Bulletin of the **S**ection of **L**ogic
Published online: December 18, 2023; 28 pages
https://doi.org/10.18778/0138-0680.2023.31

Mona Aaly Kologani
Mohammad Mohseni Takallo
Young Bae Jun
Rajab Ali Borzooei

FUZZY SUB-EQUALITY ALGEBRAS BASED ON FUZZY POINTS

Abstract

In this paper, by using the notion of fuzzy points and equality algebras, the notions of fuzzy point equality algebra, equality-subalgebra, and ideal were established. Some characterizations of fuzzy subalgebras were provided by using such concepts. We defined the concepts of $(\in, \in)$ and $(\in, \in \vee\, q)$-fuzzy ideals of equality algebras, discussed some properties, and found some equivalent definitions of them. In addition, we investigated the relation between different kinds of (α, β)-fuzzy subalgebras and (α, β)-fuzzy ideals on equality algebras. Also, by using the notion of $(\in, \in)$-fuzzy ideal, we defined two equivalence relations on equality algebras and we introduced an order on classes of X, and we proved that the set of all classes of X by these order is a poset.

Keywords: equality algebra, fuzzy set, fuzzy point, fuzzy ideal, sub-equality algebras, $(\in, \in)$-fuzzy sub-equality algebras, $(\in, \in \vee\, q)$-fuzzy sub-equality algebras, $(q, \in \vee\, q)$-fuzzy sub-equality algebras.

2020 Mathematical Subject Classification: 03G25, 06A12, 06B99, 06D72.

1. Introduction

EQ-algebras were introduced by Novák et al [15]. Equality algebras were introduced by Jenei [12] by removing the multiplication operation and as

Presented by: Janusz Ciuciura
Received: June 4, 2023

an extension of EQ-algebras. In [9, 13] the authors investigated the relation between equality algebra and BCK-meet-semilattice. Dvurečenskij et al. in [10] defined pseudo-equality algebra as an extention of equality algebra and study some properties of it. Borzooei et al. [7] introduced some types of filters of equality algebras and studied the relation between them and moreover, they considered relations among equality algebras and some of the other logical algebras such as residuated lattice, MTL-algebra, BL-algebra, MV-algebra, and etc., in [19]. Since ideal theory is an important notion in logical algebras, Paad [16] introduced the notion of the ideal in bounded equality algebras and showed that there is a reciprocal correspondence between ideals and congruence relation.

Fuzzy sets were first introduced by Zadeh [18] and then studied by many mathematicians. Some mathematicians tried to overcome its shortcomings by presenting various extensions of fuzzy sets, and some other mathematicians studied fuzzy sets on various algebraic structures such as logical algebraic structures, groups, and rings. In [8] the notion of fuzzy ideal in bounded equality algebras is defined, and several properties are studied. Fuzzy ideal generated by a fuzzy set is established, and a fuzzy ideal is made by using the collection of ideals. Characterizations of fuzzy ideal were discussed. Conditions for a fuzzy ideal to attained its infimum on all ideals are provided. Homomorphic image and preimage of fuzzy ideal were considered. Quotient structures of equality algebra induced by (fuzzy) ideal were studied. The idea of the quasi-coincidence of a fuzzy point with a fuzzy set has played a very important role in generating fuzzy subalgebras of BCK/BCI-algebras, called (α, β)-fuzzy subalgebras of BCK/BCI-algebras, introduced by Jun [14]. Moreover, $(\in, \in \vee\, q)$-fuzzy subalgebra is a useful generalization of a fuzzy subalgebra in BCK/BCI-algebras. Many researchers applied the fuzzy structures on logical algebras [2, 1, 3, 4, 5, 6, 11, 17]. Then studied point fuzzy on various algebraic structures, such as hoop, BCK/BCI-algebra, different kinds of hyperstructures, and so on.

In this paper, by using the notion of fuzzy points and equality algebras, the notions of fuzzy point equality algebra, equality-subalgebra, and ideal are established. Some characterizations of fuzzy subalgebras are provided by using such concepts. We define the concepts of $(\in, \in)$ and $(\in, \in \vee\, q)$-fuzzy ideals of equality algebras, discuss some properties, and find some equivalent definitions of them. In addition, we investigate the relation between different kinds of (α, β)-fuzzy subalgebras and (α, β)-fuzzy ideals on equality algebras. Also, by using the notion of $(\in, \in)$-fuzzy ideal we define two equivalence relations on equality algebras and we introduce an order on classes of X, and we prove that the set of all classes of X by these order is a poset.

2. Preliminaries

This section lists the known default contents that will be used later.

DEFINITION 2.1 ([12]). By an *equality algebra*, we mean an algebraic structure $(X, \wedge, \sim, 1)$ satisfying the following conditions.

(E1) $(X, \wedge, 1)$ is a commutative idempotent integral monoid,

(E2) The operation "$\sim$" is commutative,

(E3) $(\forall a \in X)(a \sim a = 1)$,

(E4) $(\forall a \in X)(a \sim 1 = a)$,

(E5) $(\forall a, b, c \in X)(a \leq b \leq c \;\Rightarrow\; a \sim c \leq b \sim c,\; a \sim c \leq a \sim b)$,

(E6) $(\forall a, b, c \in X)(a \sim b \leq (a \wedge c) \sim (b \wedge c))$,

(E7) $(\forall a, b, c \in X)(a \sim b \leq (a \sim c) \sim (b \sim c))$,

where $a \leq b$ if and only if $a \wedge b = a$.

In an equality algebra $(X, \wedge, \sim, 1)$, we define two operations "$\rightarrow$" and "$\leftrightarrow$" on X as follows:

$$a \rightarrow b := a \sim (a \wedge b), \tag{2.1}$$

$$a \leftrightarrow b := (a \rightarrow b) \wedge (b \rightarrow a). \tag{2.2}$$

PROPOSITION 2.2 ([12]). Let $(X, \wedge, \sim, 1)$ be an equality algebra. Then for all $a, b, c \in X$, the following assertions are valid:

$$a \to b = 1 \;\Leftrightarrow\; a \le b, \tag{2.3}$$
$$a \to (b \to c) = b \to (a \to c), \tag{2.4}$$
$$1 \to a = a,\; a \to 1 = 1,\; a \to a = 1, \tag{2.5}$$
$$a \le b \to c \;\Leftrightarrow\; b \le a \to c, \tag{2.6}$$
$$a \le b \to a, \tag{2.7}$$
$$a \le (a \to b) \to b, \tag{2.8}$$
$$a \to b \le (b \to c) \to (a \to c), \tag{2.9}$$
$$b \le a \;\Rightarrow\; a \leftrightarrow b = a \to b = a \sim b, \tag{2.10}$$
$$a \sim b \le a \leftrightarrow b \le a \to b, \tag{2.11}$$
$$a \le b \;\Rightarrow \begin{cases} b \to c \le a \to c, \\ c \to a \le c \to b \end{cases} \tag{2.12}$$

An equality algebra $(X, \wedge, \sim, 1)$ is said to be *bounded* if there exists an element $0 \in X$ such that $0 \le a$ for all $a \in X$. In a bounded equality algebra $(X, \wedge, \sim, 1)$, we define the negation "$\neg$" on X by $\neg a = a \to 0 = a \sim 0$ for all $a \in X$.

Definition 2.3 ([16]). Let X be a bounded equality algebra. A subset A of X is called an *ideal* of X if it satisfies:

$$(\forall x, y \in X)(x \le y,\, y \in A \;\Rightarrow\; x \in A), \tag{2.13}$$
$$\neg x \to y \in A, \text{ for all } x, y \in A. \tag{2.14}$$

Lemma 2.4 ([16]). *Let X be a bounded equality algebra. A subset A of X is an ideal of X if and only if it satisfies in the following conditions:*

$$0 \in A, \tag{2.15}$$
$$(\forall x, y \in X)(\neg(\neg y \to \neg x) \in A,\; y \in A \;\Rightarrow\; x \in A). \tag{2.16}$$

Definition 2.5 ([16]). Let X be a bounded equality algebra and P be an ideal of X. Then P is called a *prime ideal* of X if it satisfies for any $x, y \in X$, $\neg(x \to y) \in P$ or $\neg(y \to x) \in P$.

Let X be a non-empty set. The function $\lambda : X \longrightarrow [0, 1]$ is called a *fuzzy set*.

Let X and Y be sets and $f : X \to Y$ be a function. If μ is a fuzzy set in X, then the *image* of μ under f is denoted by $f(\mu)$ and is defined as follows:

$$f(\mu) : Y \to [0,1],\ y \mapsto \begin{cases} \sup\limits_{x \in f^{-1}(y)} \mu(x) & \text{if } f^{-1}(y) \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

If ν is a fuzzy set in $f(X)$, then the *preimage* of ν under f is denoted by $f^{-1}(\nu)$ and is defined by

$$f^{-1}(\nu) : X \to [0,1],\ x \mapsto \nu(f(x)).$$

Definition 2.6. A fuzzy set λ in X is said to be a *fuzzy ideal of* X if for any $x, y \in X$:

$$\lambda(0) \geq \lambda(x), \quad \text{and} \quad \lambda(x) \geq \min\{\lambda(\neg(\neg y \to \neg x)), \lambda(y)\}.$$

A fuzzy set λ in a set X of the form

$$\lambda(y) := \begin{cases} t \in (0,1] & \text{if } y = x, \\ 0 & \text{if } y \neq x, \end{cases}$$

is said to be a *fuzzy point with support x and value t* and is denoted by x_t.

For a fuzzy point x_t and a fuzzy set λ in a set X, we have the symbol $x_t \alpha \lambda$, where $\alpha \in \{\in, q, \in\vee q, \in\wedge q\}$.

To say that $x_t \in \lambda$ (resp. $x_t q \lambda$) means that $\lambda(x) \geq t$ (resp. $\lambda(x) + t > 1$), and in this case, x_t is said to *belong to* (resp. *be quasi-coincident with*) a fuzzy set λ.

