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## INTERPOLATION IN NORMAL EXTENSIONS OF THE BROUWER LOGIC

### Abstract

The Craig interpolation property and interpolation property for deducibility are considered for special kind of normal extensions of the Brouwer logic.

*Keywords:* normal extensions of the Brouwer logics, Kripke frames, interpolation property, amalgamation property

### 1. Introduction

In this paper we continue research on modal logics with and without the interpolation property within the family of normal extensions of Brouwer logic. The Brouwer logic is defined as follows:  $\mathbf{KTB} := \mathbf{K} \oplus T \oplus B$  where:

$$T := \Box p \rightarrow p$$

$$B := p \rightarrow \Box \Diamond p$$

By a normal extension we mean a logic which is closed under the rules of modus ponens (MP), substitution and the Gödel rule of necessitation (RN). The Brouwer logic  $\mathbf{KTB}$  is called to be non-transitive as it is characterized by the class of reflexive and symmetric (admitting non-transitive) frames.

In the paper [9] a class of logics without interpolation is described. The described logics are weakly transitive. In this paper we present some results for non-transitive logics determined by reflexive and symmetric Kripke frames being chains of points. So, we shall study the Brouwerian modal logic  $\mathbf{KTB.Alt(3)} := \mathbf{KTB} \oplus alt_3$  where

$$alt_3 := \Box p \vee \Box(p \rightarrow q) \vee \Box((p \wedge q) \rightarrow r) \vee \Box((p \wedge q \wedge r) \rightarrow s).$$

Let us emphasize that the logic **KTB.Alt(3)** is complete with respect to the class of reflexive and symmetric Kripke frames (possibly infinite) being chains of points.

**THEOREM 1.** [Byrd and Ullrich, 1977; Byrd, 1978] *Every normal modal logic which is a proper extension of **KTB.Alt(3)** has the finite model property and is finitely axiomatizable (and hence - decidable).*

It is easily seen by the above theorem that the cardinality of the class  $NEXT(\mathbf{KTB.Alt(3)})$  is only countably infinite.

We may also consider logics determined by reflexive and symmetric Kripke frames with a larger degree of branching. The axiom ( $alt_3$ ) is a special case of more general axiom ( $alt_n$ ),  $n \geq 3$ :

$$alt_n := \Box p_1 \vee \Box(p_1 \rightarrow p_2) \vee \dots \vee \Box((p_1 \wedge \dots \wedge p_n) \rightarrow p_{n+1}).$$

In contrast to  $NEXT(\mathbf{KTB.Alt(3)})$ , the family of logics  $NEXT(\mathbf{KTB.Alt(4)}) := \mathbf{KTB} \oplus alt_4$  is uncountably infinite, see [10].

## 2. Preliminaries

Let us recall some definitions. The symbol  $Var(\alpha)$  means the set of all propositional variables of the formula  $\alpha$ .

**DEFINITION 1.** *A logic  $L$  has the Craig interpolation property (CIP) if for every implication  $\alpha \rightarrow \beta$  in  $L$ , there exists a formula  $\gamma$  (interpolant for  $\alpha \rightarrow \beta$  in  $L$ ) such that*

$$\alpha \rightarrow \gamma \in L \quad \text{and} \quad \gamma \rightarrow \beta \in L$$

and  $Var(\gamma) \subseteq Var(\alpha) \cap Var(\beta)$ .

The weaker notion of interpolation for deducibility is defined as follows:

**DEFINITION 2.** *A logic  $L$  has interpolation for deducibility (IPD) if for any  $\alpha$  and  $\beta$  the condition  $\alpha \vdash_L \beta$  implies that there exists a formula  $\gamma$  such that*

$$\alpha \vdash_L \gamma \quad \text{and} \quad \gamma \vdash_L \beta$$

$Var(\gamma) \subseteq Var(\alpha) \cap Var(\beta)$ .

It is a logical folklore that (CIP) together with (MP) and deduction theorem implies (IPD). It is also known that **K**, **T**, **K4** and **S4** have (CIP),

see Gabbay [6]. Also the logics from  $NEXT(\mathbf{S4})$  are well characterized as regards interpolation (see [14], also [4], p.462-463). It is also known that  $\mathbf{S5}$  has (CIP). The last fact can be proven by applying a very general method of construction of inseparable tableaux (see i.e. [4], p. 446-449). The same method can be applied in the case of  $\mathbf{KTB}$ . Therefore, without getting into details, we get:

**THEOREM 2.** *The logic  $\mathbf{KTB}$  has (CIP).*

The method of construction of inseparable tableaux is not applicable in the case of  $\mathbf{KTB.Alt(3)}$  and its normal extensions. The following questions arise:

