulation of the BCK-propositional calculus system of C. A. Meredith [13], and generalize the concept of implicative algebras (see [1]). To solve some problems on BCK algebras, Y. Komori [12] introduced BCC algebras. These algebras (also called BIK$^+$-algebras) are an algebraic model of BIK$^+$-logic. In [10], as a generalization of BCK

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algebras, H. S. Kim and Y. H. Kim defined BE algebras. In 2008, A. Walendziak [15] defined commutative BE algebras and proved that they are BCK algebras. Later on, in 2010, D. Buşneag and S. Rudeanu [3] introduced the notion of pre-BCK algebra. A BCK algebra is just a pre-BCK with the antisymmetry. In 2016, A. Iorgulescu [6] introduced new generalizations of BCK and Hilbert algebras (RML, aBE, pi-BE, pimpl-RML algebras and many others).

In the paper, we define pre-Hilbert algebras in such a way that a Hilbert algebra is just a pre-Hilbert algebra satisfying the property of antisymmetry. It is a solution to Open problem 6.30 of [6]. We give basic properties and examples of pre-Hilbert algebras. We also give some characterizations of these algebras. Moreover, we introduce and investigate positive implicative pre-Hilbert algebras and present their connections with some algebras of logic. We show the hierarchies existing between all classes of algebras considered here.

The motivation of this study consists algebraic and logical arguments. Pre-Hilbert algebras introduced and investigated in the paper belong to a wide class of algebras of logic, they are a natural generalization of well-known Hilbert algebras. The definition of a pre-Hilbert algebra presented here is inspired by Henkin's Positive Implicative Logic [5].

2. Preliminaries

Let $\mathcal{A} = (A, \to, 1)$ be an algebra of type $(2, 0)$. We define the binary relation $\leq$ by: for all $x, y \in A$,

$$x \leq y \Longleftrightarrow x \to y = 1.$$

We consider the following list of properties ([6]) that can be satisfied by $\mathcal{A}$:

(An) (Antisymmetry) $x \to y = 1 = y \to x \Longrightarrow x = y$,

(An') (Antisymmetry) $(x \leq y \text{ and } y \leq x) \Longrightarrow x = y$,

(B) $(y \to z) \to [(x \to y) \to (x \to z)] = 1$,

(B') $y \to z \leq (x \to y) \to (x \to z)$,

(BB) $(y \to z) \to [(z \to x) \to (y \to x)] = 1$, (BB') $y \to z \leq (z \to x) \to (y \to x)$,

(C) $[x \to (y \to z)] \to [y \to (x \to z)] = 1$,

(C') $x \to (y \to z) \leq y \to (x \to z)$,

(D) $y \to ((y \to x) \to x) = 1$,

(D') $y \leq (y \to x) \to x$,

(Ex) (Exchange) $x \to (y \to z) = y \to (x \to z)$,

(K) $x \to (y \to x) = 1$,

(K') $x \leq y \to x$,

(L) (Last element) $x \to 1 = 1$,

(L') (Last element) $x \leq 1$,

(M) $1 \to x = x$,

(Re) (Reflexivity) $x \to x = 1$,

(Re') (Reflexivity) $x \leq x$,

(Tr) (Transitivity) $x \to y = 1 = y \to z \Longrightarrow x \to z = 1$,

(Tr') (Transitivity) $(x \leq y$ and $y \leq z) \Longrightarrow x \leq z$,

(*) $y \to z = 1 \Longrightarrow (x \to y) \to (x \to z) = 1$,

(*') $y \leq z \Longrightarrow x \to y \leq x \to z$,

(**) $y \to z = 1 \Longrightarrow (z \to x) \to (y \to x) = 1$,

(**') $y \leq z \Longrightarrow z \to x \leq y \to x$.

Remark 2.1. The properties in the list are the most important properties satisfied by a BCK algebra.

Lemma 2.2 ([6]). *Let* $\mathcal{A} = (A, \to, 1)$ *be an algebra of type* $(2, 0)$. *Then the following hold:*

(i) (M) + (K) $\Longrightarrow$(L);

(ii) (M) + (B) $\Longrightarrow$(*), (**);

(iii) (M) + (*) $\Longrightarrow$ (Tr);

(iv) (M) + (**) $\Longrightarrow$ (Tr);

(v) (M) + (BB) $\Longrightarrow$ (Re), (B), (C);

(vi) (C) + (An) $\Longrightarrow$ (Ex);

(vii) (M) + (L) + (**) $\Longrightarrow$ (K);

(viii) (M) + (B) + (C) $\Longrightarrow$ (BB).

Proof: (i)–(vii) follow from Proposition 2.1 and Theorem 2.7 of [6].

(viii) Let $x, y, z \in A$. By (B) and (C), $1 = (z \to x) \to ((y \to z) \to (y \to x)) \leq (y \to z) \to ((z \to x) \to (y \to x))$. From (M) we conclude that $(y \to z) \to ((z \to x) \to (y \to x)) = 1$, that is, (BB) holds in $\mathcal{A}$. $\square$

Following Iorgulescu [6], we say that $(A, \to, 1)$ is an *RML algebra* if it verifies the axioms (Re), (M), (L). We introduce now the following definition.

Definition 2.3. ([6]) Let $\mathcal{A} = (A, \to, 1)$ be an RML algebra. The algebra $\mathcal{A}$ is said to be:

1. an *aRML algebra* if it verifies (An),
2. a *pre-BCC algebra* if it verifies (B),
3. a *pre-BBBCC algebra* if it verifies (BB),
4. a *BCC algebra* if it verifies (B), (An), that is, it is a pre-BCC algebra with (An),
5. a *BE algebra* if it verifies (Ex),
6. an *aBE algebra* if it verifies (Ex), (An), that is, it is a BE algebra with (An),

7. a *pre-BCK algebra* if it verifies (B), (Ex), that is, it is a pre-BCC algebra with (Ex) or, equivalently, it is a BE algebra with (B),

8. a *BCK algebra* if it is a pre-BCK algebra verifying (An).