To say that $x_t \in\vee q\, \lambda$ (resp. $x_t \in\wedge q\, \lambda$) means that $x_t \in \lambda$ or $x_t q \lambda$ (resp. $x_t \in \lambda$ and $x_t q \lambda$).

If $x_t \alpha \lambda$ is not established for $\alpha \in \{\in, q\}$, it is written by $x_t \overline{\alpha} \lambda$.

3. $(\in, \in)$-fuzzy sub-equality algebras

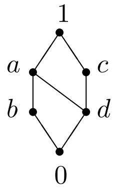
In this section, we define a sub-equality of an equality algebra X and investigate that intersection and union of family of sub-equality algebra of X is a sub-equality algebra. Then, we investigate the properties of the $(\in, \in)$-fuzzy sub-equality algebras.

Note. In what follows, let $(X, \wedge, \sim, 1)$ or X denote as an equality algebra unless otherwise specified.

Definition 3.1. A *sub-equality algebra* of an equality algebra X is a nonempty subset S of X, closed under the operations of X and equipped with the restriction to S at these operations. It means that a subset S of X is called a *sub-equality algebra* of X if $x \sim y \in S$ and $x \wedge y \in S$, for all $x, y \in S$.

Note. Note that every non-empty sub-equality algebra contains the element 1.

Example 3.2. Let $X = \{0, a, b, c, d, 1\}$ be a set with the following Hasse diagram.



Then $(X, \wedge, 1)$ is a meet semilattice with top element 1. Define an operation $\sim$ on X by Table 1.

Table 1. Cayley table for the binary operation "$\sim$"

$\sim$	0	a	b	c	d	1
0	1	d	c	b	a	0
a	d	1	a	d	c	a
b	c	a	1	0	d	b
c	b	d	0	1	a	c
d	a	c	d	a	1	d
1	0	a	b	c	d	1

Then $\mathcal{E} = (X, \wedge, \sim, 1)$ is a bounded equality algebra, and the implication "$\rightarrow$" is given by Table 2. Let $S_1 = \{1, b\}, S_2 = \{1, c\}$, $S_3 = \{1, a, b\}$ and $S_4 = \{1, a, c\}$. Clearly, S_1, S_2 and S_3 are sub-equality algebras of X, but S_4 isn't, since $a \sim c = d \notin S_4$.

Proposition 3.3. Let $\{X_i \mid i \in I\}$ be a family of sub-equality algebras of X. Then $\bigcap_{i \in I} X_i$ is a sub-equality algebras of X.

In the following example, we show that the union of a family of sub-equality algebras may not be a sub-equality algebra, in general.

Table 2. Cayley table for the implication "$\rightarrow$"

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	a	c	c	1
b	c	1	1	c	c	1
c	b	a	b	1	a	1
d	a	1	a	1	1	1
1	0	a	b	c	d	1

Example 3.4. Let X be the equality algebra as in Example 3.2. We show that, S_1 and S_2 are two sub-equality algebras of X, but $S = S_1 \cup S_2 = \{b, c, 1\}$ is not a sub-equality algebra of X, because $b \sim c = 0 \notin S$.

In the following proposition we investigate that under which condition, the union of a family of sub-equality algebras is a sub-equality algebra.

Proposition 3.5. Let $\{X_i \mid i \in I\}$ be a family of sub-equality algebra of X. If for any $i, j \in I$, $X_i \subseteq X_j$ or $X_j \subseteq X_i$, then $\bigcup_{i\in I} X_i$ is a sub-equality algebra of X.

Proposition 3.6. Let S be a sub-equality algebra of X. Then for any $x, y \in S$, $x \rightarrow y \in S$.

In the following example, we show that the reverse of the above proposition may not be true, in general.

Example 3.7. Let X be an equality algebra as in Example 3.2. Obviously, $S = \{1, a, c\}$ is closed under the operation $\rightarrow$. But S is not a sub-equality algebra of X, because $a \wedge c = d \notin S$ and $a \sim c = d \notin S$.

In the following proposition, we investigate that under which condition, close under the operation $\rightarrow$ is equal with property of sub-equality algebra.

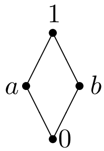
Proposition 3.8. Let X be bounded. If S is an ideal of X which is closed under $\rightarrow$, then S is a sub-equality algebra of X.

Proof: Suppose $x, y \in S$. Since $x \wedge y \leq x$, S is an ideal of X and $x \in S$, we have $x \wedge y \in S$. Also, by (2.11) $x \sim y \leq x \rightarrow y$. Since by assumption, S

is an ideal of X and $x \to y \in S$, we get $x \sim y \in S$. Thus, S is a sub-equality algebra of X. □

In the following example, we show that the ideal of equality algebra is not close under the operation $\to$.

Example 3.9. Let $X = \{0, a, b, 1\}$ be a set with the following Hasse diagram.



We define a binary operation $\sim$ and $\to$ on X by Tables 3 and 4, respectively. Then X is an equality algebra. Clearly, $S = \{0, a\}$ is an ideal of $\mathcal{E}$, but it

Table 3. Cayley table for the binary operation "$\sim$"

$\sim$	0	a	b	1
0	1	b	a	0
a	b	1	b	a
b	a	a	1	b
1	0	a	b	1

Table 4. Cayley table for the binary operation "$\to$"

$\to$	0	a	b	1
0	1	1	1	1
a	b	1	b	1
b	a	a	1	1
1	0	a	b	1

isn't close under the operation $\to$, because $a \to 0 = b \notin S$.

DEFINITION 3.10. A fuzzy set λ in X is called an $(\in, \in)$-*fuzzy sub-equality algebra* of X if the following assertion is valid.

$$(\forall x, y \in X)(\forall t, k \in [0,1]) \left(x_t \in \lambda,\ y_k \in \lambda \Rightarrow \left\{ \begin{array}{l} (x \sim y)_{\min\{t,k\}} \in \lambda \\ (x \wedge y)_{\min\{t,k\}} \in \lambda) \end{array} \right. \right). \tag{3.1}$$

Example 3.11. Let X be the equality algebra as in Example 3.9. Define a fuzzy set λ in X as follows:

$$\lambda : X \to [0,1],\ x \mapsto \begin{cases} 0.5 & \text{if } x = 0, \\ 0.4 & \text{if } x = a, \\ 0.4 & \text{if } x = b, \\ 0.8 & \text{if } x = 1 \end{cases}$$

Then λ is an $(\in, \in)$-fuzzy sub-equality algebra of X.

We consider characterizations of an $(\in, \in)$-fuzzy sub-equality algebra.

THEOREM 3.12. *A fuzzy set λ in X is an $(\in, \in)$-fuzzy sub-equality algebra of X if and only if the following assertion is valid.*

$$(\forall x, y \in X) \left(\begin{array}{l} \lambda(x \sim y) \geq \min\{\lambda(x), \lambda(y)\} \\ \lambda(x \wedge y) \geq \min\{\lambda(x), \lambda(y)\} \end{array} \right). \tag{3.2}$$

PROOF: Assume that λ is an $(\in, \in)$-fuzzy sub-equality algebra of X. Note that $x_{\lambda(x)} \in \lambda$ and $y_{\lambda(y)} \in \lambda$ for all $x, y \in X$. By (3.1), we have $(x \sim y)_{\min\{\lambda(x),\lambda(y)\}} \in \lambda$ and $(x \wedge y)_{\min\{\lambda(x),\lambda(y)\}} \in \lambda$. Then $\lambda(x \sim y) \geq \min\{\lambda(x), \lambda(y)\}$ and $\lambda(x \wedge y) \geq \min\{\lambda(x), \lambda(y)\}$ for all $x, y \in X$.

Conversely, suppose λ satisfies the condition (3.2). Let $x, y \in X$ and $t, k \in [0,1]$ such that $x_t \in \lambda$ and $y_k \in \lambda$. Then $\lambda(x) \geq t$ and $\lambda(y) \geq k$, which imply from (3.2) that

$$\lambda(x \sim y) \geq \min\{\lambda(x), \lambda(y)\} \geq \min\{t, k\}$$

and

$$\lambda(x \wedge y) \geq \min\{\lambda(x), \lambda(y)\} \geq \min\{t, k\}.$$

Hence $(x \sim y)_{\min\{t,k\}} \in \lambda$ and $(x \wedge y)_{\min\{t,k\}} \in \lambda$. Therefore λ is an $(\in, \in)$-fuzzy sub-equality algebra of X. $\square$

Theorem 3.13. *If a fuzzy set λ in X is an $(\in, \in)$-fuzzy sub-equality algebra of X, then*

$$(\forall x, y \in X)(\forall t, k \in [0,1]) \left(x_t \in \lambda,\ y_k \in \lambda \ \Rightarrow (x \to y)_{\min\{t,k\}} \in \lambda \right). \tag{3.3}$$

Proof: Assume that λ is an $(\in, \in)$-fuzzy sub-equality algebra of X. Let $x, y \in X$ and $t, k \in [0,1]$ be such that $x_t \in \lambda$ and $y_k \in \lambda$. Then $\lambda(x) \geq t$ and $\lambda(y) \geq k$, which implies from (3.2) that

$$\begin{aligned}\lambda(x \to y) &= \lambda(x \sim (x \wedge y)) \\ &\geq \min\{\lambda(x), \lambda(x \wedge y)\} \\ &\geq \min\{\lambda(x), \min\{\lambda(x), \lambda(y)\}\} \\ &\geq \min\{\lambda(x), \lambda(y)\} \\ &\geq \min\{t, k\}\end{aligned}$$

Hence, $(x \to y)_{\min\{t,k\}} \in \lambda$. □

In the following example, we show that the converse of the above theorem may not be true, in general.