**QUESTION 1.** *Does the logic  $\mathbf{KTB.Alt(3)}$  has (CIP) or (IDP)?*

**QUESTION 2.** *Which logic from the family  $NEXT(\mathbf{KTB.Alt(3)})$  has (CIP) or (IDP)?*

We shall answer question 1 in section 3, whereas question 2 in section 4. In the second case, our approach is purely semantic. We shall consider logics determined by class of Kripke frames  $\mathcal{K}$ . Formally, the logic determined by  $\mathcal{K}$  is defined as follows:

$$L(\mathcal{K}) := \{\alpha \in Form : \mathfrak{F} \models \alpha \text{ for each } \mathfrak{F} \in \mathcal{K}\}.$$

Note that the class  $\mathcal{K}$  may consist of one frame only.

The properties (CIP) and (IPD) have an appropriate algebraic characterization, (see [14], [5]). The symbol  $V(L)$  denotes the variety of algebras characterizing the logic  $L$ .

**THEOREM 3.** *For any logic  $L \in NEXT(\mathbf{K})$  the following are equivalent:*

- $L$  possesses (CIP),
- $V(L)$  has the superamalgamation property.

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- $L$  possesses (IPD),
- $V(L)$  has the amalgamation property.

By theory of duality between finite Kripke frames and finite modal algebras, the superamalgamation property and amalgamation property is

transformed into appropriate properties (APK) and (SAPK) for class  $\mathcal{K}$  of Kripke frames. We recall the notion of  $p$ -morphism first.

DEFINITION 3. Let  $\mathfrak{F}_1 := \langle W_1, R_1 \rangle$  and  $\mathfrak{F}_2 := \langle W_2, R_2 \rangle$  be Kripke frames. A map  $f : W_1 \rightarrow W_2$  is a  $p$ -morphism from  $\mathfrak{F}_1$  to  $\mathfrak{F}_2$ , if it satisfies the following conditions:

- (p1)  $f$  maps  $W_1$  onto  $W_2$ ,
- (p2) for all  $x, y \in W_1$ ,  $xR_1y$  implies  $f(x)R_2f(y)$ ,
- (p3) for each  $x \in W_1$  and for each  $a \in W_2$ , if  $f(x)R_2a$  then there exists  $y \in W_1$  such that  $xR_1y$  and  $f(y) = a$ .

It is said that the frame  $\mathfrak{F}_1$  is reducible to  $\mathfrak{F}_2$  or that the frame  $\mathfrak{F}_2$  is a  $p$ -morphic reduct of  $\mathfrak{F}_1$ .

A reduction  $f$  of  $\mathfrak{F}_1$  to  $\mathfrak{F}_2$  is called a reduction of a model  $\mathfrak{M}_1 = \langle \mathfrak{F}_1, V_1 \rangle$  to a model  $\mathfrak{M}_2 = \langle \mathfrak{F}_2, V_2 \rangle$  if, for every variable  $p$  and every point  $x$  in  $\mathfrak{F} :$

$$(\mathfrak{M}_1, x) \models p \text{ iff } (\mathfrak{M}_2, f(x)) \models p.$$

Second, we give the definitions of (APK) and (SAPK) for frames.

DEFINITION 4. For any  $\mathfrak{F}_0, \mathfrak{F}_1$  and  $\mathfrak{F}_2$  in class  $K$  and for any  $p$ -morphism  $f_1 : \mathfrak{F}_1 \rightarrow \mathfrak{F}_0$  and  $f_2 : \mathfrak{F}_2 \rightarrow \mathfrak{F}_0$  there exist  $\mathfrak{F}$  in  $K$  and  $p$ -morphisms  $g_1 : \mathfrak{F} \rightarrow \mathfrak{F}_1$  and  $g_2 : \mathfrak{F} \rightarrow \mathfrak{F}_2$  such that  $f_1 \circ g_1 = f_2 \circ g_2$  (see Figure 1).

Superamalgamation property for frames except (APK) requires the additional condition (SAPK):

$$\forall x \in \mathfrak{F}_1 \forall y \in \mathfrak{F}_2 [f_1(x) = f_2(y) \Rightarrow \exists z \in \mathfrak{F} g_1(z) = x \wedge g_2(z) = y].$$

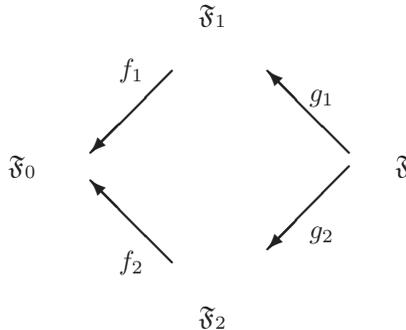


Figure 1.

