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L-MODULES

Abstract

In this paper, considering L-algebras, which include a significant number of other algebraic structures, we present a definition of modules on L-algebras (L-*modules*). Then we provide some examples and obtain some results on L-modules. Also, we present definitions of *prime ideals* of L-algebras and L-*submodules* (*prime* L-*submodules*) of L-modules, and investigate the relationship between them. Finally, by proving a number of theorems, we provide some conditions for having prime L-submodules.

Keywords: L-algebra, L-module, L-submodule, prime L-submodule.

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1. Introduction

In the study of set-theoretical solutions of the Yang-Baxter equation, the cycloid equation, $(x \cdot y) \cdot (x \cdot z) = (y \cdot x) \cdot (y \cdot z)$, plays a fundamental role, see for example [6, 15]. Finding a solution to the Young-Baxter equation is a research topic for many authors. Rump's research in order to find a solution for that equation led to the introduction of L-algebras [16]. L-algebras are related to algebraic logic and quantum structures. They are closely related to non-classical logical algebras and quantum Yang-Baxter equation solutions. It was shown that many non-classical logical algebras can be unified into L-algebras. For instance, the pseudo MV-algebras can be characterized as semiregular L-algebras with negation [21]; Orthomodular

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lattices can be characterized as L-algebras [20], and every lattice-ordered effect algebra gives rise to an L-algebra [19]. Also, Rump showed that an L-algebra can be represented as an interval in a lattice ordered group if and only if it is semiregular with an smallest element and bijective negation [18]. In short, there are effective relationships between L-algebras and other algebraic structures. For example, we can consider them as Hilbert algebras, locales, hoops, pseudo MV-algebras, etc. Other recent results on the structure of the category of L-algebras can be found in [8].
Discussions about modular structures on algebraic structures have long been of interest to scientists. For instance, the notion of BCK-module was introduced in 1994 as an action of a BCK-algebra over a commutative group [2], and it was extended in 2014 [3]; The notion of MV-modules was introduced as an action of a PMV-algebra over an MV-algebra in 2003 [1]; Also, the notion of MV-semimodules was introduced in 2013 [14], and the new definition of MV-semimodules was presented in 2021 [13]. As mentioned, there are effective connections between most algebraic structures. These connections show a relationship between the modular structures associated with these algebras. L- Algebras under conditions can be equivalent to other algebras such as BCK-algebras, MV-algebras, etc. Considering that we have spent a relatively large amount of time studying modular structures (for instance, see [3, 4, 9, 10, 11, 12, 13]), in order to complete and consolidate our study in this field, we have decided to define L-modules as an action of an L-algebra over an Abelian group. We hope that this definition can help us to clarify the structure of L-algebras.

2. Preliminaries

In this section, we review the material that we will use in the paper.

Definition 2.1 ([7]). An *L-algebra* is an algebra $(L; \to, 1)$ of type $(2,0)$ satisfying

$(L1)$ $x \to x = x \to 1 = 1$, $1 \to x = x$;

$(L2)$ $(x \to y) \to (x \to z) = (y \to x) \to (y \to z)$;

$(L3)$ $x \to y = y \to x = 1$ implies $x = y$, for all $x, y, z \in L$.

The relation $x \leq y$ if and only if $x \to y = 1$, defines a partial order for any L-algebra L. If L admits a smallest element 0, then it is called a bounded L-algebra.

Moreover, in the bounded L-algebra L, if the map $' : L \longrightarrow L$ defined, by $x \longrightarrow x' = x \to 0$ for every $x \in L$, is bijective, then we say that L has negation.

Definition 2.2 ([17]). A *KL-algebra* is an L-algebra $(L, \to, 1)$ such that

$$x \to (y \to x) = 1 \qquad (K)$$

for every $x, y \in L$.
A *CL-algebra* is an L-algebra $(L, \to, 1)$ such that

$$(x \to (y \to z)) \to (y \to (x \to z)) = 1 \quad (C)$$

for every $x, y, z \in L$.

Definition 2.3 ([16]). Let $(L; \to, 1)$ be an L-algebra. Then a subset K of L is called an *L-subalgebra* if $x \to y, y \to x \in K$, for all $x, y \in K$.
A subset I of L is called an *ideal* if the following hold for all $x, y \in L$:

$(I1)$ $1 \in I$,

$(I2)$ $x, x \to y \in I$ implies $y \in I$,

$(I3)$ $x \in I$ implies $(x \to y) \to y \in I$,

$(I4)$ $x \in I$ implies $y \to x$, $y \to (x \to y) \in I$. Denote by $\mathcal{ID}(L)$ the set of all ideals of L.

If L satisfies condition (K), then (I_4) can be omitted. Also, if L satisfies condition (C), then , (I_3) and (I_4) can be omitted.

Definition 2.4 ([5]). For every subset $Y \subseteq L$, the smallest ideal of L containing Y (i.e. the intersection of all ideals $I \in \mathcal{ID}(L)$ such that $Y \subseteq I$) is called the ideal generated by Y and it will be denoted by $[Y)$. If $Y = \{x\}$ we write $[x)$ instead of $[\{x\})$. In this case $[x)$ is called a principal ideal of L.

3. L-modules

In this section, we present our definition of L-modules, and obtain some results on them. Then we introduce the concepts of L-submodules and prime L-submodules in L-modules. Finally, we investigate some conditions for having a prime L-submodule.

Note. If L is an L-algebra, then we denote $(l \to u) \to u$ by $l \uparrow u$, for every $l, u \in L$.

