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A SYNTACTIC PROOF OF THE DECIDABILITY OF FIRST-ORDER MONADIC LOGIC

Abstract

Decidability of monadic first-order classical logic was established by Löwenheim in 1915. The proof made use of a semantic argument and a purely syntactic proof has never been provided. In the present paper we introduce a syntactic proof of decidability of monadic first-order logic in innex normal form which exploits **G3**-style sequent calculi. In particular, we introduce a cut- and contraction-free calculus having a (complexity-optimal) terminating proof-search procedure. We also show that this logic can be faithfully embedded in the modal logic **T**.

Keywords: proof theory, classical logic, decidability, Herbrand theorem.

1. Introduction

A cornerstone result in the field of classical logic is the undecidability of first-order logic (FOL) [3]. Indeed, the set of first-order (FO) logical truths is recursively enumerable and so semidecidable, but essentially undecidable. Even before the discovery of this crucial feature, some decidable fragments have been isolated and investigated.

One of the most representative ones is the monadic fragment obtained by restricting the language to one-place predicates, thus excluding relations therefrom. A first proof of the decidability of monadic classical FOL (MFOL) was given by Löwenheim [6]. The proof employed semantic arguments (in particular, a form of finite model property) and it can thus be

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regarded as partially satisfactory, as it uses a semantic method to establish a syntactic result.[1]

Other proofs were provided by Quine [7] and, later, by Boolos [1]. A key ingredient in these arguments is the reduction of formulas of MFOL to a kind of normal form, which pushes quantifiers inside formulas. Hence, validity of the formulas thus obtained—to be called *innex formulas*—is checked via semantic arguments. However, a purely syntactic and proof-theoretic version of decidability has not been presented yet. In the present paper, we aim at filling this gap.

The design of a terminating sequent calculus for monadic logic is not a trivial task. Indeed, we need to observe that the rule of contraction cannot be *a priori* dispensed with. An example is the formula $\exists x(P(x) \supset \forall y P(y))$ which is a monadic valid formula that is not provable without a (possibly implicit in the rule) step of contraction.

Therefore we focus on a specific fragment of MFOL, i.e. the innex one, and we show that we can give a terminating sequent calculus in which every rule is height-preserving invertible without the need for any form of contraction. The calculus **G3INT** is obtained by combining a form of focusing—i.e., a specific ordering in the application of the rules [5]—with a new rule for the existential quantifier.

These aspects contribute to complicating the structural analysis of the system which has some peculiar traits. Furthermore, we offer an extremely simple syntactic proof of cut-elimination which is based on a single inductive parameter, the degree of the cut formula, instead of two parameters—e.g., the degree and the height of the cut—as in calculi for FOL [8].

Finally, we offer another perspective on the decidability of the innex fragment of monadic logic. In particular, we show that it can be soundly and faithfully embedded in the modal system **T** enhanced with a first-order language (but without quantifiers). This reduction highlights some specific characteristics of the fragment by identifying $\forall$ and $\exists$ with the modal operators $\Box$ and $\Diamond$, respectively.

The plan of the paper is as follows. Section 2 introduces innex normal form and a preliminary calculus for MFOL. Section 3 is devoted to the calculus **G3INT** whose properties are thoroughly investigated in Section 4.

[1] As observed by a reviewer, under the completeness of monadic logic, the decidability result might be considered a semantic as well as syntactic problem. In our opinion, the problem of whether a logic is decidable concerns derivability in a formal system and thus it has a more intrinsically proof-theoretic content.

Soundness and completeness are discussed in Section 5 and Section 6 deals with the modal interpretation of the system. Finally, Section 7 adds some concluding remarks and sketches some themes which may be an object of future research.

2. MFOL and innex normal form

Let us fix a signature $\mathcal{S}$ containing a countable and non-empty set of monadic predicates. Given a denumerable set of variables $\mathcal{V}$, the language of MFOL (in negative normal form) is defined by the following grammar:

$$A ::= P(x) \mid \overline{P}(x) \mid A \wedge A \mid A \vee A \mid \exists x A \mid \forall x A$$

where $P \in \mathcal{S}$ and $x \in \mathcal{V}$.

Parentheses are used as usual and negation is defined via (De Morgan's) dualities and double-negation elimination—e.g., $\neg\overline{P}(x) \equiv \overline{\overline{P}} \equiv P(x)$ and $\neg\forall x A \equiv \exists x \neg A$. A *literal* is a formula of shape $P(x)$ or $\overline{P}(x)$. We use the following metavariables: x, y, z for variables, P, R, S for literals, and A, B, C for arbitrary formulas. $A[y/x]$ stands for the formula obtained by replacing in A each free occurrence of x with an occurrence of y, provided that y is free for x in A. When convenient, we use $B(y)$ for the formula obtained from QxB by removing the quantifier Qx and substituting y for x.

In (classical) FOL it is often preferred to work with formulas which have a precise shape. In this sense, a normal form for FOL is the so-called *prenex normal form*. A formula is in *prenex normal form* whenever it is of the form: $Q_{x_1}...Q_{x_n}A$, where $Q_{x_1}...Q_{x_n}$ is a finite string of quantifiers and A is a quantifier-free formula—i.e., A contains only propositional connectives.

Proposition 2.1. Each first-order formula A is logically equivalent to a formula A' in prenex normal form: A and A' are satisfied by the same FO-models.

Proof: A standard induction on the structure of A making use of De Morgan's dualities, of the distributivity of $\vee$ over $\wedge$ and of the following FO-validities:[2]

[2] Without loss of generality, we are assuming that each quantifier binds a different variable, no variable has both free and bound occurrences in a formula, and $x \notin B$.

- $\neg\exists xA \supset\subset \forall x\neg A$
- $\neg\forall xA \supset\subset \exists x\neg A$
- $\exists xA\vee\exists yC \supset\subset \exists x(A\vee C[x/y])$
- $\forall xA \wedge \forall yC \supset\subset \forall x(A \wedge C[x/y])$
- $\exists xA \vee B \supset\subset \exists x(A \vee B)$
- $\forall xA \vee B \supset\subset \forall x(A \vee B)$
- $\exists xA \wedge B \supset\subset \exists x(A \wedge B)$
- $\forall xA \wedge B \supset\subset \forall x(A \wedge B)$ □

This is a property which is specific of classical FOL which does not usually extend to non-classical logics or modal logics. In particular, neither FO-intuitionistic nor FO-modal logics do validate the prenexation laws.