### 3. Interpolation of **KTB.Alt(3)**

In this section we shall prove that the logic **KTB.Alt(3)** does not have (CIP). We define the suitable formula  $\alpha \rightarrow \beta$  as follows:

$$\begin{aligned}\alpha &:= \diamond(p \wedge q) \wedge \diamond(p \wedge \neg q), \\ \beta &:= [\diamond(\neg p \wedge r) \wedge \diamond(\neg p \wedge \neg r)] \rightarrow \perp.\end{aligned}$$

One may see that  $Var(\alpha) \cap Var(\beta) = \{p\}$ . First, we prove that

LEMMA 1. *The formula  $\alpha \rightarrow \beta$  is a theorem of **KTB.Alt(3)**.*

PROOF: Suppose, on the contrary, that  $\alpha \rightarrow \beta \notin \mathbf{KTB.Alt(3)}$ . There is some reflexive, symmetric and linear Kripke frame  $\mathfrak{F} = \langle W, R \rangle$ , a point  $x_0 \in W$  and a valuation  $V$ , such that  $x_0 \not\models_V \alpha \rightarrow \beta$ . Then

$$x_0 \models_V \alpha \tag{3.1}$$

$$x_0 \not\models_V \beta. \tag{3.2}$$

From (3.1) and (3.2) we get:

$$x_0 \models_V \diamond(p \wedge q) \wedge \diamond(p \wedge \neg q), \tag{3.3}$$

$$x_0 \models_V \diamond(\neg p \wedge r) \wedge \diamond(\neg p \wedge \neg r). \tag{3.4}$$

Ten we get:

$$x_0 \models_V \diamond(p \wedge q), \tag{3.5}$$

$$x_0 \models_V \diamond(p \wedge \neg q), \tag{3.6}$$

$$x_0 \models_V \diamond(\neg p \wedge r), \tag{3.7}$$

$$x_0 \models_V \diamond(\neg p \wedge \neg r). \tag{3.8}$$

From (3.5)-(3.8) we conclude that there are four points  $x_i \in W$ ,  $i := 1, \dots, 4$  such that  $x_0 R x_i$ , and:

$$x_1 \models_V p \wedge q, \quad x_2 \models_V p \wedge \neg q, \quad x_3 \models_V \neg p \wedge r, \quad x_4 \models_V \neg p \wedge \neg r,$$

and we conclude that  $x_i \neq x_j$  if  $i \neq j$  for  $i, j := 1, \dots, 4$ . Since the relation  $R$  is reflexive then we allow that  $x_i = x_0$  for some  $i$ . We have constructed a model in which one point  $x_0$  sees at least three others (excluding itself). Hence we get a contradiction with the axiom (*alt*<sub>3</sub>). ■

Second, we prove that there is no interpolant for  $\alpha \rightarrow \beta$ .

LEMMA 2. For the defined above formula  $\alpha \rightarrow \beta$  there is no formula  $\gamma$  such that  $\text{Var}(\gamma) \subset \text{Var}(\alpha) \cap \text{Var}(\beta) = \{p\}$ ,  $\alpha \rightarrow \gamma \in \mathbf{KTB.Alt(3)}$  and  $\gamma \rightarrow \beta \in \mathbf{KTB.Alt(3)}$ .

PROOF: Suppose, on the contrary, that there is a formula  $\gamma$ , written in one variable  $p$ , such that  $\alpha \rightarrow \gamma$  and  $\gamma \rightarrow \beta$  are theorems  $\mathbf{KTB.Alt(3)}$ . Then in each reflexive, symmetric and linear Kripke frame  $\mathfrak{F} = \langle W, R \rangle$ , at any point  $x \in W$  and for all valuations  $V_j$  we get:

$$x \models_{V_j} \alpha \rightarrow \gamma, \quad \text{and} \quad x \models_{V_j} \gamma \rightarrow \beta.$$

Let  $\mathfrak{F} = \langle \mathbb{Z}, R \rangle$ , where  $\mathbb{Z}$  - set of integers, and  $R$  is defined in the following way:  $nRm$  iff  $|n - m| \leq 1$ .

Let us consider all valuations  $V$  such that  $0 \models_V \alpha$ . Then four possible situations may hold.

1.  $V(p) \supset \{-1, 0, 1\}$  or
2.  $V(p) \supset \{-1, 0\}$  and  $1 \notin V(p)$  or
3.  $V(p) \supset \{0, 1\}$  and  $-1 \notin V(p)$  or
4.  $V(p) \supset \{-1, 1\}$  and  $0 \notin V(p)$ .

