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SOME LOGICS IN THE VICINITY OF INTERPRETABILITY LOGICS

Abstract

In this paper we shall define semantically some families of propositional modal logics related to the interpretability logic **IL**. We will introduce the logics **BIL** and $\mathbf{BIL}^+$ in the propositional language with a modal operator $\Box$ and a binary operator $\Rightarrow$ such that $\mathbf{BIL} \subseteq \mathbf{BIL}^+ \subseteq \mathbf{IL}$. The logic **BIL** is generated by the relational structures $\langle X, R, N\rangle$, called basic frames, where $\langle X, R\rangle$ is a Kripke frame and $\langle X, N\rangle$ is a neighborhood frame. We will prove that the logic $\mathbf{BIL}^+$ is generated by the basic frames where the binary relation R is definable by the neighborhood relation N and, therefore, the neighborhood semantics is suitable to study the logic $\mathbf{BIL}^+$ and its extensions. We shall also study some axiomatic extensions of **BIL** and we will prove that these extensions are sound and complete with respect to a certain classes of basic frames. Finally, we prove that the logic $\mathbf{BIL}^+$ and some of its extensions are complete respect with the class of neighborhood frames.

Keywords: interpretability logic, Kripke frames, neighbourhood frames, Veltman semantics.

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1. Introduction

The logic **GL** is known as the logic of provability and it is well known that **GL** is complete with respect to the class of all transitive and conversely well-founded finite Kripke frames [1, 2]. Interpretability logics is a family of classical propositional logics that extends the provability logic **GL** with

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a binary modal operator $\rhd$ used for study formal interpretability . Among these logics, the interpretability logic **IL** plays an important role [5, 6, 9]. The logic **IL** extends the provability logic **GL** by adding the binary modal operator connective $\rhd$ and the following axioms:

J1 $\Box(A \to B) \to (A \rhd B)$;

J2 $(A \rhd B) \wedge (B \rhd C) \to (A \rhd C)$;

J3 $(A \rhd C) \wedge (B \rhd C) \to ((A \vee B) \rhd C)$;

J4 $(A \rhd B) \to (\Diamond A \to \Diamond B)$;

J5 $\Diamond A \rhd A$.

Here the connective $\Diamond$ is defined as $\Diamond A := \neg\Box\neg A$.

The most commonly used semantics for **IL** and its extensions is the Veltman semantics (or ordinary Veltman semantics) [5, 6, 9, 10]. A Veltman frame is a relational structure $\langle X, R, \{S_x : x \in X\}\rangle$, where X is a non-empty set, R is a transitive and converse well-founded binary relation on X, and for each $x \in X$, S_x is a binary relation on $R(x) = \{y \in X : (x, y) \in R\}$ satisfying additional conditions. The relation R is used to interpret modal formulas $\Box A$, and the collection of binary relations $\{S_x : x \in X\}$, together with the binary relation R, is used to interpret formulas of type $A \rhd B$. De Jongh and Veltman proved that the logic **IL** is sound and complete with respect to all Veltman models [5]. Other semantics utilized for the study of **IL** is the called generalized Veltman semantics or Verbrugge semantics [8, 6]. In Verbrugge semantics the modal operator $\Box$ is interpreted as before, but the binary modality $\rhd$ is interpreted by means of a collection of neighborhood relations $\{N_x : N_x \subseteq R(x) \times \mathcal{P}(R(x)) \setminus \{\emptyset\}\}_{x \in X}$ satisfying additional conditions.

One of the main objectives of this paper is to present a family of logics that extends the normal modal logic **K** in the vicinity of the interpretability logic **IL**. We will study a logic, called *basic interpretability logic* (**BIL**), defined semantically by means of structures $\langle X, R, N\rangle$, called basic frames, where X is a non-empty set, R is a binary relation defined on X, and N is a neighborhood relation, i.e, $N \subseteq X \times \mathcal{P}(X)$ [4, 7]. The binary relation R is used to interpret the modal operator $\Box$, and the neighborhood relation N is used to interpret a binary operator $\Rightarrow$ defined as $A \Rightarrow B := \neg B \rhd \neg A$.

An important difference from Verbrugge's semantics is that we will not define a neighborhood relation for each point. We will use a single neighborhood relation for all points. We will treat initially the modalities $\Box$ and $\rhd$ independently. Thus, in principle, there is no connection between the relations R and N. In the interpretability logic **IL** the formulas $\Box A$ and $\bot \rhd \neg A$ are deductively equivalent, that is $\Box A \to (\bot \rhd \neg A)$ and $(\bot \rhd \neg A) \to \Box A$ are theorems of **IL**. In this paper these formulas are theorems in the extension $\mathbf{BIL}^+ = \mathbf{BIL} + \{\mathrm{J1}, \mathrm{J4}\}$. We will see that $\mathbf{BIL}^+$ is complete with respect to special basic frames $\langle X, R, N\rangle$ satisfying the condition: for all $x, y \in X$, $(x, y) \in R$ iff there exists $Y \in N(x)$ such that $y \in Y$. In other words, in the basic frames $\langle X, R, N\rangle$ of $\mathbf{BIL}^+$ the binary relation R is definable by means of the neighborhood relation N as $R(x) := \bigcup\{Y : Y \in N(x)\}$. This condition corresponds precisely to the fact that in this logic the formulas $\Box A$ and $\bot \rhd \neg A$ are deductively equivalents. Therefore to study extensions of $\mathbf{BIL}^+$ is enough to consider neighborhood frames instead of basic frames.

This paper is organized as follows. In Section 2 will define the basic interpretability logic **BIL**, and the basic frames. We will prove that **BIL** is sound with respect to the class of all basic frames. We shall study some axiomatic extensions of **BIL** and we will prove that these extensions are sound with respect to a certain classes of basic frames. In Section 3 we will prove that the logic **BIL** and the extensions defined in section 2 are complete. In Section 5 we shall prove that the logic $\mathbf{BIL}^+$ and some of its extensions are complete respect with the class of neighborhood frames [3, 7].

2. The basic logic BIL and some extensions

We consider a language $\mathcal{L}$ which consists of a set Var of countably many propositional variables $p, q, r, \ldots$, logical constants $\bot, \top$, and propositional connectives $\neg$, $\wedge$, $\vee$, and $\to$. The language $\mathcal{L}(\Box)$ of modal logic consists of the language $\mathcal{L}$ and a unary modal operator $\Box$. The language $\mathcal{L}(\Box, \rhd)$ of interpretability logic is the language $\mathcal{L}$ with a unary modal operator $\Box$, and a binary operator $\rhd$. In the usual interpretability logics the modal operator $\Box$ can be defined as $\neg A \rhd \bot$. But in our basic logic the connectives $\rhd$ and $\Box$ are primitives , i.e, $\Box$ is not definable by $\bot$ and $\rhd$. In the presence of classical negation, we can define a binary connective $\Rightarrow$ as

$A \Rightarrow B := \neg B \rhd \neg A$. We can also work with the language $\mathcal{L}(\Box, \Rightarrow)$, and in this case the connective $\rhd$ is defined as $A \rhd B := \neg B \Rightarrow \neg A$. In view of this interdefinability, it is necessary to consider only one of the connectives. In this paper we are going to work mainly with the language $\mathcal{L}(\Box, \Rightarrow)$. The set of all formulas is denoted by Fm.

We consider the following list of formulas and rules:

C All tautologies of Propositional Calculus;

K $\Box(A \to B) \to (\Box A \to \Box B)$;

L $\Box(\Box A \to A) \to \Box A$;

J1 $\Box(A \to B) \to (A \Rightarrow B)$;

J2 $(A \Rightarrow B) \wedge (B \Rightarrow C) \to (A \Rightarrow C)$;

J3 $(A \Rightarrow B) \wedge (A \Rightarrow C) \to (A \Rightarrow (B \wedge C))$;

J4 $(A \Rightarrow B) \to (\Box A \to \Box B)$;

J5 $A \Rightarrow \Box A$;

M $(A \Rightarrow B) \to ((\Box C \to A) \Rightarrow (\Box C \to B))$;

M0 $(A \Rightarrow B) \to ((\Box C \to A) \Rightarrow (\Box C \to \Box B))$;

P $(A \Rightarrow B) \to \Box(A \Rightarrow B)$;

P0 $(\Box A \Rightarrow B) \to \Box(A \Rightarrow B)$;

MP $\dfrac{A \quad A \to B}{B}$;

N $\dfrac{A}{\Box A}$;

RI $\dfrac{A \to B}{A \Rightarrow B}$.