In this paper we are actually interested in a sort of converse transformation which pushes quantifiers inside the formulas.

Definition 2.2. A first-order formula is in *innex normal form* (INF) if it is a boolean combination of formulas A and QxB, where A is a quantifier-free formula and QxB is a formula of the form $\exists x(P_1(x) \wedge ... \wedge P_n(x))$ or $\forall x(P_1(x) \vee ... \vee P_n(x))$ where P_i is a literal.

In general, FO-formulas are not equivalent to formulas in INF, but this holds if we consider the *monadic fragment* of the language—i.e, a FO-language containing only unary predicates.

Proposition 2.3. Each formula A of the monadic fragment of the FO-language is logically equivalent to a formula A' in INF.

Proof: Analogous to the proof of Prop. 2.1, applying the same equivalences in reverse direction, cf. [1, Lemma 21.12]. □

MFOL has been shown to be decidable already in [6] by means of a semantic argument. In particular, we have the following results:

Theorem 2.4.

1. *If a monadic sentence containg k predicates is satisfiable, it has a model of size no greater than 2^k [1, Lemma 21.8];*
2. *The satisfiability problem for monadic FO-logic is* NEXP*-complete [4].*

In this paper we are interested in giving a proof-theoretic proof of decidability of MFOL. A key ingredient for this result will be Proposition

Initial Sequents: $\dfrac{}{\Gamma, P, \overline{P}}\ Ax$

Rules: $\dfrac{\Gamma, A \qquad \Gamma, B}{\Gamma, A \wedge B}\ \wedge \qquad \dfrac{\Gamma, A, B}{\Gamma, A \vee B}\ \vee \qquad \dfrac{\Gamma, A[y/x]}{\Gamma, \forall x A}\ \forall, y \text{ fresh} \qquad \dfrac{\Gamma, \exists x A, A[z/x]}{\Gamma, \exists x A}\ \exists$

Figure 1. The sequent calculus **G3S**

2.3: the possibility of defining an innex normal form for monadic formulas is crucial in order to develop a terminating calculus. Indeed, innex formulas remove the nesting of quantifiers and allow for a full elimination of contraction which is harmful for proof search. Interestingly, also Quine [7] has given a proof of decidability for MFOL exploiting Prop. 2.3. However, his method uses truth tables which are arguably less immediate than the method of terminating sequent calculi adopted in this paper.

2.1. Decidability of MFOL

The one-sided sequent calculus **G3S** for MFOL is given in Figure 1, we refer the reader to [8] for its properties and to Section 3 for some basic definition. The difficulty in directly establishing a decidability proof of MFOL within **G3S** is due to the formulation of the rule $\exists$ in which the principal formula is repeated in the premise of the rule. This design choice is necessary in order to make the rule invertible, but it has a hidden contraction. In principle, there is no bound on the possible number of applications of the rule $\exists$.

In order to prove the decidability result we need to show a weak version of Herbrand's theorem which will be essential in order to obtain the proof.

Lemma 2.5. *For every finite multisets of quantifier-free formulas* Γ *and all quantifier-free formulas* B_i*, if* $\Gamma, \exists x_1 B_1, ..., \exists x_\ell B_\ell$ *is* **G3S***-derivable, then, for some* m *and* n *in* $\mathbb{N}$*, there is a derivation of the same height of*

$$\Gamma, \{B_1[y_i/x_1] : i \leq m\}, ..., \{B_\ell[y_j/x_\ell] : j \leq n\}$$

Proof: We argue by induction on the height h of the derivation. If $h = 0$, the proof is immediate, because quantified formulas cannot be principal.

If $h > 0$, then we distinguish cases according to the last rule applied. If it is a propositional rule, we apply the induction hypothesis to the premise(s) and then the rule again. If it is a quantifier rule, then it can only be the rule $\exists$. It is enough to apply the induction hypothesis. □

THEOREM 2.6. *For every sequent* Γ*, where* Γ *is a finite multiset of formulas of MFOL in INF, there is a procedure outputting either a* **G3S***-proof or a finite failed attempt to it.*

PROOF: The decision procedure consists in applying the invertibility of every propositional rule. This will imply that the derivability of the sequent Γ is equivalent to that of the sequents Γ_1,...,Γ_n, where, for each $i \in \{1, ..., n\}$, Γ_i is of the form (for Γ'_i, D_j, and B_ℓ quantifier-free):

$$\Gamma'_i, \forall x_1 D_1, \ldots, \forall x_m D_m, \exists z_1 B_1, \ldots, \exists z_k B_k$$

for some $m \geq 0$ and $k \geq 0$.

We now apply the invertibility of the rule $\forall$ to get:

$$\Gamma'_i, D_1[y_1/x_1], ..., D_m[y_m/x_r], \exists z_1 B_1, \ldots, \exists z_k B_k$$

Each sequent thus obtained satisfies the hypotheses of Lemma 2.5 and therefore we can reduce its derivability to that of a sequent Γ^*_i which does not contain any quantified formula. The derivability of each of these sequents is decidable. □

Theorem 2.6 shows that if we restrict our attention to formulas of MFOL that are in INF then we can bound the number of contractions hidden inside of the rule $\exists$ so as to obtain a decision procedure for MFOL. Observe that Theorem 2.6 does not mean that **G3S** is a terminating calculus for MFOL. Even if we have a sequent (whose formulas are) in INF, proof-search is non-terminating because of the contraction hidden in the repetition of the principal formula in the premise of the rule $\exists$. More precisely, we have defined a strategy to halt the search for a derivation or a countermodel, but the decidability is not intrinsic to the calculus **G3S**.

The system **G3S** represents a bridge towards a terminating calculus for MFOL. In order to obtain it, we will impose that the rule for the existential quantifier can be applied only when we already know all variables over which it can be instantiated, so that it can be instantiated over each of them at the same time. As it will be shown in Section 4.1, this terminating calculus has all the rules invertible without having to resort to any (hidden or explicit) instance of contraction and, hence, it will allow for a decision procedure for MFOL that is optimal complexity-wise.