In all these situations the formula  $\gamma$  must be true at the point 0. We conclude that  $0 \models_V \gamma$  if  $V$  fulfils one of the conditions (1)-(4). Since  $0 \models_V \gamma \rightarrow \beta$  then for these valuations  $V$  we also get  $0 \models_V \beta$ .

Let us consider the case (4). Without losing generality we may take  $V_1$  such that  $V_1(p) = \{-1, 1\}$  and  $V_1(\alpha) = \{0\}$ , and  $V_1(\gamma) \supset \{0\}$ .

On the other side if  $\beta$  is false for some valuation then  $\gamma$  is false either.  $\beta$  may be falsified at the point 0, for example, for the following valuation  $V_2$ :  $V_2(p) = \{-2, 1\}$ . Then  $V_2(\gamma) \not\supset \{0\}$ .

We restrict ourselves to formulas of one variable  $p$ . In this way we have defined two different models  $\mathfrak{M}_1 := \langle \mathbb{Z}, R, V_1 \rangle$  and  $\mathfrak{M}_2 := \langle \mathbb{Z}, R, V_2 \rangle$ .

There are two different p-morphisms for these models:  $f_1(k) = |k|$  for all  $k \in \mathbb{Z}$  and  $f_2(-k-1) = f_2(k) = k$  for  $k \geq 0$ . We see that in the images  $f_1(\mathfrak{M}_1)$  and  $f_2(\mathfrak{M}_2)$  the valuations  $V_1$  and  $V_2$  of variable  $p$  will be change as follows:  $V_1^*(p) = \{1\}$  and  $V_2^*(p) = \{1\}$  what means that, in fact, they are identical.

Since p-morphism for models preserves the truth of formulas then we get  $V_1^*(\gamma) \supset \{0\}$  as well as  $V_2^*(\gamma) \not\supset \{0\}$ . This is a contradiction. Then the interpolant  $\gamma$  for  $\alpha \rightarrow \beta$  does not exist. ■

From Lemmas 1 and 2 we get

**THEOREM 5.** *The logic  $\mathbf{KTB.Alt(3)}$  does not have (CIP).*

■

One may ask a question if the above counterexample can be applied to show that  $\mathbf{KTB.Alt(3)}$  does not have (IPD). We shall leave this as an open question.

#### 4. Interpolation of tabular logics from $NEXT(\mathbf{KTB.Alt(3)})$

It occurred that there is an important connection between (CIP) and Halldén completeness of modal logics. So, we recall definition of the second notion.

**DEFINITION 5.** *A logic  $L$  is Halldén complete if*

$$\varphi \vee \psi \in L \text{ implies } \varphi \in L \text{ or } \psi \in L$$

*for all  $\varphi$  and  $\psi$  containing no common variables.*

Also, we need to recall the definition of the Post completeness for logic.

**DEFINITION 6.** *A logic  $L$  is said to be Post complete if it is consistent and has no proper consistent extension.*

One may notice that each logic from  $NEXT(\mathbf{KTB})$  has only one Post complete extension; namely it is the trivial logic  $\mathbf{Triv} = \mathbf{K4} \oplus \Box p \leftrightarrow p$ .

An important connection between (CIP) and Halldén completeness is given by G. F. Schumm in [16] in the following lemma:

**LEMMA 3.** *If  $L$  has only one Post-complete extension and is Halldén-incomplete, then interpolation fails in  $L$ .*

The above lemma concerns non-normal modal logics. They are logics axiomatized without the rule (RN). Semantically, they are determined by Kripke frames with distinguished points (the so-called unnormal worlds). However we shall consider special kind of Kripke frames in which the choice of distinguished points is completely unimportant. Our key tool to recognize logics with interpolation is a recognition of Halldén complete modal logics and the following lemma due to van Benthem and Humberstone from [1].

LEMMA 4. *If a modal logic  $L$  is determined by one Kripke frame, which is homogeneous, then  $L$  is Halldén complete.*

DEFINITION 7. *A Kripke frame  $\mathfrak{F} := \langle W, R \rangle$  is homogeneous if for any  $x, y \in W$  there exists an automorphism  $f$  of  $\langle W, R \rangle$  with  $f(x) = y$ .*

In the paper [11] it is proven that

THEOREM 6. *Let  $\mathfrak{F} := \langle W, R \rangle$  be  $KT\mathcal{B}$ -Kripke frame, which is finite and connected. Logic  $L(\mathfrak{F})$  is Halldén complete iff the frame  $\mathfrak{F}$  is homogeneous.*

Defining Halldén complete logics, we are bounded by another theorem due to Lemmon [13]. For non-normal modal logics, the theorem is an equivalence. For normal extensions it has the form of implication only. Following Lemmon we say, that two logics  $L_1, L_2 \in NEXT(L)$  are incomparable, if  $L_1 \not\subseteq L_2$  and  $L_2 \not\subseteq L_1$ .