A basic interpretability logic is any consistent set of formulas Λ of $\mathcal{L}(\Box, \Rightarrow)$ which contains the axioms C, K, J2, J3, and is closed under the rules MP, N and RI, and uniform substitution. The minimal basic interpretability logic is denoted by **BIL**. We also consider the logic $\mathbf{BIL}^{+} := \mathbf{BIL} + \{\mathrm{J1}, \mathrm{J4}\}$.

As we will see later, **BIL** is the set of all valid formulas in the basic frames defined in Definition 2.1. The logic defined as $\mathbf{IL} := \mathbf{BIL}^+ + \{\mathrm{L}, \mathrm{J5}\}$ is known as the interpretability logic [9].

Let Λ be a basic interpretability logic. If A is a theorem of Λ we write $A \in \Lambda$ or $\vdash_\Lambda A$. If there is no risk of confusion we will write $\vdash$ instead of $\vdash_\Lambda$. If Γ is a set of formulas we write $\Gamma \vdash A$ iff there exist $A_1, \ldots, A_n \in \Gamma$ such that $\vdash (A_1 \wedge \ldots \wedge A_n) \to A$. We shall say that two formulas A and B are deductively equivalents, in symbols $A \leftrightarrow B$, if $\vdash A \to B$ and $\vdash B \to A$. It is easy to see the following equivalences and derived rules

1. $(A \Rightarrow (B \wedge C)) \leftrightarrow (A \Rightarrow B) \wedge (A \Rightarrow C)\,;$

2. $\Box(A \wedge B) \leftrightarrow \Box A \wedge \Box B;$

3. $\Box\top \leftrightarrow \top;$

4. $\dfrac{A \to B}{(C \Rightarrow A) \to (C \Rightarrow B)};$

5. $\dfrac{A \to B}{(B \Rightarrow C) \to (A \Rightarrow C)}.$

6. If $A_1 \leftrightarrow B_1$ and $A_2 \leftrightarrow B_2$, then $(A_1 \Rightarrow A_2) \leftrightarrow (B_1 \Rightarrow B_2)$.

Consider the logic $\mathbf{BIL}^+$. By the axiom J1 we have that $\vdash_{\mathsf{BIL}^+} \Box(\top \to A) \to (\top \Rightarrow A)$, and by the axiom J4, we get $\vdash_{\mathsf{BIL}^+} (\top \Rightarrow A) \to \Box(\top \to A)$. As $(\top \to A) \leftrightarrow A$, we get that $\Box A \leftrightarrow (\top \Rightarrow A)$.

Let X be a non-empty set. The power set of a set X is denoted by $\mathcal{P}(X)$. Given a binary relation R on a set X, let $R(x) = \{y \in X \mid (x, y) \in R\}$, for $x \in X$. For $Y \subseteq X$, let $R[Y] = \bigcup\{R(y) : y \in Y\}$. Define the operator $\Box_R : \mathcal{P}(X) \to \mathcal{P}(X)$ as

$$\Box_R(U) = \{x \in X \mid R(x) \subseteq U\}\,,$$

for each $U \subseteq X$. A *Kripke frame* is a pair $\langle X, R\rangle$ where X is a non-empty set and R is a binary relation on X.

A *neighbourhood frame* is a structure $\mathcal{F} = \langle X, N \rangle$, where X is a non-empty set and $N \subseteq X \times \mathcal{P}(X)$. Neighbourhood frame were initially introduced to define a semantics for non-normal modal logics [7]. The elements of $N(x)$ are called neighbourhoods.

Given a neighborhood frame $\langle X, N \rangle$ we define the binary operator

$$\Rightarrow_N : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathcal{P}(X)$$

as

$$U \Rightarrow_N V := \{x \in X : \forall Y \in N(x)\, (\text{if } Y \subseteq U \text{ then } Y \subseteq V)\}\,.$$

We note that the structure $\langle \mathcal{P}(X), \cup, \cap, \to, ^c, \Rightarrow_N, \Box_R \rangle$ is a Boolean algebra with a unary modal operator $\Box_R$ and with a binary operator $\Rightarrow_N$, where the boolean negation is defined as $U^c := X \setminus U$, and the implication $\to$ is defined as $U \to V := U^c \cup V$, for all $U, V \in \mathcal{P}(X)$. Thus, $\langle \mathcal{P}(X), \Rightarrow_N, \Box_R \rangle$ is a particular case of Boolean algebras with operators [1].

Definition 2.1. We say that a triple $\mathcal{F} = \langle X, R, N \rangle$ is a *basic interpretability frame* if $\langle X, R \rangle$ is a Kripke frame and $\langle X, N \rangle$ is a neighborhood frame.

Lemma 2.2. *Let $\langle X, R, N \rangle$ be a basic frame. Then the algebra $\langle \mathcal{P}(X), \Rightarrow_N, \Box_R \rangle$ satisfies the following identities*

(1) $\Box_R(X) = X$;

(2) $\Box_R(U \to V) \subseteq \Box_R(U) \to \Box_N(V)$;

(3) $U \Rightarrow_N U = X$;

(4) $(U \Rightarrow_N V) \cap (V \Rightarrow_N W) \subseteq U \Rightarrow_N W$;

(5) $(U \Rightarrow_N V) \cap (U \Rightarrow_N W) = U \Rightarrow_N (V \cap W)$;

(6) *If* $U \subseteq V$, *then* $W \Rightarrow_N U \subseteq W \Rightarrow_N V$ *and* $V \Rightarrow_N W \subseteq U \Rightarrow_N W$.

Proof: As example, we will prove the condition (4). Let $x \in (U \Rightarrow_N V) \cap (V \Rightarrow_N W)$. Let $Y \in N(x)$ and such that $Y \subseteq U$. As $x \in U \Rightarrow_N V$, we get $Y \subseteq V$, and since $x \in V \Rightarrow_N W$ we have $Y \subseteq W$. Thus, $(U \Rightarrow_N V) \cap (V \Rightarrow_N W) \subseteq U \Rightarrow_N W$. $\Box$

DEFINITION 2.3. A *valuation* on a basic frame $\mathcal{F} = \langle X, R, N\rangle$ is a function $V : Var \to \mathcal{P}(X)$. A valuation V can be extended recursively to the set of all formulas Fm by means of the following clauses:

1. $V(\top) = X$, $V(\bot) = \emptyset$,
2. $V(p \wedge q) = V(p) \cap V(q)$,
3. $V(p \vee q) = V(p) \cup V(q)$,
4. $V(p \to q) = V(p)^c \cup V(q)$,
5. $V(\Box p) = \{x \in X \mid R(x) \subseteq V(p)\}$,
6. $V(p \Rightarrow q) = \{x : \forall Y \in N(x)\,(Y \subseteq V(p) \text{ implies } Y \subseteq V(q))\}$.

A *model* is a pair $\mathcal{M} = \langle \mathcal{F}, V\rangle$, where $\mathcal{F}$ is a basic frame and V is a valuation on it.