Initial Sequents $\dfrac{}{\Gamma, P, \overline{P}}$ Ax, Γ innex

Logical Rules $\dfrac{\Gamma, A \qquad \Gamma, B}{\Gamma, A \wedge B}\ R\wedge$ $\qquad \dfrac{\Gamma, A, B}{\Gamma, A \vee B}\ R\vee$

$$\dfrac{\Gamma, P_1[y/x] \vee \ldots \vee P_n[y/x]}{\Gamma, \forall x(P_1 \vee \ldots \vee P_n)}\ R\forall,\ y \text{ fresh} \qquad \dfrac{\Gamma, \{P_1[z_i/x] \wedge \ldots \wedge P_n[z_i/x] : z_i \in Var(\Gamma)\}}{\Gamma, \exists x(P_1 \wedge \ldots \wedge P_n)}\ R\exists,\ \Gamma \text{ reduced}$$

Figure 2. The sequent calculus **G3INT**.

3. The calculus G3INT

To define a contraction-free calculus, we shall introduce another sequent calculus for MFOL in INF. In particular, we will use a **G3**-style calculus to obtain the result. The reason of the choice lies in the fact that **G3**-style calculi have good structural properties and they are suitable for backward reasoning, due to the invertibility of every rule.

The rules of the (one-sided) calculus **G3INT** are given in Figure 2. In particular, initial sequents have the side condition that Γ is a multiset of (monadic) formulas in INF. By $Var(\Gamma)$ we denote the set of free variables occurring in the multiset Γ, if any, otherwise the singleton containing some fixed variable y. Rule $R\exists$ has the side condition that Γ is a multiset of reduced formulas, where the notion of *reduced multiset* is defined as follows:

Definition 3.1. A multiset Γ is *reduced* whenever it does not contain universal quantifiers.

A *derivation* is a finite rooted tree where the leaves are initial sequents and every node is constructed by applications of the rules. The *height* of a derivation is the number of nodes in a branch of maximal length in the derivation minus one. The *degree* of a formula is the number of logical symbols occurring in the formula. A rule is *(height-preserving) admissible* if, whenever each premise of the rule is derivable (with a derivation of height $\leq n$), so is the conclusion (with a derivation of height $\leq n$). Without loss of generality, we always assume to be working up to renaming of bound variables, i.e. *modulo* α-conversion.

We briefly recall some properties of **G3INT**. We start by proving that the rules are such that the property of being in innex normal form propagates from the leaves to each node of a derivation. This allows us to restrict attention to (finite multisets of) formulas in innex normal form.

Lemma 3.2. *If* Γ *is derivable in* **G3INT**, *then (each formula occurring in) it is in innex normal form.*

Proof: The proof is by induction on the height of the derivation. If Γ is an initial sequent, the proof is trivial. Otherwise we distinguish cases according to the last rule applied. In each case it is enough to apply the induction hypothesis to (each of) the premise(s) of the rule and then observing that the rules preserve the innex normal form. □

Lemma 3.3. *The rules* $R\wedge, R\vee$ *and* $R\exists$ *are height-preserving invertible.*

Proof: The proof runs by induction on the height of the derivation. We discuss the case of $R\exists$ (the other cases are as for **G3S**). If $\Gamma, \exists x(P_1 \wedge \dots P_n)$ is an initial sequent, so is $\Gamma, \{P_1[z_i/x] \wedge \dots \wedge P_n[z_i/x] : z_i \in Var(\Gamma)\}$. If the last rule applied is $R\exists$ and $\exists x(P_1 \wedge \dots \wedge P_n)$ is principal, the proof is immediate. Otherwise, the last rule applied cannot be R$\forall$ since $\Gamma, \exists x(P_1 \wedge \dots \wedge P_n)$ is reduced. Therefore, we simply apply the induction hypothesis to each of the premises and then the rule again. □

Lemma 3.4. *The sequent* $\Gamma, A, \overline{A}$ *is provable in* **G3INT**.

Proof: We argue by induction on the degree of A. If A is a literal then there is nothing to prove. If A (or $\overline{A}$) is of the shape $B \vee C$, the proof is immediate. If it is of the shape $\forall x B$, we first apply root-first the rules to obtain sequents with a reduced context Γ' and then we proceed as follows:[3]

$$\cfrac{\cfrac{\overline{\overline{\qquad\qquad\qquad\qquad}}\ \text{IH}}{\Gamma', A[y/x], \overline{A}[z_1/x], \dots, \overline{A}[z_n/x], \overline{A}[y/x]}\ R\exists}{\cfrac{\Gamma', A[y/x], \exists x\overline{A}}{\Gamma', \forall x A, \exists x \overline{A}}\ R\forall}$$

Where $z_1, \dots, z_n, y$ are all variables occurring free in $\Gamma', A[y/x]$. □

[3]The doubleline derivation symbol marks a step that is admissible.

4. Structural analysis of G3INT

Lemma 4.1. *The rule:*

$$\frac{\Gamma, \Delta, \Delta}{\Gamma, \Delta}\ \mathit{RedC}$$

where Γ, Δ, Δ *is reduced, is height-preserving admissible.*

Proof: We proceed by induction on the height of the derivation of the sequent Γ, Δ, Δ. If it is an initial sequent, then so is Γ, Δ.

If no formula in Δ is principal, we apply the induction hypothesis and then the rule again.

If a formula A is principal in Δ, we distinguish cases according to the last rule applied. The strategy consists in applying Lemma 3.3, the induction hypothesis and then the rule again. We consider the case of the rule $R\exists$.

$$\frac{\Gamma, A[z_1/x], ..., A[z_n/x], \exists x A, \Delta', \Delta'}{\Gamma, \exists x A, \exists x A, \Delta', \Delta'}\ R\exists$$

where A is a finite conjunction of atomic formulas. We construct the following derivation:

$$\dfrac{\dfrac{\dfrac{\Gamma, A[z_1/x], ..., A[z_n/x], \exists x A, \Delta', \Delta'}{\Gamma, A[z_1/x], ..., A[z_n/x], A[z_1/x], ..., A[z_n/x], \Delta', \Delta'}\ \text{Lemma 3.3}}{\Gamma, A[z_1/x], ..., A[z_n/x], \Delta'}\ IH}{\Gamma, \exists x A, \Delta'}\ R\exists$$

□

Proceeding in a slightly unusual order, we now prove the admissibility of substitution. As usual, we extend substitutions to multisets of formulas.

