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L-MODULES

Abstract

In this paper, considering $L$-algebras, which include a significant number of other algebraic structures, we present a definition of modules on $L$-algebras ($L$-modules). Then we provide some examples and obtain some results on $L$-modules. Also, we present definitions of prime ideals of $L$-algebras and $L$-submodules (prime $L$-submodules) of $L$-modules, and investigate the relationship between them. Finally, by proving a number of theorems, we provide some conditions for having prime $L$-submodules.

Keywords: $L$-algebra, $L$-module, $L$-submodule, prime $L$-submodule.

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1. Introduction

In the study of set-theoretical solutions of the Yang-Baxter equation, the cycloid equation, $(x \cdot y) \cdot (x \cdot z) = (y \cdot x) \cdot (y \cdot z)$, plays a fundamental role, see for example [6, 15]. Finding a solution to the Young-Baxter equation is a research topic for many authors. Rump’s research in order to find a solution for that equation led to the introduction of $L$-algebras [16]. $L$-algebras are related to algebraic logic and quantum structures. They are closely related to non-classical logical algebras and quantum Yang-Baxter equation solutions. It was shown that many non-classical logical algebras can be unified into $L$-algebras. For instance, the pseudo MV-algebras can be characterized as semiregular $L$-algebras with negation [21]: Orthomodular
lattices can be characterized as L-algebras [20], and every lattice-ordered effect algebra gives rise to an L-algebra [19]. Also, Rump showed that an L-algebra can be represented as an interval in a lattice ordered group if and only if it is semiregular with an smallest element and bijective negation [18]. In short, there are effective relationships between L-algebras and other algebraic structures. For example, we can consider them as Hilbert algebras, locales, hoops, pseudo MV-algebras, etc. Other recent results on the structure of the category of L-algebras can be found in [8].

Discussions about modular structures on algebraic structures have long been of interest to scientists. For instance, the notion of BCK-module was introduced in 1994 as an action of a BCK-algebra over a commutative group [2], and it was extended in 2014 [3]; The notion of MV-modules was introduced as an action of a PMV-algebra over an MV-algebra in 2003 [1]; Also, the notion of MV-semimodules was introduced in 2013 [14], and the new definition of MV-semimodules was presented in 2021 [13]. As mentioned, there are effective connections between most algebraic structures. These connections show a relationship between the modular structures associated with these algebras. L-Algebras under conditions can be equivalent to other algebras such as BCK-algebras, MV-algebras, etc. Considering that we have spent a relatively large amount of time studying modular structures (for instance, see [3, 4, 9, 10, 11, 12, 13]), in order to complete and consolidate our study in this field, we have decided to define L-modules as an action of an L-algebra over an Abelian group. We hope that this definition can help us to clarify the structure of L-algebras.

2. Preliminaries

In this section, we review the material that we will use in the paper.

Definition 2.1 ([7]). An L-algebra is an algebra \((L; \to, 1)\) of type \((2, 0)\) satisfying

\[(L1) \; x \to x = x \to 1 = 1, \; 1 \to x = x;\]

\[(L2) \; (x \to y) \to (x \to z) = (y \to x) \to (y \to z);\]

\[(L3) \; x \to y = y \to x = 1 \text{ implies } x = y, \text{ for all } x, y, z \in L.\]

The relation \(x \leq y\) if and only if \(x \to y = 1\), defines a partial order for any L-algebra \(L\). If \(L\) admits a smallest element 0, then it is called a bounded
\textbf{L-Modules}

\textit{L}-algebra.
Moreover, in the bounded \textit{L}-algebra \( L \), if the map \( ' : L \to L \) defined, by 
\[ x \to x' = x \to 0 \] for every \( x \in L \), is bijective, then we say that \( L \) has negation.

\textbf{Definition 2.2 ([17])}. A \textit{KL-algebra} is an \( L \)-algebra \((L, \to, 1)\) such that
\[ x \to (y \to x) = 1 \quad (K) \]
for every \( x, y \in L \).
A \textit{CL-algebra} is an \( L \)-algebra \((L, \to, 1)\) such that
\[ (x \to (y \to z)) \to (y \to (x \to z)) = 1 \quad (C) \]
for every \( x, y, z \in L \).