Definition 3.1. Let $L = (L; \to, 0, 1)$ be a bounded L-algebra, and $M = (M, +)$ be an Abelian group. Then M is called an *L-module*, if there is an operation $\cdot : L \times M \longrightarrow M$ by $(l, m) \longmapsto l \cdot m$ such that for every $l, u \in L$ and $m, n \in M$, we have:

($LM1$) $1 \cdot m = m$;

($LM2$) $l \cdot (m + n) = l \cdot m + l \cdot n$;

($LM3$) $(l \to u) \cdot m = l' \cdot m + u \cdot m$, for all pairs (l, u) with $u \neq 1$. Moreover, if we have

($LM4$) $(l \uparrow u) \cdot m = l \cdot (u \cdot m)$, for all pairs (l, u) with $l \neq 0$,

then M is called an *Extended L-module* (or briefly EL-module).

Example 3.2. (i) Let $L = \{0, 1\}$ and define an operation " $\to$ " on L by

$\to$	0	1
0	1	1
1	0	1

Then $L = (L; \to, 0, 1)$ is a bounded L-algebra. The map $' : L \longrightarrow L$ by $0' = 1$ and $1' = 0$ is bijective. Consider the operation $\cdot : L \times \mathbb{Z} \longrightarrow \mathbb{Z}$ by $0 \cdot n = 0$ and $1 \cdot n = n$, for every $n \in \mathbb{Z}$. Then ($L\mathbb{Z}1$) and ($L\mathbb{Z}2$) are clear. ($L\mathbb{Z}3$) We have $(0 \to 0).n = 0'.n + 0.n$, $(1 \to 1).n = 1'.n + 1.n$ and $(1 \to 0).n = 1'.n + 0.n$, for every $n \in \mathbb{Z}$. Then $\mathbb{Z}$ is an L-module. Moreover, ($L\mathbb{Z}4$) We have $(0 \uparrow 0).n = 0.(0.n)$ and $(1 \uparrow 1).n = 1.(1.n)$, for every $n \in \mathbb{Z}$. Therefore, $\mathbb{Z}$ is an EL-module.

(ii) Let A be a non-empty set. Then it is routine to see that $(\rho(A); \to, \emptyset, A)$ is a bounded L-algebra, where $X \to Y = X' \cup Y$, for every $X, Y \in \rho(A)$. Since $\emptyset \to \emptyset = \emptyset \to A = A \to A = A$ and $A \to \emptyset = \emptyset$, we get $L = \{\emptyset, A\}$ is an L-subalgebra of $\rho(A)$ and so it is an L-algebra. Consider $M = (\rho(A), \Delta)$, where $X \Delta Y = X \cup Y \setminus X \cap Y$, for every $X, Y \in \rho(A)$. It is easy to see that M is an abelian group. Now, let the operation $\cdot : L \times M \to M$ be defined by $T \cdot Y = T \cap Y$, for any $T \in L$ and $Y \in M$. Then

($LM1$) $A \cdot Y = A \cap Y = Y$, for every $Y \in M$;

($LM2$) It is routine to see that

$$T \cdot (X + Y) = T \cap (X \Delta Y) = (T \cap X)\Delta(T \cap Y) = (T \cdot X) + (T \cdot Y),$$

for every $T \in L$ and $X, Y \in M$;

$(LM3)$ We have

$$\begin{aligned}(A \to A).X &= (A \cup A') \cap X = X = X \cap A = X \cap (A' \Delta A)\\ &= (A' \cap X)\Delta(A \cap X) = A' \cdot X + A \cdot X,\end{aligned}$$

for every $X \in M$. By the similar way, we have

$(\emptyset \to \emptyset) \cdot X = \emptyset' \cdot X + \emptyset \cdot X$ and $(A \to \emptyset) \cdot X = A' \cdot X + \emptyset \cdot X,$ for every $X \in M$.

Hence, M is an L-module. Moreover,

$(LM4)$ Since

$$A \uparrow A = (A \to A) \to A = (A' \cup A) \to A = (A \cap A') \cup A = A,$$

we have $(A \uparrow A) \cdot X = A \cdot (A \cdot X)$, for every $X \in M$. By the similar way, we have $(\emptyset \uparrow \emptyset) \cdot X = \emptyset \cdot (\emptyset \cdot X)$, for every $X \in M$. Therefore, M is an EL-module.

Note. From now on, in this paper, we let $L = (L; \to, 1)$ be an L-algebra.

DEFINITION 3.3. If $l \uparrow u = u \uparrow l$, for every $l, u \in L$, then we say that L is L-commutative.

Example 3.4. (i) Let $L = \{0, l, u, 1\}$ and define an operation " $\to$ " on L by

$\to$	0	l	u	1
0	1	1	1	1
l	u	1	u	1
u	l	l	1	1
1	0	l	u	1

Then $(L; \to, 1)$ is an L-algebra. Moreover, L is L-commutative.
(ii) According to Example 3.2 (i), L is L-commutative.

(iii) Let $L = \{0, l, u, t, 1\}$ and define operation " $\rightarrow$ " on L by

$\rightarrow$	0	l	u	t	1
0	1	1	1	1	1
l	0	1	l	t	1
u	0	l	1	t	1
t	t	1	1	1	1
1	0	l	u	t	1

Then $(L; \rightarrow, 1)$ is an L-algebra. Since $l \uparrow t = (l \rightarrow t) \rightarrow t = 1 \neq l = (t \rightarrow l) \rightarrow l = t \uparrow l$, L is not L-commutative.

In the following, we present a general example of L-module.