Lemma 4.2 (Substitution). *The rule:*

$$\frac{\Gamma}{\Gamma[y/x]}\ \mathit{Sub}$$

is height-preserving admissible

Proof: By induction on the height of the derivation $\mathcal{D}$ of the premise Γ.

If $\mathtt{h}(\mathcal{D}) = 0$ then the lemma obviously holds. If $\mathtt{h}(\mathcal{D}) = n + 1$ then we have cases according to the last rule applied in $\mathcal{D}$. If the last rule is an instance of rule $R\wedge$ or $R\vee$, the proof follows from the induction hypothesis.

If the last step in $\mathcal{D}$ is the following instance of rule $R\exists$:

$$\frac{\Gamma, A[z_1/z], \ldots, A[z_n/z]}{\Gamma, \exists z A}\ R\exists$$

where, w.l.o.g. $z \notin \{x, y\}$, then we transform the derivation as follows:

$$\dfrac{\dfrac{\dfrac{\Gamma, A[z_1/z], \ldots, A[z_n/z]}{\Gamma[y/x], (A[z_1/z])[y/x], \ldots, (A[z_n/z])[y/x]}\ \text{IH}}{\Gamma[y/x], (A[y/x])[(z_1[y/x])/z], \ldots, (A[y/x])[(z_n[y/x])/z]}\ \star}{\dfrac{\Gamma[y/x], \exists z(A[y/x])}{\Gamma[y/x], (\exists z A)[y/x]}\ \star\star}\ R\exists$$

where the steps marked with $\star$ and $\star\star$ are syntactic rewritings that do not increase the height of the derivation. The application of the rule $R\exists$ is justified as the set of terms occurring in $\Gamma[y/x]$ is a subset of the set of terms occurring in $(A[y/x])[(z_1[y/x])/z], \ldots, (A[y/x])[(z_n[y/x])/z]$.

Furthermore, note that if z is free in $A[y/x]$ and, for some $j, k \leq n$, $x \equiv z_j$ and $y \equiv z_k$, then $(A[y/x])[(z_j[y/x])/z] \equiv (A[y/x])[(z_k[y/x])/z]$. This is not a problem since by the design of the rules the sequent

$$\Gamma[y/x], (A[y/x])[(z_1[y/x])/z], \ldots, (A[y/x])[(z_n[y/x])/z]$$

is reduced and so we can safely apply Lemma 4.1.

Finally, suppose the last step in $\mathcal{D}$ is the following instance of rule:

$$\frac{\Gamma, A[y_2/y_1]}{\Gamma, \forall y_1 A}\ R\forall;\ y_2!$$

where neither y nor x is y_1. We apply the inductive hypothesis (IH) twice to the derivation of the premise, the first time to replace y_2 with a variable y_3 that is new to the premise and the second time to replace x with y. By applying an instance of rule $R\forall$ we conclude $(\Gamma, \forall y_1 A)[y/x]$. □

Lemma 4.3 (Invertibility). *All rules of* **G3INT** *are height-preserving invertible.*

Proof: The case of rules $R\vee$, $R\wedge$ and $R\exists$ has been proved in Lemma 3.3. We show, by induction on the height of the derivation $\mathcal{D}$, that rule $R\forall$ is height-preserving invertible.

If $\mathtt{h}(\mathcal{D}) = 0$, or if the formula we are inverting is principal in the last step of $\mathcal{D}$, then the proof is trivial.

If $\mathcal{D}$ is

$$\frac{\Delta', \forall y_1 A \qquad (\Delta'', \forall y_1 A)}{\Delta, \forall y_1 A}\,\mathrm{R}$$

we know that R is not an instance of rule $R\exists$. Once again we apply IH to the premise(s) (possibly with a height-preserving admissible step of substitution to avoid clashes of variables) and then an instance of R to conclude $\Delta, A[y_2/y_1]$. □

THEOREM 4.4. *The rules of contraction are height-preserving admissible in* **G3INT**.

PROOF: The proof is by induction on the height of the derivation $\mathcal{D}$. If Γ, A, A is an initial sequent, so is Γ, A. If A is not principal in the last rule applied, we apply the induction hypothesis to each of the premises of the rule and then the rule again, e.g., if $\mathcal{D}$ is

$$\frac{\Delta', A, A \qquad (\Delta'', A, A)}{\Delta, A, A}\,\mathrm{R}$$

We construct the following derivation:

$$\frac{\dfrac{\Delta', A, A}{\Delta', A}\,\mathrm{IH} \qquad \dfrac{(\Delta'', A, A)}{(\Delta'', A)}\,\mathrm{IH}}{\Delta, A}\,\mathrm{R}$$

If, instead, A is principal, we distinguish cases according to its shape. The strategy consists in applying the invertibility lemma followed by the induction hypothesis. We focus on the case of the existential quantifier. We suppose that the set of variables free in Γ is not empty and we have:

$$\frac{\Gamma, \exists x(P_1 \wedge \dots P_n), \{P_1[z_i/x] \wedge \dots \wedge P_n[z_i/x] : z_i \in Var(\Gamma)\}}{\Gamma, \exists x(P_1 \wedge \dots \wedge P_n), \exists x(P_1 \wedge \dots \wedge P_n)}\,\mathrm{R}\exists$$

We proceed as follows:

$$\dfrac{\dfrac{\dfrac{\Gamma, \exists x(P_1 \wedge \dots P_n), \{P_1[z_i/x] \wedge \dots \wedge P_n[x_i/x] : z_i \in Var(\Gamma)\}}{\Gamma, \{P_1[z_i/x] \wedge \dots \wedge P_n[z_i/x] : z_i \in Var(\Gamma)\}, \{P_1[z_i/x] \wedge \dots \wedge P_n[z_i/x] : z_i \in Var(\Gamma)\}}\,\mathrm{inv}\ \exists}{\Gamma, \{P_1[z_i/x] \wedge \dots \wedge P_n[z_i/x] : z_i \in Var(\Gamma)\}}\,\mathrm{IH}}{\Gamma, \exists x(P_1 \wedge \dots P_n)}\,\mathrm{R}\exists$$

Where IH stands for possibly multiple applications of the inductive hypothesis. □

Finally, we can prove the admissibility of the rule of weakening. Contrarily to usual **G3**-style systems, weakening is admissible without preservation of height. We start by discussing a specific case, i.e. weakening for reduced sequents.