\textbf{Definition 2.3 ([16])}. Let \((L; \to, 1)\) be an \( L \)-algebra. Then a subset \( K \) of \( L \) is called an \textit{L-subalgebra} if
\[ x \to y, y \to x \in K \] for all \( x, y \in K \).
A subset \( I \) of \( L \) is called an \textit{ideal} if the following hold for all \( x, y \in L \):

\begin{enumerate}
\item[(I1)] \( 1 \in I \),
\item[(I2)] \( x, x \to y \in I \) implies \( y \in I \),
\item[(I3)] \( x \in I \) implies \( (x \to y) \to y \in I \),
\item[(I4)] \( x \in I \) implies \( y \to x, y \to (x \to y) \in I \).
\end{enumerate}
Denote by \( \mathcal{ID}(L) \) the set of all ideals of \( L \).

If \( L \) satisfies condition \((K)\), then \((I4)\) can be omitted. Also, if \( L \) satisfies condition \((C)\), then \((I3)\) and \((I4)\) can be omitted.

\textbf{Definition 2.4 ([5])}. For every subset \( Y \subseteq L \), the smallest ideal of \( L \) containing \( Y \) (i.e. the intersection of all ideals \( I \in \mathcal{ID}(L) \) such that \( Y \subseteq I \)) is called the ideal generated by \( Y \) and it will be denoted by \([Y]\). If \( Y = \{x\} \) we write \([x]\) instead of \([\{x\}]\). In this case \([x]\) is called a principal ideal of \( L \).

\section{L-modules}

In this section, we present our definition of \( L \)-modules, and obtain some results on them. Then we introduce the concepts of \( L \)-submodules and prime \( L \)-submodules in \( L \)-modules. Finally, we investigate some conditions for having a prime \( L \)-submodule.
**Note.** If $L$ is an $L$-algebra, then we denote $(l \to u) \to u$ by $l \uparrow u$, for every $l, u \in L$.

**Definition 3.1.** Let $L = (L; \to, 0, 1)$ be a bounded $L$-algebra, and $M = (M, +)$ be an Abelian group. Then $M$ is called an $L$-module, if there is an operation $\cdot : L \times M \to M$ by $(l, m) \mapsto l \cdot m$ such that for every $l, u \in L$ and $m, n \in M$, we have:

\begin{align*}
\text{(LM1)} & \quad 1 \cdot m = m; \\
\text{(LM2)} & \quad l \cdot (m + n) = l \cdot m + l \cdot n; \\
\text{(LM3)} & \quad (l \to u) \cdot m = l' \cdot m + u \cdot m, \text{ for all pairs } (l, u) \text{ with } u \neq 1.
\end{align*}

Moreover, if we have

\begin{align*}
\text{(LM4)} & \quad (l \uparrow u) \cdot m = l \cdot (u \cdot m), \text{ for all pairs } (l, u) \text{ with } l \neq 0,
\end{align*}

then $M$ is called an Extended $L$-module (or briefly EL-module).

**Example 3.2.**

(i) Let $L = \{0, 1\}$ and define an operation $\to$ on $L$ by

\[
\begin{array}{c|cc}
\to & 0 & 1 \\
\hline
0 & 0 & 1 \\
1 & 1 & 1
\end{array}
\]

Then $L = (L; \to, 0, 1)$ is a bounded $L$-algebra. The map $' : L \to L$ by $0' = 1$ and $1' = 0$ is bijective. Consider the operation $\cdot : L \times Z \to Z$ by $0 \cdot n = 0$ and $1 \cdot n = n$, for every $n \in Z$. Then $\text{(LZ1)}$ and $\text{(LZ2)}$ are clear.

\begin{align*}
\text{(LZ3)} & \quad \text{We have } (0 \to 0).n = 0'.n + 0.n, \quad (1 \to 1).n = 1'.n + 1.n \text{ and } \quad (1 \to 0).n = 1'.n + 0.n, \text{ for every } n \in Z. \text{ Then } Z \text{ is an } L\text{-module. Moreover, } \\
\text{(LZ4)} & \quad \text{We have } (0 \uparrow 0).n = 0.(0.n) \text{ and } (1 \uparrow 1).n = 1.(1.n), \text{ for every } n \in Z.
\end{align*}

Therefore, $Z$ is an EL-module.

