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MONADIC FRAGMENTS OF INTUITIONISTIC CONTROL LOGIC

Abstract

We investigate monadic fragments of Intuitionistic Control Logic (ICL), which is obtained from Intuitionistic Propositional Logic (IPL) by extending language of IPL by a constant distinct from intuitionistic constants. In particular we present the complete description of purely negational fragment and show that most of monadic fragments are finite.

Keywords: Intuitionistic Control Logic, Intuitionistic Logic, Combining Logic, Control Operators

1. Introduction

The interplay between classical and intuitionistic logic has been investigated for many years now. In this area the translations of Gödel, Kolmogorov and others, which embed classical propositional logic into intuitionistic logic, or the Glivenko Theorem are the best known examples. Still the combination of these two systems is a relatively new notion and yet to be fully discovered. Some attempts to extend intuitionistic logic with classical connectives, in particular with classical negation, were undertaken (cf. Nelson, Rasiowa, Sendlewski and others). Another important attempt was Girard's Linear Logic. This logic contains two significant properties: a fully involutive negation, which is characteristic for classical logic, and a strong constructive interpretation, which is a core of intuitionistic logic. Nevertheless, linear logic has its limitations and main drawbacks are complicated language and undecidability of the full system.

This was the inspiration for Ch. Liang and D. Miller who in [3], [4] and [5] presented several systems that to some extent can be treated as a combination of classical and intuitionistic logic. In this paper we present Intuitionistic Control Logic (ICL) which was introduced in [3]. The language of ICL results from espanding that of Intuitionistic Propositional Logic (IPL) by additional new constant for falsum $\perp$, distinct from intuitionistic falsum 0. In [3] Liang and Miller defined sequent calculus LJC for ICL. This proof system is isomorphic to the usual natural deduction system for intuitionistic logic except for two rules which concern the additional constant $\perp$. The authors defined topological and Kripke style semantics for ICL and proved soundness and completeness of LJC with respect to these semantics. In this paper, however, we shall focus on the Kripke model approach.

Having two different falsum constant entails having two kinds of negation: intuitionistic $\sim A = A \to 0$ and "classical" $\neg A = A \to \perp$. The second negation plays crucial role in emulating within ICL control operators such as $\mathcal{C}$ or *call/cc* (function existing in many functional programming langueages, for example Scheme), which was the main impetus for creating this logic. In this paper we investigate monadic fragments of ICL, that is, the language restricted to one variable only. Moreover, we will investigate fragments resulting in restriction of connectives, for example we deal with purely negational monadic fragment i.e., the fragment in the laguage of $(p, \sim, \neg)$.

2. Preliminaria

We start with recalling some facts about ICL from [3]. Formulae of ICL are build from atomic formulae (propositional variables), intuitionistic connectives $\vee, \wedge, \to$ and of three constants $0, 1, \perp$. Constants denoted by 0 and 1 corresponds respectively to intuitionistic falsum and verum, thus if a formula of ICL does not contain the additional constant $\perp$, then it is an intuitionistic formula. We use the expression $A \leftrightarrow B$ as a shorthand for $(A \to B) \wedge (B \to A)$.

A Kripke model for ICL will be called an *r-model* and is defined as follows.

Definition 2.1. A *Kripke r-model* is a quadruple $\langle W, \mathbf{r}, \leq, \Vdash \rangle$ where W is a finite, non-empty set, $\leq$ is a reflexive and transitive relation on the set

W and $\Vdash$ is a binary relation between elements of W and atomic formulae called *forcing*. Elements of the set W are called *worlds* or *nodes*. The element $\mathbf{r} \in W$ is the *root* of the model. It is the least element of the set W ($\mathbf{r} \leq u$ for every world $u \in W$). As usual, the pair $\langle W, \leq \rangle$ is called a *Kripke frame*.

