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ON COMBINING INTUITIONISTIC AND S4 MODAL LOGIC

Abstract

We address the problem of combining intuitionistic and S4 modal logic in a non-collapsing way inspired by the recent works in combining intuitionistic and classical logic. The combined language includes the shared constructors of both logics namely conjunction, disjunction and falsum as well as the intuitionistic implication, the classical implication and the necessity modality. We present a Gentzen calculus for the combined logic defined over a Gentzen calculus for the host S4 modal logic. The semantics is provided by Kripke structures. The calculus is proved to be sound and complete with respect to this semantics. We also show that the combined logic is a conservative extension of each component. Finally we establish that the Gentzen calculus for the combined logic enjoys cut elimination.

Keywords: combination of logics, intuitionistic logic, modal logic, cut elimination.

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1. Introduction

Prawitz was the first to recognize the relevance of tolerance when combining intuitionistic and classical first-order logic [12] (see also [13, 2]). Therein Prawitz proposes a combined logic where the intuitionistic logician accepts that the *tertium non datur* $A \vee_c \neg A$ holds even when A is an intuitionistic formula. On the other hand, the classical logician must also accept that

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the *tertium non datur* $A \vee_i \neg A$ does not hold even when A is a classical formula.

This non-collapsing combination of intuitionistic and classical logic was obtained by enriching intuitionistic logic with classical constructors while sharing falsum, conjunction, negation and the universal quantifier. This logic was endowed with a natural deduction calculus. An equivalent sequent calculus presentation was discussed in [11], under the name Ecumenical sequent calculus system (using indirect translations via cuts, see [17, 9]).

The interest on combining intuitionistic and classical logic has been around namely in fibring of logics (see [3, 4]). Fibring is a combination technique that given two logics defines another one by putting together the deductive components of each logic while sharing or not some constructors. Soon after its initial proposal, the collapsing problem of intuitionistic into classical logic was identified and a proposal for avoiding this problem appeared in [1]. Later on in [16] a general solution called modulated fibring was proposed for avoiding any such collapse. Furthermore, it is worthwhile to refer to the unified calculus LU presented in [5] where a common non-collapsing single sequent calculus for classical, intuitionistic and linear logics is proposed.

Inspired by the tolerance principle identified by Prawitz in [12], we propose a non-collapsing combination between propositional intuitionistic and propositional classical modal logic S4 sharing $\perp$, $\wedge$ and $\vee$. The idea is to embed intuitionistic logic into modal logic S4 in such a way that intuitionistic logic does not loose its identity (following [6, 8]). The properties of false, conjunction and disjunction are the same for both logics and so they share these constructors. On the other hand we have an intuitionistic implication and a classical implication because these constructors have different properties. There are also an intuitionistic and a classical negation defined by abbreviation from intuitionistic and classical implication, respectively. We consider a set of (classical) propositional variables that are also used to define (intuitionistic) propositional constructors in such a way that hereditariness (necessity) holds. In this way we work with pure Kripke structures for S4 and accommodate intuitionistic constructors in this framework.

As far as we know there are no efforts on fibring intuitionistic and modal logic S4. Nevertheless we expect that such a combination would lead to a collapse of the intuitionistic part into the classical propositional part of S4. In [7] a intuitionistic modal logic (the host) is enriched with classical constructors. This approach is different from the one we adopt herein namely because the host of our combination is classical modal logic $\mathsf{S4}$.

The paper is organized as follows. In Section 2 we present the language and the Gentzen calculus for the combination of intuitionistic and $\mathsf{S4}$ modal logic. We show that reasoning in this combination extends reasoning in the components. In Section 3 we prove that the Gentzen calculus for the combined logic enjoys cut elimination. We introduce the Kripke semantics for the combination in Section 4 and prove that the combination is conservative over the combined logics. In Section 5 we establish soundness and completeness of the Gentzen calculus with respect to the semantics. Finally in Section 6 we give an overview of the paper and discuss future work.

2. Gentzen calculus

The main objective of this section is to present a Gentzen calculus for the combination of the propositional intuitionistic logic J and propositional modal logic $\mathsf{S4}$ that we denote by $\mathsf{J} \sqcup \mathsf{S4}$. We start by presenting the language $L_{\mathsf{J} \sqcup \mathsf{S4}}$ and then the sequent calculus rules and axioms. After presenting the notion of derivation we provide some examples and establish that reasoning over the combined logic extends reasoning over each component.

We consider fixed a denumerable set P. Let $P_s = \{p_s : p \in P\}$ be the set of (classical) propositional variables. The combined logic has the following sets of constructors $C_0 = \{\bot\} \cup P_i$ where P_i is the set $\{p_i : p \in P\}$, $C_1 = \{\Box_s\}$ and $C_2 = \{\wedge, \vee, \supset_i, \supset_s\}$. We denote by $L_{\mathsf{J} \sqcup \mathsf{S4}}$ the set of formulas inductively defined by the constructors in C_1 and C_2 over $C_0 \cup P_s$. We may use $\neg_i \varphi$ and $\neg_s \varphi$ as abbreviations of $\varphi \supset_i \bot$ and $\varphi \supset_s \bot$, respectively. Moreover we use $\Diamond_s \varphi$ as an abbreviation of $\neg_s \Box_s \neg_s \varphi$. We denote by L_{J} the set of formulas inductively generated by $\wedge$, $\vee$ and $\supset_i$ over $\{\bot\} \cup P_i$. Similarly we denote by L_{S4} the set of formulas inductively generated by $\Box_s$, $\wedge$, $\vee$ and $\supset_s$ over $\{\bot\} \cup P_s$.

A *sequent* is a pair (Γ, Δ), denoted by $\Gamma \to \Delta$, where Γ and Δ are finite multisets of formulas in $L_{\mathsf{J} \sqcup \mathsf{S4}}$. The Gentzen calculus $\mathrm{G}_{\mathsf{J} \sqcup \mathsf{S4}}$ is composed of the following rules for constructors:

$$(\mathrm{LP}_i) \quad \frac{\Box_s p_s, \Gamma \to \Delta}{p_i, \Gamma \to \Delta} \qquad\qquad (\mathrm{RP}_i) \quad \frac{\Gamma \to \Delta, \Box_s p_s}{\Gamma \to \Delta, p_i}$$

$$(\mathrm{L}\wedge)\quad \frac{\beta_1, \beta_2, \Gamma \to \Delta}{\beta_1 \wedge \beta_2, \Gamma \to \Delta} \qquad (\mathrm{R}\wedge)\quad \frac{\Gamma \to \Delta, \beta_1 \quad \Gamma \to \Delta, \beta_2}{\Gamma \to \Delta, \beta_1 \wedge \beta_2}$$

$$(\mathrm{L}\vee)\quad \frac{\beta_1, \Gamma \to \Delta \quad \beta_2, \Gamma \to \Delta}{\beta_1 \vee \beta_2, \Gamma \to \Delta} \qquad (\mathrm{R}\vee)\quad \frac{\Gamma \to \Delta, \beta_1, \beta_2}{\Gamma \to \Delta, \beta_1 \vee \beta_2}$$

$$(\mathrm{L}\supset_s)\quad \frac{\Gamma \to \Delta, \beta_1 \qquad \beta_2, \Gamma \to \Delta}{\beta_1 \supset_s \beta_2, \Gamma \to \Delta} \qquad (\mathrm{R}\supset_s)\quad \frac{\beta_1, \Gamma \to \Delta, \beta_2}{\Gamma \to \Delta, \beta_1 \supset_s \beta_2}$$

$$(\mathrm{L}\supset_i)\quad \frac{\Box_s(\beta_1 \supset_s \beta_2), \Gamma \to \Delta}{\beta_1 \supset_i \beta_2, \Gamma \to \Delta} \qquad (\mathrm{R}\supset_i)\quad \frac{\Gamma \to \Delta, \Box_s(\beta_1 \supset_s \beta_2)}{\Gamma \to \Delta, \beta_1 \supset_i \beta_2}$$

$$(\mathrm{L}\Box_s)\quad \frac{\beta, \Box_s\beta, \Gamma \to \Delta}{\Box_s\beta, \Gamma \to \Delta} \qquad (\mathrm{R}\Box_s)\quad \frac{\Box_s\Gamma \to \Diamond_s \Delta, \beta}{\Gamma', \Box_s\Gamma \to \Diamond_s \Delta, \Delta', \Box_s\beta}$$

the following axioms

$$(\mathrm{Ax})\quad p_s, \Gamma \to \Delta, p_s \qquad (\mathrm{L}\bot)\quad \bot, \Gamma \to \Delta$$

and

$$(\mathrm{Cut})\quad \frac{\Gamma \to \Delta, \beta \quad \beta, \Gamma \to \Delta}{\Gamma \to \Delta}$$

known as the *cut rule*.