(ii) Let $A$ be a non-empty set. Then it is routine to see that $(\rho(A); \to, \emptyset, A)$ is a bounded $L$-algebra, where $X \to Y = X' \cup Y$, for every $X, Y \in \rho(A)$. Since $\emptyset \to \emptyset = \emptyset \to A = A \to A = A$ and $A \to \emptyset = \emptyset$, we get $L = \{\emptyset, A\}$ is an $L$-subalgebra of $\rho(A)$ and so it is an $L$-algebra. Consider $M = (\rho(A), \Delta)$, where $X \Delta Y = X \cup Y - X \cap Y$, for every $X, Y \in \rho(A)$. It is easy to see that $M$ is an abelian group. Now, let the operation $\cdot : L \times M \to M$ be defined by $T \cdot Y = T \cap Y$, for any $T \in L$ and $Y \in M$. Then

- **(LM1)** $A \cdot Y = A \cap Y = Y$, for every $Y \in M$;
- **(LM2)** It is routine to see that
\[ T \cdot (X + Y) = T \cap (X \Delta Y) = (T \cap X) \Delta (T \cap Y) = (T \cdot X) + (T \cdot Y), \]

for every \( T \in L \) and \( X, Y \in M \);

\((LM3)\) We have

\[
(A \rightarrow A).X = (A \cup A') \cap X = X \cap A = X \cap (A' \Delta A)
\]

\[
= (A' \cap X) \Delta (A \cap X) = A' \cdot X + A \cdot X,
\]

for every \( X \in M \). By the similar way, we have

\[
(\emptyset \rightarrow \emptyset).X = \emptyset' \cdot X + \emptyset \cdot X \quad \text{and} \quad (A \rightarrow \emptyset).X = A' \cdot X + \emptyset \cdot X,
\]

for every \( X \in M \). Hence, \( M \) is an \( L \)-module. Moreover,

\((LM4)\) Since

\[
A \uparrow A = (A \rightarrow A) \rightarrow A = (A' \cup A) \rightarrow A = (A \cap A') \cup A = A,
\]

we have \((A \uparrow A) \cdot X = A \cdot (A \cdot X)\), for every \( X \in M \). By the similar way, we have \((\emptyset \uparrow \emptyset) \cdot X = \emptyset \cdot (\emptyset \cdot X)\), for every \( X \in M \). Therefore, \( M \) is an \( EL \)-module.

**Note.** From now on, in this paper, we let \( L = (L; \rightarrow, 1) \) be an \( L \)-algebra.

**Definition 3.3.** If \( l \uparrow u = u \uparrow l \), for every \( l, u \in L \), then we say that \( L \) is \( L \)-commutative.

**Example 3.4.** (i) Let \( L = \{0, l, u, 1\} \) and define an operation “\( \rightarrow \)” on \( L \) by

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Then \((L; \rightarrow, 1)\) is an \( L \)-algebra. Moreover, \( L \) is \( L \)-commutative.

(ii) According to Example 3.2 (i), \( L \) is \( L \)-commutative.
(iii) Let $L = \{0, l, u, t, 1\}$ and define operation “$\rightarrow$” on $L$ by

$$
\begin{array}{c|ccccc}
\rightarrow & 0 & l & u & t & 1 \\
\hline
0 & 1 & 1 & 1 & 1 & 1 \\
l & 0 & 1 & l & t & 1 \\
u & 0 & l & 1 & t & 1 \\
t & t & 1 & 1 & 1 & 1 \\
1 & 0 & l & u & t & 1 \\
\end{array}
$$

Then $(L; \rightarrow, 1)$ is an $L$-algebra. Since $l \uparrow t = (l \rightarrow t) \rightarrow t = 1 \neq l = (t \rightarrow l) \rightarrow l = t \uparrow l$, $L$ is not $L$-commutative.

In the following, we present a general example of $L$-module.

**Proposition 3.5.** Let $L = (L; \rightarrow, 0, 1)$ be bounded and $L$-commutative. Then $(L, +)$ is an Abelian group, where

$$
l + u = (l \rightarrow u)' \uparrow (u \rightarrow l)' \quad \text{for every } l, u \in L.
$$

**Proof:** At first, we show that $0 + l = l + 0 = l$, for every $l \in L$. We have

$$
l + 0 = (l \rightarrow 0)' \uparrow (0 \rightarrow l)' = (l')' \uparrow 1' = l \uparrow 1 = (l \rightarrow 0) \rightarrow 0 = (0 \rightarrow 0) \rightarrow 0 = 0 = l.
$$