THEOREM 7. *Let  $L_1, L_2 \in NEXT(L)$  be two incomparable logics. Then the logic  $L_0 = L_1 \cap L_2$  is Halldén incomplete.*

From Theorems 6 and 7 we conclude:

COROLLARY 1. *A Kripke complete and tabular logic from  $NEXT(\mathbf{KT\mathcal{B}})$ , which is Halldén complete must be determined by one connected and homogeneous Kripke frame.*

In paper [11] we have described a class of Halldén complete logics within the family of  $NEXT(\mathbf{KT\mathcal{B}.Alt(3)})$ . They are determined by so-called circular frames. Formally, we define:

DEFINITION 8. *A circular frame  $\mathfrak{C}_n := \langle W_n, R_n \rangle$ ,  $n \geq 3$  is defined as follows:*

$$\begin{aligned} W_n &:= \{x_1, x_2, \dots, x_n\}, \\ R_n &:= \{(x_i, x_j), i, j = 1, 2, \dots, n, |i - j|[\text{mod}(n - 1)] \leq 1\}. \end{aligned}$$

We also need a definition of a chain frame.

DEFINITION 9. *A chain frame  $\mathfrak{Ch}_n := \langle W_n, R_n \rangle$ ,  $n \geq 2$  is defined as follows:*

$$\begin{aligned} W_n &:= \{x_1, x_2, \dots, x_n\}, \\ R_n &:= \{(x_i, x_j), i, j = 1, 2, \dots, n, |i - j| \leq 1\}. \end{aligned}$$

*We also add to the class of chain frames the one point frame  $\circ$ .*

It is easy to notice that circular frames are the only non-trivial homogeneous Kripke frames characterizing logics from  $NEXT(\mathbf{KT\mathcal{B}.Alt(3)})$ .

So, we will study logics  $L(\mathfrak{C}_n)$ ,  $n \geq 3$  as well as two trivial cases  $L(\circ)$  and  $L(\circ - \circ)$  which are logics determined by one reflexive point or two reflexive points being in symmetric relation, appropriately.

**THEOREM 8.** *The logics  $L(\circ)$  and  $L(\circ - \circ)$  have (CIP).*

*Proof.* We shall consider amalgamation and superamalgamation properties for frames.

Case 1. Logic  $L(\circ)$ . The one-element class of frames  $\{\circ\}$  after closing under p-morphisms does not change. So, we take as  $\mathfrak{F}_0, \mathfrak{F}_1, \mathfrak{F}_2$  and  $\mathfrak{F}$ , the same frame  $\circ$ . All the needed p-morphisms are identities. Obviously, (SAPK) also holds.

Case 2. Logic  $L(\circ - \circ)$ . The one-element class of frames  $\{\circ - \circ\}$  after closing under p-morphisms enlarges to  $\{\circ - \circ, \circ\}$ . Suppose we choose as  $\mathfrak{F}_0$  the frame  $\circ$ , and as  $\mathfrak{F}_1$  and  $\mathfrak{F}_2$  twice the frame  $\circ - \circ$ . Then the p-morphisms  $f_1$  and  $f_2$  will glue  $\circ - \circ$  onto  $\circ$ . The needed frame  $\mathfrak{F}$  is  $\circ - \circ$  and the p-morphisms onto  $\mathfrak{F}_1$  and  $\mathfrak{F}_2$  are identities. Also (SAPK) holds. For other choices the proofs are similarly trivial. ■

All the possible reductions for circular frames are described in [12]. Each circular frame  $\mathfrak{C}_{2n-1}$ ,  $n \geq 2$  is reducible to some chain frame  $\mathfrak{C}\mathfrak{h}_n$ . The p-morphism may be described as gluing ‘in half’ the circle, see [12], Lemma 15. Further, each chain frame  $\mathfrak{C}\mathfrak{h}_{2n-1}$  is reducible to the chain frame  $\mathfrak{C}\mathfrak{h}_n$ , again by gluing ‘in half’. A chain frame with an even number of points  $\mathfrak{C}\mathfrak{h}_{2n}$  is reducible to both frames:  $\mathfrak{C}\mathfrak{h}_n$  and  $\mathfrak{C}\mathfrak{h}_{n+1}$ , see [12], Lemmas 13–14. We may conclude, by superposition of p-morphisms, that eventually each circle frame is reducible to  $\circ - \circ$ .