A *derivation* for $\Psi \to \Lambda$ is a sequence $\Psi_1 \to \Lambda_1 \ldots \Psi_n \to \Lambda_n$ such that $\Psi_1 \to \Lambda_1$ is $\Psi \to \Lambda$ and for $j = 1, \ldots, n$

- either $\Psi_j \to \Lambda_j$ is an axiom
- or $\Psi_j \to \Lambda_j$ is the conclusion of a rule and the premises appear from $j+1$ to n.

When there is a derivation for $\Psi \to \Lambda$ we may write

$$\vdash_{\mathsf{G}_{\mathsf{J}\sqcup\mathsf{S4}}} \Psi \to \Lambda.$$

We say that φ is a *theorem* in $\mathsf{J}\sqcup\mathsf{S4}$, written $\vdash_{\mathsf{J}\sqcup\mathsf{S4}} \varphi$, whenever $\vdash_{G_{\mathsf{J}\sqcup\mathsf{S4}}} \to \varphi$.

Observe that the rules applied in a derivation are such that the premiss(es) is (are) below the line of inference.

We now establish useful proof-theoretical results concerning weakening, cut, inversion and contraction.

Proposition 2.1. If there is a derivation $\mathcal{D}$ for $\Psi \to \Lambda$ in $\mathrm{G}_{\mathsf{J} \sqcup \mathsf{S4}}$ then there is a derivation $\mathcal{D}[\Psi' \to \Lambda']$ for $\Psi', \Psi \to \Lambda, \Lambda'$ in $\mathrm{G}_{\mathsf{J} \sqcup \mathsf{S4}}$ using the same rules by the same order over the same formulas.

The previous result follows immediately by a straightforward induction. We also omit the proof of the following proposition because it follows straightforwardly.

Proposition 2.2. The multiplicative cut rule

$$\frac{\Gamma \to \Delta, \beta \quad \beta, \Gamma' \to \Delta'}{\Gamma, \Gamma' \to \Delta, \Delta'}$$

is derivable in $\mathrm{G}_{\mathsf{J} \sqcup \mathsf{S4}}$.

The following result is needed for proving that the contraction rules are derivable.

Proposition 2.3. The inversion lemma holds for all rules of $\mathrm{G}_{\mathsf{J} \sqcup \mathsf{S4}}$.

We now state that the left and right contraction rules are derivable in $\mathrm{G}_{\mathsf{J} \sqcup \mathsf{S4}}$. The proof is similar to the proof of Proposition 2.17 of [14].

Proposition 2.4. If there is a derivation for $\varphi, \varphi, \Psi \to \Lambda$ in $\mathrm{G}_{\mathsf{J} \sqcup \mathsf{S4}}$ then there is a derivation for $\varphi, \Psi \to \Lambda$ in $\mathrm{G}_{\mathsf{J} \sqcup \mathsf{S4}}$ with at most the same length and with the same cut formulas. Moreover, if there is a derivation for $\Psi \to \Lambda, \varphi, \varphi$ in $\mathrm{G}_{\mathsf{J} \sqcup \mathsf{S4}}$ then there is a derivation for $\Psi \to \Lambda, \varphi$ in $\mathrm{G}_{\mathsf{J} \sqcup \mathsf{S4}}$ with at most the same length and with the same cut formulas.

We now provide derived rules for the negations $\neg_s$ and $\neg_i$.

Proposition 2.5. *Let* $\beta \in L_{\mathsf{J} \sqcup \mathsf{S4}}$. *Then*

$$(L\neg_i) \quad \frac{\Box_s(\neg_s \beta), \Gamma \to \Delta}{\neg_i \beta, \Gamma \to \Delta} \qquad (R\neg_i) \quad \frac{\Gamma \to \Delta, \Box_s(\neg_s \beta)}{\Gamma \to \Delta, \neg_i \beta}$$

and

$$(L\neg_s) \quad \frac{\Gamma \to \Delta, \beta}{\neg_s \beta, \Gamma \to \Delta} \qquad (R\neg_s) \quad \frac{\beta, \Gamma \to \Delta}{\Gamma \to \Delta, \neg_s \beta}$$

PROOF: The rules for $\neg_i$ follow from the following sequences:

1.	$\beta \supset_i \bot, \Gamma \to \Delta$	L$\supset_i$ 2
2.	$\Box_s(\beta \supset_s \bot), \Gamma \to \Delta$	

and

1.	$\Gamma \to \Delta, \beta \supset_i \bot$	R$\supset_i$ 2
2.	$\Gamma \to \Delta, \Box_s(\beta \supset_s \bot)$	

using the abbreviations of $\neg_i$ and $\neg_s$. Similarly for the rules for $\neg_s$. $\Box$

Observe that

$$\vdash_{\mathsf{G}_{\mathsf{J} \sqcup \mathsf{S4}}} \beta, \Gamma \to \Delta, \beta$$

and so we use this fact when presenting derivations under the name gAx.

The reader may wonder whether the rules of a sequent calculus for J are derivable in the Gentzen calculus for the combination $\mathsf{J} \sqcup \mathsf{S4}$. The answer is that it is not always the case. For instance the usual intuitionistic rule for introducing $\supset_i$ in the succedent

$$\frac{\Gamma, \beta_1 \to \beta_2}{\Gamma \to \beta_1 \supset_i \beta_2}$$

is not always derivable in $\mathrm{G}_{\mathsf{J} \sqcup \mathsf{S4}}$. It is true that if Γ is empty, we could obtain

1	$\to \beta_1 \supset_i \beta_2$
2	$\to \Box_s(\beta_1 \supset_s \beta_2)$
3	$\to \beta_1 \supset_s \beta_2$
4	$\beta_1 \to \beta_2$

But if Γ is not empty, the application of (R$\Box_s$) would not be possible in general.

Example 2.6. The following derivation

1.	$\to p_i \supset_i (\Box_s p_i)$	$R{\supset_i}$ 2
2.	$\to \Box_s(p_i \supset_s (\Box_s p_i))$	$R\Box_s$ 3
3.	$\to p_i \supset_s (\Box_s p_i)$	$R{\supset_s}$ 4
4.	$p_i \to \Box_s p_i$	LP_i 5
5.	$\Box_s p_s \to \Box_s p_i$	$R\Box_s$ 6
6.	$\Box_s p_s \to p_i$	RP_i 7
7.	$\Box_s p_s \to \Box_s p_s$	gAx

shows that $\vdash_{\mathsf{J}\sqcup\mathsf{S4}} p_i \supset_i (\Box_s p_i)$ expressing that hereditariness holds for any constructor p_i in P_i. The derivation

1.	$\to \varphi \vee (\neg_s \varphi)$	$R\vee$ 2
2.	$\to \varphi, \neg_s \varphi$	$R\neg_s$ 3
3.	$\varphi \to \varphi$	gAx

proves that $\vdash_{\mathsf{J}\sqcup\mathsf{S4}} \varphi \vee_s (\neg_s \varphi)$ asserting that *tertium non datur* holds when using classical negation. Finally, the derivation

1.	$\to (\varphi_1 \supset_i \varphi_2) \supset_i (\varphi_1 \supset_s \varphi_2)$	$R{\supset_i}$ 2
2.	$\to \Box_s((\varphi_1 \supset_i \varphi_2) \supset_s (\varphi_1 \supset_s \varphi_2))$	$R\Box_s$ 3
3.	$\to (\varphi_1 \supset_i \varphi_2) \supset_s (\varphi_1 \supset_s \varphi_2)$	$R{\supset_s}$ 4
4.	$\varphi_1 \supset_i \varphi_2 \to \varphi_1 \supset_s \varphi_2$	$R{\supset_s}$ 5
5.	$\varphi_1, \varphi_1 \supset_i \varphi_2 \to \varphi_2$	$L{\supset_i}$ 6
6.	$\varphi_1, \Box_s(\varphi_1 \supset_s \varphi_2) \to \varphi_2$	$L\Box_s$ 7
7.	$\varphi_1, \varphi_1 \supset_s \varphi_2, \Box_s(\varphi_1 \supset_s \varphi_2) \to \varphi_2$	$L{\supset_s}$ 8,9
8.	$\varphi_1, \varphi_2, \Box_s(\varphi_1 \supset_s \varphi_2) \to \varphi_2$	gAx
9.	$\varphi_1, \Box_s(\varphi_1 \supset_s \varphi_2) \to \varphi_2, \varphi_1$	gAx

proves that $\vdash_{\mathsf{J}\sqcup\mathsf{S4}} (\varphi_1 \supset_i \varphi_2) \supset_i (\varphi_1 \supset_s \varphi_2)$ expressing the intuitionistic relationship between $\supset_i$ and $\supset_s$.

Next result shows that the combined logic is an extension of intuitionistic logic, that is, every theorem in intuitionistic logic J is a theorem in the combination $\mathsf{J}\sqcup\mathsf{S4}$.

PROPOSITION 2.7. Let $\varphi \in L_{\mathsf{J}}$ and H_{J} be the Hilbert calculus for intuitionistic logic presented in [15] over L_{J}. Then $\vdash_{\mathrm{H}_{\mathsf{J}}} \varphi$ in H_{J} implies $\vdash_{\mathrm{G}_{\mathsf{J}\sqcup\mathsf{S4}}} \varphi$.