Example 3.14. Let X be the equality algebra as in Example 3.9. Define a fuzzy set λ in X as follows:

$$\lambda : X \to [0,1],\ x \mapsto \begin{cases} 0.3 & \text{if } \ x = 0, \\ 0.4 & \text{if } \ x = a, \\ 0.4 & \text{if } \ x = b, \\ 0.8 & \text{if } \ x = 1 \end{cases}$$

Then λ is satisfies in (3.3). But, it isn't an $(\in, \in)$-fuzzy sub-equality algebra of X, since $0.3 = \lambda(a \wedge b) \ngeq \min\{\lambda(a), \lambda(b)\} = 0.4$.

Theorem 3.15. *If λ is a non-zero $(\in, \in)$-fuzzy sub-equality algebra of X, then the set*

$$X_0 := \{x \in X \mid \lambda(x) \neq 0\} \tag{3.4}$$

is a sub-equality algebra of X.

Proof: Let $x, y \in X_0$. Then $\lambda(x) > 0$ and $\lambda(y) > 0$. Note that $x_{\lambda(x)} \in \lambda$ and $y_{\lambda(y)} \in \lambda$. If $\lambda(x \sim y) = 0$ or $\lambda(x \wedge y) = 0$, then $\lambda(x \sim y) =$

$0 < \min\{\lambda(x), \lambda(y)\}$ or $\lambda(x \wedge y) = 0 < \min\{\lambda(x), \lambda(y)\}$, that is, $(x \sim y)_{\min\{\lambda(x),\lambda(y)\}}\overline{\in}\lambda$ or $(x \wedge y)_{\min\{\lambda(x),\lambda(y)\}}\overline{\in}\lambda$, which is a contradiction. Thus $\lambda(x \sim y) \neq 0$ and $\lambda(x \wedge y) \neq 0$. Hence $x \sim y \in X_0$ and $x \wedge y \in X_0$. Therefore X_0 is a sub-equality algebra of X. □

DEFINITION 3.16. Let X be bounded. A fuzzy set λ in X is called an $(\in, \in)$-*fuzzy ideal* of X if the following assertions are valid.

$$(\forall x \in X)(\forall t \in [0,1])(x_t \in \lambda \ \Rightarrow \ 0_t \in \lambda), \tag{3.5}$$

$$(\forall x, y \in X)(\forall t, k \in [0,1])(x_t \in \lambda, \ \neg(\neg x \to \neg y)_k \in \lambda \ \Rightarrow y_{\min\{t,k\}} \in \lambda). \tag{3.6}$$

Example 3.17. Let X be the equality algebra as in Example 3.9. Define a fuzzy set λ in X by $\lambda(0) = 0.8, \lambda(a) = 0.6$ and $\lambda(b) = \lambda(1) = 0.5$. Then λ is an $(\in, \in)$-fuzzy sub-equality algebra of X.

THEOREM 3.18. *The following are equivalent.*

(i) *A fuzzy set λ is a fuzzy ideal of X.*

(ii) *A fuzzy set λ is an $(\in, \in)$-fuzzy ideal of X.*

PROOF: $(i) \Rightarrow (ii)$: Let λ be a fuzzy ideal of X and $x_t \in \lambda$. Then $\lambda(x) \geq t$. Since by Definition 2.6 $\lambda(0) \geq \lambda(x)$, for any $x \in X$, we have $\lambda(0) \geq \lambda(x) \geq t$ and so $0_t \in \lambda$. Now, suppose $x_t \in \lambda$ and $\neg(\neg x \to \neg y)_s \in \lambda$. Then $\lambda(x) \geq t$ and $\lambda(\neg(\neg x \to \neg y)) \geq s$. Since λ is a fuzzy ideal, we get

$$\lambda(y) \geq \min\{\lambda(x), \lambda(\neg(\neg x \to \neg y))\} \geq \min\{t, s\},$$

Hence $y_{\min\{t,s\}\in\lambda}$. Therefore, λ is an $(\in, \in)$-fuzzy ideal of X.
$(ii) \Rightarrow (i)$: Let λ be an $(\in, \in)$-fuzzy ideal of X and $\lambda(x) = t$, for $x \in X$. Then $x_t \in \lambda$. By (3.5), $0_t \in \lambda$ and so $\lambda(0) \geq t = \lambda(x)$. Hence, $\lambda(0) \geq \lambda(x)$. Let $x, y \in X$ such that $\lambda(x) = t$ and $\lambda(\neg(\neg x \to \neg y)) = s$. Then $x_t \in \lambda$ and $\neg(\neg x \to \neg y)_s \in \lambda$. By (3.6), we have $y_{\min\{t,s\}} \in \lambda$ and so, $\lambda(y) \geq \min\{t, s\} = \min\{\lambda(x), \lambda(\neg(\neg x \to \neg y))\}$. □

PROPOSITION 3.19. Let λ be an $(\in, \in)$-fuzzy ideal of X. Then for all $x, y \in X$, the following assertions are valid:

(1) $\forall x \in X, \quad \lambda(1) \leq \lambda(x)$

(2) $\forall x, y \in X, \quad \lambda(x \to y) \geq \min\{\lambda(x), \lambda(y)\}$

(3) $\forall x, y \in X, \quad$ if $x \leq y$, then $\lambda(x) \geq \lambda(y)$

Proof: (1), (2) The proof is clear.
(3) Let $x \leq y$. Then $\neg y \leq \neg x$, so $\neg(\neg y \to \neg x) = \neg 1 = 0$. Since λ is an $(\in, \in)$-fuzzy ideal of X by Theorem 3.18 , we have

$$\lambda(x) \geq \min\{\lambda(\neg(\neg y \to \neg x)), \lambda(y)\} = \min\{\lambda(0), \lambda(y)\} = \lambda(y).$$

Thus λ is order reversing. $\square$

Proposition 3.20. If λ is a non-zero $(\in, \in)$-fuzzy ideal of X, then $X_0 = \{x \in X | \lambda(x) \neq 0\}$ is an ideal of X.

Proof: Since λ is non-zero, there exists $x \in X$ such that $\lambda(x) \neq 0$ and so $X_0 \neq \emptyset$. Suppose $x \in X_0$. Then $\lambda(x) > 0$. By Theorem 3.18, $\lambda(0) \geq \lambda(x) > 0$. Thus, $0 \in X_0$. Now, consider $x, \neg(\neg x \to \neg y) \in X_0$. Then by Theorem 3.18, we have

$$\lambda(y) \geq \min\{\lambda(x), \lambda(\neg(\neg x \to \neg y))\} > 0$$

Hence $\lambda(y) > 0$, and so $y \in \lambda_0$. Therefore, X_0 is an ideal of X. $\square$

In the following theorem, we investigate that under which condition, the converse of Theorem 3.13 is true, in general.

Theorem 3.21. *Let X be bounded and a fuzzy set λ in X be an $(\in, \in)$-fuzzy ideal of X. If the following assertion is valid,*

$$(\forall x, y \in X)(\forall t, k \in [0, 1]) \left(\; x_t \in \lambda, \; y_k \in \lambda \; \Rightarrow (x \to y)_{\min\{t,k\}} \in \lambda \; \right), \tag{3.7}$$

then, the fuzzy set λ in X is an $(\in, \in)$-fuzzy sub-equality algebra of X.

Proof: Let $x_t \in \lambda$ and $y_k \in \lambda$. Since for any $x, y \in X$, by (2.11) $x \sim y \leq x \to y$ and λ in X is an $(\in, \in)$-fuzzy ideal, by Proposition 3.19(3), we have

$$\lambda(x \sim y) \geq \lambda(x \to y) \geq \min\{\lambda(x), \lambda(y)\} \geq \min\{t, k\}$$

Thus $(x \sim y)_{\min\{t,k\}} \in \lambda$. Also, we know that $x \wedge y \leq x, y$. Then $\lambda(x), \lambda(y) \leq \lambda(x \wedge y)$, by Proposition 3.19(3). Hence, $\min\{t, k\} \leq \min\{\lambda(x), \lambda(y)\} \leq \lambda(x \wedge y)$ and so, $(x \wedge y)_{\min\{t,k\}} \in \lambda$. Therefore, S is an $(\in, \in)$-fuzzy sub-equality algebra of X $\square$

Given a fuzzy set λ in X, we consider the set

$$U(\lambda;t) := \{x \in X \mid \lambda(x) \geq t\}, \tag{3.8}$$

which is called an $\in$-*level set* of λ (related to t).

THEOREM 3.22. *A fuzzy set λ in X is an $(\in,\in)$-fuzzy sub-equality algebra of X if and only if the non-empty $\in$-level set $U(\lambda;t)$ of λ is a sub-equality algebra of X for all $t \in [0,1]$.*

PROOF: Let λ be a fuzzy set in X such that $U(\lambda;t)$ is a non-empty sub-equality algebra of X for all $t \in [0,1]$. Let $x, y \in X$ and $t, k \in [0,1]$ be such that $x_t \in \lambda$ and $y_k \in \lambda$. Then $\lambda(x) \geq t$ and $\lambda(y) \geq k$, and so $x, y \in U(\lambda; \min\{t,k\})$. By hypothesis, we have $x \sim y \in U(\lambda; \min\{t,k\})$ and $x \wedge y \in U(\lambda; \min\{t,k\})$. Then $(x \sim y)_{\min\{t,k\}} \in \lambda$ and $(x \wedge y)_{\min\{t,k\}} \in \lambda$. Therefore λ is an $(\in,\in)$-fuzzy sub-equality algebra of X.