Denote by **RML**, **aRML**, **pre-BCC**, **pre-BBBCC**, **BCC**, **BE**, **aBE**, **pre-BCK**, **BCK** the classes of RML, aRML, pre-BCC, pre-BBBCC, BCC, BE, aBE, pre-BCK, BCK algebras respectively. By definitions, we have
pre-BCC = **RML** + (B), **pre-BBBCC** = **RML** + (BB),
BE = **RML** + (Ex), **pre-BCK** = **pre-BCC** + (Ex) = **BE** + (B),
aRML = **RML** + (An), **BCC** = **pre-BCC** + (An),
aBE = **BE** + (An) = **aRML** + (Ex), **BCK** = **pre-BCK** + (An).

Remark 2.4. By Lemma 2.2 (v), (viii), **pre-BBBCC** = **pre-BCC** + (C). Since (C) + (An) $\Longrightarrow$ (Ex), we have **BCK** = **BCC** + (Ex) = **pre-BCC** + (Ex) + (An) = **pre-BCC** + (C) + (An) = **pre-BBBCC** + (An).

The interrelationships between the classes of algebras mentioned before are visualized in Figure 1.
It is known that $\leq$ is an order relation in BCC and BCK algebras. By definition, in RML and BE algebras, $\leq$ is a reflexive relation; in aRML and aBE algebras, $\leq$ is reflexive and antisymmetric. By Lemma 2.2 (ii)–(iv), in pre-BCC, pre-BBBCC and pre-BCK algebras, $\leq$ is reflexive and transitive (i.e., it is a pre-order relation).

3. Definition and properties of pre-Hilbert algebras

Let $\mathcal{A} = (A, \rightarrow, 1)$ be an algebra of type $(2, 0)$. Now, we consider the following properties:

(pi) $x \rightarrow (x \rightarrow y) = x \rightarrow y$,

(p-1) $x \rightarrow (y \rightarrow z) \leq (x \rightarrow y) \rightarrow (x \rightarrow z)$,

(p-2) $(x \rightarrow y) \rightarrow (x \rightarrow z) \leq x \rightarrow (y \rightarrow z)$,

(pimpl) $x \rightarrow (y \rightarrow z) = (x \rightarrow y) \rightarrow (x \rightarrow z)$.

Remark 3.1. The properties above are the most important properties satisfied by Hilbert algebras. Recall that an algebra $(A, \rightarrow, 1)$ *is called a* Hilbert

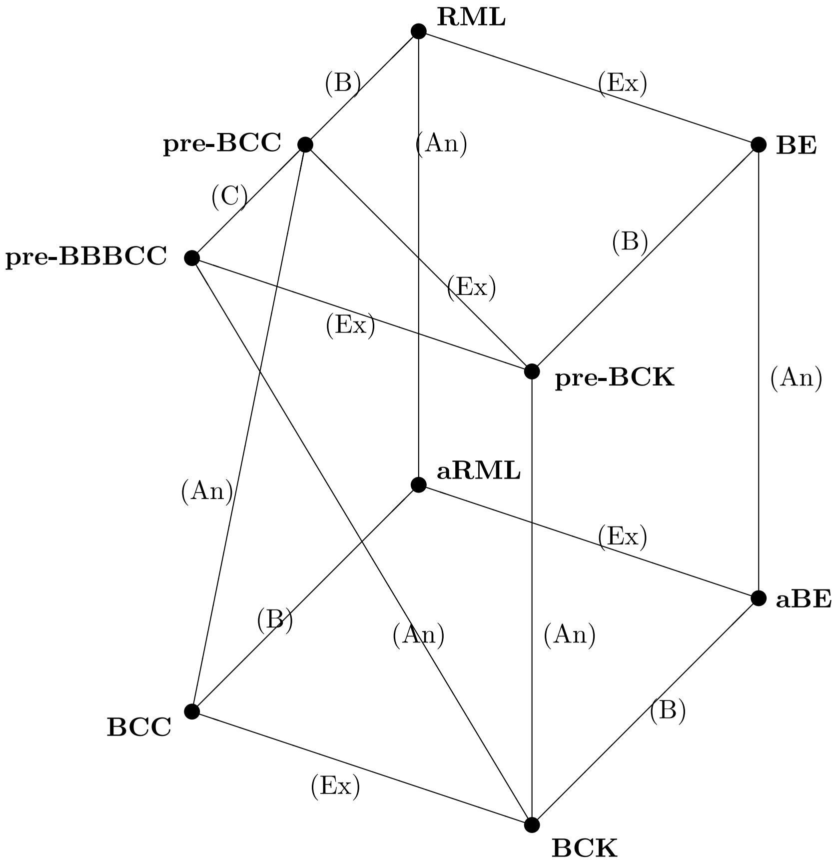

RML
(B)
(Ex)
pre-BCC
(An)
BE
(C)
(B)
pre-BBBCC
(Ex)
(Ex)
pre-BCK
(An)
aRML
(An)
(Ex)
aBE
(B)
(An)
(An)
BCC
(B)
(Ex)
BCK

Figure 1.

algebra if it verifies the axioms (An), (K), (p-1). In [4], A. Diego proved that Hilbert algebras satisfy (Re), (M), (L), (pi), (p-2), (pimpl). Moreover, he showed that the class of all Hilbert algebras is a variety.

PROPOSITION 3.2. Let $(A, \to, 1)$ be an algebra of type $(2, 0)$. Then the following are true:

(i) (Re) + (M) + (pimpl) $\Longrightarrow$ (pi),

(ii) (p-1) + (p-2) + (An) $\Longrightarrow$ (pimpl),

(iii) (Re) + (pi) $\Longrightarrow$ (L).