It is easy to see that $V(\Box A) = \Box_R V(A)$, and $V(A \Rightarrow B) = V(A) \Rightarrow_N V(B)$, for any formulas A and B. A formula A is *valid* in a model $\langle \mathcal{F}, V\rangle$ if $V(A) = X$. A formula A is valid in a basic frame $\mathcal{F}$, in symbols $\mathcal{F} \vDash A$, if $V(A) = X$, for all valuation V defined on it. A set of formulas Γ is valid in a basic frame $\mathcal{F}$, in symbols $\mathcal{F} \models \Gamma$, if $\mathcal{F} \models A$ for all $A \in \Gamma$. The class of all basic frames validating the set of formulas Γ will be denoted by $\mathrm{Fr}(\Gamma)$. For any class of basic frames $\mathbf{F}$, a formula A is valid in $\mathbf{F}$, notation $\models_{\mathbf{F}} A$, if $\mathcal{F} \models A$ for all $\mathcal{F} \in \mathbf{F}$. The set of all formulas valid in $\mathbf{F}$ is $\mathrm{Th}(\mathbf{F}) = \{A \in Fm : \models_{\mathbf{F}} A\}$. If $\mathbf{F} = \{\mathcal{F}\}$,we write $\mathrm{Th}(\mathcal{F})$ instead of $\mathrm{Th}(\{\mathcal{F}\})$.

Let P be a first or higher-order frame condition in the language $\{R, N\}$. We say that the condition P is *valid* in a basic frame $\mathcal{F}$, in notation $\mathcal{F} \Vdash P$, if it is valid in the sense of a first or higher order structure. We shall that a frame condition P characterizes a formula A if for every basic frame $\mathcal{F}$, $\mathcal{F} \Vdash P$ iff $\mathcal{F} \models A$.

A logic Λ is *sound* with respect to a class of basic frames $\mathbf{F}$ if $\Lambda \subseteq \mathrm{Th}(\mathbf{F})$. A logic Λ is complete with respect to a class of basic frames $\mathbf{F}$ if $\mathrm{Th}(\mathbf{F}) \subseteq \Lambda$. A logic Λ is characterized by a class $\mathbf{F}$ of basic frames or is complete relative to a class of basic frames $\mathbf{F}$ if $\Lambda = \mathrm{Th}(\mathbf{F})$. Moreover, it is frame complete if $\Lambda = \mathrm{Th}(\mathrm{Fr}(\Lambda))$. It is clear that a logic Λ is frame complete if and only if it is characterized by some class of frames.

We first prove that the logic **BIL** is sound with respect to the class of all basic frames.

Proposition 2.4 (Soundness). Let **Fr** be the class of all basic frames. Then $\mathbf{BIL} \subseteq \mathrm{Th}(\mathbf{Fr})$ and $\mathrm{Fr}(\mathbf{BIL}) = \mathbf{Fr}$.

Proof: By Lemma 2.2 (4) and (5) we have that J2 and J3 are valid in all basic frames, and it is clear that the rules Modus Ponens preserve the validity. Then it suffices to prove that the rule RI preserve the validity. But this also follows from Lemma 2.2 (6). Thus, we have that every theorem of **BIL** is valid in every basic frames, i.e., $\mathbf{BIL} \subseteq \mathrm{Th}(\mathbf{Fr})$. On the other hand, it is clear that $\mathrm{Fr}(\mathbf{BIL}) = \mathbf{Fr}$. □

Now we are going to introduce certain relational conditions defined in basic frames and we are going to prove soundness of extensions of **BIL** respect to these relational conditions. Let us consider the following relational conditions:

RJ1 If $(x, Y) \in N$, then $Y \subseteq R(x)$.

RJ4 If $(x, y) \in R$, then there exists $Y \subseteq X$ such that $(x, Y) \in N$, $Y \subseteq R(x)$ and $y \in Y$.

RJ5 If $(x, Y) \in N$, then $R(y) \subseteq Y$ for any $y \in Y$.

RP If $(x, y) \in R$ and $(y, Y) \in N$, then $(x, Y) \in N$.

RP0 If $(x, y) \in R$ and $(y, Y) \in N$, then there exists $Z \subseteq X$ such that $y \in Z$, $R[Z] \subseteq Y \subseteq Z$ and $(x, Z) \in N$.

RM If $(x, Y) \in N$ and $y \in Y$, then there exists $Z \subseteq X$ such that $(x, Z) \in N$, $y \in Z \subseteq Y$ and $R[Z] \subseteq R(y)$.

RM0 If $(x, Y) \in N$, $y \in Y$ and $(y, z) \in R$, then there exists $Z \subseteq X$ such that $(x, Z) \in N$, $z \in Z \subseteq Y$ and $R[Z] \subseteq R(y)$.

Theorem 2.5 (Soundness of extensions of **BIL**). *Let $\mathcal{F} = \langle X, R, N\rangle$ be a basic frame.*

1. $\mathcal{F} \Vdash$ RJ1 *iff* $\mathcal{F} \models \Box(A \to B) \to (A \Rightarrow B)$.
2. $\mathcal{F} \Vdash$ RJ4 *iff* $\mathcal{F} \models (A \Rightarrow B) \to (\Box A \to \Box B)$.
3. $\mathcal{F} \Vdash$ RJ5 *iff* $\mathcal{F} \models A \Rightarrow \Box A$.
4. $\mathcal{F} \Vdash$ RP *implies that* $\mathcal{F} \models (A \Rightarrow B) \to \Box(A \Rightarrow B)$.

5. $\mathcal{F} \Vdash$ RP0 *implies that* $\mathcal{F} \models (\Box A \Rightarrow B) \rightarrow \Box(A \Rightarrow B)$.

6. $\mathcal{F} \Vdash$ RM *implies that* $\mathcal{F} \models (A \Rightarrow B) \rightarrow ((\Box C \rightarrow A) \Rightarrow (\Box C \rightarrow B))$.

7. $\mathcal{F} \Vdash$ RM0 *implies that* $\mathcal{F} \models (A \Rightarrow B) \rightarrow ((\Box C \rightarrow A) \Rightarrow (\Box C \rightarrow \Box B))$.

PROOF: 1. Suppose that $\mathcal{F} \models \Box(A \rightarrow B) \rightarrow (A \Rightarrow B)$. Let $Y \in N(x)$. Consider the subset $U = R(x)$. Then $x \in \Box_R(U) = \Box_R(X \rightarrow U) \subseteq X \Rightarrow_N U$. As $Y \subseteq X$, we get $Y \subseteq U = R(x)$.

Assume that $\mathcal{F}$ satisfies RJ1. Let $U, V \in \mathcal{P}(X)$. Let $x \in \Box_R(U \rightarrow V)$ and $Y \in N(x)$ such that $Y \subseteq U$. Then $Y \subseteq R(x)$. As $Y \subseteq R(x) \cap U \subseteq V$, we have $Y \subseteq V$. Thus, $x \in U \Rightarrow_N V$. Then $\mathcal{F} \models \Box(A \rightarrow B) \rightarrow (A \Rightarrow B)$.

2. Assume that $\mathcal{F} \models (A \Rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$. Let $x, y \in X$ such that $(x, y) \in R$. Consider the subsets $U = R(x)$ and $V = \{y\}^c = X - \{y\}$. Then $x \in \Box_R(U)$ and as $R(x) \nsubseteq \{y\}^c$, we get that $x \notin \Box_R(V)$. So, $x \notin \Box_R(U)^c \cup \Box_R(V) = \Box_R(U) \rightarrow \Box_R(V)$. As $U \Rightarrow_N V \subseteq \Box_R(U) \rightarrow \Box_R(V)$, we have $x \notin U \Rightarrow_N V$. Then there exists $Y \in N(x)$ such that $Y \subseteq U$ and $Y \nsubseteq V = \{y\}^c$, i.e., $Y \subseteq R(x)$ and $y \in Y$.

Assume that $\mathcal{F}$ satisfies RJ4. Let $U, V \in \mathcal{P}(X)$ and $x \in U \Rightarrow_N V$. Suppose that $x \in \Box_R(U)$. We prove that $x \in \Box_R(V)$. Let $y \in R(x)$. By hypothesis there exists $Y \in N(x)$ such that $Y \subseteq R(x)$ and $y \in Y$. As $R(x) \subseteq U$, we have $Y \subseteq U$, and as $x \in U \Rightarrow_N V$, we get $Y \subseteq V$. Thus, $y \in V$, i.e., $x \in \Box_R(V)$.