Conversely, assume that λ is an $(\in,\in)$-fuzzy sub-equality algebra of X. Let $x, y \in U(\lambda;t)$ for all $t \in [0,1]$. Then $\lambda(x) \geq t$ and $\lambda(y) \geq t$, that is, $x_t \in \lambda$ and $y_t \in \lambda$. By (3.1) we have $(x \sim y)_t \in \lambda$ and $(x \wedge y)_t \in \lambda$. Then $x \sim y \in U(\lambda;t)$ and $x \wedge y \in U(\lambda;t)$. Therefore $U(\lambda;t)$ of λ is a sub-equality algebra of X for all $t \in [0,1]$. □

COROLLARY 3.23. Consider a fuzzy set λ in X is an $(\in,\in)$-fuzzy sub-equality algebra of X. Then λ is closed under the operation $\rightarrow$ if and only if the non-empty $\in$-level set $U(\lambda;t)$ of λ is closed under the operation $\rightarrow$.

PROOF: Let a fuzzy set λ in X be an $(\in,\in)$-fuzzy sub-equality algebra of X. For any $x, y \in U(\lambda;t)$, $\lambda(x), \lambda(y) \geq t$ and we get $\lambda(x \rightarrow y) = \lambda(x \sim (x \wedge y)) \geq \min\{\lambda(x), \lambda(y)\} \geq t$. Hence $x \rightarrow y \in U(\lambda;t)$.
Conversely, suppose $U(\lambda;t)$ is closed under the operation $\rightarrow$. Let $x, y \in X$ and $t, k \in [0,1]$ be such that $x_t \in \lambda$ and $y_k \in \lambda$. Then $\lambda(x) \geq t$ and $\lambda(y) \geq k$, and so $\lambda(x), \lambda(y) \geq \min\{t,k\}$. Thus $x, y \in U(\lambda; \min\{t,k\})$. Since $U(\lambda;t)$ is closed under $\rightarrow$, we get $x \rightarrow y \in U(\lambda; \min\{t,k\})$. Hence $\lambda(x \rightarrow y) \geq \min\{t,k\}$. □

THEOREM 3.24. *Let λ be an $(\in,\in)$-fuzzy sub-equality algebra of X. Then the following are equivalent.*

(i) *λ is an $(\in,\in)$-fuzzy ideal of X.*

(ii) *The nonempty set $U(\lambda;t)$ is an ideal of X.*

PROOF: $(i) \Rightarrow (ii)$: Let λ be an $(\in, \in)$-fuzzy ideal of X such that $x \in U(\lambda;t)$. Then $\lambda(x) \geq t$. By (i), since $x_t \in \lambda$, we have $0_t \in \lambda$ and so $\lambda(0) \geq t$. Hence, $0 \in U(\lambda;t)$. Now, suppose $x, \neg(\neg x \to \neg y) \in U(\lambda;t)$. Then $\lambda(x) \geq t$ and $\lambda(\neg(\neg x \to \neg y)) \geq t$ and so $x_t \in \lambda$ and $\neg(\neg x \to \neg y)_t \in \lambda$. By (i), we have $y_t \in \lambda$ and so $\lambda(y) \geq t$. Thus, $y \in U(\lambda;t)$. Therefore $U(\lambda;t)$ is an ideal of X.
$(ii) \Rightarrow (i)$: Let $x_t \in \lambda$. Then $\lambda(x) \geq t$ and so $x \in U(\lambda;t)$. By (ii), $0 \in U(\lambda;t)$ and so $\lambda(0) \geq t$. Hence $0_t \in \lambda$. Suppose $x_t \in \lambda$ and $\neg(\neg x \to \neg y)_k \in \lambda$. Then $x, \neg(\neg x \to \neg y) \in U(\lambda; \min\{t,k\})$. By (ii), $y \in U(\lambda; \min\{t,k\})$. Hence $y_{\min\{t,k\}} \in \lambda$. Therefore, λ is an $(\in, \in)$-fuzzy ideal of X. □

THEOREM 3.25. *Let S be an ideal of X. For any $t \in [0,1]$, there exists an $(\in,\in)$-fuzzy ideal λ of X such that $U(\lambda;t) = S$.*

PROOF: Let $t \in [0,1]$ and $\lambda : X \longrightarrow [0,1]$ is defined by $\lambda(x) = t$, for any $x \in S$ and $\lambda(x) = 0$, otherwise. By definition, clearly $U(\lambda;t) = S$. So it is enough to prove that λ is an $(\in, \in)$-fuzzy ideal of X. Let $x \in X$. Then $\lambda(x) = 0$ or $\lambda(x) = t$. Since S is an ideal of X, we have $0 \in S$ and so $\lambda(0) = t$. Hence, $\lambda(0) \geq \lambda(x)$, for any $x \in X$.
Now, suppose $x_t \in \lambda$ and $\neg(\neg x \to \neg y)_k \in \lambda$. Then, we have the following cases:

Case 1: If $\lambda(x) = \lambda(\neg(\neg x \to \neg y)) = t$. Then $x, \neg(\neg x \to \neg y) \in S$. Since S is an ideal of X, we have $y \in S$ and so $\lambda(y) = t$. Hence, $\lambda(y) \geq \min\{\lambda(x), \lambda(\neg(\neg x \to \neg y))\}$.

Case 2: If $\lambda(x) = t$ and $\lambda(\neg(\neg x \to \neg y)) = 0$. Then $x \in S$ and $\neg(\neg x \to \neg y) \notin S$. Then $\lambda(y) = 0$ or $\lambda(y) = t$ and in both case $\lambda(y) \geq \min\{\lambda(x), \lambda(\neg(\neg x \to \neg y))\} = 0$.

Case 3: If $\lambda(x) = 0$ and $\lambda(\neg(\neg x \to \neg y)) = t$, then similar to Case2, $\lambda(y) \geq \min\{\lambda(x), \lambda(\neg(\neg x \to \neg y))\}$.

Case 4: If $\lambda(x) = \lambda(\neg(\neg x \to \neg y)) = 0$, then $x, \neg(\neg x \to \neg y) \notin S$. Clearly,

$$\lambda(y) \geq \min\{\lambda(x), \lambda(\neg(\neg x \to \neg y))\} = 0.$$

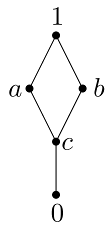
Therefore, λ is an $(\in, \in)$-fuzzy ideal of X. □

DEFINITION 3.26. Let X be bounded. A fuzzy set λ in X is called a *fuzzy prime ideal* of X if the following assertions are valid.

$$(\forall x \in X)(\lambda(0) \geq \lambda(x)) \tag{3.9}$$

$$(\forall x, y \in X) \begin{cases} \lambda(\neg(x \to y)) \geq \min\{\lambda(x), \lambda(y)\} \\ \text{or} \\ \lambda(\neg(y \to x)) \geq \min\{\lambda(y), \lambda(x)\} \end{cases} \tag{3.10}$$

Example 3.27. Let $X = \{0, a, b, c, 1\}$ be a set with the following Hasse diagram.



Then $(X, \wedge, 1)$ is a commutative idempotent integral monoid. We define a binary operation $\sim$ on X by Table 5. Then $(X, \wedge, \sim, 1)$ is an equality

Table 5. Cayley table for the implication "$\sim$"

$\sim$	0	a	b	c	1
0	1	0	0	0	0
a	0	1	b	a	c
b	0	b	1	c	a
c	0	a	c	1	b
1	0	c	a	b	1

algebra, and the implication "$\to$" is given by Table 6. We define a fuzzy set λ in X as follows:

$$\lambda : X \to [0, 1],\ x \mapsto \begin{cases} 0.7 & \text{if } x = 0, \\ 0.6 & \text{if } x = c, \\ 0.5 & \text{if } x = a, \\ 0.3 & \text{if } x \in \{b, 1\}. \end{cases}$$

Then λ is a fuzzy prime ideal of X.

Table 6. Cayley table for the implication "$\to$"

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	1	1	1
b	0	b	1	b	1
c	0	a	a	1	1
1	0	c	a	b	1

DEFINITION 3.28. Let X be bounded. A fuzzy set λ in X is called an $(\in, \in)$-*fuzzy prime ideal* of X if the following assertions are valid.

$$(\forall x, y \in X)(\forall t, k \in [0,1]) \left(x_t \in \lambda,\ y_k \in \lambda \ \Rightarrow \begin{cases} \neg(x \to y)_{\min\{t,k\}} \in \lambda \\ \text{or} \\ \neg(y \to x)_{\min\{t,k\}} \in \lambda \end{cases} \right). \tag{3.11}$$

Example 3.29. Let X be an equality algebra in as Example 3.27. Obviously, λ is an $(\in, \in)$-fuzzy prime ideal of X.

THEOREM 3.30. *Let X be bounded. Then, λ is a fuzzy prime ideal of X if and only if $U(\lambda; t)$ is a prime ideal of X.*

PROOF: Let $x \in U(\lambda; t)$. Then $\lambda(x) \geq t$. Since λ is a fuzzy prime ideal of X, we have $\lambda(0) \geq \lambda(x) \geq t$. Thus, $0 \in U(\lambda; t)$. Suppose $x, y \in U(\lambda; t)$. Then $\lambda(x), \lambda(y) \geq t$. Since $\lambda(\neg(x \to y)) \geq \min\{\lambda(x), \lambda(y)\} \geq t$ or $\lambda(\neg(y \to x)) \geq t$, we have $\neg(x \to y) \in U(\lambda; t)$ or $\neg(y \to x) \in U(\lambda; t)$. Hence $U(\lambda; t)$ is a prime ideal of X.
Conversely, assume $\lambda(x) = t$. Then $\lambda(x) \geq t$ and so $x \in U(\lambda; t)$. Since $U(\lambda; t)$ is a prime ideal of X, $0 \in U(\lambda; t)$. Thus $\lambda(0) \geq t = \lambda(x)$. Suppose $x_t \in \lambda$ and $y_k \in \lambda$. Then $x, y \in U(\lambda; \min\{t, k\})$. Since $U(\lambda; \min\{t, k\})$ is a prime ideal, we have $\neg(x \to y) \in U(\lambda; \min\{t, k\})$ or $\neg(y \to x) \in U(\lambda; \min\{t, k\})$. Hence $\neg(x \to y)_{\min\{t,k\}} \in \lambda$ or $\neg(y \to x)_{\min\{t,k\}} \in \lambda$, so λ is a fuzzy prime ideal of X. $\square$