PROOF: We start by proving that if φ is an axiom of H_{J} then $\vdash_{\mathrm{G}_{\mathsf{J}\sqcup\mathsf{S4}}} \varphi$.

We just consider the axiom

$$(\varphi_1 \supset_i \varphi_2) \supset_i ((\varphi_1 \supset_i (\neg_i \varphi_2)) \supset_i (\neg_i \varphi_1)).$$

The sequence

1.	$\to (\varphi_1 \supset_i \varphi_2) \supset_i ((\varphi_1 \supset_i (\neg_i \varphi_2)) \supset_i (\neg_i \varphi_1))$	R$\supset_i$ 2
2.	$\to \Box_s((\varphi_1 \supset_i \varphi_2) \supset_s ((\varphi_1 \supset_i (\neg_i \varphi_2)) \supset_i (\neg_i \varphi_1)))$	R$\Box_s$ 3
3.	$\to (\varphi_1 \supset_i \varphi_2) \supset_s ((\varphi_1 \supset_i (\neg_i \varphi_2)) \supset_i (\neg_i \varphi_1))$	R$\supset_s$ 4
4.	$\varphi_1 \supset_i \varphi_2 \to (\varphi_1 \supset_i (\neg_i \varphi_2)) \supset_i (\neg_i \varphi_1)$	L$\supset_i$ 5
5.	$\Box_s(\varphi_1 \supset_s \varphi_2) \to (\varphi_1 \supset_i (\neg_i \varphi_2)) \supset_i (\neg_i \varphi_1)$	R$\supset_i$ 6
6.	$\Box_s(\varphi_1 \supset_s \varphi_2) \to \Box_s((\varphi_1 \supset_i (\neg_i \varphi_2)) \supset_s (\neg_i \varphi_1))$	R$\Box_s$ 7
7.	$\Box_s(\varphi_1 \supset_s \varphi_2) \to (\varphi_1 \supset_i (\neg_i \varphi_2)) \supset_s (\neg_i \varphi_1)$	R$\supset_s$ 8
8.	$\varphi_1 \supset_i (\neg_i \varphi_2), \Box_s(\varphi_1 \supset_s \varphi_2) \to \neg_i \varphi_1$	L$\supset_i$ 9
9.	$\Box_s(\varphi_1 \supset_s (\neg_i \varphi_2)), \Box_s(\varphi_1 \supset_s \varphi_2) \to \neg_i \varphi_1$	R$\neg_i$ 10
10.	$\Box_s(\varphi_1 \supset_s (\neg_i \varphi_2)), \Box_s(\varphi_1 \supset_s \varphi_2) \to \Box_s \neg_s \varphi_1$	R$\Box_s$ 11
11.	$\Box_s(\varphi_1 \supset_s (\neg_i \varphi_2)), \Box_s(\varphi_1 \supset_s \varphi_2) \to \neg_s \varphi_1$	(L$\Box_s$)^2 12
12.	$\varphi_1 \supset_s (\neg_i \varphi_2), \varphi_1 \supset_s \varphi_2, \Box_s(\varphi_1 \supset_s (\neg_i \varphi_2)),$ $\Box_s(\varphi_1 \supset_s \varphi_2) \to \neg_s \varphi_1$	R$\neg_s$ 13
13.	$\varphi_1, \varphi_1 \supset_s (\neg_i \varphi_2), \varphi_1 \supset_s \varphi_2, \Box_s(\varphi_1 \supset_s (\neg_i \varphi_2)),$ $\Box_s(\varphi_1 \supset_s \varphi_2) \to$	L$\supset_s$ 14,15
14.	$\varphi_1, \varphi_1 \supset_s (\neg_i \varphi_2), \Box_s(\varphi_1 \supset_s (\neg_i \varphi_2)),$ $\Box_s(\varphi_1 \supset_s \varphi_2) \to \varphi_1$	gAx
15.	$\varphi_1, \varphi_1 \supset_s (\neg_i \varphi_2), \varphi_2, \Box_s(\varphi_1 \supset_s (\neg_i \varphi_2)),$ $\Box_s(\varphi_1 \supset_s \varphi_2) \to$	L$\supset_s$ 16,19
16.	$\varphi_1, \neg_i \varphi_2, \varphi_2, \Box_s(\varphi_1 \supset_s (\neg_i \varphi_2)), \Box_s(\varphi_1 \supset_s \varphi_2) \to$	L$\neg_i$ 17
17.	$\varphi_1, \Box_s \neg_s \varphi_2, \varphi_2, \Box_s(\varphi_1 \supset_s (\neg_i \varphi_2)), \Box_s(\varphi_1 \supset_s \varphi_2) \to$	L$\Box_s$ 18
18.	$\varphi_1, \neg_s \varphi_2, \Box_s \neg_s \varphi_2, \varphi_2, \Box_s(\varphi_1 \supset_s (\neg_i \varphi_2)),$ $\Box_s(\varphi_1 \supset_s \varphi_2) \to$	L$\neg_s$ 20
19.	$\varphi_1, \varphi_2, \Box_s(\varphi_1 \supset_s (\neg_i \varphi_2)), \Box_s(\varphi_1 \supset_s \varphi_2) \to \varphi_1$	gAx
20.	$\varphi_1, \Box_s \neg_s \varphi_2, \varphi_2, \Box_s(\varphi_1 \supset_s (\neg_i \varphi_2)),$ $\Box_s(\varphi_1 \supset_s \varphi_2) \to \varphi_2$	gAx

is a derivation for $\vdash_{\mathsf{G}_{\mathsf{J} \sqcup \mathsf{S4}}} (\varphi_1 \supset_i \varphi_2) \supset_i ((\varphi_1 \supset_i (\neg_i \varphi_2)) \supset_i (\neg_i \varphi_1))$.

It remains to show that $\vdash_{\mathrm{G}_{\mathsf{J}\sqcup\mathsf{S4}}} \varphi_1, \varphi_1 \supset_i \varphi_2 \to \varphi_2$. Indeed consider the sequence

1.	$\varphi_1, \varphi_1 \supset_i \varphi_2 \to \varphi_2$	$\mathrm{L}\supset_i$ 2
2.	$\varphi_1, \Box_s(\varphi_1 \supset_s \varphi_2) \to \varphi_2$	$\mathrm{L}\Box_s$ 3
3.	$\varphi_1, \varphi_1 \supset_s \varphi_2, \Box_s(\varphi_1 \supset_s \varphi_2) \to \varphi_2$	$\mathrm{L}\supset_s$ 4,5
4.	$\varphi_1, \varphi_2, \Box_s(\varphi_1 \supset_s \varphi_2) \to \varphi_2$	gAx
5.	$\varphi_1, \Box_s(\varphi_1 \supset_s \varphi_2) \to \varphi_2, \varphi_1$	gAx

The fact that there is a derivation for $\mathrm{G}_{\mathsf{J}\sqcup\mathsf{S4}}$ for $\to \varphi$ follows by a straightforward induction on the length of a derivation for φ in H_{J}. □

We provide an example of the use of *Modus Ponens* (MP) in the context of a derivation in $\mathrm{G}_{\mathsf{J}\sqcup\mathsf{S4}}$. We now show that

$$\begin{cases} \vdash_{\mathrm{G}_{\mathsf{J}\sqcup\mathsf{S4}}} \to \varphi_1 \\ \vdash_{\mathrm{G}_{\mathsf{J}\sqcup\mathsf{S4}}} \to \varphi_1 \supset_i \varphi_2 \end{cases} \quad \text{implies} \quad \vdash_{\mathrm{G}_{\mathsf{J}\sqcup\mathsf{S4}}} \to \varphi_2.$$

We start by observing that there are derivations for

$$\begin{cases} (\dagger) & \vdash_{\mathrm{G}_{\mathsf{J}\sqcup\mathsf{S4}}} \to \varphi_1, \varphi_2 \\ (\ddagger) & \vdash_{\mathrm{G}_{\mathsf{J}\sqcup\mathsf{S4}}} \varphi_1 \to \varphi_2, \varphi_1 \supset_i \varphi_2 \end{cases}$$

using Proposition 2.1. Then the sequence

1.	$\to \varphi_2$	Cut 2,3
2.	$\varphi_1 \to \varphi_2$	Cut 4,5
3.	$\to \varphi_1, \varphi_2$	($\dagger$)
4	$\varphi_1 \supset_i \varphi_2, \varphi_1 \to \varphi_2$	MP
5.	$\varphi_1 \to \varphi_2, \varphi_1 \supset_i \varphi_2$	($\ddagger$)

is a derivation for $\vdash_{\mathrm{G}_{\mathsf{J}\sqcup\mathsf{S4}}} \to \varphi_2$.

Similarly to the previous result it is straightforward to show that reasoning over the combined logic is an extension of the reasoning in S4 modal logic.