3. Assume that $\mathcal{F} \models A \Rightarrow \Box A$. Let $Y \in N(x)$ and $y \in Y$. Suppose that $R(y) \nsubseteq Y$. Then there exists $z \in R(y)$ such that $z \notin Y$. Let $U = \{z\}^c$. So, $Y \subseteq U$, and as $x \in X = U \Rightarrow_N \Box_R(U)$, we get $Y \subseteq \Box_R(U)$. Then $R(y) \subseteq U = \{z\}^c$, which is a contradiction. Thus, $R(y) \subseteq Y$.

Assume that $\mathcal{F}$ satisfies RJ5. We prove that $U \Rightarrow_N \Box_R(U) = X$ for any $U \subseteq X$. Let $x \in X$,$U \subseteq X$ and $Y \in N(x)$ such that $Y \subseteq U$. Let $y \in Y$. Then $R(y) \subseteq Y \subseteq U$, i.e., $y \in \Box_R(U)$. Thus, $Y \subseteq \Box_R(U)$.

4. Assume that $\mathcal{F} \models$ RP. Let $U, V \in \mathcal{P}(X)$, $x \in X$, and suppose that $x \in U \Rightarrow_N V$. We prove that $R(x) \subseteq U \Rightarrow_N V$. Let $y \in X$ and $Y \subseteq X$ such that $(x, y) \in R$, $(y, Y) \in N$ and $Y \subseteq U$. Then $(x, Y) \in N$, and as $x \in U \Rightarrow_N V$, we have $Y \subseteq V$. Thus, $x \in \Box_R(U \Rightarrow_N V)$.

5. Assume that $\mathcal{F} \Vdash$ RP0. Let $U, V \in \mathcal{P}(X)$ and $x \in X$. Suppose that $x \in \Box_R U \Rightarrow_N V$. We prove that $x \in \Box_R(U \Rightarrow_N V)$. Let $y \in X$ and $Y \subseteq X$ such that $(x, y) \in R$, $(y, Y) \in N$, and $Y \subseteq U$. By hypothesis there exist $Z \subseteq X$ such that $y \in Z$, $R[Z] \subseteq Y \subseteq Z$ and $(x, Z) \in N$. Since

$R[Z] \subseteq Y \subseteq U$, we have $Z \subseteq \Box_R U$. As $x \in \Box_R U \Rightarrow_N V$, $Z \subseteq V$. Now, by the inclusion $Y \subseteq Z$ we get $Y \subseteq V$. Thus, $y \in U \Rightarrow_N V$.

6. Assume that $\mathcal{F} \Vdash$ RM. Let $U, V, W \in \mathcal{P}(X)$ and $x \in X$ such that $x \in U \Rightarrow_N V$. Let $Y \subseteq X$ such that $(x, Y) \in N$ and $Y \subseteq \Box_R(W) \to U$, i.e., $Y \cap \Box_R(W) \subseteq U$. We prove that $Y \cap \Box_R(W) \subseteq V$. Let $y \in Y \cap \Box_R(W)$. By hypothesis, there exists $Z \subseteq X$ such that $(x, Z) \in N$, $y \in Z \subseteq Y$ and $R[Z] \subseteq R(y)$. As $y \in \Box_R(W)$, we have $R[Z] \subseteq R(y) \subseteq W$, i.e., $Z \subseteq \Box_R(W)$. Then $Z \subseteq Y \cap \Box_R(W) \subseteq U$. Since $(x, Z) \in N$ and $x \in U \Rightarrow_N V$, we get $Z \subseteq V$. Finally, as $y \in Z$, we have $y \in V$.

7. Assume that $\mathcal{F} \Vdash$ RM0. Let $U, V, W \in \mathcal{P}(X)$ and $x \in X$ such that $x \in U \Rightarrow_N V$. Let $Y \subseteq X$ such that $(x, Y) \in N$ and $Y \subseteq \Box_R(W) \to U$, i.e., $Y \cap \Box_R(W) \subseteq U$. We prove that $Y \cap \Box_R(W) \subseteq \Box_R(V)$. Let $y \in Y \cap \Box_R(W)$. We need to prove that $y \in \Box_R(V)$. Let $z \in X$ such that $(y, z) \in R$. By hypothesis, there exists $Z \subseteq X$ such that $(x, Z) \in N$, $z \in Z \subseteq Y$ and $R[Z] \subseteq R(y)$. Since, $y \in \Box_R(W)$, we have $R[Z] \subseteq R(y) \subseteq W$, i.e., $Z \subseteq \Box_R(W)$. Then $Z \subseteq Y \cap \Box_R(W) \subseteq U$. Since $(x, Z) \in N$ and $x \in U \Rightarrow_N V$, we get $Z \subseteq V$. Finally, as $z \in Z$, we have $z \in V$, i.e., $y \in \Box_R(V)$. $\Box$

From Theorem 2.5 we have that a logic Λ obtained by extending **BIL** by any subset of formulas of the set $\{\mathrm{J1}, \mathrm{J4}, \mathrm{J5}, \mathrm{M}, \mathrm{M0}, \mathrm{P}, \mathrm{P0}\}$ is sound respect with an adequate class of basic frames.

3. Canonical models and completeness theorem

In this section we introduce the canonical basic frame and model for **BIL** and some its extensions. Throughout this section Λ will denote any logic such that **BIL** $\subseteq \Lambda$.

We follow the standard strategy: in order to prove completeness of a logic Λ with respect to a class of models **M**, we define the canonical frame $\mathcal{F}_\Lambda$ and the canonical model $\langle \mathcal{F}_\Lambda, V_\Lambda \rangle$ and we prove that $\langle \mathcal{F}_\Lambda, V_\Lambda \rangle \in$ **M**, and for any formula A, $A \in \Lambda$ iff A is valid in $\langle \mathcal{F}_\Lambda, V_\Lambda \rangle$. This means that logic Λ is *canonical*. From this fact we have that the completeness of Λ with respect the class **M** immediately follows.

A set of formulas Γ is a theory of Λ, or an Λ-theory, if $\Lambda \subseteq \Gamma$, it is closed under $\vdash_\Lambda$, i.e., $A \in \Gamma$ and $A \vdash_\Lambda B$, then $B \in \Gamma$, and it is closed under $\wedge$, i.e., if $A, B \in \Gamma$, then $A \wedge B \in \Gamma$. A theory Γ is Λ-consistent if $\bot \notin \Gamma$. When there is no risk of confusion, we will directly say that Γ is a theory instead of Γ is a Λ-theory. The set of all theories of Λ is denoted by $\mathcal{T}(\Lambda)$. A theory Γ is complete if it is consistent and for every formula A, $A \in \Gamma$ or $\neg A \in \Gamma$. A consistent theory Γ is maximal if for any consistent theory Δ such that $\Gamma \subseteq \Delta$ we have that $\Gamma = \Delta$. It is clear that a theory Γ is complete if and only if it is maximal if and only if it is consistent and for all formulas A, B, if $A \vee B \in \Gamma$ then $A \in \Gamma$ or $B \in \Gamma$.

Let X_Λ be the set of all maximal Λ-theories. By the Lindenbaum's lemma, for every consistent theory T there exists a maximal theory Γ such that $T \subseteq \Gamma$. Moreover, for each formula A, if $A \notin T$, then there exists a maximal theory Γ such that $T \subseteq \Gamma$ and $A \notin \Gamma$. The set of maximal theories determined by a theory T is the set

$$\hat{T} := \{\Gamma \in X_\Lambda : T \subseteq \Gamma\}.$$

Similarly, the set of maximal theories determined by a formula A is the set $\hat{A} = \{\Gamma \in X_\Lambda : A \in \Gamma\}$. We note that if T and H are two theories, $T \subseteq H$ iff $\hat{H} \subseteq \hat{T}$. This fact will be used in several proofs.