Similarly as in Kripke models for intuitionistic logic, the forcing relation $\Vdash$ is monotone, that is if $u \leq v$ then $u \Vdash p$ implies $v \Vdash p$. The $\Vdash$ relation is extended to all formulae in the following way. Let $u, v, i \in W$.

- $u \Vdash 1$ and $u \nVdash 0$
- $\mathbf{r} \nVdash \bot$
- $i \Vdash \bot$ for all $i > \mathbf{r}$
- $u \Vdash A \vee B$ iff $u \Vdash A$ or $u \Vdash B$
- $u \Vdash A \wedge B$ iff $u \Vdash A$ and $u \Vdash B$
- $u \Vdash A \to B$ iff for all $v \geq u$ if $v \Vdash A$ then $v \Vdash B$.

Satisfiability of a formula A in an r-model $\mathcal{M}$ (in symbols $\mathcal{M} \models A$) and validity of a formula A (in symbols $\models A$) are defined for the class of all r-models as usual.

We consider only models based on finite, rooted frames or even finite trees. It is known that Kripke models for intuitionistic propositional logic can be assumed to have this restriction. Forcing of the additional constant $\bot$ distinguishes between the root of an r-model and the rest of worlds. Worlds properly above the root, that is worlds in which $\bot$ is forced, will be called *imaginary*. We use symbols u, v, w to represent arbitrary worlds in W and the symbol i to represent imaginary worlds.

Having two different constants for falsum, enables to define two different negations

Intuitionistic negation: $\sim p = p \to 0$

Classical negation: $\neg p = p \to \bot$

It is well-known that the set of theorems of intuitionistic propositional logic is the proper subset of the set of classical theorems. For example it does not contain neither the law of excluded middle nor the double negation principle. Similarly in ICL none of the two formulae $\sim p \vee p$ nor $\sim\sim p \to p$ are theorems. However, the law of excluded middle with respect to the second negation, that is $\neg p \vee p$, is an ICL tautology, hence the term *classical* in the name of this negation. Nevertheless, being defined using intuitionistic implication, this negation is not involutive as $\neg\neg p$ does not imply p. Therefore we prefer to call it $\bot$-*negation*, instead of "classical" negation.

3. Negational fragment

The impetus for ICL came from the search of a logic that would enable to type programming language control operators without collapsing intuitionistic implication into classical one. Within ICL we can emulate control operator $\mathcal{C}$ using formula $\sim\neg A \to A$ and control operator *call/cc* using a version of Peirce's formula namely $((A \to\bot) \to A) \to A$. In both formulae the $\bot$-negation plays an important role. From this, and the fact that the combination of the two negations is not involutive, arises the question of the actual number of nonequivalent negational formulae in ICL. This was investigated in details in [2], in this section we quote the most important results.

We consider the fragment in the language of $\sim$, $\neg$ and p only and look into properties of these two negations and the interplay between them. To stress that the only connectives are negations, we call this fragment *monadic purely negational fragment.* We treat both negations as primitive connectives, not defined by means of constants and implication. Every r-model defined as in Definition 2.1 is a model for this fragment of ICL as well but we define interpretation of negations independently.

Definition 3.1. A Kripke model for the negational fragment of ICL is a tuple $\mathcal{M} = \langle W, \mathbf{r}, \leq, \Vdash\rangle$ where W, r and $\leq$ are defined as in Definition 2.1 and the forcing relation $\Vdash$ is restricted to the variable p. Additionally we define the interpretation of negations:

- $u \Vdash \sim A$ iff $w \nVdash A$ for all $w \geq u$,
- $u \Vdash \neg A$ iff $w \nVdash A$ or $w > \mathbf{r}$ for all $w \geq u$.

It is according to the definition of forcing for constants $0, \bot$ and intuitionistic implication in the case of full language. Forcing of intuitionistic negation is standard. For $\bot$-negation we have

$$u \Vdash \neg p \text{ iff } w \nVdash p \text{ or } w \Vdash\bot, \text{ for all } w \geq u.$$

The condition $w \Vdash\bot$ means that w is an imaginary world.