PROOF: (i) By Proposition 6.4 of [6].

(ii) Obvious.

(iii) Let $x \in A$. We have $x \to 1 \stackrel{\text{(Re)}}{=} x \to (x \to x) \stackrel{\text{(pi)}}{=} x \to x \stackrel{\text{(Re)}}{=} 1$, thus (L) holds in $(A, \to, 1)$. □

Remark 3.3. From Proposition 3.2 (i) it follows that in RML algebras, (pimpl) implies (pi). For BCK algebras, (pimpl) and (pi) are equivalent (cf. Theorem 8 of [8]).

PROPOSITION 3.4. Let $(A, \to, 1)$ be an algebra of type $(2, 0)$. Then the following are true:

(i) (M) + (K) + (p-1) $\Longrightarrow$ (Re),

(ii) (M) + (L) + (p-1) $\Longrightarrow$ (*),

(iii) (K) + (Tr) + (p-1) $\Longrightarrow$ (B),

(iv) (M) + (K) + (**) + (p-1) $\Longrightarrow$ (C),

(v) (Re) + (M) + (C) $\Longrightarrow$ (D),

(vi) (M) + (K) + (**) + (C) $\Longrightarrow$ (p-2).

PROOF: (i) Let $x \in A$. We have $1 \stackrel{\text{(K)}}{=} x \to ((x \to x) \to x) \stackrel{\text{(p-1)}}{\leq} (x \to (x \to x)) \to (x \to x) \stackrel{\text{(K)}}{=} 1 \to (x \to x) \stackrel{\text{(M)}}{=} x \to x$.

(ii) Let $x, y, z \in A$ and suppose that $y \leq z$. We obtain $1 \stackrel{\text{(L)}}{=} x \to (y \to z) \stackrel{\text{(p-1)}}{\leq} (x \to y) \to (x \to z)$. Hence, by (M), $x \to y \leq x \to z$.

(iii) Let $x, y, z \in A$. By (K) and (p-1), $y \to z \leq x \to (y \to z)$ and $x \to (y \to z) \leq (x \to y) \to (x \to z)$. Applying (Tr), we get $y \to z \leq (x \to y) \to (x \to z)$.

(iv) Let $x, y, z \in A$. From (p-1) we obtain

$$x \to (y \to z) \leq (x \to y) \to (x \to z). \tag{3.1}$$

By (K), $y \leq x \to y$ and hence, by (**),

$$(x \to y) \to (x \to z) \leq y \to (x \to z). \tag{3.2}$$

From (M) and (**) it follows that (Tr) holds in $\mathcal{A}$. Using (Tr), from (3.1) and (3.2) we have $x \to (y \to z) \leq y \to (x \to z)$.

(v) We have $1 \overset{\text{(Re)}}{=} (y \to x) \to (y \to x) \overset{\text{(C)}}{\leq} y \to ((y \to x) \to x)$. Applying (M), we get (D).

(vi) Conditions (K) and (**) imply (3.2), see the proof of (iv). By (C), $y \to (x \to z) \leq x \to (y \to z)$. Then $(x \to y) \to (x \to z) \leq x \to (y \to z)$, by (Tr). $\square$

We introduce the following notion:

Definition 3.5. A *pre-Hilbert algebra* is an algebra $(A, \to, 1)$ of type $(2, 0)$ satisfying (M), (K) and (p-1).

Let us denote by **pre-H** and **H** the classes of pre-Hilbert and Hilbert algebras, respectively.

Remark 3.6. Since (An) + (K) + (p-1) imply (M) (see [4]), a Hilbert algebra is in fact a pre-Hilbert algebra verifying (An), that is, **H** = **pre-H** + (An).

Remark 3.7. A motivation for the definition of pre-Hilbert algebra is Positive (Implicative) Logic given by L. Henkin [5]. This logic is the part of intuitionistic logic corresponding to formulas in which implication occurs as the only connective. The propositional calculus of Henkin system of positive logic is specified by the following two axiom schemes:

(H1) $\quad \alpha \to (\beta \to \alpha)$,
(H2) $\quad (\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$.

and the modus ponens inference rule. Conditions (K) and (p-1) of Definition 3.5 are inspired by axioms (H1) and (H2), respectively. Moreover,

(M) is inspired by the modus ponens (indeed, from (M) it follows that if $x = 1$ and $x \to y = 1$, then $y = 1$*)*.

Remark 3.8. Note that Definition 3.5 is a solution to Open problem 6.30 of [6].

THEOREM 3.9. *Pre-Hilbert algebras satisfy* (Re), (M), (L), (K), (*), (**), (Tr), (B), (C), (D), (BB), (p-1), (p-2)*.*

PROOF: Let $\mathcal{A}$ be a pre-Hilbert algebra. By definition, $\mathcal{A}$ satisfies (M), (K) and (p-1). By Proposition 3.4 (i), (M) + (K) + (p-1) imply (Re); thus (Re) holds in $\mathcal{A}$. By Lemma 2.2 (i), (M) + (K) imply (L); thus (L) holds. From Proposition 3.4 (ii) we conclude that $\mathcal{A}$ satisfies (*), hence it also satisfies (Tr) by Lemma 2.2 (iii). Applying Proposition 3.4 (iii), we deduce that (B) holds in A. Then (**) also holds, see Lemma 2.2 (ii). By Proposition 3.4 (iv), (M) + (K) + (p-1) + (**) imply (C); thus (C) holds. From Proposition 3.4 (v) and (vi) it follows that (D) and (p-2) hold. By Lemma 2.2 (viii), (M) + (B) + (C) imply (BB); thus (BB) holds. $\square$

THEOREM 3.10. *Let* $\mathcal{A} = (A, \to, 1)$ *be an algebra of type* $(2, 0)$*. The following are equivalent:*

- (i) $\mathcal{A}$ *is a pre-Hilbert algebra;*
- (ii) $\mathcal{A}$ *is a pre-BCC algebra satisfying (C) and (p-1);*
- (iii) $\mathcal{A}$ *satisfies (M), (L), (B), (C) and (p-1);*
- (iv) $\mathcal{A}$ *satisfies (M), (L), (BB) and (p-1);*
- (v) $\mathcal{A}$ *is a pre-BBBCC algebra satisfying (p-1).*

PROOF: (i) $\Longrightarrow$ (ii) Follows from Theorem 3.9.