For each $\Gamma \in X_\Lambda$ and for each non-empty set Z of formulas we define the set of formulas:

$$\mathsf{D}_\Gamma(Z) := \{A \in Fm : \exists C_1, \ldots, C_n \in Z\ (C_1 \wedge \ldots\ldots \wedge C_n) \Rightarrow A \in \Gamma\}.$$

Lemma 3.1. *For any $\Gamma \in X_\Lambda$ and for any non-empty set Z of formulas, $\mathsf{D}_\Gamma(Z)$ is a theory such that $Z \subseteq \mathsf{D}_\Gamma(Z)$, and for all $A, B \in Fm$, if $A \Rightarrow B \in \Gamma$ and $A \in \mathsf{D}_\Gamma(Z)$, then $B \in \mathsf{D}_\Gamma(Z)$.*

Proof: Let $\Gamma \in X_\Lambda$ and let Z be a non-empty set of formulas. As $C \Rightarrow C \in \Gamma$, for each $C \in Z$, we get $Z \subseteq \mathsf{D}_\Gamma(Z)$.

Since Z is a non-empty set, there exists $C \in Z$. As $C \to \top \in \Gamma$, we have $C \Rightarrow \top \in \Gamma$. So, $\top \in \mathsf{D}_\Gamma(Z)$.

Let $A, B \in \mathsf{D}_\Gamma(Z)$. We prove that $A \wedge B \in \mathsf{D}_\Gamma(Z)$. Then there exist $C_1, \ldots, C_n, D_1, \ldots D_m \in Z$ such that $(C_1 \wedge \ldots \wedge C_n) \Rightarrow A \in \Gamma$ and $(D_1 \wedge \ldots \wedge D_m) \Rightarrow B \in \Gamma$. Let $C = C_1 \wedge \ldots \wedge C_n$ and $D = D_1 \wedge \ldots \wedge D_m$. Then $(C \wedge D) \Rightarrow C \in \Gamma$ and $(C \wedge D) \Rightarrow D \in \Gamma$. So, by axiom J2 we have $(C \wedge D) \Rightarrow A \in \Gamma$ and $(C \wedge D) \Rightarrow B \in \Gamma$. By J3, $(C \wedge D) \Rightarrow (A \wedge B) \in \Gamma$. Thus, $A \wedge B \in \mathsf{D}_\Gamma(Z)$.

We prove that $\mathsf{D}_\Gamma(Z)$ is closed under $\vdash$. Let $A \in \mathsf{D}_\Gamma(Z)$ and $A \vdash B$. Then there exist $C_1, \ldots, C_n \in Z$ such that $(C_1 \wedge \ldots \wedge C_n) \Rightarrow A \in \Gamma$. As $\vdash A \to B$, by the rule R1 and the axiom J2 we have $\vdash ((C_1 \wedge \ldots \wedge C_n) \Rightarrow A) \to ((C_1 \wedge \ldots \wedge C_n) \Rightarrow B)$. Since Γ is a theory, $(C_1 \wedge \ldots \wedge C_n) \Rightarrow B \in \Gamma$.

Therefore, $B \in \mathsf{D}_\Gamma(Z)$.

Let $A, B \in Fm$ such that $A \Rightarrow B \in \Gamma$ and $A \in \mathsf{D}_\Gamma(Z)$. Then there exist $C_1, \ldots, C_n \in Z$ such that $(C_1 \wedge \ldots \wedge C_n) \Rightarrow A \in \Gamma$. So, $((C_1 \wedge \ldots \wedge C_n) \Rightarrow A) \wedge (A \Rightarrow B) \in \Gamma$. By axiom J2, $(C_1 \wedge \ldots \wedge C_n) \Rightarrow B \in \Gamma$, i.e., $B \in \mathsf{D}_\Gamma(Z)$. □

We are now in position to define the canonical model of any logic Λ that extends the logic **BIL**.

Definition 3.2. The *canonical basic frame* of Λ is the relational structure

$$\mathcal{F}_\Lambda := \langle X_\Lambda, R_\Lambda, N_\Lambda \rangle ,$$

where

1. X_Λ is the set of all maximal theories;
2. R_Λ is a binary relation defined on X_Λ by
$$(\Gamma, \Delta) \in R_\Lambda \text{ iff } \Box^{-1}(\Gamma) \subseteq \Delta,$$
where $\Box^{-1}(\Gamma) = \{A \in Fm : \Box A \in \Gamma\}$;
3. N_Λ is a subset of $X_\Lambda \times \mathcal{P}(X_\Lambda)$ defined by
$$(\Gamma, Y) \in N_\Lambda \text{ iff } \exists T \in \mathcal{T}(\Lambda) \left(Y = \hat{T} \text{ and } \mathsf{D}_\Gamma(T) \subseteq T\right).$$

Since the image of the relation N_Λ is the family

$$\left\{Y \subseteq X_\Lambda : \exists T \in \mathcal{T}(\Lambda)\, (Y = \hat{T})\right\},$$

we can also define the relation N_Λ as

$$\left(\Gamma, \hat{T}\right) \in N_\Lambda \text{ iff } \forall A, B \in Fm \; (A \Rightarrow B \text{ and } A \in T \text{ then } B \in T)\,.$$

We define the canonical valuation V_Λ given by $V_\Lambda(p) = \{\Gamma \in X_\Lambda : p \in \Gamma\}$, for every propositional variable p. We note that $V_\Lambda(p) = \hat{p}$, for each variable p.

In the following result we need recall that for any formula A and for any consistent theory T, $A \in T$ iff $A \in \Gamma$, for any $\Gamma \in \hat{T}$.

LEMMA 3.3. *Let $A, B \in Fm$. Let Γ be a maximal theory. Then $A \Rightarrow B \notin \Gamma$ iff there exists a consistent theory T such that $(\Gamma, \hat{T}) \in N_\Lambda$, $A \in T$ and $B \notin T$.*

PROOF: Assume that $A \Rightarrow B \notin \Gamma$. Let us consider the theory

$$T = \mathsf{D}_\Gamma(\{A\}) = \{C \in Fm : A \Rightarrow C \in \Gamma\}.$$

By Lemma 3.1 $(\Gamma, \hat{T}) \in N_\Lambda$, $A \in T$, and $B \notin T$. The proof of the other direction is immediate. □

LEMMA 3.4. *For every maximal theory Γ and for any formula A,*

$$\Gamma \in V_\Lambda(A) \text{ iff } A \in \Gamma.$$

PROOF: The proof is by induction on the construction of A. For atomic and propositional formulas the proof is standard. The case of formulas $\Box A$ is usual (see for example [1]). Let $A, B \in Fm$. Let Γ be a maximal theory. Let $A \Rightarrow B \in \Gamma$. We need to show that $\Gamma \in V_\Lambda(A \Rightarrow B)$. Suppose that $\hat{T} \in N_\Lambda(\Gamma)$ and $\hat{T} \subseteq V_\Lambda(A)$. Then, $A \in T$. As $A \Rightarrow B \in \Gamma$, $A \in T$ and $\hat{T} \in N_\Lambda(\Gamma)$, we get $B \in T$. By the induction hypothesis, $\hat{T} \subseteq V_\Lambda(B)$. Thus, $\Gamma \in V_\Lambda(A \Rightarrow B)$.

On the other hand, if $A \Rightarrow B \notin \Gamma$, then by Lemma 3.3 there exists a consistent theory T such that $(\Gamma, \hat{T}) \in N_\Lambda$, $A \in T$ but $B \notin T$. By induction hypothesis, $\hat{T} \subseteq V_\Lambda(A)$ and $\hat{T} \nsubseteq V_\Lambda(B)$, i.e., $\Gamma \notin V_\Lambda(A \Rightarrow B)$. □

THEOREM 3.5 (Completeness of **BIL**). *Let* **Fr** *be the class of all basic frames. Then,* **BIL** $= \mathsf{Th}(\mathbf{Fr})$.

PROOF: If A is a formula such that $A \notin \mathbf{BIL}$, then there exists a maximal theory Γ such that $A \notin \Gamma$. By Lemma 3.4, $\Gamma \notin V_\Lambda(A)$. Then A is not valid in the canonical model $\langle \mathcal{F}_{\mathbf{BIL}}, V_{\mathbf{BIL}} \rangle$ of **BIL**. Thus, A is not valid in the canonical frame $\mathcal{F}_{\mathbf{BIL}}$ of **BIL**. i.e., $A \notin \mathrm{Th}(\mathbf{Fr})$. □

4. Completeness of extensions of BIL

Our next aim is to prove the completeness for several extensions of **BIL**. To prove the completeness of the extensions of **BIL** we will proceed in the usual way. That is, we are going to prove that the canonical basic frame of each logic Λ such that $\mathbf{BIL} \subseteq \Lambda$ is a basic frame of Λ.