LEMMA 4.5. *The rule:*

$$\frac{\Gamma}{\Gamma, \Delta}\ \mathit{Weak}_{Red}\ ,$$

where Δ *is (innex and) reduced, is height-preserving admissible in* **G3INT**.

PROOF: The proof is by induction on the height of the derivation. If Γ is an initial sequent, so is Γ, Δ. If the last rule applied is $\wedge$ or $\vee$, the proof follows from the application of the induction hypothesis and the rule again. As an example, we detail the case of $\vee$:

$$\frac{\Gamma, A, B}{\Gamma, A \vee B}\ \vee \quad \rightsquigarrow \quad \frac{\dfrac{\Gamma, A, B}{\Gamma, A, B, \Delta}\ \text{IH}}{\Gamma, A \vee B, \Delta}\ \vee$$

If the last rule applied is $R\exists$ and Δ is reduced, we proceed as follows:

$$\frac{\Gamma', B(z_1), \dots, B(z_m)}{\Gamma', \exists x B}\ R\exists \quad \rightsquigarrow \quad \frac{\dfrac{\Gamma', B(z_1), \dots, B(z_m)}{\Gamma', B(z_1), \dots, B(z_m), B(z_{m+1}), \dots, B(z_{m+n}), \Delta}\ IH}{\Gamma', \exists x B, \Delta}\ R\exists$$

where $z_{m+1}, \dots, z_{m+n}$ are the variables occurring free in Δ but not Γ. □

LEMMA 4.6. *The rule:*

$$\frac{\Gamma}{\Gamma, \Delta}\ \mathit{Weak},\ \Delta\ \mathit{innex}$$

is admissible in **G3INT**.

PROOF: The proof runs by induction on the height of the derivation. We detail—for the sake of readability—the case in which Δ consists of a single formula A.

If Γ is an initial sequent, then Γ, A is an initial sequent too. In the remaining cases except for $R\exists$, we apply the induction hypothesis (and possibly height-preserving admissibility of substitution in order to avoid clashes of variables) and then the rule again.

If the last rule applied is $R\exists$, we have:

$$\frac{\Gamma, B(z_1), \ldots, B(z_m)}{\Gamma, \exists x B} R\exists$$

By the induction hypothesis we get a derivation of $\Gamma, B(z_1), \ldots, B(z_m), A$. We decompose it into reduced sequents via height-preserving invertibility of the rules $R\forall$, $R\wedge$ and $R\vee$ to get sequents of the shape:

$$\Gamma, B(z_1), \ldots, B(z_m), A_1, \ldots, A_n$$

where A_i is reduced for $i \in \{1, \ldots, n\}$. Next, we proceed as follows:

$$\cfrac{\dfrac{\Gamma, B(z_1), \ldots, B(z_m), A_1, \ldots, A_n}{\Gamma, B(z_1), \ldots, B(z_m), B(z_{m+1}), \ldots, B(z_{m+l}), A_1, \ldots, A_n} Weak_{Red}}{\Gamma, \exists x B, A_1, \ldots, A_n} R\exists$$

The formulas $B(z_{m+1}), \ldots, B(z_{m+l})$ are instantiations of B over terms occurring in $A_1, \ldots, A_n$ (introduced by the analysis of A). The application $Weak_{Red}$ is justified by the previous lemma.

The desired conclusion is obtained from $\Gamma, \exists x B, A_1, \ldots, A_n$ via the application of the rules used to decompose A in the reverse order. □

We are now in the position to state and prove the admissibility of the cut rule. In this case, we shall argue by induction on a single parameter, the degree of the cut formula.

THEOREM 4.7. *The cut rule is admissible in* **G3INT**.

PROOF: The proof is by induction on the degree of the cut formula. We consider an upper-most instance of a context-sharing cut:

$$\frac{\Gamma, A \qquad \Gamma, \overline{A}}{\Gamma} \text{Cut}$$

The admissibility of a context-free cut follows by the admissibility of weakening and contraction.

If the cut formula is atomic, it is of the shape P and $\overline{P}$ and we have:

$$\dfrac{\begin{array}{c}\vdots\mathcal{D}\\ \Gamma, P\end{array} \qquad \Gamma, \overline{P}}{\Gamma}\ \text{Cut}$$

We consider the topmost sequents of the derivation $\mathcal{D}$. They will be the sequents Γ_i, P, for $1 \leq i \leq n$. We substitute P with Γ. We claim the resulting sequent is derivable. Indeed, if P is not principal in the initial sequent Γ_i, P, then also Γ_i, Γ is an initial sequent. Else, P is principal in Γ_i, P and $\Gamma_i \equiv \Gamma_i', \overline{P}$. The sequent $\Gamma_i', \overline{P}, \Gamma$ is cut-free derivable by applying an admissible instance of weakening to the right premise of the cut rule. We can, thus replace each premise Γ_i, P of $\mathcal{D}$ with Γ_i, Γ. We have the following cut-free derivation of Γ:

$$\dfrac{\begin{array}{c}\Gamma_1, P \quad \cdots \quad \Gamma_n, P\\ \vdots\mathcal{D}\\ \Gamma, P\end{array} \qquad \Gamma, \overline{P}}{\Gamma}\ \text{Cut} \quad \rightsquigarrow \quad \begin{array}{c} \dfrac{\dfrac{\Gamma_1, \Gamma}{\Gamma_1, \Gamma_1}\ \text{Inv}}{\Gamma_1}\ \text{Ctr} \quad \cdots \quad \dfrac{\dfrac{\Gamma_n, \Gamma}{\Gamma_n, \Gamma_n}\ \text{Inv}}{\Gamma_n}\ \text{Ctr}\\ \vdots\mathcal{D}\\ \Gamma \end{array}$$