(ii) $\Longrightarrow$ (iii) By definition.

(iii) $\Longrightarrow$ (iv) By Lemma 2.2 (viii).

(iv) $\Longrightarrow$ (v) Since (M) + (BB) imply (Re), we conclude that $\mathcal{A}$ is a pre-BBBCC algebra. Then (v) holds.

(v) $\Longrightarrow$ (i) Pre-BBBCC algebras satisfy (M), (L), (B), hence also (**) and (K) (by Lemma 2.2 (ii), (vii)). Then $\mathcal{A}$ satisfies (M), (K) and (p-1). Thus $\mathcal{A}$ is a pre-Hilbert algebra. $\square$

Example 3.11. ([6], 9.24) Let $A = \{a, b, c, d, 1\}$ and $\to$ be given by the following table:

$\to$	a	b	c	d	1
a	1	a	c	c	1
b	1	1	d	c	1
c	a	b	1	1	1
d	a	b	1	1	1
1	a	b	c	d	1

Then $(A, \to, 1)$ verifies (Re), (M), (L), (BB). It does not verify (An) for $x = c$, $y = d$; (Ex) for $x = a, y = b, z = c$; (pi) for $x = a, y = b$ and (p-1) for $x = y = a, z = b$. Therefore, $(A, \to, 1)$ is a pre-BBBCC algebra without (An), (Ex) and (p-1).

Remark 3.12. Pre-Hilbert algebras do not have to satisfy (An), (Ex), (pi); see example below.

Example 3.13. Consider the set $A = \{a, b, c, d, 1\}$ and the operation $\to$ given by the following table:

$\to$	a	b	c	d	1
a	1	c	b	d	1
b	a	1	1	d	1
c	a	1	1	d	1
d	a	c	c	1	1
1	a	b	c	d	1

We can observe that the properties (M), (K), (p-1) (hence (Re), (L), (B), (BB), (C), (D), (*), (**), (Tr), (p-2)) are satisfied. Then, $(A, \to, 1)$ is a pre-Hilbert algebra. It does not satisfy (An) for $(x, y) = (b, c)$; (Ex) and (pimpl) for $(x, y, z) = (a, d, b)$; (pi) for $(x, y) = (a, b)$.

Definition 3.14. If $\mathcal{A}$ is a pre-Hilbert algebra not satisfying (An), (Ex) and (pi), then we say that $\mathcal{A}$ *is* proper.

Remark 3.15. The algebra given in Example 3.13 is a proper pre-Hilbert algebra.

Remark 3.16. By Theorem 3.10, **pre-H** = **pre-BBBCC** + (p-1). Hence **H** = **pre-H** + (An) = **pre-BBBCC** + (An) + (p-1) = **BCK** + (p-1). From

Example 3.11 it follows that **pre-H** is a proper subclass of **pre-BBBCC**, that is, **pre-H** $\subset$ **pre-BBBCC**.

By Remark 3.16 and Figure 1, we can draw now the hierarchy between **pre-BBC** and **H**, in the next Figure 2.

PROPOSITION 3.17. Let $\mathcal{A} = (A, \to, 1)$ be a pre-Hilbert algebra. Then $\mathcal{A}$ induces a pre-order $\leq$ on A, defined by: $x \leq y \Longleftrightarrow x \to y = 1$ and 1 is the element of A satisfying the following conditions:

(L1) $x \leq 1$,

(L2) $1 \leq x \Longrightarrow x = 1$.

PROOF: Straightforward. $\square$

PROPOSITION 3.18. Let A be a non-void set of elements and $\leq$ be a pre-order relation on A and 1 be the element of A satisfying (L1) and (L2). We define the operation $\to$ by

$$x \to y = \begin{cases} 1, & \text{if} \quad x \leq y \\ y, & \text{if} \quad \text{otherwise.} \end{cases}$$

Then $\mathcal{A} = (A, \to, 1)$ is a pre-Hilbert algebra.

PROOF: It is easy to see that $\mathcal{A}$ satisfies (Re), (M), (L), (Tr) and (K). Observe that $\mathcal{A}$ also satisfies (p-1). Let $x, y, z \in A$. We shall consider three cases.

Case 1: $x \leq z$. Then $(x \to y) \to (x \to z) = (x \to y) \to 1 = 1$. Since $\mathcal{A}$ satisfies (L), we conclude that (p-1) holds for $x \leq z$.

Case 2: $x \not\leq z$ (that is, $x \leq z$ is false) and $x \leq y \to z$. In this case, we have $y \leq z$ and $x \not\leq y$. We obtain $x \to (y \to z) = 1 = y \to z = (x \to y) \to (x \to z)$.