PROPOSITION 2.8. Let $\varphi \in L_{\mathsf{S4}}$. Then φ is a theorem of $\mathsf{J} \sqcup \mathsf{S4}$ when φ is a theorem of $\mathsf{S4}$.

The next example shows that *tertium non datur* holds in the combined logic $\mathsf{J} \sqcup \mathsf{S4}$ with respect to the L_{S4} fragment.

Example 2.9. Let $\varphi \in L_{\mathsf{S4}}$. Then $\varphi \vee (\neg_s \varphi)$ is a theorem in $\mathsf{J} \sqcup \mathsf{S4}$, by Proposition 2.8, since $\varphi \vee (\neg_s \varphi)$ is a theorem in $\mathsf{S4}$.

3. Cut elimination

The main goal of this section is to prove the *Gentzen's Hauptsatz* for $\mathrm{G}_{\mathsf{J} \sqcup \mathsf{S4}}$. We follow the strategy of the proof in [14].

We start by introducing the notion of branch of a derivation. A *branch* of a derivation $\Psi_1 \to \Lambda_1 \ \cdots \ \Psi_n \to \Lambda_n$ starting at sequent $\Psi_i \to \Lambda_i$ is a finite subsequence $\Psi_{i_1} \to \Lambda_{i_1} \ \cdots \ \Psi_{i_m} \to \Lambda_{i_m}$ of the derivation such that:

- $\Psi_{i_1} \to \Lambda_{i_1}$ is $\Psi_i \to \Lambda_i$;
- for each $1 \leq j < m$, $\Psi_{i_j} \to \Lambda_{i_j}$ is the conclusion of a rule in the derivation and $\Psi_{i_{j+1}} \to \Lambda_{i_{j+1}}$ is a premise of that rule in the derivation;
- $\Psi_{i_m} \to \Lambda_{i_m}$ is either Ax or L$\perp$.

Moreover, the *depth* of a branch is the number of sequents in the branch minus 1.

Let $\mathcal{D}$ be a derivation in $\mathrm{G}_{\mathsf{J} \sqcup \mathsf{S4}}$ where the cut rule was applied in step i from premises at steps j and k.

The *level* of this cut application at i is the sum of the maximum depth of a branch starting at the premise in j with the maximum depth of a branch starting at the premise in k. The *complexity* of a formula φ denoted by $|\varphi|$ is inductively defined as follows.

- $|p_s| = |\perp| = 0$ for every $p_s \in P_s$
- $|p_i| = 2$ for every $p_i \in P_i$
- $|\varphi_1 \wedge \varphi_2| = |\varphi_1 \vee \varphi_2| = |\varphi_1 \supset_s \varphi_2| = \max(|\varphi_1|, |\varphi_2|) + 1$

- $|\varphi_1 \supset_i \varphi_2| = \max(|\varphi_1|, |\varphi_2|) + 3$
- $|\Box_s \varphi_1| = |\varphi_1| + 1$.

The *rank* of a cut application in $\mathcal{D}$ is the complexity of the respective cut formula plus 1. The *cutrank* of $\mathcal{D}$ is the maximum of the ranks of the cut applications in $\mathcal{D}$ (the cutrank of a derivation with no cut applications is 0).

PROPOSITION 3.1. Given a derivation $\mathcal{D}$ for $\vdash_{\mathsf{G_J \sqcup S4}} \Psi \to \Lambda$ where $\Psi \to \Lambda$ is obtained by a cut from derivations with a lower cutrank than $\mathcal{D}$ then there is a derivation $\mathcal{D}^\bullet$ for $\vdash_{\mathsf{G_J \sqcup S4}} \Psi \to \Lambda$ with a lower cutrank than $\mathcal{D}$.

PROOF: Let $\mathcal{D}$ be

1	$\Psi \to \Lambda$	Cut 2,n
2	$\Psi \to \Lambda, \varphi$	
	$\mathcal{D}_1$	
n	$\varphi, \Psi \to \Lambda$	
	$\mathcal{D}_2$	

The proof follows by induction on the level of the cut. The base cases are straightforward (see [17] and [14]). With respect to the inductive step we only consider the case where the lengths of $\mathcal{D}_1$ and $\mathcal{D}_2$ are greater than 1. We start by considering the case where φ is principal in both premises of the cut. There are several subcases to consider depending on the main constructor of φ. We omit the subcases where the main constructor is from $\mathsf{S4}$ (see [17]).

(1) φ is $p_i \in P_i$. Then $\mathcal{D}$ is the sequence

1.	$\Psi \to \Lambda$	Cut 2,n
2.	$\Psi \to \Lambda, p_i$	RP$_i$ 3
3.	$\Psi \to \Lambda, \Box_s p_s$	
	$\mathcal{D}'_1$	
n.	$p_i, \Psi \to \Lambda$	LP$_i$ $n+1$
$n+1$.	$\Box_s p_s, \Psi \to \Lambda$	
	$\mathcal{D}'_2$	

Hence the target $\mathcal{D}^\bullet$ can be of the form

$$\begin{array}{lll} 1. & \Psi \rightarrow \Lambda & \text{Cut } 2, n-1 \\ 2. & \Psi \rightarrow \Lambda, \Box_s p_s & \\ & \mathcal{D}'_1 & \\ n-1. & \Box_s p_s, \Psi \rightarrow \Lambda & \\ & \mathcal{D}'_2 & \end{array}$$

since this derivation has lower cutrank than $\mathcal{D}$ and it is for the same goal.
(2) φ is the formula $\varphi_1 \supset_i \varphi_2$. Then $\mathcal{D}$ is the sequence

$$\begin{array}{lll} 1. & \Psi \rightarrow \Lambda & \text{Cut } 2, n \\ 2. & \Psi \rightarrow \Lambda, \varphi_1 \supset_i \varphi_2 & \text{R}\supset_i 3 \\ 3. & \Psi \rightarrow \Lambda, \Box_s(\varphi_1 \supset_s \varphi_2) & \\ & \mathcal{D}'_1 & \\ n. & \varphi_1 \supset_i \varphi_2, \Psi \rightarrow \Lambda & \text{L}\supset_i n+1 \\ n+1. & \Box_s(\varphi_1 \supset_s \varphi_2), \Psi \rightarrow \Lambda & \\ & \mathcal{D}'_2 & \end{array}$$

Thus the target $\mathcal{D}^\bullet$ can be of the form

$$\begin{array}{lll} 1. & \Psi \rightarrow \Lambda & \text{Cut } 2, n-1 \\ 2. & \Psi \rightarrow \Lambda, \Box_s(\varphi_1 \supset_s \varphi_2) & \\ & \mathcal{D}'_1 & \\ n-1. & \Box_s(\varphi_1 \supset_s \varphi_2), \Psi \rightarrow \Lambda & \\ & \mathcal{D}'_2 & \end{array}$$

because this derivation has lower cutrank than $\mathcal{D}$ and it is for the same goal.

We now consider the case where the cut formula is not principal in the premise at step 2. Moreover we assume that the rule applied at step 2 is L$\supset_i$. So $\mathcal{D}$ is of the following form:

$$
\begin{array}{lll}
1. & \varphi_1 \supset_i \varphi_2, \Psi_1 \to \Lambda & \text{Cut } 2,n \\
2. & \varphi_1 \supset_i \varphi_2, \Psi_1 \to \Lambda, \varphi & \text{L}\supset_i 3 \\
3. & \Box_s(\varphi_1 \supset_s \varphi_2), \Psi_1 \to \Lambda, \varphi & \\
 & \mathcal{D}'_1 & \\
n. & \varphi, \varphi_1 \supset_i \varphi_2, \Psi_1 \to \Lambda & \\
 & \mathcal{D}_2 &
\end{array}
$$

Thus Cut can be applied to the premise of $\supset_i$ taking into account Proposition 2.1:

$$
\begin{array}{lll}
1. & \Box_s(\varphi_1 \supset_s \varphi_2), \varphi_1 \supset_i \varphi_2, \Psi_1 \to \Lambda & \text{Cut } 2,n \\
2. & \Box_s(\varphi_1 \supset_s \varphi_2), \varphi_1 \supset_i \varphi_2, \Psi_1 \to \Lambda, \varphi & \\
 & \mathcal{D}'_1[\varphi_1 \supset_i \varphi_2 \to] & \\
n. & \varphi, \Box_s(\varphi_1 \supset_s \varphi_2), \varphi_1 \supset_i \varphi_2, \Psi_1 \to \Lambda & \\
 & \mathcal{D}_2[\Box_s(\varphi_1 \supset_s \varphi_2) \to] &
\end{array}
$$

Then by the induction hypothesis on the level of the cut there is the following derivation

$$
\begin{array}{ll}
1. & \Box_s(\varphi_1 \supset_s \varphi_2), \varphi_1 \supset_i \varphi_2, \Psi_1 \to \Lambda \\
 & \mathcal{D}^\bullet_1
\end{array}
$$

with less cutrank than the original one. Hence we have the following derivation

$$
\begin{array}{lll}
1. & \varphi_1 \supset_i \varphi_2, \varphi_1 \supset_i \varphi_2, \Psi_1 \to \Lambda & \text{L}\supset_i 2 \\
2. & \Box_s(\varphi_1 \supset_s \varphi_2), \varphi_1 \supset_i \varphi_2, \Psi_1 \to \Lambda & \\
 & \mathcal{D}^\bullet_1 &
\end{array}
$$

and the thesis follows by Proposition 2.4. $\Box$

The next result follows straightforwardly by induction on the number of cuts with the greatest cutrank taking into account Proposition 3.1.