By the similar way, we have $0 + l = l$ and so $0 + l = l + 0 = l$, for every $l \in L$. Also, since

$$
l + l = (l \rightarrow l)' \uparrow (l \rightarrow l)' = 1' \uparrow 1' = 0 \uparrow 0 = (0 \rightarrow 0) \rightarrow 0 = 1 \rightarrow 0 = 0,
$$

we conclude that every member of $L$ has a counterpart in $L$. Now, with a long and routine method, it can be seen

$$
l + (u + t) = (l + u) + t \quad \text{for every } l, u, t \in L.
$$

Finally, since $L$ is $L$-commutative, we have

$$
l + u = (l \rightarrow u)' \uparrow (u \rightarrow l)' = (u \rightarrow l)' \uparrow (l \rightarrow u)'u + l \quad \text{for every } l, u \in L.
$$

Therefore, $(L, +)$ is an Abelian group.

**Proposition 3.6.** Let $L = (L; \wedge, \vee', 0, 1)$ be a Boolean-algebra. Then $L$ is a bounded $L$-algebra. Moreover, $L$ is $L$-commutative.

**Proof:** We define $l \rightarrow u = l' \vee u$, for every $l, u \in L$. Then
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(L1) It is clear that \(l \rightarrow l = l \rightarrow 1 = 1\) and \(1 \rightarrow l = l\), for every \(l \in L\).

(L2) For every \(l, u \in L\), we have

\[
(l \rightarrow u) \rightarrow (l \rightarrow t) = (l' \lor u) \rightarrow (l' \lor t) = (l' \lor u)' \lor (l' \lor t) = (l \land u') \lor (l \land t) = ((1 \land u') \lor l') \lor t = (u' \lor l') \lor t.
\]

On the other hand, by the similar way, we have \((u \rightarrow l) \rightarrow (u \rightarrow t) = (u' \lor l') \lor t\). Hence

\[
(l \rightarrow u) \rightarrow (l \rightarrow t) = (u \rightarrow l) \rightarrow (u \rightarrow t), \text{ for every } l, u \in L.
\]

(L3) Let \(l \rightarrow u = u \rightarrow l = 1\), for any \(l, u \in L\). Then \(l' \lor u = u' \lor l = 1\) and so

\[
l \land u = (l \land l') \lor (l \land u) = l \land (l' \lor u) = l \land 1 = l.
\]

This means that \(l \leq u\). By the similar way, we have \(u \leq l\) and so \(u = l\).
Thus, \((L, \rightarrow, 1)\) is an \(L\)-algebra. Note that \(0 \rightarrow l = 0' \lor l = 1 \lor l = 1\). So \(0 \leq l\), for every \(l \in L\) and so \(L\) is bounded. Moreover, we have

\[
l \uparrow u = (l \rightarrow u) \rightarrow u = (l' \lor u)' \lor u = (l \land u') \lor (u \land u) = l \lor (l' \lor l') = l \lor (u \lor u) = l \lor (u \lor l') = (l \lor u) \lor (u \lor l') = l \lor (u \rightarrow l)' = (u \rightarrow l) \rightarrow l = u \uparrow l, \text{ for every } u, l \in L.
\]

Therefore, \(L\) is \(L\)-commutative.

Example 3.7. Let \(L = (L; \land, \lor, 0, 1)\) be a Boolean-algebra. If \(l \rightarrow u \neq 1\) implies \(u \leq l\), for every \(u, l \in L\), then \(L\) is an \(L\)-module.

Proof: By Proposition 3.6, \(L\) is bounded and \(L\)-commutative, and by Proposition 3.5, \(M = (L, +)\) is an Abelian group, where \(l + u = (l \rightarrow u)' \uparrow (u \rightarrow l)\), for every \(l, u \in L\). We define the operation \(\cdot \) : \(L \times M \rightarrow M\) by \(l.m = l \land m\), for every \(l \in L\) and \(m \in M\). Then

(LM1) \(1 \cdot m = 1 \land m\), for every \(m \in M\);