Case 3: $x \not\leq z$ and $x \not\leq y \to z$. Then $y \not\leq z$. Therefore, $x \to (y \to z) = x \to z = z$ and $(x \to y) \to (x \to z) = (x \to y) \to z = z$, since $x \to y \not\leq z$. Thus (p-1) holds in $\mathcal{A}$. Consequently, $\mathcal{A}$ is a pre-Hilbert algebra. $\square$

In particular, we have the following

Example 3.19. Let $\mathbb{Z}$ be the set of integers and let for $x, y \in \mathbb{Z}$ the symbol $x \mid y$ means that x divides y. Then the relation $\mid$ is a pre-order on $\mathbb{Z}$ which is not an order (for example, $1 \mid -1$ and $-1 \mid 1$ but $1 \neq -1$). Moreover,

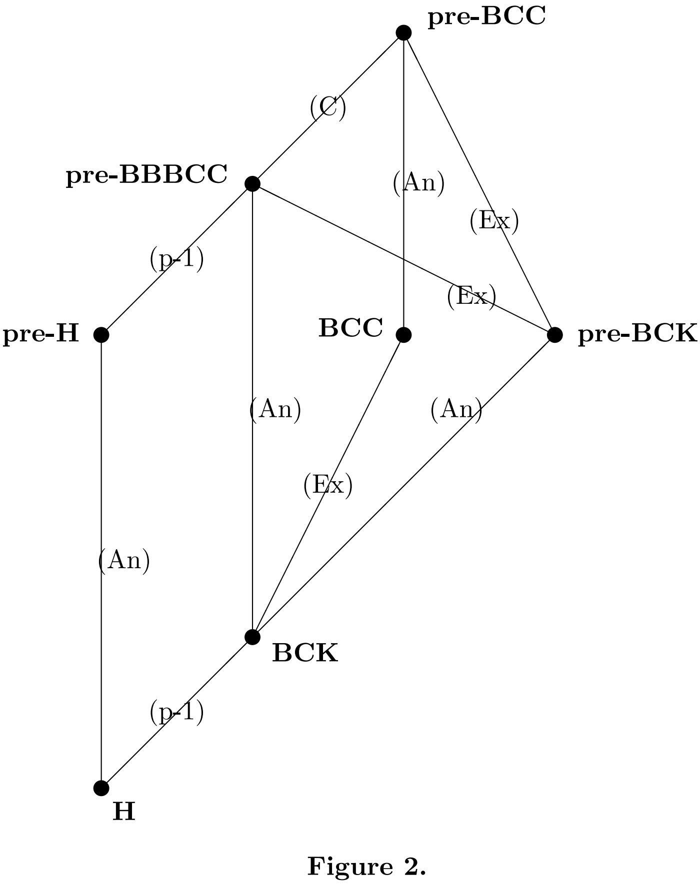

pre-BCC
(C)
pre-BBBCC
(An)
(Ex)
(p-1)
(Ex)
pre-H
BCC
pre-BCK
(An)
(An)
(Ex)
(An)
BCK
(p-1)
H

Figure 2.

$x \mid 0$ for each $x \in \mathbb{Z}$ and if $0 \mid x$, then $x = 0$. If we define the operation $\to$ by

$$x \to y = \begin{cases} 0, & \text{if} \quad x \mid y \\ y, & \text{if} \quad \text{otherwise}, \end{cases}$$

then $(\mathbb{Z}, |, 0)$ is a pre-Hilbert algebra.

Remark 3.20. The class of all pre-Hilbert algebras is a variety. Therefore, if $\mathcal{A}_1$ and $\mathcal{A}_2$ are two pre-Hilbert algebras, then the direct product $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ is also a pre-Hilbert algebra.

Let T be be any set and, for each $t \in T$, let $\mathcal{A}_t = (A_t, \to_t, 1)$ be a pre-Hilbert algebra. Suppose that $A_s \cap A_t = \{1\}$ for $s \neq t$, $s, t \in T$. Set $A = \bigcup_{t \in T} A_t$ and define the binary operation $\to$ on A via

$$x \to y = \begin{cases} x \to_t y & \text{if } x, y \in A_t;\ t \in T, \\ y & \text{if } x \in A_s,\ y \in A_t;\ s, t \in T,\ s \neq t. \end{cases}$$

It is easy to check that $\mathcal{A} = (A, \to, 1)$ is a pre-Hilbert algebra. The algebra $\mathcal{A}$ will be called the *disjont union of* $(\mathcal{A}_t)_{t \in T}$.

Proposition 3.21. Any (proper) pre-Hilbert algebra can be extended to a (proper) pre-Hilbert algebra containing one element more.

Proof: Let $\mathcal{A} = (A, \to, 1)$ be a *pre-Hilbert* algebra and let $\delta \notin A$. On the set $B = A \cup \{\delta\}$ consider the operation:

$$x \to' y = \begin{cases} x \to y & \text{if} \quad x, y \in A, \\ \delta & \text{if} \quad x \in A \text{ and } y = \delta, \\ 1 & \text{if} \quad x = \delta \text{ and } y \in B. \end{cases}$$

Obviously, $\mathcal{B} := (B, \to', 1)$ satisfies the axioms (M) and (K). Further, the axiom (p-1) is easily satisfied for all $x, y, z \in A$. Moreover, by routine calculation we can verify it in the case when at least one of x, y, z is equal to δ. Thus, by definition, $\mathcal{B}$ is a pre-Hilbert algebra. Clearly, if $\mathcal{A}$ is a proper pre-Hilbert algebra, then $\mathcal{B}$ is also a proper pre-Hilbert algebra. $\square$

4. Positive implicative pre-Hilbert algebras

Recall that any Hilbert algebra satisfies (pi) and (pimpl), but pre-Hilbert algebras do not have to satisfy these properties (see Example 3.13). From [6] we have the following definitions:

Definition 4.1 ([6]).

1. A pi-*RML algebra* is an RML algebra verifying (pi).
2. A positive implicative RML algebra, or a *pimpl-RML algebra* for short, is a RML algebra verifying (pimpl).

Remark 4.2. Note that pimpl-RML algebras are also called generalized Tarski algebras (see [11], [9], [6]).

First we give some characterizations of pi-pre-Hilbert algebras.