**LEMMA 5.** *The logic  $L(\mathfrak{C}_3)$  does not have (IPD).*

*Proof.* The one-element class of frames  $\{\mathfrak{C}_3\}$  after closing under p-morphisms enlarges to  $\{\mathfrak{C}_3, \circ - \circ, \circ\}$ . We show that the condition (APK) does not hold. We choose as follows:  $\mathfrak{F}_1 := \mathfrak{C}_3$ , and  $\mathfrak{F}_2 := \mathfrak{C}_3$ , and  $\mathfrak{F}_0 := \circ - \circ$ . We have to call the elements of the considered frames. Hence:  $\mathfrak{F}_1 := \langle \{a, b, c\}; R \rangle$ ,  $\mathfrak{F}_2 := \langle \{a', b', c'\}; R \rangle$ , and  $\mathfrak{F}_0 := \langle \{\alpha, \beta\}; R \rangle$ . In all cases  $R$  is reflexive and symmetric (what, in fact, involves here being an equivalence relation). As a frame  $\mathfrak{F}$  we have to choose  $\mathfrak{C}_3$ . Let  $\mathfrak{F} := \langle \{x, y, z\}, R \rangle$ . See Figure 2. The p-morphisms  $f_1$  and  $f_2$  are defined as follows:  $f_1(a) = f_1(c) = \beta$ ,  $f_1(b) = \alpha$  and  $f_2(a') = f_2(b') = \alpha$ ,  $f_2(c') = \beta$ .

There exists only one p-morphism  $\mathfrak{F} \rightarrow \mathfrak{F}_1$  up to renaming variables. We define  $g_1$ , for example, as follows  $g_1(x) = a$ ,  $g_1(y) = b$ ,  $g_1(z) = c$ .

Then  $x \xrightarrow{g_1} a \xrightarrow{f_1} \beta$ . Because only for  $c'$  we have  $c' \xrightarrow{f_2} \beta$  then we must define  $g_2(x) = c'$  and we get  $(f_1 \circ g_1)(x) = (f_2 \circ g_2)(x)$ .

So we try to define  $g_2$  for  $z$ . We have that  $z \xrightarrow{g_1} c \xrightarrow{f_1} \beta$ . Because only for one element  $c'$  we get  $f_2(c') = \beta$ , then we have to define  $g_2(z) = c'$ . But then  $g_2$  is not a p-morphism. We get a contradiction. ■

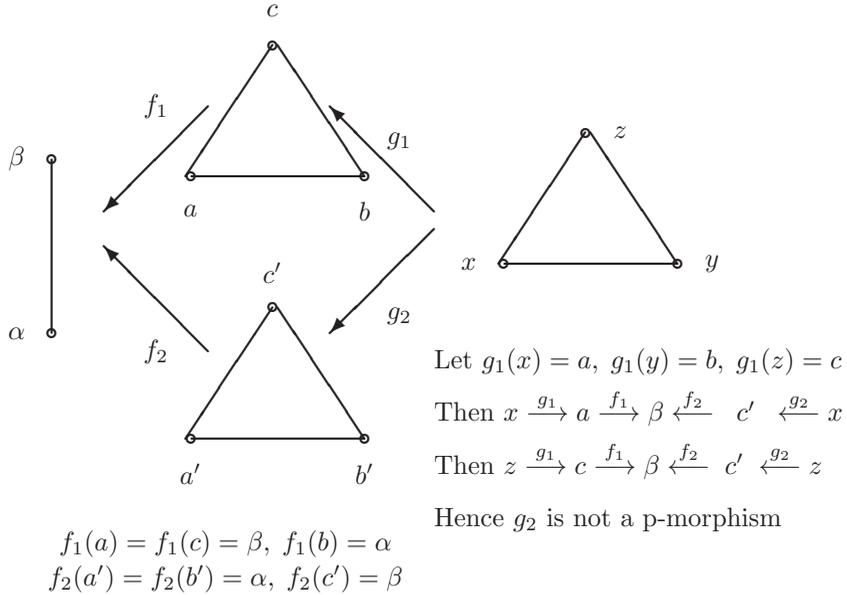


Figure 2.

We shall similarly prove that

LEMMA 6. *No logic  $L(\mathfrak{C}_{2n-1})$  with  $n \geq 3$ , has (IPD).*

*Proof.* Instead of making the full proof, we provide it for  $n = 7$ , to avoid a mess with indices.