(LM2) Since for every \(m, n \in M\),
\[ m + n = (m \rightarrow n)' \uparrow (n \rightarrow m)' = ((m \rightarrow n)' \rightarrow (n \rightarrow m)') \rightarrow (n \rightarrow m)'
\]
\[ = ((m' \lor n)' \rightarrow (n' \lor m)') \rightarrow (n' \lor m)'
\]
\[ = ((m' \lor n) \lor (n \land m'))' \lor (n \land m')
\]
\[ = ((m \land n') \lor (n' \lor m')) \lor (n \land m')
\]
\[ = ((m \land n') \lor (n \land m)) \land ((n \lor m) \lor (n' \lor m'))
\]
\[ = (n \lor m) \land (n' \lor m')
\]
\[ = (n \land m) \lor (m' \lor n')
\]
\[ = (n \land m') \lor (m' \lor n)
\]
\[ = (n \land m') \lor (m' \lor n')
\]
\[ = (n \land m') \lor (m \land n')
\]
\[ \text{we have}
\]
\[ l \cdot (m + n) = l \land ((n \land m') \lor (m \land n')) = (l \land n \land m') \lor (l \land m \land n')
\]
\[ = ((l \land m) \lor (l \land n')) \lor ((l \land m)' \land (l \land n))
\]
\[ = (l \land m) + (l \land n) = l \cdot m + l \cdot n,
\]
for every \( l \in L \) and \( m, n \in M \).

\((LM3)\) Let \( l \rightarrow u \neq 1 \) or \( l = u \), for any \( l, u \in L \). Then \( u \leq l \) and so \( u \lor l = l \) and \( u \land l = u \). Thus, for every \( m \in M \),

\[ l' \cdot m + u \cdot m = (l' \land m) + (u \land m)
\]
\[ = ((l' \land m') \land (u \land m)) \lor ((l' \land m) \land (u \land m'))
\]
\[ = ((l \lor m') \land (u \land m)) \lor ((l' \land m) \land (u' \lor m'))
\]
\[ = ((u \land m) \land l) \lor (u \land m \land m') \lor (l' \land m \land u') \lor (l' \land m \land m')
\]
\[ = (u \land m \land l) \lor (l' \land m \land u') = m \land ((u \land l) \lor (l' \land u'))
\]
\[ = ((l \lor u) \rightarrow (l \land u), m = (l \rightarrow u).m.
\]

Note that if \( l \rightarrow u = 1 \), then \( l \leq u \). So by the similar way, we have \((l \rightarrow u).m = l' \cdot m + u \cdot m\). Hence,

\[ (l \rightarrow u) \cdot m = l' \cdot m + u \cdot m, \text{ for all pairs } (l, u) \text{ with } u \neq 1.
\]

Therefore, \( L \) is an \( L \)-module. \( \square \)
**L-Modules**

**Proposition 3.8.** Let \( L = (L; \rightarrow, 0, 1) \) be bounded and \( L \)-commutative, \( I \) be an ideal of \( L \) and \( L \) be an \( L \)-module. Then \( \frac{L}{I} \) is an \( L \)-module. Moreover, if \( L \) is an \( EL \)-module, then \( \frac{L}{I} \) is an \( EL \)-module.

**Proof:** Since \((L, +)\) is an Abelian group, it is easy to see that \( \frac{L}{I}, \oplus \) is an abelian group, where \([l] \oplus [u] = [l + u] \), for every \( l, u \in L \). We define the operation \( \bullet : L \times \frac{L}{I} \longrightarrow \frac{L}{I} \) by \( l \bullet [m] = [l \cdot m] \), for every \( l \in L \) and \([m] \in \frac{L}{I} \).

Then

1. \((L^L_I1)\) By \((LL1)\), we have \( 1 \bullet [m] = [m] \), for every \([m] \in \frac{L}{I} \);

2. \((L^L_I2)\) By \((LL2)\), for every \( l \in L \) and \([m], [n] \in \frac{L}{I} \), we have \( 1 \bullet ([m] \oplus [n]) = 1 \bullet [m + n] = [l \cdot (m + n)] = [l \cdot m + l \cdot n] = [l \cdot m] \oplus [l \cdot n] = l \bullet [m] \oplus l \bullet [n] \);

3. \((L^L_I3)\) By \((LL3)\), for every \([m] \in \frac{L}{I} \) and for all pairs \((l, u)\) with \( u \neq 1 \), we have \( (l \rightarrow u) \bullet [m] = [(l \rightarrow u) \cdot m] = [l' \cdot m + u \cdot m] = [l' \cdot m] \oplus [u \cdot m] = l' \bullet [m] \oplus u \bullet [m] \).