Theorem 4.3. *Let $\mathcal{A} = (A, \to, 1)$ be an algebra of type $(2, 0)$. The following are equivalent:*

(i) *$\mathcal{A}$ is a pi-pre-Hilbert algebra;*

(ii) *$\mathcal{A}$ satisfies (M), (K), (p-1), (pi);*

(iii) *$\mathcal{A}$ satisfies (M), (BB) and (pi);*

(iv) *$\mathcal{A}$ is a pi-pre-BBBCC algebra;*

(v) *$\mathcal{A}$ satisfies (M), (B), (C) and (pi);*

(vi) *$\mathcal{A}$ is a pi-pre-BCC algebra with (C).*

Proof: (i) $\Longrightarrow$ (ii) By definition.

(ii) $\Longrightarrow$ (iii) Follows from Theorem 3.9.

(iii) $\Longrightarrow$ (iv) By Lemma 2.2 (v) and Proposition 3.2 (iii), $\mathcal{A}$ satisfies (Re) and (L). Then $\mathcal{A}$ is a pi-pre-BBBCC algebra.

(iv) $\Longrightarrow$ (v) Follows from Lemma 2.2 (v).

(v) $\Longrightarrow$ (vi) By Lemma 2.2 (viii), $\mathcal{A}$ satisfies (BB). Applying Lemma 2.2 (v), we conclude that (Re) holds in $\mathcal{A}$. From Lemma 3.2 (iii) it follows that (L) also holds in $\mathcal{A}$. Thus (vi) is satisfied.

(vi) $\Longrightarrow$ (i) Let $\mathcal{A}$ be a pi-pre-BCC algebra with (C). Then $\mathcal{A}$ satisfies (Re), (M), (L), (B) (hence, by Lemma 2.2, (*), (**), (Tr), (K)), (pi), (C). To prove (p-1), let $x, y, z \in A$. From (B) we conclude that $y \to z \leq (x \to y) \to (x \to z)$. Using (*), we get

$$x \to (y \to z) \leq x \to ((x \to y) \to (x \to z)). \tag{4.1}$$

By (C),

$$x \to ((x \to y) \to (x \to z)) \leq (x \to y) \to (x \to (x \to z)). \tag{4.2}$$

Applying (Tr) and (pi), we obtain $x \to (y \to z) \leq (x \to y) \to (x \to (x \to z)) = (x \to y) \to (x \to z)$. Consequently, $\mathcal{A}$ is a pi-pre-Hilbert algebra. □

Example 4.4 ([6, 10.17]). Let $A = \{a, b, c, d, 1\}$ and $\to$ be defined as follows:

$\to$	a	b	c	d	1
a	1	b	b	d	1
b	a	1	1	d	1
c	a	1	1	d	1
d	a	c	c	1	1
1	a	b	c	d	1

.

Properties (M), (BB) and (pi) are satisfied, as is easy to check. From Therem 4.3 we conclude that $(A, \to, 1)$ is a pi-pre-Hilbert algebra. It does not satisfy (An) for $x = b$, $y = c$; (Ex) and (pimpl) for $x = a, y = d, z = b$.

Remark 4.5. (1) Example 3.13 shows that there exists a pre-Hilbert algebra which is not a pi-pre-Hilbert algebra. Therefore, **pi-pre-H** $\subset$ **pre-H**.
(2) From Therem 4.3 we deduce that **pi-pre-H** $=$ **pi-pre-BBBCC** $=$ **pi-pre-BCC** $+$ (C).
(3) By definitions,
pi-RML $=$ **RML** $+$ (pi),
pi-pre-BCC $=$ **pre-BCC** $+$ (pi) $=$ **pi-RML** $+$ (B),
pi-BE $=$ **BE** $+$ (pi) $=$ **pi-RML** $+$ (Ex),
pi-pre-BCK $=$ **pre-BCK** $+$ (pi) $=$ **pi-BE** $+$ (B) and
pi-pre-BCK $=$ **pi-pre-BCC** $+$ (Ex) $=$ **pi-pre-H** $+$ (Ex).

By Remark 4.5, we can draw the hierarchy between classes **RML** and **pi-pre-BCK**, in the next Figure 3.

Now we give several characterizations of positive implicative pre-Hilbert algebras. We will use the following lemma:

Lemma 4.6 ([6]). *Let $\mathcal{A} = (A, \to, 1)$ be an algebra of type $(2, 0)$ satisfy (Re), (M) and (pimpl). Then $\mathcal{A}$ satisfies (L), (BB), (hence (B), (*), (**), (Tr)), (K), (C), (p-1), (p-2), (pi).*

From Lemma 4.6 we obtain

Proposition 4.7. Let $\mathcal{A} = (A, \to, 1)$ be an algebra of type $(2, 0)$. The following are equivalent:

(i) $\mathcal{A}$ is a pimpl-pre-Hilbert algebra;