First, we describe the class of possible reductions of  $\mathfrak{C}_7$ . It is not reducible to any other circle frame, but is reducible to  $\mathfrak{C}\mathfrak{h}_4$  and then to  $\mathfrak{C}\mathfrak{h}_2 := \circ - -\circ$  and finally to  $\circ$ . So we chose the needed frames as follows:  $\mathfrak{F}_1 := \mathfrak{C}_7, \mathfrak{F}_2 := \mathfrak{C}_7$  and  $\mathfrak{F}_0 := \mathfrak{C}\mathfrak{h}_4$ . Let  $\mathfrak{F}_1 := \langle \{x_1, \dots, x_7\}; R \rangle, \mathfrak{F}_2 := \langle \{x'_1, \dots, x'_7\}; R \rangle$  and  $\mathfrak{F}_0 := \langle \{a, b, c, d\}; R \rangle$ . The relation  $R$  is reflexive and symmetric.

We define the p-morphisms  $f_1 : \mathfrak{F}_1 \rightarrow \mathfrak{F}_0$  and  $f_2 : \mathfrak{F}_2 \rightarrow \mathfrak{F}_0$  as follows:

$$\begin{aligned} f_1(x_1) &= d, & f_2(x'_1) &= f_2(x'_2) = d \\ f_1(x_2) &= f_1(x_7) = c, & f_2(x'_3) &= f_2(x'_7) = c \\ f_1(x_3) &= f_1(x_6) = b, & f_2(x'_4) &= f_2(x'_6) = b \\ f_1(x_4) &= f_1(x_5) = a, & f_2(x'_5) &= a \end{aligned}$$

Then as the frame  $\mathfrak{F}$  we have to take  $\mathfrak{C}_7$  and the p-morphism  $g_1 : \mathfrak{F} \rightarrow \mathfrak{F}_1$  is a unique one up to renaming variables. Let  $\mathfrak{F} := \langle \{y_1, \dots, y_7\}; R \rangle$  and  $g_1(y_i) = x_i$  for  $i := 1, \dots, 7$ .

Then we get:  $y_4 \xrightarrow{g_1} x_4 \xrightarrow{f_1} a$ ,  $x'_5 \xrightarrow{f_2} a$  hence for  $g_2$  we must take:  $g_2(y_4) = x'_5$  and we have  $(g_1 \circ f_1)(y_4) = (g_2 \circ f_2)(y_4)$ . Then we try to define in the appropriate way  $g_2$  for  $y_5$ . We have:  $y_5 \xrightarrow{g_1} x_5 \xrightarrow{f_1} a$ , also only for  $x'_5$  we have  $x'_5 \xrightarrow{f_2} a$ . So we must take  $g_2(y_5) = x'_5$ . But then  $g_2$  is not a p-morphism.

A quite analogous proof may be provided for other odd numbers, with an analogous choice of frames  $\mathfrak{F}_1 := \mathfrak{C}_{2n-1}$ ,  $\mathfrak{F}_2 := \mathfrak{C}_{2n-1}$ ,  $\mathfrak{F}_0 := \mathfrak{C}_{\mathfrak{h}_n}$ , with  $n \geq 3$ . ■

LEMMA 7. *No logic  $L(\mathfrak{C}_{2n})$  with  $n \geq 2$  has (IPD).*

*Proof.* Instead of making the full proof, we provide it for  $n = 4$ , again to avoid a mess with indexes. The one-element class of frames  $\{\mathfrak{C}_4\}$  after closing under p-morphisms enlarges to  $\{\mathfrak{C}_4, \mathfrak{C}_{\mathfrak{h}_3}, \circ - \circ\}$ . We define a counterexample for (APK). We choose as follows:  $\mathfrak{F}_1 := \mathfrak{C}_4$ , and  $\mathfrak{F}_2 := \mathfrak{C}_{\mathfrak{h}_3}$ , and  $\mathfrak{F}_0 := \circ - \circ$ . We call the elements of the considered frames. Hence:  $\mathfrak{F}_1 := \langle \{a, b, c, d\}; R \rangle$ ,  $\mathfrak{F}_2 := \langle \{a', b', c'\}; R \rangle$ , and  $\mathfrak{F}_0 := \langle \{\alpha, \beta\}; R \rangle$ . In all considered cases  $R$  is reflexive and symmetric. As a frame  $\mathfrak{F}$  we have to choose  $\mathfrak{C}_4$ . Let  $\mathfrak{F} := \langle \{x, y, z, w\}; R \rangle$ . The p-morphisms  $f_1$  and  $f_2$  are defined as follows:  $f_1(a) = f_1(d) = \beta$ ,  $f_1(b) = f_1(c) = \alpha$  and  $f_2(a') = f_2(c') = \alpha$ ,  $f_2(b') = \beta$ . There exists only one p-morphism  $\mathfrak{F} \rightarrow \mathfrak{F}_1$  up to renaming variables. We define  $g_1$ , for example, as follows  $g_1(x) = a$ ,  $g_1(y) = b$ ,  $g_1(z) = c$  and  $g_1(w) = d$ . We obtain  $x \xrightarrow{g_1} a \xrightarrow{f_1} \beta$ . Since  $b' \xrightarrow{f_2} \beta$  then we must define  $g_2(x) = b'$  and we get  $(f_1 \circ g_1)(x) = (f_2 \circ g_2)(x)$ . Then we consider another element  $w$ .