Proposition 4.1. Let Λ be a logic such that **BIL** $\subseteq \Lambda$. Then

(1) $\Box(A \to B) \to (A \Rightarrow B) \in \Lambda$ iff $\mathcal{F}_\Lambda \Vdash$ RJ1.

(2) $(A \Rightarrow B) \to (\Box A \to \Box B) \in \Lambda$ iff $\mathcal{F}_\Lambda \Vdash$ RJ4.

(3) $A \Rightarrow \Box A \in \Lambda$ iff $\mathcal{F}_\Lambda \Vdash$ RJ5.

Proof: (1) $\Rightarrow$) Let $\Gamma \in X_\Lambda$ and let T be a theory such that $(\Gamma, \hat{T}) \in N_\Lambda$. Let $A \in \Box^{-1}(\Gamma)$. As $\Box(\top \to A) \to (\top \Rightarrow A) \in \Gamma$ and $\Box(\top \to A) \leftrightarrow \Box A$, we have $\top \Rightarrow A \in \Gamma$. Since $\top \in T$ and $(\Gamma, \hat{T}) \in N_\Lambda$, $A \in T$. Thus, $\Box^{-1}(\Gamma) \subseteq T$, and this is equivalent to the inclusion $\hat{T} \subseteq R_\Lambda(\Gamma)$.

$\Leftarrow$). Suppose that $\mathcal{F}_\Lambda \Vdash$ RJ1 and that there exist formulas A and B such that $\Box(A \to B) \to (A \Rightarrow B) \notin \Lambda$. Then there exists a maximal theory Γ such that $\Box(A \to B) \to (A \Rightarrow B) \notin \Gamma$. So, $\Box(A \to B) \in \Gamma$ and $A \Rightarrow B \notin \Gamma$. By Lemma 3.3 there exists a theory T such that $(\Gamma, \hat{T}) \in N_\Lambda$, $A \in T$ but $B \notin T$. By hypothesis, $\Box^{-1}(\Gamma) \subseteq T$. So $A \to B \in T$ and by MP, $B \in T$, which is a contradiction. Thus, $\Box(A \to B) \to (A \Rightarrow B) \in \Lambda$.

(2) $\Rightarrow$) Let $\Gamma, \Delta \in X_\Lambda$. Suppose that $(\Gamma, \Delta) \in R_\Lambda$. Let us consider the theory $T = \Box^{-1}(\Gamma)$. We prove that $(\Gamma, \hat{T}) \in N_\Lambda$. Let $A \Rightarrow B \in \Gamma$ and $A \in \Box^{-1}(\Gamma)$. So $\Box A \to \Box B \in \Gamma$ and $\Box A \in \Gamma$. Then $\Box B \in \Gamma$. Thus, $(\Gamma, \hat{T}) \in N_\Lambda$. It is clear that $\Delta \in \hat{T}$ and $\hat{T} \subseteq R_\Lambda(\Gamma)$.

$\Leftarrow$) Suppose that $\mathcal{F}_\Lambda \Vdash$ RJ4. We suppose that there exist formulas A and B such that $(A \Rightarrow B) \to (\Box A \to \Box B) \notin \Lambda$. Then there exists a maximal theory Γ such that $A \Rightarrow B \in \Gamma$, $\Box A \in \Gamma$ and $\Box B \notin \Gamma$. By Lemma 3.4 there exists $\Delta \in X_\Lambda$ such that $\Delta \in R_\Lambda(\Gamma)$ and $B \notin \Delta$. By hypothesis, there exists a theory T such that $(\Gamma, \hat{T}) \in N_\Lambda$, $\Delta \in \hat{T}$ and $\hat{T} \subseteq R_\Lambda(\Gamma)$. So, $A \in \Box^{-1}(\Gamma) \subseteq T \subseteq \Delta$. As $A \Rightarrow B \in \Gamma$, $A \in T$ and $(\Gamma, \hat{T}) \in N_\Lambda$, we get $B \in T$. So, $B \in \Delta$, which is a contradiction. Therefore, $(A \Rightarrow B) \to (\Box A \to \Box B) \in \Lambda$.

(3) $\Rightarrow$) Let $\Gamma \in X_\Lambda$ and let T be a theory such that $(\Gamma, \hat{T}) \in N_\Lambda$. We prove that for any $\Delta \in \hat{T}$, $R_\Lambda(\Delta) \subseteq \hat{T}$, $T \subseteq \Box^{-1}(\Delta)$. Let $A \in T$. As $A \Rightarrow \Box A \in \Gamma$, and $(\Gamma, \hat{T}) \in N_\Lambda$, we get $\Box A \in T \subseteq \Delta$, i.e., $A \in \Box^{-1}(\Delta)$.

The direction $\Leftarrow$) is easy and left to the reader. $\Box$

Corollary 4.2. Let Λ be any logic such that $\mathbf{BIL}^+ \subseteq \Lambda$. For all $\Gamma, \Delta \in X_\Lambda$,

$(\Gamma, \Delta) \in R_\Lambda$ iff there exists a theory T such that $(\Gamma, \hat{T}) \in N_\Lambda$ and $T \subseteq \Delta$.

According to this result we have that in any extension of the logic $\mathbf{BIL}^+$ the canonical relation R_Λ is definable by means of the canonical neighborhood relation N_Λ. This fact will be used in Section 5 to propose a simplify semantics for extension of $\mathbf{BIL}^+$.

Lemma 4.3. *Let Λ be a logic such that $\mathbf{BIL} \subseteq \Lambda$.*

(1) *If $(A \Rightarrow B) \to ((\Box C \to A) \Rightarrow (\Box C \to B))$ is an axiom schema of Λ, then $((A \wedge \Box C) \Rightarrow B) \to (A \Rightarrow (\Box C \to B)) \in \Lambda$.*

(2) *If $(A \Rightarrow B) \to ((\Box C \to A) \Rightarrow (\Box C \to \Box B))$ is an axiom schema of Λ, then $((A \wedge \Box C) \Rightarrow B) \to (A \Rightarrow (\Box C \to \Box B)) \in \Lambda$.*

Proof: We prove only (1). The proof of (2) is similar.

Suppose that $(A \Rightarrow B) \to (\Box C \to A) \Rightarrow (\Box C \to B)$ is an axiom schema of Λ. Then the following formula is an instance of this axiom

$$((A \wedge \Box C) \Rightarrow B) \to ((\Box C \to (A \wedge \Box C)) \Rightarrow (\Box C \to B)).$$

As $(\Box C \to (A \wedge \Box C)) \leftrightarrow (\Box C \to A)$, we have

$$((A \wedge \Box C) \Rightarrow B) \to ((\Box C \to A) \Rightarrow (\Box C \to B)) \in \Lambda. \tag{4.1}$$

Since $A \to (\Box C \to (A \wedge \Box C)) \in \Lambda$, by rule RI

$$A \Rightarrow (\Box C \to (A \wedge \Box C)) \in \Lambda,$$

and consequently

$$A \Rightarrow (\Box C \to A) \in \Lambda. \tag{4.2}$$

By axiom J2 we get

$$[(A \Rightarrow (\Box C \to A)) \wedge ((\Box C \to A) \Rightarrow (\Box C \to B))] \to (A \Rightarrow (\Box C \to B)) \in \Lambda,$$

and by (4.2) we have

$$((\Box C \to A) \Rightarrow (\Box C \to B)) \to (A \Rightarrow (\Box C \to B)) \in \Lambda. \tag{4.3}$$

Finally, by (4.1), (4.3) and axiom J2 we get

$$((A \wedge \Box C) \Rightarrow B) \to (A \Rightarrow (\Box C \to B)) \in \Lambda. \qquad \Box$$

For each theory T, define $\Box T := \{\Box A : A \in T\}$. The following lemma is necessary in the proof of Theorem 4.5.