PROPOSITION 3.2. Given a derivation for $\vdash_{G_{\mathsf{J} \sqcup \mathsf{S4}}} \Psi \to \Lambda$ with non null cutrank then there is a derivation for $\vdash_{G_{\mathsf{J} \sqcup \mathsf{S4}}} \Psi \to \Lambda$ with a lower cutrank than the given one.

Finally, we are ready to establish *Gentzen's Hauptsatz* for $G_{\mathsf{J} \sqcup \mathsf{S4}}$. The proof follows immediately by induction on the cutrank of the given derivation taking into account Proposition 3.2.

PROPOSITION 3.3. Given a derivation for $\vdash_{G_{\mathsf{J} \sqcup \mathsf{S4}}} \Psi \to \Lambda$, then there is a derivation with no cut applications for $\vdash_{G_{\mathsf{J} \sqcup \mathsf{S4}}} \Psi \to \Lambda$.

4. Kripke semantics

The objective of this section is to introduce the main semantic concepts for $\mathsf{J} \sqcup \mathsf{S4}$. Then we prove that the combined logic is conservative with respect to each component.

A *Kripke structure* for the combined logic $\mathsf{J} \sqcup \mathsf{S4}$ is a triple $M = (W, R, V)$ such that (W, R) is a Kripke frame where R is a reflexive and transitive relation and $V : P_s \times W \to \{0, 1\}$ is a valuation map. We denote by $\mathcal{M}_{\mathsf{J} \sqcup \mathsf{S4}}$ the class of all Kripke structures for $\mathsf{J} \sqcup \mathsf{S4}$.

We define that $M \in \mathcal{M}_{\mathsf{J} \sqcup \mathsf{S4}}$ and $w \in W$ *locally satisfies* φ written

$$M, w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \varphi$$

by induction on φ as follows:

- $M, w \nVdash_{\mathsf{J} \sqcup \mathsf{S4}} \perp$
- $M, w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} p_s$ whenever $V(p_s, w) = 1$
- $M, w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} p_i$ whenever $V(p_s, w') = 1$ for every $w' \in W$ such that wRw'
- $M, w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \varphi_1 \wedge \varphi_2$ whenever $M, w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \varphi_j$ for each $j = 1, 2$
- $M, w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \varphi_1 \vee \varphi_2$ whenever $M, w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \varphi_j$ for some $j = 1, 2$
- $M, w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \varphi_1 \supset_s \varphi_2$ whenever $M, w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \varphi_1$ implies $M, w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \varphi_2$
- $M, w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \varphi_1 \supset_i \varphi_2$ whenever $M, w' \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \varphi_1$ implies $M, w' \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \varphi_2$ for every $w' \in W$ such that wRw'

- $M, w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \Box_s \varphi_1$ whenever $M, w' \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \varphi_1$ for all $w' \in W$ such that wRw'.

Following the abbreviations we also have

- $M, w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \neg_s \varphi$ whenever $M, w \nVdash_{\mathsf{J} \sqcup \mathsf{S4}} \varphi$
- $M, w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \neg_i \varphi$ whenever $M, w' \nVdash_{\mathsf{J} \sqcup \mathsf{S4}} \varphi$ for every $w' \in W$ such that wRw'.

We extend local satisfaction to sets of formulas as follows: $M, w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \Psi$ whenever $M, w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \psi$ for every $\psi \in \Psi$.

Moreover we say that M *satisfies* φ, written

$$M \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \varphi$$

whenever $M, w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \varphi$ for every $w \in W$ and $M \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \Psi$ whenever $M \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \psi$ for every $\psi \in \Psi$. Finally, we say that Ψ *entails* φ, written

$$\Psi \vDash_{\mathsf{J} \sqcup \mathsf{S4}} \varphi$$

if $M \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \varphi$ whenever $M \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \Psi$ for every $M \in \mathcal{M}_{\mathsf{J} \sqcup \mathsf{S4}}$. When $\Psi = \emptyset$ we say that φ is *valid* and write $\vDash_{\mathsf{J} \sqcup \mathsf{S4}} \varphi$.

We now show that the combined logic $\mathsf{J} \sqcup \mathsf{S4}$ is conservative with respect to intuitionistic logic J. We assume that J is endowed with a Kripke semantics (see [15]) and denote by $\mathcal{M}_{\mathsf{J}}$ the class of all Kripke structures for J.

Proposition 4.1. Let $\varphi \in L_{\mathsf{J}}$. Then $\vDash_{\mathsf{J} \sqcup \mathsf{S4}} \varphi$ implies $\vDash_{\mathsf{J}} \varphi$.

Proof: Let $M \in \mathcal{M}_{\mathsf{J}}$ where $M = (W, R, V)$. We denote by M' the Kripke structure (W, R, V') with $V' : P_s \times W \to \{0, 1\}$ such that $V'(p_s, w) = 1$ whenever $V(p_i, w) = 1$ and $V'(p_s, w) = 0$ otherwise. Thus M' is a Kripke structure for $\mathsf{J} \sqcup \mathsf{S4}$. We start by proving by induction on φ that

$$M, w \Vdash_{\mathsf{J}} \varphi \text{ if and only if } M', w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \varphi.$$

(Base) φ is p_i. Thus $M, w \Vdash_{\mathsf{J}} p_i$ iff $V(p_i, w) = 1$ iff $V(p_i, w') = 1$ for every $w' \in W$ such that wRw' iff $V'(p_s, w') = 1$ for every $w' \in W$ such that wRw' iff $M', w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} p_i$.

(Step) We only consider the case where φ is $\varphi_1 \supset_i \varphi_2$. Hence $M, w \Vdash_{\mathsf{J}} \varphi_1 \supset_i \varphi_2$ iff for every $w' \in W$ such that wRw' if $M, w' \Vdash_{\mathsf{J}} \varphi_1$ then $M, w' \Vdash_{\mathsf{J}}$

φ_2 iff (IH) for every $w' \in W$ such that wRw' if $M', w' \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \varphi_1$ then $M', w' \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \varphi_2$ iff $M', w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \varphi_1 \supset_i \varphi_2$.

So $M' \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \varphi$ if and only if $M \Vdash_{\mathsf{J}} \varphi$.

Finally we are ready to prove the thesis. Assume that $\vDash_{\mathsf{J} \sqcup \mathsf{S4}} \varphi$ and let $M \in \mathcal{M}_{\mathsf{J}}$. Then M' as defined above is in $\mathcal{M}_{\mathsf{J} \sqcup \mathsf{S4}}$. Hence $M' \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \varphi$. Thus, as shown above $M \Vdash_{\mathsf{J}} \varphi$. □

The next example shows that *tertium non datur* does not hold in the L_{J} fragment of the combined logic $\mathsf{J} \sqcup \mathsf{S4}$.

Example 4.2. Let $\varphi \in L_{\mathsf{J}}$. Then $\nvDash_{\mathsf{J} \sqcup \mathsf{S4}} \varphi \vee (\neg_i \varphi)$ by Proposition 4.1 because $\nvDash_{\mathsf{J}} \varphi \vee (\neg_i \varphi)$.

It is straightforward to show that validity over the combined logic is a conservative extension with respect to validity in $\mathsf{S4}$ modal logic.

PROPOSITION 4.3. Let $\varphi \in L_{\mathsf{S4}}$. Then φ is valid in $\mathsf{J} \sqcup \mathsf{S4}$ if and only if φ is valid in $\mathsf{S4}$.

5. Soundness and completeness

The main objective of this section is to prove that the Gentzen calculus $\mathrm{G}_{\mathsf{J} \sqcup \mathsf{S4}}$ for the combination of intuitionistic logic J and modal logic $\mathsf{S4}$ defined in Section 2 is sound and complete with respect to the Kripke semantics introduced in Section 4

We begin by extending the semantic notions to sequents. We say that $M = (W, R, V) \in \mathcal{M}_{\mathsf{J} \sqcup \mathsf{S4}}$ *locally satisfies* in $w \in W$ the sequent $\Psi \to \Lambda$, written

$$M, w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \Psi \to \Lambda$$

whenever $M, w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \Psi$ implies that there is $\lambda \in \Lambda$ such that $M, w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \lambda$. Moreover, we say that M *satisfies* the sequent $\Psi \to \Lambda$, written

$$M \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \Psi \to \Lambda$$

whenever $M, w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \Psi \to \Lambda$ for every $w \in W$. Furthermore we say $\Psi \to \Lambda$ is *valid*, written $\vDash_{\mathsf{J} \sqcup \mathsf{S4}} \Psi \to \Lambda$, whenever $M \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \Psi \to \Lambda$ for every $M \in \mathcal{M}_{\mathsf{J} \sqcup \mathsf{S4}}$.