Directly from the above definition of forcing of negations we get four basic properties.

FACT 3.2. *For every Kripke model $\mathcal{M} = \langle W, \mathbf{r}, \leq, \Vdash \rangle$ and every negational formula A we have:*

(1) $\mathbf{r} \Vdash \neg A$ iff $\mathbf{r} \nVdash A$,
(2) $\mathbf{r} \nVdash \neg A$ iff $\mathbf{r} \Vdash A$,
(3) $u \nVdash \neg A$ iff $u = \mathbf{r}$ and $u \Vdash A$, for arbitrary $u \in W$,
(4) $i \Vdash \neg A$ for all $i > \mathbf{r}$.

In ICL the distinction between the root of the r-model and other worlds is expressed by the forcing of $\perp$, whereas in the monadic purely negational fragment the root of the model is the only world in which $\perp$-negation of a formula can be refuted. It follows that $\perp$-negation of a formula is forced in every imaginary world.

We consider monadic negational formulae, i.e., we deal with formulae of the form $p, \neg p, \sim\neg p, \sim\neg\sim p, \neg\sim\sim\neg p$ etc. Arbitrary sequence of both negations of length $k \in \{0, 1, 2, \ldots\} = \mathbb{N}$ will be denoted N_k. For iteration of n negations of given kind we write either $\sim^n$ or $\neg^n$. By *the length of an negational formula $N_k p$* we define the number k of negations of both kinds. Formulae of the form $N_{2j}p$ and $N_{2j+1}p$ will be called *even negational formula* and *odd negational formula*, respectively. We treat the variable p as a negational formula of the length 0.

Let $\mathcal{N}$ be the set of all negational formulae. We consider negational formulae up to the equivalence relation defined on the set $\mathcal{N}$ in the standard way:

$$A \equiv B \text{ iff } \models A \leftrightarrow B.$$

Later on we shall use a relation $\preceq$ defined on the quotient set $\mathcal{N}/_{\equiv}$ by

$$[A]_{\equiv} \preceq [B]_{\equiv} \text{ iff } \models A \rightarrow B,$$

where $[A]_{\equiv}$ is the equivalence class of a formula A.

It is easy to see that no negational formula with odd number of negations can be equivalent to a formula with even number of negations. Therefore all properties concerning equivalences between such formulae are divided into two cases: for odd and even number of negations. The property of reduction of negations with respect to sequences of one type of negation is straightforward and it follows from the fact that both $\sim\sim\sim p \equiv \sim p$ and $\neg\neg\neg p \equiv \neg p$.

Proposition 3.3. *For any $k \in \mathbb{N}$ we have:*

(1) $\sim^{(2k+2)}p \equiv \sim\sim p$,
(2) $\sim^{(2k+1)}p \equiv \sim p$,
(3) $\neg^{(2k+2)}p \equiv \neg\neg p$,
(4) $\neg^{(2k+1)}p \equiv \neg p$.

We already pointed out a formula $\sim\neg A \to A$ which enables to emulate the control operator $\mathcal{C}$. It turns out that the negational formula $\sim\neg p$ is a representative of a wide class of equivalent negational formulae of the form $\sim\neg N_{2k}p$. Similar role for the class of odd negational formulae plays either the formula $\sim\neg\neg p$ or the equivalent formula $\sim\neg\sim p$.

Proposition 3.4. *For any $k \in \mathbb{N}$ we have:*

(1) $\sim\neg N_{2k}p \equiv \sim\neg p$,
(2) $\sim\neg N_{2k+1}p \equiv \sim\neg\neg p$.

Proposition 3.3 and Proposition 3.4 show that in many cases we can reduce a negational formula of a greater length to a formula of length less than 4.