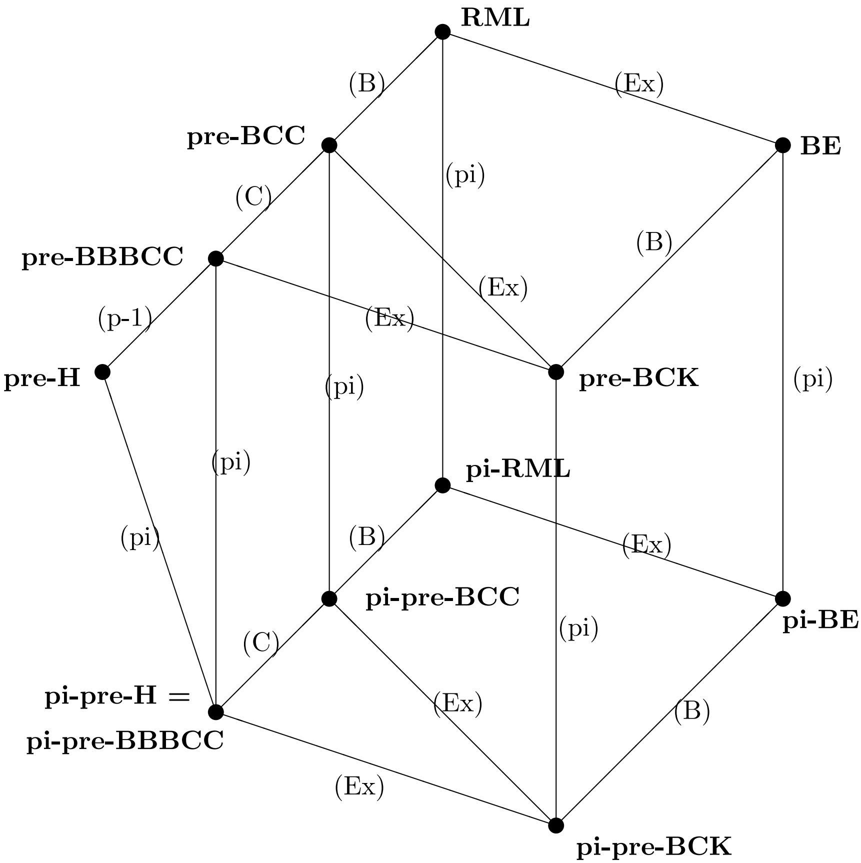

RML
(B)
(Ex)
pre-BCC
BE
(pi)
(C)
(B)
pre-BBBCC
(Ex)
(Ex)
(p-1)
pre-H
pre-BCK
(pi)
(pi)
(pi)
pi-RML
(pi)
(B)
(Ex)
pi-pre-BCC
pi-BE
(pi)
(C)
pi-pre-H =
pi-pre-BBBCC
(Ex)
(B)
(Ex)
pi-pre-BCK

Figure 3.

(ii) $\mathcal{A}$ satisfies (Re), (M), (pimpl);

(iii) $\mathcal{A}$ is a pimpl-RML algebra, that is, it is a generalized Tarski algebra;

(iv) $\mathcal{A}$ is a pimpl-pre-BCC algebra;

(v) $\mathcal{A}$ is a pimpl-pre-BBBCC algebra.

Example 4.8 ([6, 10.18]). Consider the set $A = \{a, b, c, d, 1\}$ and the operation $\to$ given by the following table:

$\to$	a	b	c	d	1
a	1	b	b	1	1
b	a	1	1	a	1
c	a	1	1	a	1
d	1	c	c	1	1
1	a	b	c	d	1

.

We can observe that the properties (Re), (M), (pimpl) (hence (L), (B), (BB), (C), (D), (*), (**), (Tr), (p-1), (p-2)) are verified. Then, $(A, \to, 1)$ is a pimpl-pre-Hilbert algebra. It does not verify (An) for $(x, y) = (b, c)$; (Ex) for $(x, y, z) = (a, d, b)$. Hence, it is not a pimpl-BE algebra.

Remark 4.9. (1) By Proposition 4.7, **pimpl-pre-H** = **pimpl-RML** = **pimpl-pre-BCC** = **pimpl-pre-BBBCC**. Since (Re) + (M) + (pimpl) imply (B), we conclude that **pimpl-BE** = **pimpl-pre-BCK**.
(2) From (1) we have **pimpl-pre-H** = **pi-pre-H** + (pimpl) = **pi-RML** + (pimpl) = **pi-pre-BCC** + (pimpl) and **pimpl-BE** = **pi-BE** + (pimpl) = **pi-pre-BCK** + (pimpl), because (Re) + (M) + (pimpl) imply (pi).
(3) Moreover, **pimpl-pre-H** + (Ex) = **pimpl-RML** + (Ex) = **pimpl-BE**.

Remark 4.10. By Remarks 6.19 and 6.19 of [6], we get **H** = **pimpl-aRML** = **pimpl-BCC** = **pimpl-BCK** = **pimpl-aBE** = **pimpl-BE** + (An).

Remark 4.11. Note that a self-distributive BE algebra (see [10]) is in fact our pimpl-BE algebra.

Example 4.12. Let $(\mathbb{Z}, |, 0)$ be the algebra given in Example 3.19. It is easy to see that $(\mathbb{Z}, |, 0)$ satisfies (Re), (M), (Ex) and (pimpl). Then, it is a pimpl-BE algebra. Since $1 \to -1 = 0 = -1 \to 1$ but $1 \neq -1$, and

$(2 \to 1) \to 2 = 1 \to 2 = 0 \neq 2$ we deduce that $(\mathbb{Z}, |, 0)$ does not satisfy (An). Therefore, it is not a Hilbert algebra.

Remark 4.13. Examples 4.4, 4.8 and 4.12 show that *the inclusions below are proper.*

pi-pre-H $\supset$ **pimpl-pre-H** $\supset$ **pimpl-BE** $\supset$ **H**.

From Remarks 4.9 and 4.10 we obtain Figure 4.

5. Summary and future work

In this paper, we introduced pre-Hilbert algebras as a generalization of well-known Hilbert algebras. We investigated basic properties of pre-Hilbert algebras and presented some examples and characterizations of these algebras. We defined and studied positive implicative pre-Hilbert algebras and obtained their connections with some other algebras of logic considered here. In particular, we proved that the class of positive implicative pre-Hilbert algebras coincides with the class of generalized Tarski algebras. Finally, we showed the interrelationships between some subclasses of the class of pi-RML algebras.

The results obtained in the paper can be a starting point for future research. We suggest the following topics:

(1) Studying pre-Hilbert algebras with the implicative property, that is, verifying the identity $(x \to y) \to x = x$.