In the cases in which the formula is compound, but not quantified, we exploit invertibility and then cuts on formulas of lesser degrees. In particular, we have:

$$\dfrac{\Gamma, A \wedge B \qquad \Gamma, \overline{A} \vee \overline{B}}{\Gamma}\ \text{Cut}$$

We transform the derivation as follows:

$$\dfrac{\dfrac{\Gamma, A \wedge B}{\Gamma, B}\ \text{Inv} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\Gamma, A \wedge B}{\Gamma, A}\ \text{Inv}}{\Gamma, A, \overline{B}}\ \text{Weak} \qquad \dfrac{\Gamma, \overline{A} \vee \overline{B}}{\Gamma, \overline{A}, \overline{B}}\ \text{Inv}}{\Gamma, \overline{B}}\ \text{Cut}}{\Gamma}\ \text{Cut}$$

The cuts are removed by induction on the degree of the cut formula.

If the formula is quantified, we have:

$$\frac{\Gamma, \exists x(P_1 \wedge \ldots \wedge P_n) \qquad \Gamma, \forall x(\overline{P}_1 \vee \ldots \vee \overline{P}_n)}{\Gamma}$$

In this case we first apply height-preserving invertibility to both premises of the cut in order to reach reduced sequents in Γ. In particular, this yields two sets of sequents:

$$\mathbf{A} = \{\Gamma_i, \exists x(P_1 \wedge \ldots \wedge P_n) \,|\, 1 \leq i \leq k\} \text{ and}$$
$$\mathbf{B} = \{\Gamma_i, \forall x(\overline{P}_1 \vee \ldots \vee \overline{P}_n) \,|\, 1 \leq i \leq k\}$$

By applying height-preserving invertibility of the rules for the existential quantifier, we get the set $\mathbf{A}'$:

$$\mathbf{A}' = \{\Gamma_i, \{P_1(z_j) \wedge \ldots \wedge P_n(z_j) \,:\, z_j \in VAR(\Gamma_i)\} \,:\, 1 \leq i \leq k\}$$

By invertibility of the rule $R\forall$, we get derivations of: $\Gamma_i, \overline{P}_1(z_j) \vee \ldots \vee \overline{P}_n(z_j)$ for each i and each z_j. For every i we proceed as follows:

$$\dfrac{\dfrac{\dfrac{\Gamma_i, \{P_1(z_j) \wedge \ldots \wedge P_n(z_j) \,:\, 1 \leq j \leq \ell\} \qquad \Gamma_i, \overline{P}_1(z_1) \vee \ldots \vee \overline{P}_n(z_1)}{\Gamma_i, \{P_1(z_j) \wedge \ldots \wedge P_n(z_j) \,:\, 2 \leq j \leq \ell\}}\text{Cut}}{\vdots}}{\dfrac{\Gamma_i, P_1(z_\ell) \wedge \ldots \wedge P_n(z_\ell) \qquad \Gamma_i, \overline{P}_1(z_\ell) \vee \ldots \vee \overline{P}_n(z_\ell)}{\Gamma_i}\text{Cut}}$$

All the cuts are eliminated invoking the induction hypothesis on the degree of the cut formula. Finally, we apply the rules in the reverse order to get a derivation of Γ. $\square$

4.1. Termination and bounds on cut-free proofs

In this subsection, we establish the termination of the proof search and we define bounds on the height of cut-free derivations. It is easy to see that each bottom-up application of a rule either decreases the number of quantifiers or the number of connectives occurring in the endsequent.

Proposition 4.8. The calculus **G3INT** is terminating.

Proof: Given a sequent Γ we argue by induction on lexicographically ordered pairs (m, n), where m is the number of quantifiers occurring in the endsequent and n is the number of connectives occurring in Γ.

It is immediate to see that the rules $R\forall$ and $R\exists$ decrease the number of quantifiers. The latter potentially increases the number of connectives, but this is not problematic, because it is the second inductive parameter.

The rules $R\wedge$ and $R\vee$ do not increase the number of quantifiers, but strictly decrease the number of connectives in the endsequent. Therefore we can infer that the proof search terminates. □

Next, we would like to compute explicit bounds on the height of cut-free derivations. As it is well-known, in classical (and intuitionistic) FOL the elimination of cuts can lead to an hyperexponential increase of the height of the proofs. In the case of the innex fragment of classical MFOL, we can show that the bounds on cut-free proofs is way lower than for FOL. Indeed, since the proof search terminates for every sequent, we can define a maximal height of any derivation.

Definition 4.9. Given a sequent Γ, we define a measure of complexity for every formula A occurring in it, in symbols $\sigma_\Gamma(A)$. If A is a literal, then $\sigma_\Gamma(A) = 0$. If A is $B \# C$, with $\# \in \{\wedge, \vee\}$, then $\sigma_\Gamma(A) = \sigma_\Gamma(B) + \sigma_\Gamma(C) + 1$. If A is $\forall x B$, then $\sigma_\Gamma(\forall x B) = \sigma_\Gamma(B) + 1$ and if A is $\exists x B$, then $\sigma_\Gamma(\exists x B) = \sigma_\Gamma(B) \cdot sw(n(\forall)_\Gamma + n(VAR)_\Gamma) + 1$, where $sw(k) = 1$ if $k = 0$ and k otherwise, $n(\forall)_\Gamma$ is the number of universal quantifiers occurring in Γ and $n(VAR)_\Gamma$ is the number of variables having free occurrences in Γ. The complexity of a sequent $\sigma(\Gamma)$ is $\Sigma_{A \in \Gamma} \sigma_\Gamma(A)$.

Proposition 4.10. Given a derivable sequent Γ, the maximal height of a cut-free derivation is $\sigma(\Gamma)$.

Proof: The proof is straightforward by observing that the maximal number of rules which are bottom-up applicable to Γ is precisely $\sigma(\Gamma)$. □

This gives us a decision procedure for the derivablity problem in **G3INT** whose complexity is in co-np. The procedure is shown in Table 1; where universal choice handles the branching caused by rule $R\wedge$ and Lemma 4.3 allows us to freely choose which rule to apply at each step.