It is worth noting that formulae $\sim\neg p$ and $\sim\neg\neg p$ imply every formulae with even and odd number of negations, accordingly. In other words, these formulae are minimal elements with respect to the relation $\preceq$ in subsets of even negational formulae and odd negational formulae, respectively. Proposition 3.5 presents also maximal elements of this kind.

Proposition 3.5. *For any $k \in \mathbb{N}$ following implications are valid:*

(1) $\sim\neg p \to N_{2k}p$,
(2) $N_{2k}p \to \neg\sim\neg\sim p$,
(3) $\sim\neg\neg p \to N_{2k+1}p$,
(4) $N_{2k+1}p \to \neg\sim\neg p$.

It is clear that for every $k \in \mathbb{N}$ there are 2^k negational formulae N_kp. By semantical argument we can show that there are at most fifteen nonequivalent formulae up to length $k = 5$. By induction we can prove that every negational formula N_kp, for $k \geq 6$ is equivalent to some negational formula N_mp of length $k \leq 5$. From this, we get the final theorem about the monadic, purely negational fragment of ICL.

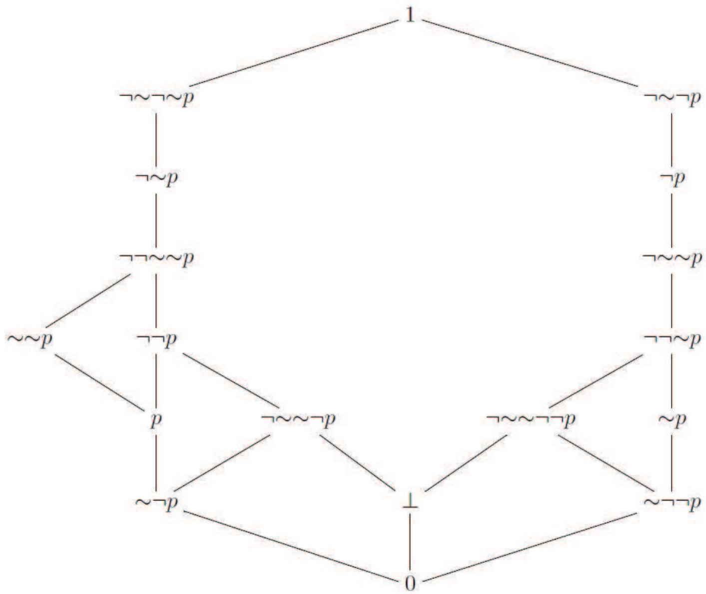

Fig. 1. Poset of monadic negational fragment of ICL

THEOREM 3.6. *The set $\mathcal{N}/_{\equiv}$ consists of fifteen elements. Representatives of its elements are for example negational formulae:*

$$p, \quad \sim\sim p, \quad \sim\neg p, \quad \neg\sim p, \quad \neg\neg p, \quad \neg\sim\neg\sim p, \quad \neg\sim\sim\neg p, \quad \neg\neg\sim\sim p,$$
$$\sim p, \quad \neg p, \quad \neg\sim\neg p, \quad \sim\neg\neg p, \quad \neg\sim\sim p, \quad \neg\neg\sim p, \quad \neg\sim\sim\neg\neg p.$$

Relations between classes of equivalent formulae are described by Lindenbaum algebra. In the case of negational monadic fragment of ICL we start with presenting the poset $(\mathcal{N}^* \cup \{0, \bot, 1\}, \preceq)$, where $\mathcal{N}^*$ is the set of chosen representatives of equivalence classes of $\mathcal{N}/_{\equiv}$ (Fig. 1). The addition of constants enables to join the two subsets of equivalence classes of even and odd negational formulae. Closing this poset under conjunction and disjunction results in a very complex structure with many elements. Thus we divide the description of it into several stages.