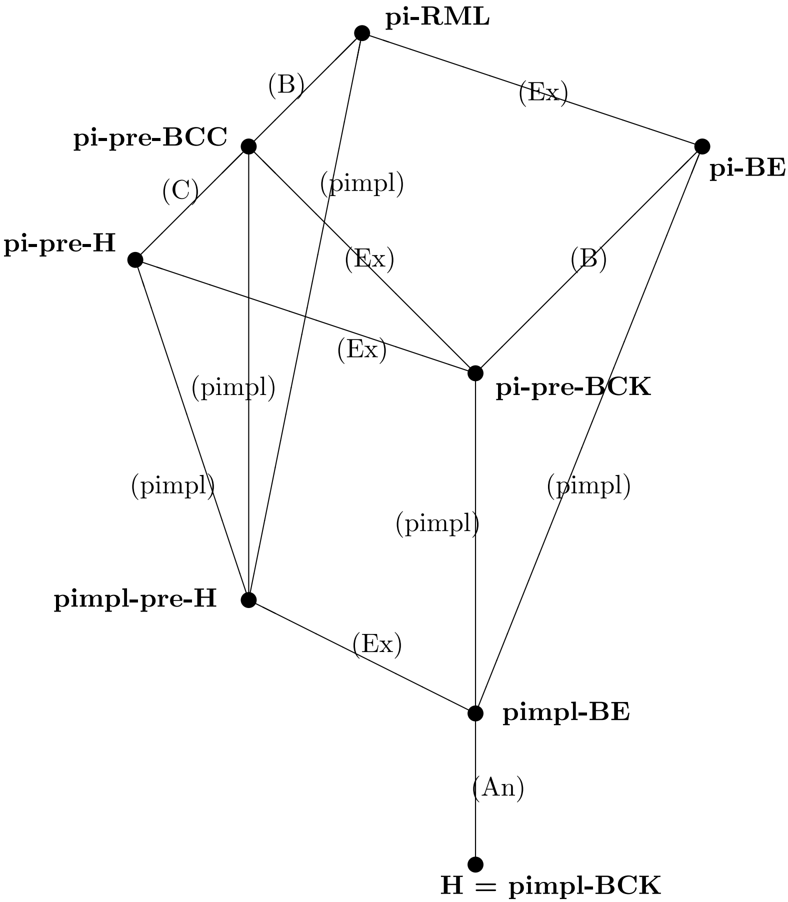
(2) Describing the deductive systems, the congruences, the quotient algebras, etc. of pre-Hilbert algebras.

(3) Investigating the connections between pre-Hilbert algebras and GE algebras (generalized exchange algebras) introduced in 2021 by R. Bandaru et al. [2].

References

[1] J. C. Abbott, *Semi-boolean algebra*, **Matematički Vesnik**, vol. 4(19) (1967), pp. 177–198, URL: https://eudml.org/doc/258960.

[2] R. Bandaru, A. B. Saeid, Y. B. Jun, *On GE-algebras*, **Bulletin of the Section of Logic**, vol. 50(1) (2021), pp. 81–96, DOI: https://doi.org/0138-0680.2020.20.



pi-RML
(B)
(Ex)
pi-pre-BCC
pi-BE
(C)
(pimpl)
pi-pre-H
(Ex)
(B)
(Ex)
pi-pre-BCK
(pimpl)
(pimpl)
(pimpl)
(pimpl)
pimpl-pre-H
(Ex)
pimpl-BE
(An)
H = pimpl-BCK

Figure 4.

[3] D. Buşneag, S. Rudeanu, *A glimpse of deductive systems in algebra*, **Central European Journal of Mathematics**, vol. 8(4) (2010), pp. 688–705, DOI: https://doi.org/10.2478/s11533-010-0041-4.

[4] A. Diego, **Sur les algébras de Hilbert**, vol. 21 of Collection de Logigue Mathématique, Serie A, Gauthier-Villars, Paris (1966).

[5] L. Henkin, *An algebraic characterization of quantifilers*, **Fundamenta Mathematicae**, vol. 37(1) (1950), pp. 63–74, URL: http://eudml.org/doc/213228.

[6] A. Iorgulescu, *New generalizations of BCI, BCK and Hilbert algebras—Part I, II*, **Journal of Multiple-Valued Logic and Soft Computing**, vol. 27(4) (2016), pp. 353–456.

[7] K. Iséki, *An algebra related with a propositional calculus*, **Proceedings of the Japan Academy**, vol. 42(1) (1966), pp. 26–29, DOI: https://doi.org/10.3792/pja/1195522171.

[8] K. Iséki, S. Tanaka, *An introduction to the theory of BCK-algebras*, **Mathematica Japonica**, vol. 23(1) (1978), pp. 1–26.

[9] Y. B. Jun, M. S. Kang, *Fuzzifications of generalized Tarski filters in Tarski algebras*, **Computers and Mathematics with Applications**, vol. 61(1) (2011), pp. 1–7, DOI: https://doi.org/10.1016/j.camwa.2010.10.024.

[10] H. S. Kim, Y. H. Kim, *On BE-algebras*, **Scientiae Mathematicae Japonicae**, vol. 66(1) (2007), pp. 113–116, DOI: https://doi.org/10.32219/isms.66.1_113.

[11] J. Kim, Y. Kim, E. H. Roh, *A note on GT-algebras*, **The Pure and Applied Mathematics**, vol. 16(1) (2009), pp. 59–69, URL: https://koreascience.kr/article/JAKO200910335351650.pdf.

[12] Y. Komori, *The class of BCC-algebras is not a variety*, **Mathematica Japonica**, vol. 29 (1984), pp. 391–394.

[13] C. A. Meredith, **Formal logics**, 2nd ed., Oxford University Press, Oxford (1962).

[14] A. Monteiro, **Lectures on Hilbert and Tarski algebras**, Insitituto de Mathemática, Universuidad Nacional del Sur, Bahía Blanca, Argentina (1960).

[15] A. Walendziak, *On commutative BE-algebras*, **Scientiae Mathematicae Japonicae**, vol. 69(2) (2009), pp. 281–284, DOI: https://doi.org/10.32219/isms.69.2_281.

On Pre-Hilbert and Positive Implicative Pre-Hilbert Algebras

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