LEMMA 4.4. *Let H be a consistent theory and let Δ be a maximal theory. Then, $R_\Lambda[\hat{H}] \subseteq R_\Lambda(\Delta)$ iff $\Box^{-1}(\Delta) \subseteq \Box^{-1}(H)$.*

PROOF: Suppose that $R_\Lambda[\hat{H}] \subseteq R_\Lambda(\Delta)$ but $\Box^{-1}(\Delta) \nsubseteq \Box^{-1}(H)$. Then there exists $\Box D \in \Delta$ such that $\Box D \notin H$. So, there are maximal theories G and K such that $H \subseteq G$, $\Box D \notin G$, $(G, K) \in R_\Lambda$ and $D \notin K$. Then $G \in \hat{H}$ and $K \in R_\Lambda(G) \subseteq R_\Lambda[\hat{H}]$. Hence, $K \in R_\Lambda(\Delta)$, i.e., $\Box^{-1}(\Delta) \subseteq K$. But this implies that $D \in K$, which is a contradiction. Thus, $\Box^{-1}(\Delta) \subseteq \Box^{-1}(H)$.

Suppose that $\Box^{-1}(\Delta) \subseteq \Box^{-1}(H)$. Let $K \in R_\Lambda[\hat{H}]$. Then there exists $G \in \hat{H}$ such that $(G, K) \in R_\Lambda$, i.e., $H \subseteq G$ and $\Box^{-1}(G) \subseteq K$. So, $\Box^{-1}(H) \subseteq \Box^{-1}(G) \subseteq K$. Thus, $\Box^{-1}(\Delta) \subseteq K$, i.e., $K \in R_\Lambda(\Delta)$. $\Box$

THEOREM 4.5. *Let Λ be a logic such that* **BIL** $\subseteq \Lambda$. *Then*

(1) $(A \Rightarrow B) \to ((\Box C \to A) \Rightarrow (\Box C \to B)) \in \Lambda$ *iff* $\mathcal{F}_\Lambda \Vdash$ RM.

(2) $(A \Rightarrow B) \to ((\Box C \to A) \Rightarrow (\Box C \to \Box B)) \in \Lambda$ *iff* $\mathcal{F}_\Lambda \Vdash$ RM0.

PROOF: (1) $\Rightarrow$) Let $\Gamma, \Delta \in X_\Lambda$ and let T be a theory such that $(\Gamma, \hat{T}) \in N_\Lambda$ and $T \subseteq \Delta$. Consider the set $\Box(\Box^{-1}(\Delta)) = \{\Box A : A \in \Box^{-1}(\Delta)\}$ and the theory $\mathsf{D}_\Gamma\left(T \cup \Box(\Box^{-1}(\Delta))\right)$. We prove that

$$\mathsf{D}_\Gamma\left(T \cup \Box(\Box^{-1}(\Delta))\right) \subseteq \Delta.$$

Let $B \in \mathsf{D}_\Gamma\left(T \cup \Box(\Box^{-1}(\Delta))\right)$. Then there exists $A \in T$ and there exists $C_1, \ldots, C_n \in \Box^{-1}(\Delta)$ such that

$$(A \wedge \Box C_1 \wedge \ldots \wedge \Box C_n) \Rightarrow B \in \Gamma.$$

Since $\Box C_1 \wedge \ldots \wedge \Box C_n \leftrightarrow \Box(C_1 \wedge \ldots \wedge C_n)$, we get

$$(A \wedge \Box(C_1 \wedge \ldots \wedge C_n)) \Rightarrow B \in \Gamma.$$

By Lemma 4.3 (1) we have $A \Rightarrow (\Box(C_1 \wedge \ldots \wedge C_n) \to B) \in \Gamma$. As $(\Gamma, \hat{T}) \in N_\Lambda$ and $A \in T$, we have $\Box(C_1 \wedge \ldots \wedge C_n) \to B \in T \subseteq \Delta$. Finally, as $\Box(C_1 \wedge \ldots \wedge C_n) \in \Delta$, we get $B \in \Delta$. Thus $Z = \mathsf{D}_\Gamma\left(T \cup \Box(\Box^{-1}(\Delta))\right)$ is a consistent theory such that $Z \subseteq \Delta$ and $(\Gamma, Z) \in N_\Lambda$. By construction, $T \subseteq Z \subseteq \Delta$. Since $\Box(\Box^{-1}(\Delta)) \subseteq Z$, we have $\Box^{-1}(\Delta) \subseteq \Box^{-1}(Z)$, i.e., $R_\Lambda[\hat{Z}] \subseteq R_\Lambda(\Delta)$. As $T \subseteq Z \subseteq \Delta$, we have that $\Delta \in \hat{Z} \subseteq \hat{T}$. Thus, we have found a theory Z such that $(\Gamma, Z) \in N_\Lambda$, $\Delta \in \hat{Z} \subseteq \hat{T}$, and $R_\Lambda[\hat{Z}] \subseteq R_\Lambda(\Delta)$, i.e. $\mathcal{F}_\Lambda \Vdash$ RM.

(1) $\Leftarrow$) Suppose that $\mathcal{F}_\Lambda \Vdash$ RM and there exists formulas A, B and C such that $(A \Rightarrow B) \to ((\Box C \to A) \Rightarrow (\Box C \to B)) \notin \Lambda$. Then there exists a maximal theory Γ such that $(A \Rightarrow B) \to ((\Box C \to A) \Rightarrow (\Box C \to B)) \notin \Gamma$. So, $A \Rightarrow B \in \Gamma$ and $(\Box C \to A) \Rightarrow (\Box C \to B) \notin \Gamma$. Then there exists a theory T such that $(\Gamma, \hat{T}) \in N_\Lambda$, $\Box C \to A \in T$ and $\Box C \to B \notin T$. So, there exists a maximal theory Δ such that $\Box C \in \Delta$ and $B \notin \Delta$. By hypothesis, there exists a theory H such that

$$(\Gamma, \hat{H}) \in N_\Lambda, \Delta \in \hat{H} \subseteq \hat{T} \text{ and } R_\Lambda[\hat{H}] \subseteq R_\Lambda(\Delta).$$

By Lemma 4.4 we have

$$(\Gamma, \hat{H}) \in N_\Lambda, T \subseteq H \subseteq \Delta \text{ and } \Box^{-1}(\Delta) \subseteq \Box^{-1}(H).$$

As $\Box C \in \Delta$, we get $\Box C \in H$. Moreover, $\Box C \to A \in T \subseteq H$, and by Modus Ponens, $A \in H$. Since $A \Rightarrow B \in \Gamma$, $(\Gamma, \hat{H}) \in N_\Lambda$ and $A \in H$, we deduce $B \in H \subseteq \Delta$, i.e., $B \in \Delta$, which is a contradiction. Thus, $(A \Rightarrow B) \to ((\Box C \to A) \Rightarrow (\Box C \to B)) \in \Lambda$.

(2) The proof is very similar to the proof of (1). We prove only the direction $\Rightarrow$). Let $\Gamma, \Delta, \Theta \in X_\Lambda$ and let T be a theory such that $(\Gamma, \hat{T}) \in N_\Lambda$, $\Delta \in \hat{T}$, and $(\Delta, \Theta) \in R_\Lambda$. We prove that there exists a theory H such that $(\Gamma, \hat{H}) \in N_\Lambda$, $\Theta \in \hat{H} \subseteq \hat{T}$ and $R_\Lambda[\hat{H}] \subseteq R_\Lambda(\Delta)$. Consider the theory $H = \mathrm{D}_\Gamma\left(T \cup \Box(\Box^{-1}(\Delta))\right)$. We prove that $H \subseteq \Theta$. Let $B \in H$.