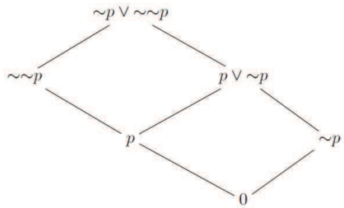

Fig. 2. Lattice of monadic negational fragment of IPC

4. Closures of the poset of negational formulae

Propositions presented in this section were proved semantically using techniques developed and presented in [2].

First let us consider two parts of the poset given in fig.1: the upper part consisting of formulae implied by the constant $\perp$, namely

$$\neg p, \neg\neg p, \neg{\sim}p, \neg\neg{\sim}p, \neg{\sim}{\sim}p, \neg{\sim}\neg p, \neg\neg{\sim}{\sim}p, \neg{\sim}\neg{\sim}p, \neg{\sim}{\sim}\neg p, \neg{\sim}{\sim}\neg\neg p,$$

and the lower part which contains five formulae that are not implied by $\perp$:

$$p, {\sim}p, {\sim}{\sim}p, {\sim}\neg p, {\sim}\neg\neg p.$$

Since all formulae in the upper part of the poset are implied by the constant $\perp$, then they are also always forced in every imaginary world of every r-model. To highlight this property, we chose for every equivalence class in the upper part of the poset a representative that is of a form $\neg N_k p$. According to point (3.2) of Fact 3.2 the upper part consists of formulae that can be refuted only in the root of some model.

Let us recall that in purely intuitionistic case the closure of the negational fragment $0, p, {\sim}p, {\sim}{\sim}p$ under conjunction and disjunction gives rise to the lattice given in Fig. 2. All these formulae are contained in the lower part of the poset. Closing this part under conjunction and disjunction results in a finite lattice given in Fig. 3.

Note that the disjunction of the form ${\sim}\neg p \vee {\sim}\neg\neg p$ is equivalent to a closed formula i.e., not depending on the variable p, namely ${\sim}\perp$. This formula can be described semantically by the set of all its r-models. Namely its only r-models are one-world models consisting of the root only and with the two possible valuations, that is r-model $\mathcal{M}_\bullet$ in which $\mathbf{r} \Vdash p$ and r-model $\mathcal{M}_\circ$ in which $\mathbf{r} \nVdash p$. The rest of the closure is straightforward.

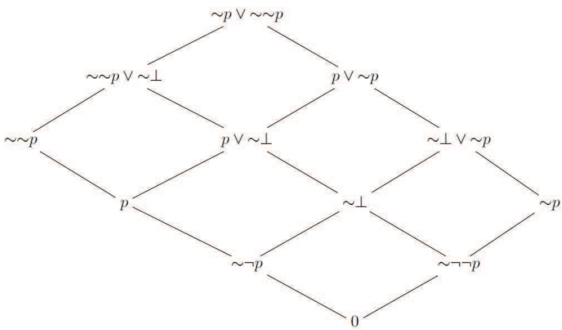

Fig. 3. Lattice of lower part of the poset

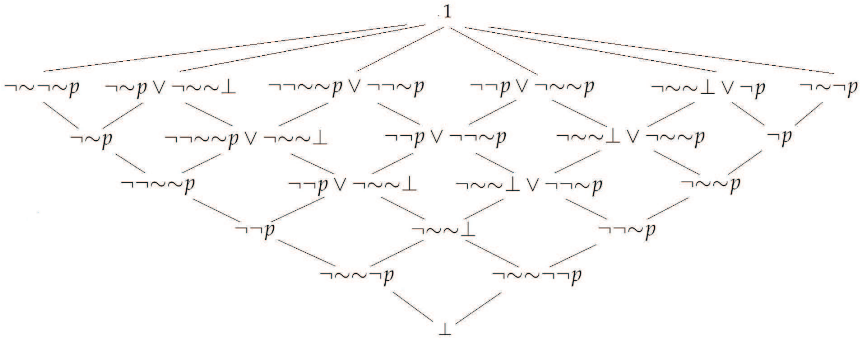

Fig. 4. Upper semilattice

It would be arduous to present similar lattice for the upper part of the poset, as closing it under disjunction and conjunction results in a very complicated structure. Nevertheless, we can present semilattices with respect to join and meet (Fig. 4 and Fig. 5). Again we obtain two closed formulae