We have that  $w \xrightarrow{g_1} d \xrightarrow{f_1} \beta$ . Because only for one element  $b'$  we get  $f_2(b') = \beta$ , then we have to define  $g_2(w) = b'$ . But then  $g_2$  is not a p-morphism. We get a contradiction. See Figure 3.

A quite analogous proof may be provided for other even numbers, with an analogous choice of frames  $\mathfrak{F}_1 := \mathfrak{C}_{2n}$ ,  $\mathfrak{F}_2 := \mathfrak{C}h_n$ ,  $\mathfrak{F}_0 := \mathfrak{C}h_2$ , with  $n \geq 3$ .

■

As a conclusion we obtain:

**COROLLARY 2.** *No tabular logic from  $NEXT(\mathbf{KTB.Alt(3)})$  distinct from  $L(\circ)$  or  $L(\circ - \circ)$  has (IPD) (and (CIP)).*

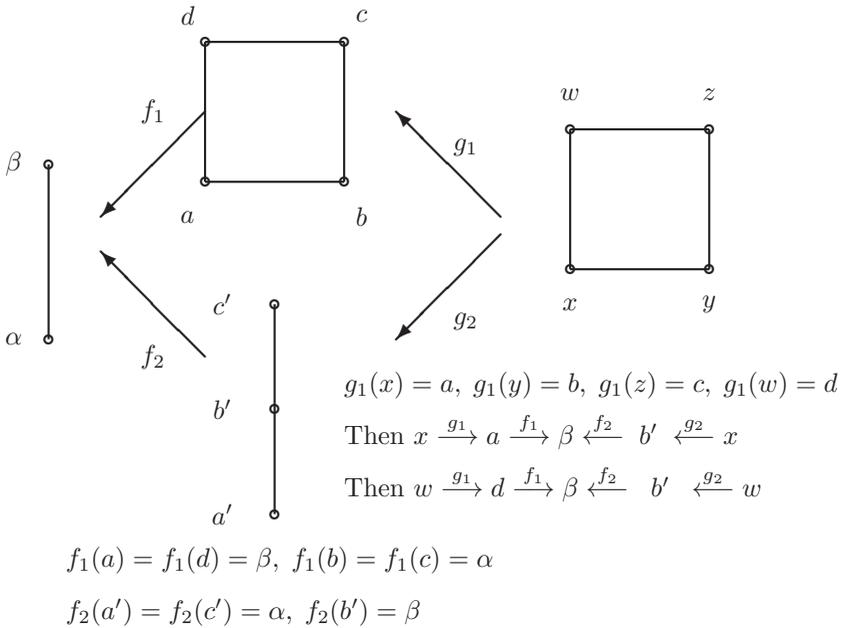


Figure 3.

## 5. Problems

In the paper we prove many negative results on interpolation in the family  $NEXT(\mathbf{KTB.Alt(3)})$ . Our future work will concern interpolation within  $NEXT(\mathbf{KTB.Alt(n)})$ , for  $n \geq 4$ . For each  $n \geq 1$ , the logic  $\mathbf{KTB.Alt(n)}$  is complete with respect to the class of reflexive and symmetric Kripke frames  $\mathfrak{F}$  such that each point in  $\mathfrak{F}$  sees at most  $n$  others (including itself).

Looking for tabular logics with interpolation among  $NEXT(\mathbf{KTB.Alt}(n))$  we have to consider homogeneous Kripke frames, that are here, for example, Platonic and Archimedean solids. But not only. The problem is very interesting; in fact describing p-morphisms in such cases is not a trivial job. In some simple cases it seems to be easier. First, we would like to prove that:

CONJECTURE 1. *The logic determined by a reflexive and symmetric Kripke frame having the structure of a Boolean cube has (IDP).*

The logic determined by such a cube belongs to  $\mathbf{KTB.Alt}(4)$ . In the area of logics determined by Kripke frames with a larger degree of branching, we also would like to show that

CONJECTURE 2. *The logic determined by a reflexive and symmetric Kripke frame having the structure of  $2^n$ -element Boolean cube,  $n \geq 3$ , has (IDP).*

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