Then there exist $A \in T$ and $C_1, \ldots, C_n \in \Box^{-1}(\Delta)$ such that $(A \wedge \Box C_1 \wedge \ldots \wedge \Box C_n) \Rightarrow B \in \Gamma$. Since $\Box C_1 \wedge \ldots \wedge \Box C_n \leftrightarrow \Box(C_1 \wedge \ldots \wedge C_n)$, we get $(A \wedge \Box(C_1 \wedge \ldots \wedge C_n)) \Rightarrow B \in \Gamma$. Then by Lemma 4.3, $A \Rightarrow (\Box(C_1 \wedge \ldots \wedge C_n) \to \Box B) \in \Gamma$. As $(\Gamma, \hat{T}) \in N_\Lambda$ and $A \in T$, we get $\Box(C_1 \wedge \ldots \wedge C_n) \to \Box B \in T \subseteq \Delta$. Moreover, as $\Box(C_1 \wedge \ldots \wedge C_n) \in \Delta$, we have $\Box B \in \Delta$. Then, $B \in \Box^{-1}(\Delta) \subseteq \Theta$. By construction $T \subseteq H \subseteq \Theta$, and as $\Box(\Box^{-1}(\Delta)) \subseteq H \subseteq \Theta$, we have $\Box^{-1}(\Delta) \subseteq \Box^{-1}(H)$, i.e., $R_\Lambda[\hat{H}] \subseteq R_\Lambda(\Delta)$. $\Box$

Proposition 4.6. Let Λ be a logic such that $\mathbf{BIL} \subseteq \Lambda$. Then

(1) $(A \Rightarrow B) \to \Box(A \Rightarrow B) \in \Lambda$ iff $\mathcal{F}_\Lambda \Vdash$ RP.

(2) $(\Box A \Rightarrow B) \to \Box(A \Rightarrow B) \in \Lambda$ iff $\mathcal{F}_\Lambda \Vdash$ RP0.

Proof: We prove only (2). The proof of (1) is similar and left to the reader.

$\Rightarrow$) Let $\Gamma, \Delta \in X_\Lambda$and let T be a theory such that $(\Gamma, \Delta) \in R_\Lambda$ and $(\Delta, \hat{T}) \in N_\Lambda$. We consider the theory

$$\mathrm{D}_\Gamma(\Box(T)) = \{B \in Fm : \exists A \in T\ (\Box A \Rightarrow B \in \Gamma)\}.$$

We prove that $\mathrm{D}_\Gamma(\Box(T)) \subseteq T$. If $B \in \mathrm{D}_\Gamma(\Box(T))$ then there exists $A \in T$ such that $\Box A \Rightarrow B \in \Gamma$. So, $\Box(A \Rightarrow B) \in \Gamma$, and as $(\Gamma, \Delta) \in R_\Lambda$, we get $A \Rightarrow B \in \Delta$. Since $(\Delta, \hat{T}) \in N_\Lambda$,we have $B \in T$. Consider the theory $H = \mathrm{D}_\Gamma(\Box(T))$. Then, $\Box(T) \subseteq H \subseteq T$. Now it is easy to see that $R_\Lambda[\hat{H}] \subseteq \hat{T} \subseteq \hat{H}$.

The direction $\Leftarrow$) it is easy and left to the reader. $\Box$

We denote by $\mathbf{BIL}(A_1, \ldots, A_n)$ the basic logic $\mathbf{BIL}$ together with the axioms schemata $A_1, \ldots, A_n$.

Theorem 4.7. *Any extension of* $\mathbf{BIL}$ *obtained by adding any subset of the following set of formulas*

$$\{\mathrm{J1}, \mathrm{J4}, \mathrm{J5}, \mathrm{M}, \mathrm{M0}, \mathrm{P}, \mathrm{P0}\}$$

is canonical and therefore frame complete.

Proof: Let $\Lambda_X = \mathbf{BIL}(X)$ be the basic interpretability logic where X is one of these subsets. Consider the properties that characterize its frames stated in Theorem 2.5. Then Propositions 4.1 and 4.6, and Theorem 4.5 establish that the canonical basic frame $\mathcal{F}_{\Lambda_X}$ has these properties. Therefore it is a frame of the logic Λ_X, that is, the logic Λ_X is canonical. $\Box$

5. Pure neighbourhood semantics

Let us consider the class $\mathbf{BFr}^+$ of basic frames satisfying the relational properties RJ1 and RJ4. By Theorem 2.5 and Theorem 4.7 the logic $\mathbf{BIL}^+$ is characterized by the class $\mathbf{BFr}^+$, i.e, $\mathbf{BIL}^+ = \mathsf{Th}(\mathbf{BFr}^+)$.

Consider the language $\mathcal{L}(\Rightarrow)$ and with the modal operator $\Box$ defined by $\Box A := \top \Rightarrow A$. Let $\langle X, N\rangle$ be a neighborhood frame. A valuation on a neighborhood frame $\langle X, N\rangle$ is any function $V : Var \to \mathcal{P}(X)$. A valuation V can be extended recursively to the set of all formulas Fm by means of the same clauses given in Definition 2.3 for the connectives $\top$,$\bot$,$\wedge$, $\vee$ and $\Rightarrow$. As $\Box A := \top \Rightarrow A$, the clause for the modal operator is $V(\Box A) = \{x \in X : \forall Y \in N(x)\ (Y \subseteq U)\}$. A *neighborhood model* is a pair $\mathcal{M} = \langle \mathcal{F}, V\rangle$, where $\mathcal{F}$ is a neighborhood frame and V is a valuation on it. The notions of formula valid in a neighborhood frame and neighborhood model are defined as in the case of basic frames and basic models (for more details see [4, 7, 3]).

Let $\langle X, N\rangle$ be a neighborhood frame. We take the binary relation $R_N \subseteq X \times X$ defined by:

$$(x, y) \in R_N \text{ iff } \exists Y \in N(x) \text{ such that } y \in Y. \tag{5.1}$$

Then it is immediate to see that $\langle X, R_N, N\rangle \in \mathbf{BFr}^+$ and

$$\langle X, N\rangle \models A \text{ iff } \langle X, R_N, N\rangle \models A,$$

for any formula A.

On the other hand, we consider a basic frame $\langle X, R, N\rangle$. We define the binary relation $R_N \subseteq X \times X$ defined by (5.1). We note that $R_N(x) = \bigcup\{Y : Y \in N(x)\}$. If $\langle X, R, N\rangle \in \mathbf{BFr}^+$, then by RJ1 we have $R_N \subseteq R$, and by RJ4 we get that $R \subseteq R_N$. Thus, in the basic frames of $\mathbf{BFr}^+$ the binary relation R and R_N are the same, i.e., R is definable by the relation N. Consequently if we work in the language $\mathcal{L}(\Rightarrow)$ and the modal operator $\Box$ is definable as $\Box A := \top \Rightarrow A$, then

$$\langle X, R, N\rangle \models A \text{ iff } \langle X, N\rangle \models A,$$

for any formula A. Consequently we can study extensions of $\mathbf{BIL}^+$ by means of neighborhood frames $\langle X, N\rangle$ where the operator $\Box$ is interpreted semantically by the relation R_N. Thus, if $\mathbf{NFr}$ is the class of all neighborhood frames and $\mathsf{Th}(\mathbf{NFr})$ is the set of all formulas valid in the class $\mathbf{NFr}$, we have the following result.

Theorem 5.1 (Soundness and Completeness). *$\mathbf{BIL}^+ = \mathsf{Th}(\mathbf{NFr})$.*

Soundness and Completeness for all axiomatic extensions of $\mathbf{BIL}^+$ by means of the formulas RJ5, M0,M,P and P0 is proved in the same way as in the Theorems 2.5, 4.1 and 4.5 but using the auxiliary relation R_N for the modality $\Box$. For example, the logic $\mathbf{BIL}^+ + \{(A \Rightarrow B) \to \Box(A \Rightarrow B)\}$ is complete with respect to the class of neighborhood frames $\langle X, N\rangle$ satisfying the relational condition RP, where $R = R_N$. For completeness we state the following result whose proof is exactly the same as the case of basic frames.

THEOREM 5.2. *Any extension of* $\mathbf{BIL}^+$ *by any subset of* {RJ5,M0,M,P,P0} *is canonical and therefore frame complete with respect to pure neighborhood frames.*

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