In the sequel we need two properties. The first one states that satisfaction of boxed formulas is preserved by the Kripke relation. The second one states that for diamond formulas non-satisfiability is preserved.

PROPOSITION 5.1. Let $M \in \mathcal{M}_{\mathsf{J} \sqcup \mathsf{S4}}$, $w \in W$ and $\varphi \in L_{\mathsf{J} \sqcup \mathsf{S4}}$. Then

- if $M, w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \Box_s \varphi$, $w' \in W$ and wRw' then $M, w' \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \Box_s \varphi$ by transitivity of R
- if $M, w \nVdash_{\mathsf{J} \sqcup \mathsf{S4}} \Diamond_s \varphi$, $w' \in W$ and wRw' then $M, w' \nVdash_{\mathsf{J} \sqcup \mathsf{S4}} \Diamond_s \varphi$ by transitivity of R.

Soundness brings to light that the host of the combination is S4 modal logic. Hence we need to translate formulas in $L_{\mathsf{J} \sqcup \mathsf{S4}}$ to equivalent formulas in L_{S4}. For that we need the following map inspired by the Gödel-McKinsey-Tarski translation [15, 8].

Let $\tau_{\mathsf{J} \sqcup \mathsf{S4}} : L_{\mathsf{J} \sqcup \mathsf{S4}} \to L_{\mathsf{S4}}$ be the map inductively defined as follows:

- $\tau_{\mathsf{J} \sqcup \mathsf{S4}}(p_s) = p_s$
- $\tau_{\mathsf{J} \sqcup \mathsf{S4}}(p_i) = \Box_s p_s$
- $\tau_{\mathsf{J} \sqcup \mathsf{S4}}(\bot) = \bot$
- $\tau_{\mathsf{J} \sqcup \mathsf{S4}}(\varphi * \psi) = \tau_{\mathsf{J} \sqcup \mathsf{S4}}(\varphi) * \tau_{\mathsf{J} \sqcup \mathsf{S4}}(\psi)$ where $* \in \{\wedge, \vee\}$
- $\tau_{\mathsf{J} \sqcup \mathsf{S4}}(\varphi_1 \supset_s \varphi_2) = \tau_{\mathsf{J} \sqcup \mathsf{S4}}(\varphi) \supset_s \tau_{\mathsf{J} \sqcup \mathsf{S4}}(\psi)$
- $\tau_{\mathsf{J} \sqcup \mathsf{S4}}(\varphi_1 \supset_i \varphi_2) = \Box_s(\tau_{\mathsf{J} \sqcup \mathsf{S4}}(\varphi) \supset_s \tau_{\mathsf{J} \sqcup \mathsf{S4}}(\psi))$
- $\tau_{\mathsf{J} \sqcup \mathsf{S4}}(\Box_s \varphi_1) = \Box_s \tau_{\mathsf{J} \sqcup \mathsf{S4}}(\varphi_1)$.

Observe that $\tau_{\mathsf{J} \sqcup \mathsf{S4}}(\neg_i \varphi) = \Box_s(\neg_s \tau_{\mathsf{J} \sqcup \mathsf{S4}}(\varphi))$ and $\tau_{\mathsf{J} \sqcup \mathsf{S4}}(\neg_s \varphi) = \neg_s \tau_{\mathsf{J} \sqcup \mathsf{S4}}(\varphi)$. We extend the definition of $\tau_{\mathsf{J} \sqcup \mathsf{S4}}$ as follows:

$$\tau_{\mathsf{J} \sqcup \mathsf{S4}}(\Psi) = \{\tau_{\mathsf{J} \sqcup \mathsf{S4}}(\psi) : \psi \in \Psi\}.$$

The following result shows that the translation of a formula is locally equivalent to the original formula.

PROPOSITION 5.2. Let $\varphi \in L_{\mathsf{J} \sqcup \mathsf{S4}}$, M be a Kripke structure and $w \in W$. Then,

$$M, w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \varphi \text{ if and only if } M, w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \tau_{\mathsf{J} \sqcup \mathsf{S4}}(\varphi).$$

PROOF: The proof is by induction on the structure of φ.

(Base) There are three cases.

(1) φ is $p_s \in P_s$. The result is immediate.

(2) φ is $p_i \in P_i$. Thus $M, w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} p_i$ iff $V(p_s, w') = 1$ for every $w' \in W$ such that wRw' iff $M, w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \Box_s p_s$ iff $M, w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \tau_{\mathsf{J} \sqcup \mathsf{S4}}(p_i)$.

(3) φ is $\perp$. The result is immediate.

(Step) There are five cases.

(1) φ is $\varphi_1 \wedge \varphi_2$. Then $M, w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \varphi_1 \wedge \varphi_2$ iff $M, w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \varphi_j$ for $j = 1, 2$ iff (by IH) $M, w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \tau_{\mathsf{J} \sqcup \mathsf{S4}}(\varphi_j)$ for $j = 1, 2$ iff $M, w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \tau_{\mathsf{J} \sqcup \mathsf{S4}}(\varphi_1) \wedge \tau_{\mathsf{J} \sqcup \mathsf{S4}}(\varphi_2)$ iff $M, w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \tau_{\mathsf{J} \sqcup \mathsf{S4}}(\varphi_1 \wedge \varphi_2)$.

(2) φ is $\varphi_1 \vee \varphi_2$. Similar to case (1) of step.

(3) φ is $\varphi_1 \supset_s \varphi_2$. So $M, w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \varphi_1 \supset_s \varphi_2$ iff if $M, w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \varphi_1$ then $M, w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \varphi_2$ iff (by IH) if $M, w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \tau_{\mathsf{J} \sqcup \mathsf{S4}}(\varphi_1)$ then $M, w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \tau_{\mathsf{J} \sqcup \mathsf{S4}}(\varphi_2)$ iff $M, w \Vdash_{\mathsf{JS4}} \tau_{\mathsf{J} \sqcup \mathsf{S4}}(\varphi_1) \supset_s \tau_{\mathsf{J} \sqcup \mathsf{S4}}(\varphi_2)$ iff $M, w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \tau_{\mathsf{J} \sqcup \mathsf{S4}}(\varphi_1 \supset_s \varphi_2)$.

(4) φ is $\varphi_1 \supset_i \varphi_2$. Thus $M, w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \varphi_1 \supset_i \varphi_2$ iff if $M, w' \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \varphi_1$ then $M, w' \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \varphi_2$ for every $w' \in W$ such that wRw' iff (by IH) if $M, w' \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \tau_{\mathsf{J} \sqcup \mathsf{S4}}(\varphi_1)$ then $M, w' \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \tau_{\mathsf{J} \sqcup \mathsf{S4}}(\varphi_2)$ for every $w' \in W$ such that wRw' iff $M, w' \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \tau_{\mathsf{J} \sqcup \mathsf{S4}}(\varphi_1) \supset_s \tau_{\mathsf{J} \sqcup \mathsf{S4}}(\varphi_2)$ for every $w' \in W$ such that wRw' iff $M, w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \Box_s(\tau_{\mathsf{J} \sqcup \mathsf{S4}}(\varphi_1) \supset_s \tau_{\mathsf{J} \sqcup \mathsf{S4}}(\varphi_2))$ iff $M, w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \tau_{\mathsf{J} \sqcup \mathsf{S4}}(\varphi_1 \supset_i \varphi_2)$.

(5) φ is $\Box_s \varphi_1$. Thus $M, w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \Box_s \varphi_1$ iff $M, w' \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \varphi_1$ for every $w' \in W$ such that wRw' iff (by IH) $M, w' \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \tau_{\mathsf{J} \sqcup \mathsf{S4}}(\varphi_1)$ for every $w' \in W$ such that wRw' iff $M, w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \Box_s \tau_{\mathsf{J} \sqcup \mathsf{S4}}(\varphi_1)$ iff $M, w \Vdash_{\mathsf{J} \sqcup \mathsf{S4}} \tau_{\mathsf{J} \sqcup \mathsf{S4}}(\Box_s \varphi_1)$. □

The next result extends to entailment the equivalence between a formula and its translation. We omit the proof since it follows straightforwardly from Propositions 4.3 and 5.2.

Proposition 5.3. Let $\varphi \in L_{\mathsf{J} \sqcup \mathsf{S4}}$. Then $\vDash_{\mathsf{J} \sqcup \mathsf{S4}} \varphi$ if and only if $\vDash_{\mathsf{S4}} \tau_{\mathsf{J} \sqcup \mathsf{S4}}(\varphi)$.

We are now ready to prove the soundness of $\mathrm{G}_{\mathsf{J} \sqcup \mathsf{S4}}$. We start by proving that the rules are sound.

A rule is said to be *sound* whenever for every Kripke structure $M \in \mathcal{M}_{\mathsf{J} \sqcup \mathsf{S4}}$, if M satisfies the premises of the rule then M also satisfies the conclusion of the rule.