Proposition 4.11. The algorithm in Table 1 runs in co-np.

Proof: The procedure is in the form of a non-deterministic Turing machine with universal choice whose computations are bounded by $\sigma(\Gamma)$. □

Observe that Prop. 4.11 entails that the satisfiability problem for monadic formulas in INF is in np. However, this result does not clash with the

Table 1. Decision procedure for **G3INT**-derivability.

Input: A sequent Γ in innex normal form.
Output: If Γ is derivable then 'yes', else a sequent.

If for some A, both A and $\overline{A}$ are in Γ **then** return 'yes' and halt;
else if some rule is applicable **then**
| pick the first rule instance applicable;
| universally choose one premise Γ' of this rule instance;
| check recursively the derivability of Γ', output the answer and halt;
else return Γ and halt;
end.

NEXP-hardness of the satisfiability problem for monadic FO-logic [4] since the conversion of an arbitrary monadic formula into an innex one can lead to an exponential explosion of $\sigma(\Gamma)$.

5. Characterisation

THEOREM 5.1 (soundness). *If* Γ *is* **G3INT**-*derivable then* $\bigvee \Gamma$ *is valid in classical FO-logic.*

PROOF: An easy induction on the height of the derivation of Γ. □

In order to prove completeness, we show that all rules of **G3INT** are semantically invertible:

LEMMA 5.2. *If there is a countermodel for all formulas in one premise of an instance of a rule of* **G3INT** *then there is a countermodel for its conclusion.*

PROOF: The case of rules $R\wedge$, $R\vee$, and $R\forall$ are immediate. For rule $R\exists$ we assume $\mathcal{M} = \langle \mathcal{D}, \mathcal{V} \rangle$ is a model and μ an assignment defined over $\mathcal{D}$ such that $\mathcal{M}, \mu$ falsifies all formulas in

$$\Gamma, P_1(z_1) \wedge \cdots \wedge P_n(z_1), \ldots, P_1(z_\ell) \wedge \cdots \wedge P_n(z_\ell) \qquad (\Delta)$$

We construct a countermodel for all formulas in $\Gamma, \exists x(P_1(x) \wedge \cdots \wedge P_n(x))$.

Initial Sequents: $\dfrac{}{\Gamma, P, \overline{P}}\,Ax$

Logical Rules: $\dfrac{\Gamma, A \qquad \Gamma, B}{\Gamma, A \wedge B}\,R\wedge \quad \dfrac{\Gamma, A, B}{\Gamma, A \vee B}\,R\vee \quad \dfrac{\Sigma, A}{\Gamma, \Diamond\Sigma, \Box A}\,R\Box \quad \dfrac{\Gamma, \Diamond A, A}{\Gamma, \Diamond A}\,R\Diamond$

Figure 3. The sequent calculus **G3T**

Given that Δ, being reduced, contains no instance of $\forall$, we can apply Lemma 4.3 to it until it becomes a multiset Δ' of literals such that $X = \{z_1, \ldots, z_\ell\}$ is the finite set of all variables occurring (free) in Δ. It is easy to see that that Δ' is falsified by $\mathcal{M}^\Delta = \langle \mathcal{D}^\Delta, \mathcal{V}^\Delta\rangle, \mu^\Delta$, where $\mathcal{D}^\Delta = \mathcal{D} \cap \mu(X)$, $\mathcal{V}^\Delta(P) = \mathcal{V}(P) \cap \mu(X)$, and μ^Δ behave like μ for all variables occurring free in Δ' and maps all other variables to $\mu(z_1)$. $\mathcal{M}^\Delta, \mu^\Delta$ falsifies also $\exists x(P_1(x) \wedge \cdots \wedge P_n(x))$ since each conjunct in $P_1(x) \wedge \cdots \wedge P_n(x)$ is false of some object in $\mathcal{D}^\Delta$. □

Theorem 5.3 (Completeness). *If $\bigvee \Gamma$ is valid then Γ is* **G3INT**-*derivable.*

Proof: By Prop. 2.3 we can assume Γ is in INF. If **G3INT** $\nvdash \Gamma$ then there is a finite proof-search tree for Γ having at least one leaf Δ that is not an initial sequent. We can easily define a countermodel for Δ from that leaf and, by Lemma 5.2, we conclude that $\bigvee \Gamma$ has a countermodel. □

6. Modal interpretation

It is well known that there is a sound and faithful interpretation of the propositional modal logic **S5** into MFOL [2]. We show in this section that the innex fragment of MFOL can be soundly and faithfully interpreted in the quantifier-free monadic fragment of the FO-modal logic **T**. This will be done by using the sequent calculus for **T** given in Figure 3, cf. [8].

Let $\mathcal{L}^\Box$ be the language obtained from the language of MFOL (cf. Section 2) by replacing $\forall$ and $\exists$ with $\Box$ and $\Diamond$, respectively. We define inductively a pair of translations τ_1, τ_2 from the language of MFOL to $\mathcal{L}^\Box$ ($\tau = \tau_2 \circ \tau_1$). Formally, given an innex sequent Γ, we have:

- $(P(y))^{\tau_1} = P(y)$
- $(\overline{P}(y))^{\tau_1} = \overline{P}(y)$
- $(A \# B)^{\tau_1} = A^\tau \# B^\tau$, with $\# \in \{\wedge, \vee\}$

- $(\forall x A)^{\tau_1} = \Box A[y/x]$, where y does not occur in Γ
- $(\exists x A)^{\tau_2} = \Diamond(A[z_1/x] \vee \ldots \vee A[z_n/x])$, where $z_1, \ldots, z_n$ are all the variables free in $(\Gamma)^{\tau_1}$.

We start by showing a preliminary lemma concerning derivability in **G3INT**.