Proposition 3.5. Let $L = (L; \rightarrow, 0, 1)$ be bounded and L-commutative. Then $(L, +)$ is an Abelian group, where

$$l + u = (l \rightarrow u)' \uparrow (u \rightarrow l)', \text{ for every } l, u \in L.$$

Proof: At first, we show that $0 + l = l + 0 = l$, for every $l \in L$. We have

$$l + 0 = (l \rightarrow 0)' \uparrow (0 \rightarrow l)' = (l')' \uparrow 1' = l \uparrow 0 = (l \rightarrow 0) \rightarrow 0 = (l')' = l.$$

By the similar way, we have $0 + l = l$ and so $0 + l = l + 0 = l$, for every $l \in L$. Also, since

$$l + l = (l \rightarrow l)' \uparrow (l \rightarrow l)' = 1' \uparrow 1' = 0 \uparrow 0 = (0 \rightarrow 0) \rightarrow 0 = 1 \rightarrow 0 = 0,$$

we conclude that every member of L has a counterpart in L. Now, with a long and routine method, it can be seen

$$l + (u + t) = (l + u) + t, \text{ for every } l, u, t \in L.$$

Finally, since L is L-commutative, we have

$$l + u = (l \rightarrow u)' \uparrow (u \rightarrow l)' = (u \rightarrow l)' \uparrow (l \rightarrow u)' u + l, \text{ for every } l, u \in L.$$

Therefore, $(L, +)$ is an Abelian group. $\square$

Proposition 3.6. Let $L = (L; \wedge, \vee, ', 0, 1)$ be a Boolean-algebra. Then L is a bounded L-algebra. Moreover, L is L-commutative.

PROOF: We define $l \to u = l' \vee u$, for every $l, u \in L$. Then

($L1$) It is clear that $l \to l = l \to 1 = 1$ and $1 \to l = l$, for every $l \in L$.

($L2$) For every $l, u \in L$, we have

$$\begin{aligned}(l \to u) \to (l \to t) &= (l' \vee u) \to (l' \vee t) = (l' \vee u)' \vee (l' \vee t)\\ &= (l \wedge u') \vee (l' \vee t) = ((l \wedge u') \vee l') \vee t\\ &= ((l \vee l') \wedge (u' \vee l')) \vee t = (1 \wedge (u' \vee l')) \vee t\\ &= (u' \vee l') \vee t.\end{aligned}$$

On the other hand, by the similar way, we have $(u \to l) \to (u \to t) = (u' \vee l') \vee t$. Hence

$$(l \to u) \to (l \to t) = (u \to l) \to (u \to t), \text{ for every } l, u \in L.$$

($L3$) Let $l \to u = u \to l = 1$, for any $l, u \in L$. Then $l' \vee u = u' \vee l = 1$ and so

$$l \wedge u = (l \wedge l') \vee (l \wedge u) = l \wedge (l' \vee u) = l \wedge 1 = l.$$

This means that $l \leq u$. By the similar way, we have $u \leq l$ and so $u = l$. Thus, $(L, \to, 1)$ is an L-algebra. Note that $0 \to l = 0' \vee l = 1 \vee l = 1$. So $0 \leq l$, for every $l \in L$ and so L is bounded. Moreover, we have

$$\begin{aligned}l \uparrow u &= (l \to u) \to u = (l' \vee u)' \vee u = (l \wedge u') \vee u = (l \vee u) \wedge (u \vee u')\\ &= l \vee u = (l \vee u) \wedge (l \vee l') = l \vee (u \wedge l') = l \vee (u' \vee l)' = l \vee (u \to l)'\\ &= (u \to l) \to l = u \uparrow l, \text{ for every } u, l \in L.\end{aligned}$$

Therefore, L is L-commutative. □

Example 3.7. Let $L = (L; \wedge, \vee, ', 0, 1)$ be a Boolean-algebra. If $l \to u \neq 1$ implies $u \leq l$, for every $u, l \in L$, then L is an L-module.

PROOF: By Proposition 3.6, L is bounded and L-commutative, and by Proposition 3.5, $M = (L, +)$ is an Abelian group, where $l + u = (l \to u)' \uparrow (u \to l)'$, for every $l, u \in L$. We define the operation $\cdot : L \times M \longrightarrow M$ by $l.m = l \wedge m$, for every $l \in L$ and $m \in M$. Then

($LM1$) $1 \cdot m = 1 \wedge m$, for every $m \in M$;

($LM2$) Since for every $m, n \in M$,

$$\begin{aligned}
m+n &= (m\to n)'\uparrow(n\to m)' = ((m\to n)'\to(n\to m)')\to(n\to m)'\\
&= ((m'\vee n)'\to(n'\vee m)')\to(n'\vee m)'\\
&= ((m'\vee n)\vee(n\wedge m'))'\vee(n\wedge m')\\
&= ((m\wedge n')\wedge(n'\vee m))\vee(n\wedge m')\\
&= ((m\wedge n')\vee(n\wedge m'))\wedge((n\wedge m')\vee(n'\vee m))\\
&= ((m\wedge n')\vee n)\wedge((m\wedge n')\vee m)\wedge((n\vee m\vee n)\wedge(n'\vee m'\vee m'))\\
&= ((n\vee m)\wedge(n\vee n'))\wedge((m\vee m')\wedge(m'\vee n'))\wedge(m\wedge m)\\
&= (n\vee m)\wedge(m'\vee n') = ((n\vee m)\wedge m')\vee((n\vee m)\wedge n')\\
&= ((n\wedge m')\vee(m\wedge m'))\vee((n\wedge n')\vee(m\wedge n'))\\
&= (n\wedge m')\vee(m\wedge n'),
\end{aligned}$$

we have

$$\begin{aligned}
l\cdot(m+n) &= l\wedge((n\wedge m')\vee(m\wedge n')) = (l\wedge n\wedge m')\vee(l\wedge m\wedge n')\\
&= ((l\wedge m)\wedge(l\wedge n)')\vee((l\wedge m)'\wedge(l\wedge n))\\
&= (l\wedge m)+(l\wedge n) = l\cdot m + l\cdot n,
\end{aligned}$$

for every $l\in L$ and $m,n\in M$.