PROPOSITION 5.4. The rules of $\mathrm{G}_{\mathsf{J}\sqcup\mathsf{S4}}$ are sound.

PROOF: Let $M \in \mathcal{M}_{\mathsf{J}\sqcup\mathsf{S4}}$.
(LP$_i$) Suppose that $M \Vdash_{\mathsf{J}\sqcup\mathsf{S4}} \Box_s p_s, \Gamma \to \Delta$. Let $w \in W$. Assume that $M, w \Vdash_{\mathsf{J}\sqcup\mathsf{S4}} \Gamma$ and $M, w \Vdash_{\mathsf{J}\sqcup\mathsf{S4}} p_i$. Then $M, w \Vdash_{\mathsf{J}\sqcup\mathsf{S4}} \tau_{\mathsf{J}\sqcup\mathsf{S4}}(p_i)$ by Proposition 5.2 and so $M, w \Vdash_{\mathsf{J}\sqcup\mathsf{S4}} \Box_s p_s$. Hence $M, w \Vdash_{\mathsf{J}\sqcup\mathsf{S4}} \delta$ for some $\delta \in \Delta$ using the hypothesis.

(RP$_i$) Assume that $M \Vdash_{\mathsf{J}\sqcup\mathsf{S4}} \Gamma \to \Delta, \Box_s p_s$. Let $w \in W$. Suppose that $M, w \Vdash_{\mathsf{J}\sqcup\mathsf{S4}} \Gamma$. There are two cases. (1) $M, w \Vdash_{\mathsf{J}\sqcup\mathsf{S4}} \delta$ for some $\delta \in \Delta$ and so $M, w \Vdash_{\mathsf{J}\sqcup\mathsf{S4}} \Gamma \to \Delta, p_i$. (2) $M, w \Vdash_{\mathsf{J}\sqcup\mathsf{S4}} \Box_s p_s$. Hence $M, w \Vdash_{\mathsf{J}\sqcup\mathsf{S4}} \tau_{\mathsf{J}\sqcup\mathsf{S4}}(p_i)$ and so $M, w \Vdash_{\mathsf{J}\sqcup\mathsf{S4}} p_i$ by Proposition 5.2.

(L$\supset_i$) Suppose that $M \Vdash_{\mathsf{J}\sqcup\mathsf{S4}} \Box_s(\beta_1 \supset_s \beta_2), \Gamma \to \Delta$. Let $w \in W$. Assume that $M, w \Vdash_{\mathsf{J}\sqcup\mathsf{S4}} \beta_1 \supset_i \beta_2$ and $M, w \Vdash_{\mathsf{J}\sqcup\mathsf{S4}} \Gamma$. Thus $M, w \Vdash_{\mathsf{J}\sqcup\mathsf{S4}} \tau_{\mathsf{J}\sqcup\mathsf{S4}}(\beta_1 \supset_i \beta_2)$ by Proposition 5.2 and so $M, w \Vdash_{\mathsf{J}\sqcup\mathsf{S4}} \Box_s(\tau_{\mathsf{J}\sqcup\mathsf{S4}}(\beta_1) \supset_s \tau_{\mathsf{J}\sqcup\mathsf{S4}}(\beta_2))$. Thus, $M, w' \Vdash_{\mathsf{J}\sqcup\mathsf{S4}} \tau_{\mathsf{J}\sqcup\mathsf{S4}}(\beta_1) \supset_s \tau_{\mathsf{J}\sqcup\mathsf{S4}}(\beta_2)$ for every $w' \in W$ such that wRw' and so $M, w' \Vdash_{\mathsf{J}\sqcup\mathsf{S4}} \tau_{\mathsf{J}\sqcup\mathsf{S4}}(\beta_1 \supset_s \beta_2)$ for every $w' \in W$ such that wRw'. Therefore, again by Proposition 5.2 $M, w' \Vdash_{\mathsf{J}\sqcup\mathsf{S4}} \beta_1 \supset_s \beta_2$ for every $w' \in W$ such that wRw'. So $M, w \Vdash_{\mathsf{J}\sqcup\mathsf{S4}} \Box_s(\beta_1 \supset_s \beta_2)$. Hence, there is $\delta \in \Delta$ such that $M, w \Vdash_{\mathsf{J}\sqcup\mathsf{S4}} \delta$ using the hypothesis.

(R$\supset_i$) Assume that $M \Vdash_{\mathsf{J}\sqcup\mathsf{S4}} \Gamma \to \Delta, \Box_s(\beta_1 \supset_s \beta_2)$. Let $w \in W$. Suppose that $M, w \Vdash_{\mathsf{J}\sqcup\mathsf{S4}} \Gamma$. There are two cases. (1) There is $\delta \in \Delta$ such that $M, w \Vdash_{\mathsf{J}\sqcup\mathsf{S4}} \delta$ and therefore $M, w \Vdash \Gamma \to \Delta, \beta_1 \supset_i \beta_2$. (2) $M, w \Vdash_{\mathsf{J}\sqcup\mathsf{S4}} \Box_s(\beta_1 \supset_s \beta_2)$. Hence $M, w' \Vdash_{\mathsf{J}\sqcup\mathsf{S4}} \beta_1 \supset_s \beta_2$ for every $w' \in W$ such that wRw' and so, by Proposition 5.2, $M, w' \Vdash_{\mathsf{J}\sqcup\mathsf{S4}} \tau_{\mathsf{J}\sqcup\mathsf{S4}}(\beta_1 \supset_s \beta_2)$ for every $w' \in W$ such that wRw'. Hence $M, w' \Vdash_{\mathsf{J}\sqcup\mathsf{S4}} \tau_{\mathsf{J}\sqcup\mathsf{S4}}(\beta_1) \supset_s \tau_{\mathsf{J}\sqcup\mathsf{S4}}(\beta_2)$ for every $w' \in W$ such that wRw' Thus, $M, w \Vdash_{\mathsf{J}\sqcup\mathsf{S4}} \Box_s(\tau_{\mathsf{J}\sqcup\mathsf{S4}}(\beta_1) \supset_s \tau_{\mathsf{J}\sqcup\mathsf{S4}}(\beta_2))$ and so $M, w \Vdash_{\mathsf{J}\sqcup\mathsf{S4}} \tau_{\mathsf{J}\sqcup\mathsf{S4}}(\beta_1 \supset_i \beta_2)$. Finally, by Proposition 5.2, $M, w \Vdash_{\mathsf{J}\sqcup\mathsf{S4}} \beta_1 \supset_i \beta_2$.

(L$\Box_s$) Suppose that $M \Vdash_{\mathsf{J}\sqcup\mathsf{S4}} \beta, \Box_s\beta, \Gamma \to \Delta$. Let $w \in W$ and assume that $M, w \Vdash_{\mathsf{J}\sqcup\mathsf{S4}} \Box_s\beta$ and $M, w \Vdash_{\mathsf{J}\sqcup\mathsf{S4}} \Gamma$. Then $M, w' \Vdash_{\mathsf{J}\sqcup\mathsf{S4}} \beta$ for every $w' \in W$ such that wRw'. Hence, $M, w \Vdash_{\mathsf{J}\sqcup\mathsf{S4}} \beta$ by reflexivity and so there is $\delta \in \Delta$ such that $M, w \Vdash_{\mathsf{J}\sqcup\mathsf{S4}} \delta$.

(R$\Box_s$) Assume that $M \Vdash_{\mathsf{J}\sqcup\mathsf{S4}} \Box_s\Gamma \to \Diamond_s \Delta, \beta$. Let $w \in W$ and suppose that $M, w \Vdash_{\mathsf{J}\sqcup\mathsf{S4}} \Box_s\Gamma$ and $M, w \Vdash_{\mathsf{J}\sqcup\mathsf{S4}} \Gamma'$. There are two cases to consider. (1) $M, w \Vdash_{\mathsf{J}\sqcup\mathsf{S4}} \Diamond_s\delta$ for some $\delta \in \Delta$ and the thesis follows. (2) Otherwise let $w' \in W$ be such that wRw'. Observe that $M, w' \Vdash_{\mathsf{J}\sqcup\mathsf{S4}} \Box_s\Gamma$ by Proposition 5.1. Moreover, $M, w' \nVdash_{\mathsf{J}\sqcup\mathsf{S4}} \Diamond_s\delta$ for for every $\delta \in \Delta$ by

Proposition 5.1. So, $M, w' \Vdash_{\mathsf{J}\sqcup\mathsf{S4}} \beta$ using the hypothesis.

The other cases follow in a similar way. □

The next step is to show that the axioms of $\mathrm{G}_{\mathsf{J}\sqcup\mathsf{S4}}$ are sound. We say that an axiom is *sound* whenever it is satisfied by every Kripke structure in $\mathcal{M}_{\mathsf{J}\sqcup\mathsf{S4}}$. The following result is straightforward.