THEOREM 3.31. *A fuzzy set λ in X is an $(\in, \in)$-fuzzy prime ideal of X if and only if λ is a fuzzy prime ideal.*

PROOF: The proof is similar to the proof of Theorem 3.18. $\square$

THEOREM 3.32. *Let λ in X be an $(\in, \in)$-fuzzy ideal on X. Define the relation*

$$x \equiv_\lambda y \Longleftrightarrow \neg(\neg x \to \neg y)_{\lambda(0)} \in \lambda \text{ and } \neg(\neg y \to \neg x)_{\lambda(0)} \in \lambda,$$

for any $x, y \in X$. Then $\equiv_\lambda$ is an equivalence relation on X

PROOF: Let $x, y, z \in X$. Clearly, the relation $\equiv_\lambda$ is reflexive and symmetric. Suppose $x \equiv_\lambda y$ and $y \equiv_\lambda z$. Then $\neg(\neg x \to \neg y)_{\lambda(0)} \in \lambda, \neg(\neg y \to \neg x)_{\lambda(0)} \in \lambda, \neg(\neg y \to \neg z)_{\lambda(0)} \in \lambda$ and $\neg(\neg z \to \neg y)_{\lambda(0)} \in \lambda$. Thus by (2.9) and (2.12) we have

$$\neg y \to \neg z \leq (\neg x \to \neg y) \to (\neg x \to \neg z) \leq \neg\neg(\neg x \to \neg y) \to \neg\neg(\neg x \to \neg z),$$

and so,

$$\neg(\neg\neg(\neg x \to \neg y) \to \neg\neg(\neg x \to \neg z)) \leq \neg(\neg y \to \neg z).$$

Since λ is an $(\in, \in)$-fuzzy ideal of X, by Proposition 3.19(3), we get

$$\begin{aligned}\lambda(0) &= \lambda(\neg(\neg y \to \neg z)) \\ &\leq \lambda(\neg(\neg\neg(\neg x \to \neg y) \to \neg\neg(\neg x \to \neg z))) \\ &\leq \lambda(0).\end{aligned}$$

Hence $(\neg(\neg\neg(\neg x \to \neg y) \to \neg\neg(\neg x \to \neg z)))_{\lambda(0)} \in \lambda$. In addition by assumption and Theorem 3.18, we have

$$\begin{aligned}\lambda(\neg(\neg x \to \neg z)) \geq \min\{&\lambda(\neg(\neg\neg(\neg x \to \neg y) \to \neg\neg(\neg x \to \neg z))), \\ &\lambda(\neg(\neg x \to \neg y))\} \\ = \min\{&\lambda(0), \lambda(0)\} = \lambda(0)\end{aligned}$$

Hence $\neg(\neg x \to \neg z)_{\lambda(0)} \in \lambda$. By similar way, $\neg(\neg z \to \neg x)_{\lambda(0)} \in \lambda$, and so $\equiv_{\lambda(0)}$ is transitive. Therefore, $\equiv_{\lambda(0)}$ is an equivalence relation on X. $\square$

Note. Denote by $[x]_\lambda$ the set $\{y \in X | x \equiv_\lambda y\}$ and $\frac{X}{\equiv_\lambda}$ the set $\{[x]_\lambda | x \in X\}$.

PROPOSITION 3.33. Let λ in X be an $(\in, \in)$-fuzzy ideal on X. Then $[0] = \{x \in X | \lambda(x) = \lambda(0)\}$ and $[1] = \{x \in X | \lambda(\neg x) = \lambda(0)\}$.

Proof: Let λ be an $(\in,\in)$-fuzzy ideal on X. Then $[0]=\{x\in X|x\equiv_{\lambda(0)} 0\}=\{x\in X|\lambda(\neg(\neg x\to\neg 0))=\lambda(0)$ and $\lambda(\neg(\neg 0\to\neg x))=\lambda(0)\}=\{x\in X|\lambda(\neg\neg x)=\lambda(0)\}$. Since $x\leq\neg\neg x$ and λ is an $(\in,\in)$-fuzzy ideal of X, by Proposition 3.19(3), $\lambda(0)=\lambda(\neg\neg x)\leq\lambda(x)\leq\lambda(0)$. Hence, $\lambda(x)=\lambda(0)$. So $[0]=\{x\in X|\lambda(\neg\neg x)=\lambda(0)\}=\{x\in X|\lambda(x)=\lambda(0)\}$. The proof of other case is similar. □

Proposition 3.34. Let λ be an $(\in,\in)$-fuzzy ideal of X. Define

$$[x]\leq[y]\Longleftrightarrow\neg(\neg x\to\neg y)_{\lambda(0)}\in\lambda,$$

for any $[x],[y]\in\frac{X}{\equiv_{\lambda(0)}}$. Then $(\frac{X}{\equiv_{\lambda(0)}},\leq)$ is a poset.

Proof: Let $[x],[y]\in\frac{X}{\equiv_{\lambda(0)}}$. Obviously, $\leq$ is reflexive. Suppose $[x]\leq[y]$ and $[y]\leq[x]$. Then $\neg(\neg x\to\neg y)_{\lambda(0)}\in\lambda$ and $\neg(\neg y\to\neg x)_{\lambda(0)}\in\lambda$. Thus $x\equiv_{\lambda(0)} y$ and so $[x]=[y]$. Assume that $[x]\leq[y]$ and $[y]\leq[z]$ for any $x,y,z\in X$. Then $\neg(\neg x\to\neg y)_{\lambda(0)}\in\lambda$ and $\neg(\neg y\to\neg z)_{\lambda(0)}\in\lambda$. By similar to the proof of Theorem 3.32, we have

$$\lambda(\neg(\neg y\to\neg z))\leq\lambda(\neg(\neg\neg(\neg x\to\neg y)\to\neg\neg(\neg x\to\neg z))).$$

From $\neg(\neg y\to\neg z)_{\lambda(0)}\in\lambda$, we get $\neg(\neg\neg(\neg x\to\neg y)\to\neg\neg(\neg x\to\neg z))_{\lambda(0)}\in\lambda$, and so by Theorem 3.18, we have

$$\begin{aligned}\lambda(\neg(\neg x\to\neg z))&\geq\min\{\lambda(\neg(\neg\neg(\neg x\to\neg y)\to\neg\neg(\neg x\to\neg z)),\\&\qquad\lambda(\neg(\neg x\to\neg y))\}\\&=\min\{\lambda(0),\lambda(0)\}\\&=\lambda(0).\end{aligned}$$

Hence $\neg(\neg x\to\neg z)_{\lambda(0)}\in\lambda$ and so $[x]\leq[z]$. Therefore, $(\frac{X}{\equiv_{\lambda(0)}},\leq)$ is a poset. □

Theorem 3.35. *Let λ be an $(\in,\in)$-fuzzy ideal of X. Define*

$$x\sim_\lambda y\Longleftrightarrow\neg(x\to y)_{\lambda(0)}\in\lambda \text{ and } \neg(y\to x)_{\lambda(0)}\in\lambda$$

Then for any $x,y\in X$, $\sim_\lambda$ is an equivalence relation on X.

PROOF: Let λ be an $(\in, \in)$-fuzzy ideal of X. Clearly the relation $\sim_\lambda$ is a reflexive and symetric relation on X. Suppose $x, y, z \in X$ such that $x \sim_\lambda y$ and $y \sim_\lambda z$. Then $\neg(x \to y)_{\lambda(0)} \in \lambda, \neg(y \to x)_{\lambda(0)} \in \lambda, \neg(y \to z)_{\lambda(0)} \in \lambda$ and $\neg(z \to y)_{\lambda(0)} \in \lambda$. Then by (2.11) and (2.9) we have,

$$x \to y \leq (y \to z) \to (x \to z) \leq \neg(x \to z) \to \neg(y \to z).$$

Thus $\neg(\neg(x \to z) \to \neg(y \to z)) \leq \neg(x \to y)$. Since λ is an $(\in, \in)$-fuzzy ideal of X, by Proposition 3.19(3), we get $\lambda(0) = \lambda(\neg(x \to y)) \leq \lambda(\neg(\neg(x \to z) \to \neg(y \to z))) \leq \lambda(0)$. Hence $\neg(\neg(x \to z) \to \neg(y \to z))_{\lambda(0)} \in \lambda$. Since

$$\begin{aligned} \neg(\neg(x \to z) \to \neg(y \to z)) &= \neg(\neg(x \to z) \to \neg\neg\neg(y \to z)) \\ &= \neg(\neg\neg(y \to z) \to \neg\neg(x \to z)), \end{aligned}$$

by Theorem 3.18, we have $\neg(\neg\neg(y \to z) \to \neg\neg(x \to z))_{\lambda(0)} \in \lambda$ and $\neg(y \to z)_{\lambda(0)} \in \lambda$, we get $\neg(x \to z)_{\lambda(0)} \in \lambda$. By similar way, $\neg(z \to x)_{\lambda(0)} \in \lambda$, and so $x \sim_\lambda z$. Therefore $\sim_\lambda$ is an equivalence relation on X. $\square$

Also, similar to Proposition 3.34, we can define an order $\leq$ on X as follows:

$$[x] \leq [y] \Longleftrightarrow \neg(x \longrightarrow y)_{\lambda(0)} \in \lambda,$$

and prove that $(\frac{X}{\sim}, \leq)$ is a poset.

4. $(\in, \in \vee q)$-fuzzy sub-equality algebra

In this section, we define an $(\in, \in \vee q)$-fuzzy sub-equality of an equality algebra X and investigate that the properties of the $(\in, \in \vee q)$-fuzzy sub-equality algebras.