$(LM3)$ Let $l\to u\neq 1$ or $l=u$, for any $l,u\in L$. Then $u\leq l$ and so $u\vee l=l$ and $u\wedge l=u$. Thus, for every $m\in M$,

$$\begin{aligned}
l'.m+u.m &= (l'\wedge m)+(u\wedge m)\\
&= ((l'\wedge m)'\wedge(u\wedge m))\vee((l'\wedge m)\wedge(u\wedge m)')\\
&= ((l\vee m')\wedge(u\wedge m))\vee((l'\wedge m)\wedge(u'\vee m'))\\
&= ((u\wedge m\wedge l)\vee(u\wedge m\wedge m')\vee(l'\wedge m\wedge u')\vee(l'\wedge m\wedge m'))\\
&= (u\wedge m\wedge l)\vee(l'\wedge m\wedge u') = m\wedge((u\wedge l)\vee(l'\wedge u'))\\
&= ((l\vee u)\to(l\wedge u)).m = (l\to u).m.
\end{aligned}$$

Note that if $l\to u=1$, then $l\leq u$. So by the similar way, we have $(l\to u).m = l'.m+u.m$. Hence,

$$(l\to u)\cdot m = l'\cdot m + u\cdot m, \text{ for all pairs (l,u) with } u\neq 1.$$

Therefore, L is an L-module. □

PROPOSITION 3.8. Let $L = (L; \rightarrow, 0, 1)$ be bounded and L-commutative, I be an ideal of L and L be an L-module. Then $\frac{L}{I}$ is an L-module. Moreover, if L is an EL-module, then $\frac{L}{I}$ is an EL-module.

PROOF: Since $(L, +)$ is an Abelian group, it is easy to see that $(\frac{L}{I}, \oplus)$ is An abelian group, where $[l] \oplus [u] = [l + u]$, for every $l, u \in L$. We define the operation $\bullet : L \times \frac{L}{I} \longrightarrow \frac{L}{I}$ by $l \bullet [m] = [l \cdot m]$, for every $l \in L$ and $[m] \in \frac{L}{I}$. Then

$(L\frac{L}{I}1)$ By $(LL1)$, we have $1 \bullet [m] = [m]$, for every $[m] \in \frac{L}{I}$;

$(L\frac{L}{I}2)$ By $(LL2)$, for every $l \in L$ and $[m], [n] \in \frac{L}{I}$, we have

$$1 \bullet ([m] \oplus [n]) = l \bullet [m+n] = [l \cdot (m+n)] = [l \cdot m + l \cdot n] = [l \cdot m] \oplus [l \cdot n] = l \bullet [m] \oplus l \bullet [n];$$

$(L\frac{L}{I}3)$ By $(LL3)$, for every $[m] \in \frac{L}{I}$ and for all pairs (l, u) with $u \neq 1$, we have

$$(l \rightarrow u) \bullet [m] = [(l \rightarrow u) \cdot m] = [l' \cdot m + u \cdot m] = [l' \cdot m] \oplus [u \cdot m] = l' \bullet [m] \oplus u \bullet [m].$$

Then $\frac{L}{I}$ is an L-module. Moreover,

$(L\frac{L}{I}4)$ By $(LL4)$, for every $[m] \in \frac{L}{I}$ and for all pairs (l, u) with $l \neq 0$, we have

$$(l \uparrow u) \bullet [m] = [(l \uparrow u) \cdot m] = [l \cdot (u \cdot m)] = l \bullet [u \cdot m] = l \bullet (u \bullet [m]).$$

Therefore, $\frac{L}{I}$ is an EL-module. $\square$

Note. From now on, in this paper, we let M be an Abelian group.

Let $I \in \mathcal{ID}(L)$. The relation $\sim$ on L is defined by

$$u \sim l \Leftrightarrow u \rightarrow l, l \rightarrow u \in I, \text{ for every } u, l \in L.$$

It was proved that $\sim$ is a congruence on L. Then $(\frac{L}{I};\rightarrow,[1])$ is an L-algebra, where $[u]\rightarrow[l]=[u\rightarrow l]$, for every $u,l\in L$ (see [16]).