Proposition 5.5. The axioms of $\mathrm{G}_{\mathsf{J}\sqcup\mathsf{S4}}$ are sound.

Finally we have the soundness result.

Proposition 5.6. Let $\varphi \in L_{\mathsf{J}\sqcup\mathsf{S4}}$. Then $\vdash_{\mathsf{J}\sqcup\mathsf{S4}} \varphi$ implies $\vDash_{\mathsf{J}\sqcup\mathsf{S4}} \varphi$.

Proof: We must start by proving that

$$(\dagger) \qquad \vdash_{\mathrm{G}_{\mathsf{J}\sqcup\mathsf{S4}}} \Psi \to \Lambda \text{ implies } \vDash_{\mathsf{J}\sqcup\mathsf{S4}} \Psi \to \Lambda.$$

The proof follows by a straightfoward induction on the length of a derivation for $\Psi \to \Lambda$ using Proposition 5.5 and Proposition 5.4. Hence assuming $\vdash_{\mathsf{J}\sqcup\mathsf{S4}} \varphi$ then $\vdash_{\mathrm{G}_{\mathsf{J}\sqcup\mathsf{S4}}} \to \varphi$. Thus, by (†), $\vDash_{\mathsf{J}\sqcup\mathsf{S4}} \to \varphi$. Therefore, $\vDash_{\mathsf{J}\sqcup\mathsf{S4}} \varphi$. □

Completeness We start by showing that the sequent derivation in $\mathrm{G}_{\mathsf{J}\sqcup\mathsf{S4}}$ is a conservative extension of the sequent derivation in G_{S4} modulo the translation $\tau_{\mathsf{J}\sqcup\mathsf{S4}}$ (see [17] for the Gentzen calculus for $\mathsf{S4}$). The strategy of proving completeness that we follow is similar to the one in [10].

Proposition 5.7. Let $\Psi \cup \Lambda \subseteq L_{\mathsf{J}\sqcup\mathsf{S4}}$. Then

$$\vdash_{\mathrm{G}_{\mathsf{J}\sqcup\mathsf{S4}}} \Psi \to \Lambda \quad \text{if and only if} \quad \vdash_{\mathrm{G}_{\mathsf{S4}}} \tau_{\mathsf{J}\sqcup\mathsf{S4}}(\Psi) \to \tau_{\mathsf{J}\sqcup\mathsf{S4}}(\Lambda).$$

Proof:
($\to$) Let $\Psi_1 \to \Lambda_1 \ldots \Psi_n \to \Lambda_n$ be a derivation for $\Psi \to \Lambda$ in $\mathrm{G}_{\mathsf{J}\sqcup\mathsf{S4}}$. The proof follows by induction on n.

(Basis) $n = 1$. There are two cases. (1) $\Psi_1 \to \Lambda_1$ is justified by (Ax), that is, it is of the form $p_s, \Gamma \to \Delta, p_s$. Hence $\tau_{\mathsf{J}\sqcup\mathsf{S4}}(p_s), \tau_{\mathsf{J}\sqcup\mathsf{S4}}(\Gamma) \to \tau_{\mathsf{J}\sqcup\mathsf{S4}}(\Delta), \tau_{\mathsf{J}\sqcup\mathsf{S4}}(p_s)$ is also justified by (Ax) in G_{S4}. (2) $\Psi_1 \to \Lambda_1$ is justified by (L$\perp$), that is, it is of the form $\perp, \Gamma \to \Delta$ and so $\tau_{\mathsf{J}\sqcup\mathsf{S4}}(\perp), \tau_{\mathsf{J}\sqcup\mathsf{S4}}(\Gamma) \to \tau_{\mathsf{J}\sqcup\mathsf{S4}}(\Delta)$ is also justified by (L$\perp$) in G_{S4} because $\tau_{\mathsf{J}\sqcup\mathsf{S4}}(\perp)$ is $\perp$.

(Step) There are several cases. We only present the proof for (LP$_i$) and (R$\supset_i$). The other proofs follow in a similar way.

(1) $\Psi_1 \to \Lambda_1$ is the conclusion of rule (LP$_i$), that is, is of the form $p_i, \Psi'_1 \to \Lambda_1$ and so there is $j = 2, \ldots, n$ such that $\Psi_j \to \Lambda_j$ is $\Box_s p_s, \Psi'_1 \to \Lambda_1$. Hence $\vdash_{\mathrm{G}_{\mathsf{J} \sqcup \mathsf{S4}}} \Box_s p_s, \Psi'_1 \to \Lambda_1$ and so by (IH) $\vdash_{\mathrm{G}_{\mathsf{S4}}} \tau_{\mathsf{J} \sqcup \mathsf{S4}}(\Box_s p_s), \tau_{\mathsf{J} \sqcup \mathsf{S4}}(\Psi'_1) \to \tau_{\mathsf{J} \sqcup \mathsf{S4}}(\Lambda_1)$. So there is a derivation in G_{S4} for $\Box_s p_s, \tau_{\mathsf{J} \sqcup \mathsf{S4}}(\Psi'_1) \to \tau_{\mathsf{J} \sqcup \mathsf{S4}}(\Lambda_1)$. The thesis follows since $\tau_{\mathsf{J} \sqcup \mathsf{S4}}(p_i)$ is $\Box_s p_s$.

(2) $\Psi_1 \to \Lambda_1$ is the conclusion of rule (R$\supset_i$), that is, is of the form $\Psi_1 \to \Lambda'_1, \varphi_1 \supset_i \varphi_2$ and therefore there is $j = 2, \ldots, n$ such that $\Psi_1 \to \Lambda'_1, \Box_s(\varphi_1 \supset_s \varphi_2)$. Thus $\vdash_{\mathrm{G}_{\mathsf{J} \sqcup \mathsf{S4}}} \Psi_1 \to \Lambda'_1, \Box_s(\varphi_1 \supset_s \varphi_2)$ and so $\vdash_{\mathrm{G}_{\mathsf{S4}}} \tau_{\mathsf{J} \sqcup \mathsf{S4}}(\Psi_1) \to \tau_{\mathsf{J} \sqcup \mathsf{S4}}(\Lambda'_1), \Box_s(\tau_{\mathsf{J} \sqcup \mathsf{S4}}(\varphi_1) \supset_s \tau_{\mathsf{J} \sqcup \mathsf{S4}}(\varphi_2))$ by (IH). The thesis follows because $\Box_s(\tau_{\mathsf{J} \sqcup \mathsf{S4}}(\varphi_1) \supset_s \tau_{\mathsf{J} \sqcup \mathsf{S4}}(\varphi_2))$ is $\tau_{\mathsf{J} \sqcup \mathsf{S4}}(\varphi_1 \supset_i \varphi_2)$. □

The previous result can be extended straightforwardly to derivation of formulas.

Proposition 5.8. Let $\varphi \in L_{\mathsf{J} \sqcup \mathsf{S4}}$. Then $\vdash_{\mathsf{J} \sqcup \mathsf{S4}} \varphi$ if and only if $\vdash_{\mathsf{S4}} \tau_{\mathsf{J} \sqcup \mathsf{S4}}(\varphi)$.

We are ready to prove completeness of $\mathrm{G}_{\mathsf{J} \sqcup \mathsf{S4}}$ with respect to $\mathcal{M}_{\mathsf{J} \sqcup \mathsf{S4}}$.

Proposition 5.9. Let $\varphi \in L_{\mathsf{J} \sqcup \mathsf{S4}}$. Then $\vDash_{\mathsf{J} \sqcup \mathsf{S4}} \varphi$ implies $\vdash_{\mathsf{J} \sqcup \mathsf{S4}} \varphi$.

Proof: Suppose that $\vDash_{\mathsf{J} \sqcup \mathsf{S4}} \varphi$. Hence $\vDash_{\mathsf{S4}} \tau_{\mathsf{J} \sqcup \mathsf{S4}}(\varphi)$ by Proposition 5.3. Thus $\vdash_{\mathsf{S4}} \tau_{\mathsf{J} \sqcup \mathsf{S4}}(\varphi)$ by completeness of S4 (see [17]) and so, by Proposition 5.8, $\vdash_{\mathsf{J} \sqcup \mathsf{S4}} \varphi$. □

6. Concluding remarks

Inspired by the works of [12] and [11], we propose a logic combining intuitionistic and S4 modal logic in a tolerant way. That is, the intuitionistic logician accepts that the classical principles are present for the modal language fragment of the logic and the modal logician accepts that the intuitionistic principles hold in the intuitionistic language fragment of the logic.

We endow the logic with a Gentzen calculus and with a Kripke semantics and show that the combined logic is sound and complete. We prove that the combined logic extends conservatively intutionistic and S4 modal logic. Moreover we show that the cut rule can be eliminated.

We want to study other metaproperties of the combined logic namely decidability, Craig interpolation and definability. Moreover, we would like to investigate combinations of intuitionistic and other modal logics. Furthermore, it would interesting to leave the realm of Kripke semantics and analyze for example the combination of paraconsistent logics with intuitionistic or classical logic.

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