DEFINITION 4.1. A fuzzy set λ in X is called an $(\in, \in \vee q)$-*fuzzy sub-equality algebra* of X if the following assertion is valid.

$$(\forall x, y \in X)(\forall t, k \in [0,1]) \left(x_t \in \lambda,\ y_k \in \lambda \ \Rightarrow \begin{cases} (x \sim y)_{\min\{t,k\}} \in \vee\, q\, \lambda \\ (x \wedge y)_{\min\{t,k\}} \in \vee\, q\, \lambda \end{cases} \right). \tag{4.1}$$

Example 4.2. Consider the equality algebra $(X, \sim, \wedge, 1)$ which is described in Example 3.9. Define a fuzzy set λ in X as follows:

$$\lambda : X \to [0,1],\ x \mapsto \begin{cases} 0.8 & \text{if } \ x = 1, \\ 0.3 & \text{if } \ x = a, \\ 0.71 & \text{if } \ x = 0, \\ 0.73 & \text{if } \ x = b. \end{cases}$$

Then λ is an $(\in, \in\vee q)$-fuzzy sub-equality algebra of X.

Note. Every $(\in, \in)$-fuzzy sub-equality algebra is an $(\in, \in\vee q)$-fuzzy sub-equality algebra.

The converse of Note 4 is not true in general as seen in the following example.

Example 4.3. The $(\in, \in\vee q)$-fuzzy sub-equality algebra μ in Example 4.2 is not an $(\in, \in)$-fuzzy sub-equality algebra of X since

$$0.3 = \lambda(a) = \lambda(b \sim 0) \ngeq \min\{\lambda(b), \lambda(0)\} = 0.7$$

We consider characterizations of $(\in, \in\vee q)$-fuzzy sub-equality algebra.

THEOREM 4.4. *A fuzzy set λ in X is an $(\in, \in\vee q)$-fuzzy sub-equality algebra of X if and only if the following assertion is valid.*

$$(\forall x, y \in X) \left(\begin{array}{l} \lambda(x \sim y) \geq \min\{\lambda(x), \lambda(y), 0.5\} \\ \lambda(x \wedge y) \geq \min\{\lambda(x), \lambda(y), 0.5\} \end{array} \right). \tag{4.2}$$

PROOF: Assume λ is an $(\in, \in\vee q)$-fuzzy sub-equality algebra of X and $x, y \in X$. Suppose $\min\{\lambda(x), \lambda(y)\} < 0.5$. If $\lambda(x \sim y) < \min\{\lambda(x), \lambda(y)\}$, then $\lambda(x \sim y) < t \leq \min\{\lambda(x), \lambda(y)\}$ for some $t \in [0, 0.5)$, since $t \leq \min\{\lambda(x), \lambda(y)\} < 0.5$. It follows that $x_t \in \lambda$ and $y_t \in \lambda$. By assumption, $(x \sim y) \in\vee q\lambda$ and so $\lambda(x \sim y) \geq t$ or $\lambda(x \sim y) + t > 1$. If $\lambda(x \sim y) \geq t$, then is a contradiction, since $\lambda(x \sim y) < t$. If $\lambda(x \sim y) + t > 1$, then $\lambda(x \sim y) > 1 - t > 0.5$, is contradiction, since $\lambda(x \sim y) < 0.5$. Hence, $\lambda(x \sim y) \geq \min\{\lambda(x), \lambda(y)\}$, and so $(x \sim y)_t \in\vee q\lambda$. By the similar discussion, we get $\lambda(x \wedge y) \geq \min\{\lambda(x), \lambda(y)\}$ whenever $\min\{\lambda(x), \lambda(y)\} < 0.5$. Assume that $\min\{\lambda(x), \lambda(y)\} \geq 0.5$. Then $x_{0.5} \in \lambda$ and $y_{0.5} \in \lambda$. It follows from (4.1) that $(x \sim y)_{0.5} = (x \sim y)_{\min\{0.5, 0.5\}} \in\vee q\,\lambda$ and $(x \wedge y)_{0.5} = (x \wedge y)_{\min\{0.5, 0.5\}} \in\vee q\,\lambda$. Thus $\lambda(x \sim y) \geq 0.5$ and $\lambda(x \wedge$

$y) \geq 0.5$. Consequently, $\lambda(x \sim y) \geq \min\{\lambda(x), \lambda(y), 0.5\}$ and $\lambda(x \wedge y) \geq \min\{\lambda(x), \lambda(y), 0.5\}$.

Conversely, suppose λ satisfies the condition (4.2). Let $x, y \in X$ and $t, k \in [0,1]$ be such that $x_t \in \lambda$ and $y_k \in \lambda$. Then $\lambda(x) \geq t$ and $\lambda(y) \geq k$. If $\lambda(x \sim y) < \min\{t, k\}$, then $\min\{\lambda(x), \lambda(y)\} \geq 0.5$ because if $\min\{\lambda(x), \lambda(y)\} < 0.5$, then

$$\lambda(x \sim y) \geq \min\{\lambda(x), \lambda(y), 0.5\} \geq \min\{\lambda(x), \lambda(y)\} \geq \min\{t, k\}$$

which is a contradiction. Hence, $\lambda(x \sim y) \geq 0.5$. Similarly, if $\lambda(x \wedge y) < \min\{t, k\}$, then $\min\{\lambda(x), \lambda(y)\} \geq 0.5$. It follows that

$$\lambda(x \sim y) + \min\{t, k\} > 2\lambda(x \sim y) \geq 2\min\{\lambda(x), \lambda(y), 0.5\} \geq 1$$

and

$$\lambda(x \wedge y) + \min\{t, k\} > 2\lambda(x \wedge y) \geq 2\min\{\lambda(x), \lambda(y), 0.5\} = 1.$$

Hence $(x \sim y)_{\min\{t,k\}} q\lambda$ and $(x \wedge y)_{\min\{t,k\}} q\lambda$, and so $(x \sim y)_{\min\{t,k\}} \in\vee\, q\lambda$ and $(x \wedge y)_{\min\{t,k\}} \in\vee\, q\lambda$. Therefore, λ is an $(\in, \in\vee\, q)$-fuzzy sub-equality algebra of X. $\square$

THEOREM 4.5. *If a fuzzy set λ in X is an $(\in, \in\vee\, q)$-fuzzy sub-equality algebra of X, then the following assertion is valid.*

$$(\forall x, y \in X)(\forall t, k \in [0,1])\left(\ x_t \in \lambda,\ y_k \in \lambda\ \Rightarrow (x \to y)_{\min\{t,k\}} \in\vee\, q\,\lambda\ \right). \tag{4.3}$$

PROOF: Assume λ is an $(\in, \in\vee\, q)$-fuzzy sub-equality algebra of X. Let $x, y \in X$ and $t, k \in [0,1]$ be such that $x_t \in \lambda$ and $y_k \in \lambda$. Then $\lambda(x) \geq t$ and $\lambda(y) \geq k$, thus by Theorem 3.13, we have $(x \to y)_{\min\{t,k\}} \in \lambda$. Hence, $(x \to y)_{\min\{t,k\}} \in\vee\, q\,\lambda$. $\square$

In the following example, we show that the converse of the above theorem may not be true, in general.

Example 4.6. According to Example 3.14, we have $0.3 = \lambda(a \wedge b) = \lambda(0) \ngeq \min\{\lambda(a), \lambda(b)\} = 0.4$, also $\lambda(a \wedge b) + \min\{\lambda(a), \lambda(b)\} \ngeq 1$. Hence, λ is not an $(\in, \in\vee\, q)$-fuzzy sub-equality.

THEOREM 4.7. *A fuzzy set λ in X is an $(\in, \in\vee\, q)$-fuzzy sub-equality algebra of X if and only if the non-empty $\in$-level set $U(\lambda; t)$ of λ is a sub-equality algebra of X for all $t \in (0, 0.5]$.*

PROOF: Assume that λ is an $(\in, \in \vee q)$-fuzzy sub-equality algebra of X. Let $x, y \in U(\lambda; t)$ for $t \in (0, 0.5]$. Then $\lambda(x) \geq t$ and $\lambda(y) \geq t$. It follows from Theorem 4.4 that $\lambda(x \sim y) \geq \min\{\lambda(x), \lambda(y), 0.5\} \geq \min\{t, 0.5\} = t$ and $\lambda(x \wedge y) \geq \min\{\lambda(x), \lambda(y), 0.5\} \geq \min\{t, 0.5\} = t$. Hence $x \sim y \in U(\lambda; t)$ and $x \wedge y \in U(\lambda; t)$. Therefore, $U(\lambda; t)$ is a sub-equality algebra of X.

Conversely, suppose the non-empty $\in$-level set $U(\lambda; t)$ of λ is a sub-equality algebra of X for all $t \in (0, 0.5]$. If there exists $x, y \in X$ such that $\lambda(x \sim y) < \min\{\lambda(x), \lambda(y), 0.5\}$ or $\lambda(x \wedge y) < \min\{\lambda(x), \lambda(y), 0.5\}$, then $\lambda(x \sim y) < t \leq \min\{\lambda(x), \lambda(y), 0.5\}$ or $\lambda(x \wedge y) < t \leq \min\{\lambda(x), \lambda(y), 0.5\}$ for some $t \in (0, 1]$. Hence $t \leq 0.5$ and $x, y \in U(\lambda; t)$, and by assumption $x \sim y \in U(\lambda; t)$ and $x \wedge y \in U(\lambda; t)$, and so $\lambda(x \sim y) \geq t$ and $\lambda(x \wedge y) \geq t$ which is a contradiction. Hence, $\lambda(x \sim y) \geq \min\{\lambda(x), \lambda(y), 0.5\}$ and $\lambda(x \wedge y) \geq \min\{\lambda(x), \lambda(y), 0.5\}$. Therefore, by Theorem 4.4, λ is an $(\in, \in \vee q)$-fuzzy sub-equality algebra of X. □

We provide a condition for an $(\in, \in \vee q)$-fuzzy sub-equality algebra to be an $(\in, \in)$-fuzzy sub-equality algebra.