LEMMA 6.1. *Let Γ, Π and Σ be multisets of quantifier-free, universal and existential formulas in innex normal form, respectively. If Γ, Π, Σ, is derivable, then Γ, Σ or Σ, A, where $A \in \Pi$, is derivable with at most the same height.*

PROOF: The proof runs by induction on the height of the derivation. Every case is trivial with the exception of the case in which the last rule applied is $R\forall$. In the latter case we have:

$$\frac{\Gamma, \Pi', \Sigma, P_1(y) \vee \ldots \vee P_n(y)}{\Gamma, \Pi', \Sigma, \forall x(P_1(x) \vee \ldots \vee P_n(x))}\, R\forall$$

The induction hypothesis yields the derivability of $\Gamma, \Sigma, P_1(y) \vee \ldots \vee P_n(y)$ or of A, Σ for some A in Π'. In the second case, we already have obtained the desired conclusion. In the first one, due to the eigenvariable condition, we observe that $\Sigma, P_1(y) \vee \ldots \vee P_n(y)$ is derivable or Γ, Σ is derivable. In the first case we get the desired conclusion via an application of the rule $R\forall$, the other case is trivial. $\Box$

The previous lemma allows us to prove the soundness of the embedding.

THEOREM 6.2. *If* **G3INT** *proves Δ, then* **G3T** *proves $(\Delta)^{\tau}$.*

PROOF: The proof is by induction on the height of the derivation. We detail the case of the quantifiers. Let Γ, Π and Σ be multisets of quantifier-free, universal and existential formulas in innex normal form, respectively. If the last rule applied is $R\forall$, we have:

$$\frac{\Gamma, \Sigma, \Pi, P_1(y) \vee \ldots \vee P_n(y)]}{\Gamma, \Sigma, \Pi, \forall x(P_1(x) \vee \ldots \vee P_n(x))}\, R\forall$$

Since $\Gamma, \Sigma, \Pi, P_1(y) \vee \ldots \vee P_n(y)$ is derivable, then Lemma 6.1 entails that $\Gamma, \Sigma, P_1(y) \vee \ldots \vee P_n(y)$ is derivable or Σ, C is derivable, where C is a formula in Π. The latter case is trivial and the conclusion follows from the induction hypothesis and an application of weakening. In the first case, due to the eigenvariable condition either $\Sigma, P_1(y) \vee \ldots \vee P_n(y)$ is derivable or Γ, Σ is derivable. Once again, in the latter subcase the conclusion can be obtained by the induction hypothesis and weakening. In the first subcase, we first apply the height-preserving invertibility of the rule $R\exists$ to get $A_1[y/x_1], \ldots, A_m[y/x_m], P_1(y) \vee \ldots \vee P_n(y)$, where $\Sigma = \exists x_1 A_1, \ldots, \exists x_m A_m$. Next, we have the following **G3T**-derivation:

$$\dfrac{\dfrac{\dfrac{A_1[y/x_1], \ldots, A_m[y/x_m], P_1(y) \vee \ldots \vee P_n(y)}{A_1^\tau[y/x], \ldots, A_m^\tau[y/x], P_1(y) \vee \ldots \vee P_n(y)}\text{ IH}}{\Diamond\Sigma^\tau, P_1(y) \vee \ldots \vee P_n(y)}\text{ several Weak and } R\Diamond}{(\Gamma, \Pi)^\tau, \Diamond\Sigma^\tau, \Box(P_1(y) \vee \ldots \vee P_n(y))}\ R\Box$$

If the last rule applied is $R\exists$, we proceed as follows:

$$\dfrac{\Gamma, \Sigma, A[z_1/x], \ldots, A[z_n/x]}{\Gamma, \Sigma, \exists x A}\ R\exists \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\Gamma, \Sigma, A[z_1/x], \ldots, A[z_n/x]}{\Gamma^\tau, \Sigma^\tau, A^\tau[z_1/x], \ldots, A^\tau[z_n/x]}\text{ IH}}{\Gamma^\tau, \Sigma^\tau, A^\tau[z_1/x] \vee \ldots \vee A^\tau[z_n/x]}\ R\vee}{\Gamma^\tau, \Sigma^\tau, \Diamond(A^\tau[z_1/x] \vee \ldots \vee A^\tau[z_n/x])}\text{ Weak and } R\Diamond$$

Where $A^\tau \equiv A$ since A is conjunction of literals. $\Box$

We can also prove the faithfulness of the embedding.

THEOREM 6.3. *Given a sequent Δ of monadic formulas in innex normal form, if $(\Delta)^\tau$ is derivable in* **G3T**, *then Δ is derivable in* **G3INT**.

PROOF: If the sequent is initial, the proof is immediate. If it is the conclusion of $R\wedge$, $R\vee$, the proof is straightforward by the induction hypothesis. If the last rule applied is $R\Box$, we have:

$$\dfrac{\Sigma^\tau, A^\tau[y/x]}{\Gamma^\tau, \Diamond\Sigma^\tau, \Box A^\tau}\ \Box$$

We apply the induction hypothesis, the rules $R\exists$ and $R\forall$ and weakening.

If the last rule applied is $R\Diamond$, we proceed as follows (where A is $B[z_1/x] \vee \cdots \vee B[z_n/x]$ and $\{z_1, \ldots z_n\}$ are all variables free in Γ):

$$\dfrac{A, \Diamond A^\tau, \Gamma^\tau}{\Diamond A^\tau, \Gamma^\tau} \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{A^\tau, \Diamond A^\tau, \Gamma^\tau}{B[z_1/x], \ldots, B[z_n/x], \exists x B, \Gamma}\text{ IH+ }\vee\text{-inv}}{\exists x B, \exists x B, \Gamma}\ R\exists}{\exists x B, \Gamma}\text{ Ctr}$$

$\Box$

Let us observe that the structural properties established for **G3INT**—including cut elimination—can now be proved indirectly via the embedding in the modal system.

7. Concluding remarks and future work

We have introduced a terminating sequent calculus for a fragment of MFOL. This, combined with a normal form theorem, gives a fully syntactic decision procedure for monadic classical first-order logic.

It is natural to ask whether it is possible to design a sequent calculus for the full language of monadic logic. One way to do so is to define rules which directly convert formulas in innex normal form and then to proceed as for **G3INT**. We leave this theme for future investigations.

Finally, we would like to generalize the cut-elimination strategy to other classes of logics, showing how to eliminate the cuts by induction on the degree of the cut formula. Particularly promising would be to spell out sufficient conditions for cut-elimination.

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