Theorem 3.9. *Let M be an L-module, and I be an ideal of L such that $I\subseteq Ann_L(M)$, where $Ann_L(M)=\{l\in L : l\cdot m=0,$ for every $m\in M\}$. Then M is an $\frac{L}{I}$-module. Moreover, if M is an EL-module, then M is an $E\frac{L}{I}$-module.*

Proof: Consider $'$: $\frac{L}{I}\longrightarrow\frac{L}{I}$ by $([l])'=[l']$, for every $l\in L$ which is a bijective mapping. Define the operation $\bullet$: $\frac{L}{I}\times M\longrightarrow M$ by $[l]\bullet m=l\cdot m$, for every $[l]\in\frac{L}{I}$ and $m\in M$. Let $[l]=[u]$ and $m=n$, for every $[l],[u]\in\frac{L}{I}$ and $m,n\in M$. Then $l\rightarrow u,u\rightarrow l\in I\subseteq Ann_L(M)$ and so $(l\rightarrow u)\cdot m=(u\rightarrow l)\cdot m=0$, for every $m\in M$. It results that $l'\cdot m+u\cdot m=u'\cdot m+l\cdot m=0$ and so $l\cdot m-u\cdot m=l'\cdot m-u'\cdot m$ and $l\cdot m=-u'\cdot m$. Hence $l\cdot m-u\cdot m=l'\cdot m+l\cdot m=(l\rightarrow l)\cdot m=1\cdot m$ and so $l\cdot m-u\cdot m=1\cdot m$. By the similar way, we have $u\cdot m-l\cdot m=1\cdot m$. It results that $l\cdot m-u\cdot m=u\cdot m-l\cdot m$ and so $l\cdot m=u\cdot m$. It means that $\bullet$ is well defined. Now, we have:

$(\frac{L}{I}M1)$ By $(LM1)$, it is clear that $[1]\bullet m=m$, for every $m\in M$;

$(\frac{L}{I}M2)$ By $(LM2)$, we have

$$[l]\bullet(m+n)=l\cdot(m+n)=l\cdot m+l\cdot n=[l]\bullet m+l\bullet n,$$

for every $[l]\in\frac{L}{I}$ and $m,n\in M$;

$(\frac{L}{I}M3)$ By $(LM3)$, for every $m\in M$ and for all pairs $([l],[u])$ with $[u]\neq[1]$, we have

$$([l]\rightarrow[u])\bullet m=[l\rightarrow u]\bullet m=(l\rightarrow u)\cdot m=l'\cdot m+u\cdot m=[l]'\bullet m+[u]\bullet m.$$

Note that $l \neq 1$ implies $[l] \neq [1]$. Hence, M is an $\dfrac{L}{I}$-module. Moreover,

$(\dfrac{L}{I}M4)$ by $(LM4)$, for every $m \in M$ and for all pairs $([l],[u])$ with $[l] \neq [0]$, we have

$$([l] \uparrow [u]) \bullet m = [l \uparrow u] \bullet m = (l \uparrow u) \cdot m = l \cdot (u \cdot m) = [l] \bullet (u \cdot m) = [l] \bullet ([u] \bullet m).$$

Note that $l = 0$ implies $[l] = [0]$. Therefore, M is an $E\dfrac{L}{I}$-module. $\square$

Definition 3.10. Let M be an L-module, and S be a subgroup of M. If S satisfies

$$l \cdot s \in S, \text{ for every } l \in L \text{ and } s \in S,$$

then it is called an L-*submodule* of M.

Example 3.11. (i) By Example 3.2 (i), $2\mathbb{Z}$ is an L-submodule of M.
(ii) According to Example 3.2 (ii), consider $A = \{a, b\}$. Then $S_1 = \{\emptyset, \{a\}\}$ and $S_2 = \{\emptyset, \{b\}\}$ are L-submodules of M.

Let M be an L-module, and S be an L-submodule of M. Since $(M, +)$ is an Abelian group and S is a subgroup of M, we can apply the module theory to present quotient L-module. So it is clear that $(\dfrac{M}{S}, \oplus)$ is an Abelian group, where $(m+S) \oplus (n+S) = (m+n) \oplus S$, for every $m, n \in M$.

Proposition 3.12. Let M be an L-module, and S be an L-submodule of M. Then $\dfrac{M}{S}$ is an L-module. Moreover, if M is an EL-module, then $\dfrac{M}{S}$ is an EL-module.

Proof: We define the operation $\bullet : L \times \dfrac{M}{S} \longrightarrow \dfrac{M}{S}$ by $l \bullet (m+S) = l \cdot m + S$, for every $l \in L$ and $m + S \in \dfrac{M}{S}$. It is routine to see that $\bullet$ is well defined. By $(LM1)$ and $(LM2)$, the proofs of $(L\dfrac{M}{S}1)$ and $(L\dfrac{M}{S}2)$ are routine.
$(L\dfrac{M}{S}3)$ By $(LM3)$, for all pairs (l, u) with $u \neq 1$, we have

$$\begin{aligned}(l \to u) \bullet (m+S) &= (l \to u) \cdot m + S = (l' \cdot m + u \cdot m) + S \\ &= (l' \cdot m + S) \oplus (u \cdot m + S) \\ &= l' \bullet (m+S) \oplus u \bullet (m+S),\end{aligned}$$

for every $m + S \in \dfrac{M}{S}$. Then $\dfrac{M}{S}$ is an L-module. Moreover,
$(L\dfrac{M}{S}4)$ by $(LM4)$, for all pairs (l, u) with $l \neq 0$, we have

$$\begin{aligned}(l \uparrow u) \bullet (m+S) &= (l \uparrow u) \cdot m + S = l \cdot (u \cdot m) + S \\ &= l \bullet (u \cdot m + S) = l \bullet (u \bullet (m+S)),\end{aligned}$$

for every $m + S \in \dfrac{M}{S}$. Therefore, $\dfrac{M}{S}$ is an EL-module. □

LEMMA 3.13. *Let M be an EL-module, and I be an ideal of L. Then*

$$I_L(M) = \{\Sigma_{i=1}^{n} t_i \cdot m_i : 0 \neq t_i \in I, m_i \in M, n \in \mathbb{N}\}$$

is an L-submodule of M.