THEOREM 4.8. *If an $(\in, \in \vee q)$-fuzzy sub-equality algebra λ of X satisfies the condition*

$$(\forall x \in X)(\lambda(x) < 0.5), \tag{4.4}$$

then λ is an $(\in, \in)$-fuzzy sub-equality algebra of X.

PROOF: Let $x, y \in X$ and $t, k \in [0, 1]$ be such that $x_t \in \lambda$ and $y_k \in \lambda$. Then $\lambda(x) \geq t$ and $\lambda(y) \geq k$. By assumption, (4.4), and Theorem 4.4, we have

$$\lambda(x \sim y) \geq \min\{\lambda(x), \lambda(y), 0.5\} = \min\{\lambda(x), \lambda(y)\} \geq \min\{t, k\}$$

and

$$\lambda(x \wedge y) \geq \min\{\lambda(x), \lambda(y), 0.5\} = \min\{\lambda(x), \lambda(y)\} \geq \min\{t, k\}.$$

Hence $(x \sim y)_{\min\{t,k\}} \in \lambda$ and $(x \wedge y)_{\min\{t,k\}} \in \lambda$. Therefore λ is an $(\in, \in)$-fuzzy sub-equality algebra of X. □

PROPOSITION 4.9. If λ is a non-zero $(\in, \in \vee q)$-fuzzy sub-equality algebra of X, then $\lambda(1) > 0$.

PROOF: Assume that $\lambda(1) = 0$. Since λ is non-zero, there exists $x \in X$ such that $\lambda(x) = t \neq 0$, and so for any $t \in (0, 1]$, $x_t \in \lambda$. Then $\lambda(x \sim x) =$

$\lambda(1) = 0$ and $\lambda(x \sim x) + t = \lambda(1) + t = t \leq 1$, that is, $(x \sim x)_t \overline{\in} \lambda$ and $(x \sim x)_t \, \overline{q} \, \lambda$. Thus $(x \sim x)_t \, \overline{\in \vee q} \, \lambda$, which is a contradiction. Therefore $\lambda(1) > 0$. $\square$

COROLLARY 4.10. If λ is a non-zero $(\in, \in)$-fuzzy sub-equality algebra of X, then $\lambda(1) > 0$.

THEOREM 4.11. *For any sub-equality algebra S of X and $t \in [0, 0.5)$, there exists an $(\in, \in \vee q)$-fuzzy sub-equality algebra λ of X such that $U(\lambda; t) = S$.*

PROOF: Let λ be a fuzzy set in X defined by

$$\lambda : X \to [0,1], \; x \mapsto \begin{cases} t & \text{if } \; x \in S, \\ 0 & \text{otherwise}, \end{cases} \tag{4.5}$$

where $t \in [0, 0.5)$. Obviously, $U(\lambda; t) = S$. Suppose that $\lambda(x \sim y) < \min\{\lambda(x), \lambda(y), 0.5\}$ or $\lambda(x \wedge y) < \min\{\lambda(x), \lambda(y), 0.5\}$ for some $x, y \in X$. Since $|Im(\lambda)| = 2$, it follows that $\lambda(x \sim y) = 0$ or $\lambda(x \wedge y) = 0$, and $\min\{\lambda(x), \lambda(y), 0.5\} = t$. Since $t < 0.5$, we have $\lambda(x) = t = \lambda(y)$ and so $x, y \in S$. Then $x \sim y \in S$ and $x \wedge y \in S$, which imply that $\lambda(x \sim y) = t$ and $\lambda(x \wedge y) = t$, which is a contradiction, and so $\lambda(x \sim y) \geq \min\{\lambda(x), \lambda(y), 0.5\}$ and $\lambda(x \wedge y) \geq \min\{\lambda(x), \lambda(y), 0.5\}$. Hence, by Theorem 4.4, we know that λ is an $(\in, \in \vee q)$-fuzzy sub-equality algebra of X. $\square$

For any fuzzy set λ in X and $t \in [0, 1]$, we consider the following sets and we call then *q-level set* and *$\in \vee q$-level set*, respectively.

$$\lambda_q^t := \{x \in X \mid x_t \, q \, \lambda\} \text{ and } \lambda_{\in \vee q}^t := \{x \in X \mid x_t \in \vee \, q \lambda\}$$

Clearly, $\lambda_{\in \vee q}^t = \lambda_{\in}^t \cup \lambda_q^t$.

THEOREM 4.12. *A fuzzy set λ in X is an $(\in, \in \vee q)$-fuzzy sub-equality algebra of X if and only if $\lambda_{\in \vee q}^t$ is a sub-equality algebra of X for all $t \in [0, 1]$.*

PROOF: Assume that λ is an $(\in, \in \vee q)$-fuzzy sub-equality algebra of X. Let $x, y \in \lambda_{\in \vee q}^t$ for $t \in [0, 1]$. Then $x_t \in \vee \, q \lambda$ and $y_t \in \vee \, q \lambda$, i.e., $\lambda(x) \geq t$ or $\lambda(x) + t > 1$, and $\lambda(y) \geq t$ or $\lambda(y) + t > 1$. It follows from (4.2) that $\lambda(x \sim y) \geq \min\{t, 0.5\}$ and $\lambda(x \wedge y) \geq \min\{t, 0.5\}$. In fact, if $\lambda(x \sim y) < \min\{t, 0.5\}$ or $\lambda(x \wedge y) < \min\{t, 0.5\}$, then $x_t \, \overline{\in \vee q} \, \lambda$ or $y_t \, \overline{\in \vee q} \, \lambda$, a contradiction. If $t \leq 0.5$, then $\lambda(x \sim y) \geq \min\{t, 0.5\} = t$ and $\lambda(x \wedge y) \geq$

$\min\{t, 0.5\} = t$. Hence $x \sim y \in U(\lambda; t) \subseteq \lambda^t_{\in\vee\, q}$ and $x \wedge y \in U(\lambda; t) \subseteq \lambda^t_{\in\vee\, q}$. If $t > 0.5$, then $\lambda(x \sim y) \geq \min\{t, 0.5\} = 0.5$ and $\lambda(x \wedge y) \geq \min\{t, 0.5\} = 0.5$. Hence $\lambda(x \sim y) + t > 0.5 + 0.5 = 1$ and $\lambda(x \wedge y) + t > 0.5 + 0.5 = 1$, that is, $(x \sim y)_t\, q\, \lambda$ and $(x \wedge y)_t\, q\, \lambda$. It follows that $x \sim y \in \lambda^t_q \subseteq \lambda^t_{\in\vee\, q}$ and $x \wedge y \in \lambda^t_q \subseteq \lambda^t_{\in\vee\, q}$. Therefore $\lambda^t_{\in\vee\, q}$ is a sub-equality algebra of X for all $t \in (0, 1]$.

Conversely, let λ be a fuzzy set in X and $t \in [0, 1]$ such that $\lambda^t_{\in\vee\, q}$ is a sub-equality algebra of X. Suppose that $\lambda(x \sim y) < \min\{\lambda(x), \lambda(y), 0.5\}$ or $\lambda(x \wedge y) < \min\{\lambda(x), \lambda(y), 0.5\}$ for some $x, y \in X$. Then $\lambda(x \sim y) < k < \min\{\lambda(x), \lambda(y), 0.5\}$ or $\lambda(x \wedge y) < k < \min\{\lambda(x), \lambda(y), 0.5\}$ for some $k \in (0, 0.5)$. Hence $x, y \in U(\lambda; k) \subseteq \lambda^k_{\in\vee\, q}$, and so $x \sim y \in \lambda^k_{\in\vee\, q}$ and $x \wedge y \in \lambda^k_{\in\vee\, q}$. Thus $\lambda(x \sim y) \geq k$ or $\lambda(x \sim y) + k > 1$, and $\lambda(x \wedge y) \geq k$ or $\lambda(x \wedge y) + k > 1$. This is a contradiction, and therefore $\lambda(x \sim y) \geq \min\{\lambda(x), \lambda(y), 0.5\}$ and $\lambda(x \wedge y) \geq \min\{\lambda(x), \lambda(y), 0.5\}$ for all $x, y \in X$. Consequently, λ is an $(\in, \in\vee\, q)$-fuzzy sub-equality algebra of X by Theorem 4.4. $\square$

THEOREM 4.13. *If λ is an $(\in, \in\vee\, q)$-fuzzy sub-equality algebra of X, then the q-set λ^t_q is a sub-equality algebra of X for all $t \in (0.5, 1]$.*

PROOF: Let $x, y \in \lambda^t_q$ for $t \in (0.5, 1]$. Then $\lambda(x) + t > 1$ and $\lambda(y) + t > 1$, and so $\lambda(x) > 1 - t$, and $\lambda(y) > 1 - t$. By assumption, we have $(x \sim y)_{1-t} \in\vee\, q\lambda$ and $(x \wedge y)_{1-t} \in\vee\, q\lambda$. Thus, by Theorem 4.4 that

$$\lambda(x \sim y) \geq \min\{\lambda(x), \lambda(y), 0.5\} > \min\{1 - t, 0.5\},$$

since $t \in (0.5, 1]$, we have $1 - t \in [0, 0.5)$ and so $1 - t < 0.5$. Thus, $\lambda(x \sim y) \geq \min\{1 - t, 0.5\} = 1 - t$ and so $\lambda(x \sim y) + t > 1$. Hence $x \sim y \in \lambda^t_q$. Similarly, we have $x \wedge y \in \lambda^t_q$. $\square$

THEOREM 4.14. *Let $f : X \to Y$ be a homomorphism of equality algebras. If λ and μ are $(\in, \in\vee\, q)$-fuzzy sub-equality algebras of X and Y, respectively, then*

(1) *$f^{-1}(\mu)$ is an $(\in, \in\vee\, q)$-fuzzy sub-equality algebra of X.*

(2) *If f is onto and λ satisfies the condition*

$$(\forall T \subseteq X)(\exists x_0 \in T)\left(\lambda(x_0) = \sup_{x\in T}\lambda(x)\right), \tag{4.6}$$

then $f(\lambda)$ is an $(\in, \in\vee\, q)$-fuzzy sub-equality algebra of Y.