$$\neg\sim\sim\bot \equiv \neg\sim\sim\neg p \vee \neg\sim\sim\neg\neg p,$$

$$\sim\sim\bot \equiv \neg\sim\neg\sim p \wedge \neg\sim\neg p.$$

As before these formulae can be characterized semantically via their models. Models for formula $\sim\sim\bot$ are all r-models except for two models consisting of the root only. Models for the formula $\neg\sim\sim\bot$ are the same as

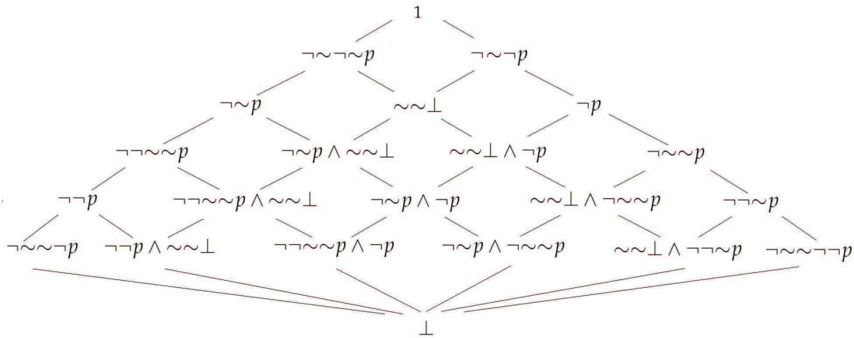

Fig. 5. Lower semilattice

for the formula $\sim\perp$, namely $\mathcal{M}_\bullet$ and $\mathcal{M}_\circ$. Nevertheless these two formulae are not equivalent, due to the fact that $\sim\perp$ is refuted in any imaginary world, whereas formula $\neg\sim\sim\perp$ is forced in every imaginary world of every model.

5. "Classical" fragment of ICL

One of the best known fragments of IPL is its monadic fragment. It is characterized by the so-called Rieger-Nishimura lattice which is the Lindenbaum algebra of the fragment in question and consists of infinitely many distinct equivalence classes of formulae. It was already mentioned that when an ICL formula does not contain $\perp$ as a subformula, then it is an intuitionistic formula, thus the monadic fragment of ICL in the language of $(0, 1, \vee, \wedge, \rightarrow, p)$ is simply the Rieger-Nishimura lattice.

The additional constant $\perp$ in ICL is treated as an additional falsum. However, in the fragment with $\perp$ instead of intuitionistic falsum, the "classical" falsum $\perp$ is not the least element of the lattice and we cannot expect an immediate analogue of the Rieger-Nishimura lattice.

THEOREM 5.1. *The fragment of* ICL *consisting of* $(\perp, p, \rightarrow, \vee, \wedge)$ *is finite.*

The proof and will be omitted here. Let us point out that contrary to the intuitionistic example, it is not the case that $p \wedge \perp \equiv \perp$, since $\not\models \perp \rightarrow p$. Moreover this fragment is identical to the fragment of $(\perp, p, \rightarrow, \wedge)$, because of the equivalence

$$p \vee \bot \equiv \neg\neg p.$$

Let us conclude with one more observation about the monadic implicational fragment of ICL.

THEOREM 5.2. *The fragment of* ICL *consisting of* $(0, \bot, p, \to)$ *is finite.*

Note that due to equivalences

$$\bot \to p \equiv \neg\neg p \to p,$$

$$\sim\bot \equiv \neg\neg p \to \sim\neg p,$$

the fragment in the language of $(p, \neg, \sim, \to)$ is equivalent with the fragment of ICL in the language of $(0, \bot, p, \to)$.

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