PROOF: It is clear that $I_L(M)$ is a subgroup of M. Now, for every $l \in L$ and $\Sigma_{i=1}^{n} t_i \cdot m_i \in I_L(S)$, by $(LM2)$, we have

$$l \cdot \Sigma_{i=1}^{n} t_i \cdot m_i = l \cdot (t_1 \cdot m_1) + l \cdot (t_2 \cdot m_2) + \cdots + l \cdot (t_n \cdot m_n)$$

and so by $(LM4)$,

$$l \cdot \Sigma_{i=1}^{n} t_i \cdot m_i = (l \uparrow t_1) \cdot m_1 + (l \uparrow t_2) \cdot m_2 + \cdots + (l \uparrow t_n) \cdot m_n.$$

Since by (I_3), $t_i \cdot m_i \in I$, for every $1 \leq i \leq n$, we get $l \cdot \Sigma_{i=1}^{n} t_i \cdot m_i \in I_L(M)$. Therefore, $I_L(M)$ is an L-submodule of M. □

DEFINITION 3.14. Let I be a proper ideal of L. Then I is called a *prime ideal* of L, if $l \uparrow u \in I$ implies $l \in I$ or $u \in I$, where $l, u \in L$.

Example 3.15. According to Example 3.4 (i), it is easy to see that $I_1 = \{1, l\}$ and $I_2 = \{1, u\}$ are prime ideals of L.

THEOREM 3.16. *Let M be an EL-module, S be an L-submodule of M and P be a prime ideal of L. Then*

$$S_{N,P} = \{m \in M : c \cdot m \in P_L(M) + S, \ \exists\, 0 \neq c \in (L \setminus P) \cup \{1\}\}$$

is an L-submodule of M and $P_L(M) + S \subseteq S_{N,P}$.

PROOF: Let $m, n \in S_{N,P}$. Then there are $c_1, c_2 \in (L \setminus P) \cup \{1\}$ such that $0 \neq c_1$, $0 \neq c_2$ and $c_1 \cdot m$, $c_2 \cdot n \in P \cdot M + S$. Consider $c = c_1 \uparrow c_2$. It is clear that $c \in (L \setminus P) \cup \{1\}$. Then by $(LM4)$, we have

$$\begin{aligned} c \cdot (m - n) &= (c_1 \uparrow c_2) \cdot (m - n) = c_1 \cdot (c_2 \cdot (m - n)) \\ &= c_1 \cdot (c_2 \cdot m - c_2 \cdot n) = c_1 \cdot (c_2 \cdot m) - c_1 \cdot (c_2 \cdot n) \end{aligned}$$

and so by Lemma 3.13, $c \cdot (m - n) \in P_L(M) + S$. Now, for every $l \in L$ and $m \in S_{N,P}$, we show that $l \cdot m \in S_{N,P}$. Since $m \in S_{N,P}$, there is $0 \neq c \in (L \setminus P) \cup \{1\}$ such that $c \cdot m \in P_L(M)$. Then by Lemma 3.13 and $(LM4)$,

$$c \cdot (l \cdot m) = (c \uparrow l) \cdot m = (l \uparrow c) \cdot m = l \cdot (c \cdot m) \in P_L(M).$$

Hence, $S_{N,P}$ is an L-submodule of M. Finally, let $t \cdot m \in P_L(M)$. Then we have $1 \cdot (t \cdot m) \in P_L(M) + S$, where $c = 1 \in (L \setminus P) \cup \{1\}$. Therefore, $t \cdot m \in S_{N,P}$ and so $P_L(M) \subseteq S_{N,P}$. $\square$

THEOREM 3.17. *Let I be an ideal of L, and M be an EL-module. Then $\dfrac{M}{I_L(M)}$ is an $E\dfrac{L}{I}$-module. Moreover, if M is an EL-module, then $\dfrac{M}{I_L(M)}$ is an $E\dfrac{L}{I}$-module.*

PROOF: The module $\dfrac{M}{I_L(M)}$ can be defined by Lemma 3.13. Then we define the operation
$\bullet : \dfrac{L}{I} \times \dfrac{M}{I_L(M)} \longrightarrow \dfrac{M}{I_L(M)}$ by $[l] \bullet (m + I_L(M)) = l \cdot m + I_L(M)$, for every $[l] \in \dfrac{L}{I}$ and $m + I_L(M) \in \dfrac{M}{I_L(M)}$. Since

$$\begin{aligned} I \bullet \frac{M}{I_L(M)} &= \{l \bullet (m + I_L(M)) : l \in L, m \in M\} \\ &= \{l \cdot m + I_L(M) : l \in L, m \in M\} = I_L(M), \end{aligned}$$

we have $I \subseteq Ann_L(\frac{M}{I_L(M)})$ and so with a proof similar to the proof of Theorem 3.9, $\bullet$ is well defined.