PROOF: (1) Let $x, y \in X$ and $t, k \in [0,1]$ be such that $x_t \in f^{-1}(\mu)$ and $y_k \in f^{-1}(\mu)$. Then $(f(x))_t \in \mu$ and $(f(y))_k \in \mu$. Since μ is an $(\in, \in \vee q)$-fuzzy sub-equality algebra of Y, we have

$$(f(x \sim y))_{\min\{t,k\}} = (f(x) \sim f(y))_{\min\{t,k\}} \in \vee\, q\, \mu$$

and

$$(f(x \wedge y))_{\min\{t,k\}} = (f(x) \wedge f(y))_{\min\{t,k\}} \in \vee\, q\, \mu.$$

Hence $(x \sim y)_{\min\{t,k\}} \in \vee\, q\, f^{-1}(\mu)$ and $(x \wedge y)_{\min\{t,k\}} \in \vee\, q\, f^{-1}(\mu)$. Therefore, $f^{-1}(\mu)$ is an $(\in, \in \vee q)$-fuzzy sub-equality algebra of X.

(2) Let $a, b \in Y$ and $t, k \in [0,1]$ be such that $a_t \in f(\lambda)$ and $b_k \in f(\lambda)$. Then $(f(\lambda))(a) \geq t$ and $(f(\lambda))(b) \geq k$. Using the condition (4.6), there exist $x \in f^{-1}(a)$ and $y \in f^{-1}(b)$ such that

$$\lambda(x) = \sup_{z \in f^{-1}(a)} \lambda(z) \text{ and } \lambda(y) = \sup_{w \in f^{-1}(b)} \lambda(w).$$

Then $x_t \in \lambda$ and $y_k \in \lambda$, which imply that $(x \sim y)_{\min\{t,k\}} \in \vee\, q\, \lambda$ and $(x \wedge y)_{\min\{t,k\}} \in \vee\, q\, \lambda$, since λ is an $(\in, \in \vee q)$-fuzzy sub-equality algebra of X. Now, $x \sim y \in f^{-1}(a \sim b)$ and $x \wedge y \in f^{-1}(a \wedge b)$, and so $(f(\lambda))(a \sim b) \geq \lambda(x \sim y)$ and $(f(\lambda))(a \wedge b) \geq \lambda(x \wedge y)$. Hence,

$$(f(\lambda))(a \sim b) \geq \min\{t,k\} \text{ or } (f(\lambda))(a \sim b) + \min\{t,k\} > 1$$

and

$$(f(\lambda))(a \wedge b) \geq \min\{t,k\} \text{ or } (f(\lambda))(a \wedge b) + \min\{t,k\} > 1,$$

that is, $(a \sim b)_{\min\{t,k\}} \in \vee\, q\, f(\lambda)$ and $(a \wedge b)_{\min\{t,k\}} \in \vee\, q\, f(\lambda)$. Therefore, $f(\lambda)$ is an $(\in, \in \vee q)$-fuzzy sub-equality algebra of Y. □

5. Conclusion

Our aim was to define the concepts of an $(\in, \in)$-fuzzy sub-equality algebra, an $(\in, \in \vee q)$-fuzzy sub-equality algebra and we discussed some properties and found some equivalent definitions of them. Then, we discussed characterizations of an $(\in, \in)$-fuzzy sub-equality algebra and an $(\in, \in \vee q)$-fuzzy sub-equality algebra. Also, we found relations between an $(\in, \in)$-fuzzy sub-equality algebra and an $(\in, \in \vee q)$-fuzzy sub-equality algebra.

References

[1] M. Aaly Kologani, M. Mohseni Takallo, R. Borzooei, Y. Jun, *Implicative equality algebras and annihilators in equality algebras*, **Journal of Discrete Mathematical Sciences and Cryptography**, (2021), pp. 1–18, DOI: https://doi.org/10.1080/09720529.2021.1876293.

[2] M. Aaly Kologani, M. Mosheni Takallo, R. A. Borzooei, Y. B. Jun, *Right and left mappings in equality algebras*, **Kragujevac Journal of Mathematics**, vol. 46(5) (2022), pp. 815–832, DOI: https://doi.org/10.46793/KgJMat2205.815K.

[3] M. Aaly Kologani, X. Xin, Y. Jun, M. Mohseni Takallo, *Positive implicative equality algebras and equality algebras with some types*, **Journal of Algebraic Hyperstructures and Logical Algebras**, vol. 3(2) (2022), pp. 69–86, DOI: https://doi.org/10.52547/HATEF.JAHLA.3.2.6.

[4] P. Allen, H. S. Kim, J. Neggers, *Smarandache disjoint in BCK/d-algebras*, **Scientiae Mathematicae Japonicae**, vol. 61(3) (2005), pp. 447–450.

[5] P. J. Allen, H. S. Kim, J. Neggers, *Companion d-algebras*, **Mathematica Slovaca**, vol. 57(2) (2007), pp. 93–106, DOI: https://doi.org/10.2478/s12175-007-0001-z.

[6] P. J. Allen, H.-S. Kim, J. Neggers, *Deformations of d/BCK-algebras*, **Bulletin of the Korean Mathematical Society**, vol. 48(2) (2011), pp. 315–324, DOI: https://doi.org/10.4134/BKMS.2011.48.2.315.

[7] R. Borzooei, F. Zebardast, M. Aaly Kologani, *Some types of filters in equality algebras*, **Categories and General Algebraic Structures with Applications**, vol. 7 (Special Issue on the Occasion of Banaschewski's 90th Birthday (II)) (2017), pp. 33–55.

[8] R. A. Borzooei, M. Mohseni Takallo, M. Aaly Kologani, Y. B. Jun, *Quotient Structures in Equality Algebras*, **Journal of Algebraic Systems**, vol. 11(2) (2024), pp. 65–82, DOI: https://doi.org/10.22044/JAS.2022.11919.1608.

[9] L. C. Ciungu, *On pseudo-equality algebras*, **Archive for Mathematical Logic**, vol. 53(5–6) (2014), pp. 561–570, DOI: https://doi.org/10.1007/s00153-014-0380-0.

[10] A. Dvurečenskij, O. Zahiri, *Pseudo equality algebras: revision*, **Soft Computing**, vol. 20 (2016), pp. 2091–2101, DOI: https://doi.org/10.1007/s00500-015-1888-x.

[11] B. Ganji Saffar, *Fuzzy n-fold obstinate and maximal (pre) filters of EQ-algebras*, **Journal of Algebraic Hyperstructures and Logical Algebras**, vol. 2(1) (2021), pp. 83–98, DOI: https://doi.org/10.52547/HATEF.JAHLA.2.1.6.

[12] S. Jenei, *Equality algebras*, **Studia Logica**, vol. 100(6) (2012), pp. 1201–1209, DOI: https://doi.org/10.1007/s11225-012-9457-0.

[13] S. Jenei, L. Kóródi, *On the variety of equality algebras*, [in:] **Proceedings of the 7th conference of the European Society for Fuzzy Logic and Technology**, Atlantis Press (2011), pp. 153–155, DOI: https://doi.org/10.2991/eusflat.2011.1.

[14] Y. B. Jun, *On (α, β)-fuzzy subalgebras of BCK/BCI-algebras*, **Bulletin of the Korean Mathematical Society**, vol. 42(4) (2005), pp. 703–711, DOI: https://doi.org/10.4134/BKMS.2005.42.4.703.

[15] V. Novák, B. De Baets, *EQ-algebras*, **Fuzzy sets and systems**, vol. 160(20) (2009), pp. 2956–2978, DOI: https://doi.org/10.1016/j.fss.2009.04.010.

[16] A. Paad, *Ideals in bounded equality algebras*, **Filomat**, vol. 33(7) (2019), pp. 2113–2123, DOI: https://doi.org/10.2298/FIL1907113P.

[17] M. M. Takallo, M. A. Kologani, *MBJ-neutrosophic filters of equality algebras*, **Journal of Algebraic Hyperstructures and Logical Algebras**, vol. 1(2) (2020), pp. 57–75, DOI: https://doi.org/10.29252/HATEF.JAHLA.1.2.6.

[18] L. A. Zadeh, *Fuzzy sets*, **Information and control**, vol. 8(3) (1965), pp. 338–353, DOI: https://doi.org/10.1016/S0019-9958(65)90241-X.

[19] F. Zebardast, R. A. Borzooei, M. A. Kologani, *Results on equality algebras*, **Information Sciences**, vol. 381 (2017), pp. 270–282, DOI: https://doi.org/10.1016/j.ins.2016.11.027.

Mona Aaly Kologani

Hatef Higher Education
Zahedan, Iran

e-mail: mona4011@gmail.com

Mohammad Mohseni Takallo
Shahid Beheshti University
Department of Mathematics
Faculty of Mathematical Sciences
Tehran, Iran
e-mail: mohammad.mohseni1122@gmail.com

Young Bae Jun
Gyeongsang National University
Department of Mathematics Education
Jinju 52828
Jinju, Amsterdam, Korea
e-mail: skywine@gmail.com

Rajab Ali Borzooei
Shahid Beheshti University
Department of Mathematics
Faculty of Mathematical Sciences
Tehran, Iran
Istinye University
Department of Mathematics
Faculty of Engineering and Natural Sciences
Istanbul, Turkiye
e-mail: borzooei@sbu.ac.ir