$(\frac{L}{I}\frac{M}{I_L(M)}1)$ By $(LM1)$, $[1]\bullet(m+I_L(M)) = 1\cdot m+I_L(M) = m+I_L(M)$, for every $m \in M$;

$(\frac{L}{I}\frac{M}{I_L(M)}2)$ By $(LM2)$, we have

$$\begin{aligned}
[l]\bullet((m+I_L(M))\oplus(n+I_L(M))) &= [l]\bullet(m+n+I_L(M))\\
&= l\cdot(m+n)+I_L(M)\\
&= l\cdot m+l\cdot n+I_L(M)\\
&= (l\cdot m+I_L(M))\oplus(l\cdot n+I_L(M))\\
&= [l]\bullet(m+I_L(M))\oplus[l]\bullet(n+I_L(M)),
\end{aligned}$$

for every $[l] \in \frac{L}{I}$ and $(m+I_L(M)),(n+I_L(M)) \in \frac{M}{I_L(M)}$;

$(\frac{L}{I}\frac{M}{I_L(M)}3)$ By $(LM3)$, for every $m+I_L(M) \in \frac{M}{I_L(M)}$ and for all pairs $([l],[u])$ with $[u] \neq [1]$, we have

$$\begin{aligned}
([l]\to[u])\bullet(m+I_L(M)) &= [l\to u]\bullet(m+I_L(M))\\
&= (l\to u)\cdot m+I_L(M)\\
&= (l'\cdot m+u\cdot m)+I_L(M)\\
&= (l'\cdot m+I_L(M))\oplus(u\cdot m+I_L(M))\\
&= [l]'\bullet(m+I_L(M))\oplus[u]\bullet(m+I_L(M));
\end{aligned}$$

Hence, M is an $\frac{L}{I}$-module. Moreover,

$(\frac{L}{I}\frac{M}{I_L(M)}4)$ by $(LM4)$, for every $m+I_L(M) \in \frac{M}{I_L(M)}$ and for all pairs $([l],[u])$ with $[l] \neq [0]$, we have

$$\begin{aligned}
([l]\uparrow[u])\bullet(m+I_L(M)) &= [l\uparrow u]\bullet(m+I_L(M)) = (l\uparrow u)\cdot m+I_L(M)\\
&= l\cdot(u\cdot m)+I_L(M) = [l]\bullet(u\cdot m+I_L(M))\\
&= [l]\bullet([u]\bullet(m+I_L(M)).
\end{aligned}$$

Therefore, $\dfrac{M}{I_L(M)}$ is an $E\dfrac{L}{I}$-module. $\square$

DEFINITION 3.18. Let M be an L-module and S be a proper L-submodule of M. Then S is called a *prime L-submodule* of M, if by $l \cdot m \in S$, we have $m \in S$ or $l \in (S : M) = \{l \in L : l \cdot M \subseteq S\}$.

Example 3.19. By Example 3.2(i), $2\mathbb{Z}$ is a prime L-submodule of $\mathbb{Z}$.

Note. Let M be an L-module, $I \subseteq L$ and $D \subseteq M$. Then we set $ID = \{i \cdot d : i \in I \text{ and } d \in D\}$, and $I_t = \{\alpha \in L : t \to \alpha = 1\}$, for every $t \in L$. It is clear that $1, t \in I_t$ and so $I_t \neq \emptyset$.

THEOREM 3.20. *Let L be bounded and L-commutative, M be an L-module and S be a proper L-submodule of M. Then S is a prime L-submodule of M if and only if $I_tD \subseteq S$ implies $D \subseteq S$ or $I_t \subseteq (S : M)$, for any L-submodule D of M and $t \in L$.*

PROOF: ($\Rightarrow$) Let S be a prime L-submodule of M and $I_tD \subseteq S$, where D is an L-submodule of M and $t \in L$. We show that $D \subseteq S$ or $I_t \subseteq (S : M)$. Let $I_t \nsubseteq (S : M)$ and $D \nsubseteq S$. Then there are $x \in I_t$ and $d \in D$ such that $x \cdot M \nsubseteq S$ and $d \notin S$. Since $ID \subseteq S$, we have $x \cdot d \in S$ and so by $d \notin S$, we get $x \in (S : M)$, which is a contradiction.
($\Leftarrow$) Let by $I_tD \subseteq S$, we have $D \subseteq S$ or $I_t \subseteq (S : M)$, for any L-submodule D of M and $t \in L$. Suppose $x \cdot m \in S$ and $m \notin S$, for any $x \in L$ and $m \in M$. For every $\alpha \in I_x$, we have

$$\begin{aligned} \alpha \cdot m &= (1 \to \alpha) \cdot m = ((x \to \alpha) \to \alpha) \cdot m = (x \uparrow \alpha) \cdot m = (\alpha \uparrow x) \cdot m \\ &= \alpha \cdot (x \cdot m) \in S. \end{aligned}$$

Now, consider $D = \prec m \succ = \{y \cdot m : y \in L\}$. Then

$$I_xD = \{\alpha \cdot (y \cdot m) : \alpha, y \in L\} = \{y \cdot (\alpha \cdot m) : \alpha, y \in L\} \subseteq S$$

and so $I_x \subseteq (S : M)$ or $D \subseteq S$. Since $m \notin S$, we have $I_x \subseteq (S : M)$ and so $x \in (S : M)$. Therefore, S is a prime L-submodule of M. $\square$

PROPOSITION 3.21. For every $x, y \in L$,

(i) $x' \to (x \to y) = 1$;

(ii) $(x \to y) \to x' = (y \to x) \to y'$.

Proof: (i) By $(L2)$, we have
$x' \to (x \to y) = (x \to 0) \to (x \to y) = (0 \to x) \to (0 \to y) = 1 \to 1 = 1$, for every $x, y \in L$. (ii) By $(L2)$, we have
$(x \to y) \to x' = (x \to y) \to (x \to 0) = (y \to x) \to (y \to 0) = (y \to x) \to y'$, for every $x, y \in L$. □

Lemma 3.22. *Let L be a bounded KL-algebra, M be an EL-module and S be a proper L-submodule of M. Then $P_S = (S : M) \cup \{1\}$ is an ideal of L.*

Proof: $(I1)$ It is clear that $1 \in P_S$.
$(I2)$ Let $x, x \to y \in P_S$. Because of the nature of the definition of P_S, we need to consider three cases:

(1) If $x = 1$, then $y = 1 \to y = x \to y \in P_S$.

(2) Let $x \to y = 1$. Then for $y = 1$, the problem is solved. Thus, let $y \neq 1$. In this case, if $x = 0$, then by $(LM3)$, $m = 1 \cdot m = (0 \to y) \cdot m = 1 \cdot m + y \cdot m = m + y \cdot m$ and so $y \cdot m = 0$, for every $m \in M$. It means that $y \in (S : M)$ and so $y \in P_S$. Hence, suppose $x \neq 0$ and $y \neq 1$. Since $y = 1 \to y = (x \to y) \to y = x \uparrow y$, by $(LM4)$, we have

$$y \cdot m = (x \uparrow y) \cdot m = (y \uparrow x) \cdot m = y \cdot (x \cdot m) \in S, \text{ for every } m \in M.$$

Thus, $y \in (S : M)$ and so $y \in P_S$.

(3) Let $x \neq 1$ and $x \to y \neq 1$. Then $x \cdot m, (x \to y) \cdot m \in S$, for every $m \in M$. It results that $x \cdot m + (x \to y) \cdot m \in S$, for every $m \in M$. Now, by Proposition 3.21(i) and $(LM3)$, for every $m \in M$, we have

$$m = 1 \cdot m = (x' \to (x \to y)) \cdot m = x \cdot m + (x \to y) \cdot m \in S,$$

which is a contradiction.

Therefore, $P_S = (S : M) \cup \{1\}$ is an ideal of L. □

Definition 3.23. Let L be bounded and M be an L-module. Then M is called a *torsion free* L-module, if $l \cdot m = 0$ implies $l = 0$ or $m = 0$, for every $l \in L$ and $m \in M$.

Example 3.24. By Example 3.2(ii), M is a torsion free L-module.

THEOREM 3.25. *Let L be a bounded KL-algebra, M be an EL-module and S be a proper L-submodule of M. Then S is a prime L-submodule of M if and only if $P_S = (S : M) \cup \{1\}$ is a prime ideal of L and $\frac{M}{S}$ is a torsion free $\frac{L}{P_S}$-module.*

PROOF: ($\Rightarrow$) Let S is a prime L-submodule of M. By Lemma 3.22, P_S is an ideal of L. At first, we show that P_S is a prime ideal of L. Let $x \uparrow y \in P_S$, for any $x, y \in P_S$. We consider three cases:

(1) If $x = 1$ or $y = 1$, then $x \in P_S$ or $y \in P_S$.

(2) If $x \uparrow y \neq 1$, $x \neq 1$ and $y \neq 1$, then by ($LM4$), we have $x \cdot (y \cdot m) = (x \uparrow y) \cdot m \in S$, for every $m \in S$. Hence, $x \in (S : M)$ or $y \cdot m \in S$, for every $m \in M$. It results that $x \in P_S$ or $y \in P_S$.

(3) Let $x \uparrow y = 1$, $x \neq 1$ and $y \neq 1$. Then $(x \to y) \to y = x \uparrow y = 1$ and so $x \to y \leq y$. Since $y \leq x \to y$, we have $x \to y = y$ and so by ($LM3$),

$$(x \to y) \cdot m = x' \cdot m + y \cdot m = y \cdot m, \text{ for every, } m \in M.$$

Then $x' \cdot m = 0 \in S$ and so $x' \in (S : M)$ or $m \in S$, for every $m \in M$. If $m \in S$, for every $m \in M$, then $M = S$, which is a contradiction. Thus, $x' \in (S : M) \subseteq P_S$ and so by ($I3$), we have $y = x \to y = y' \to x' \in P_S$. Hence, P_S is a prime ideal of L.

Now, we define the operation $\bullet : \frac{L}{P_S} \times \frac{M}{S} \longrightarrow \frac{M}{S}$ by $[l] \bullet (m+S) = l \cdot m + S$, for every $[l] \in \frac{L}{P_S}$ and $m + S \in \frac{M}{S}$. By the similar way to the proof of Theorem 3.17, $\frac{M}{S}$ is an $\frac{L}{P_S}$-module. Finally, let $[l] \bullet (m + S) = S$, for any $[l] \in \frac{L}{P_S}$ and $m + S \in \frac{M}{S}$. Then $l \cdot m + S = S$ and so $l \cdot m \in S$. It results that $l \in (S : M) \subseteq P_S$ or $m \in S$ and so $[l] = P_S$ or $m + S = S$. Therefore, $\frac{M}{S}$ is a torsion free $\frac{L}{P_S}$-module.

($\Leftarrow$) Let $P_S = (S : M) \cup \{1\}$ be a prime ideal of L and $\dfrac{M}{S}$ be a torsion free $\dfrac{L}{P_S}$-module. If $l \cdot m \in S$, for any $l \in L$ and $m \in S$, then $[l] \bullet (m + S) = l \cdot m + S = S$ and so $[l] = P_S = [1]$ or $m + S = S$. It means that $l = 1 \to l \in P_S$. Therefore, S is a prime L-submodule of M. $\square$

4. Conclusions and future works

In this paper, we have presented the definitions of L-modules, L-submodules and prime L-submodules, and some results about prime L-submodules. We intend to study L-modules in specific cases, too. For examples, free L-modules, projective(injective) L-modules, and so on. Because L-algebras cover a number of algebraic structures (such as BCK-algebras, etc.), the results of this paper can be generalized to those algebraic structures. We hope that we have taken an effective step in this regard.

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