Then \( \frac{L}{I} \) is an \( L \)-module. Moreover,

4. \((L^L_I4)\) By \((LL4)\), for every \([m] \in \frac{L}{I} \) and for all pairs \((l, u)\) with \( l \neq 0 \), we have \( (l \uparrow u) \bullet [m] = [(l \uparrow u) \cdot m] = [l \cdot (u \cdot m)] = l \cdot [u \cdot m] = l \bullet (u \bullet [m]) \).

Therefore, \( \frac{L}{I} \) is an \( EL \)-module. \( \square \)

**Note.** From now on, in this paper, we let \( M \) be an Abelian group.

Let \( I \in ID(L) \). The relation \( \sim \) on \( L \) is defined by

\[ u \sim l \iff u \rightarrow l, l \rightarrow u \in I, \text{ for every } u, l \in L. \]
It was proved that $\sim$ is a congruence on $L$. Then $(\frac{L}{I}; \to, [1])$ is an $L$-algebra, where $[u] \to [l] = [u \to l]$, for every $u, l \in L$ (see [16]).

**Theorem 3.9.** Let $M$ be an $L$-module, and $I$ be an ideal of $L$ such that $I \subseteq \text{Ann}_L(M)$, where $\text{Ann}_L(M) = \{l \in L : l \cdot m = 0, \text{ for every } m \in M\}$. Then $M$ is an $\frac{L}{I}$-module. Moreover, if $M$ is an $EL$-module, then $M$ is an $E \frac{L}{I}$-module.

**Proof:** Consider $\prime : \frac{L}{I} \to \frac{L}{I}$ by $([l])' = [l']$, for every $l \in L$ which is a bijective mapping. Define the operation $\cdot : \frac{L}{I} \times M \to M$ by $[l] \cdot m = l \cdot m$, for every $[l] \in \frac{L}{I}$ and $m \in M$. Let $[l] = [u]$ and $m = n$, for every $[l], [u] \in \frac{L}{I}$ and $m, n \in M$. Then $l \to u, u \to l \in I \subseteq \text{Ann}_L(M)$ and so $(l \to u) \cdot m = (u \to l) \cdot m = 0$, for every $m \in M$. It results that $l' \cdot m + u \cdot m = u' \cdot m + l \cdot m = 0$ and so $l \cdot m - u \cdot m = l' \cdot m - u' \cdot m$ and $l \cdot m = -u' \cdot m$. Hence $l \cdot m - u \cdot m = l' \cdot m = l \cdot m = l \cdot m = 1 \cdot m$. By the similar way, we have $u \cdot m - l \cdot m = 1 \cdot m$. It results that $l \cdot m - u \cdot m = u \cdot m - l \cdot m$ and so $l \cdot m = u \cdot m$. It means that $\cdot$ is well defined. Now, we have:

$(\frac{L}{I}M1)$ By $(LM1)$, it is clear that $[1] \cdot m = m$, for every $m \in M$;

$(\frac{L}{I}M2)$ By $(LM2)$, we have

$$[l] \cdot (m + n) = l \cdot (m + n) = l \cdot m + l \cdot n = [l] \cdot m + l \cdot n,$$

for every $[l] \in \frac{L}{I}$ and $m, n \in M$;

$(\frac{L}{I}M3)$ By $(LM3)$, for every $m \in M$ and for all pairs $([l], [u])$ with $[u] \neq [1]$, we have

$$([l] \to [u]) \cdot m = (l \to u) \cdot m = (l \to u) \cdot m = l' \cdot m + u \cdot m = [l'] \cdot m + [u] \cdot m.$$

Note that $l \neq 1$ implies $[l] \neq [1]$. Hence, $M$ is an $\frac{L}{I}$-module. Moreover,
Let $M$ be an $L$-module, and $S$ be a subgroup of $M$. If $S$ satisfies
\[ l \cdot s \in S, \text{ for every } l \in L \text{ and } s \in S, \]
then it is called an $L$-submodule of $M$.

Example 3.11. (i) By Example 3.2 (i), $2\mathbb{Z}$ is an $L$-submodule of $M$.
(ii) According to Example 3.2 (ii), consider $A = \{a, b\}$. Then $S_1 = \emptyset, \{a\}$ and $S_2 = \emptyset, \{b\}$ are $L$-submodules of $M$.

Let $M$ be an $L$-module, and $S$ be an $L$-submodule of $M$. Since $(M, +)$ is an Abelian group and $S$ is a subgroup of $M$, we can apply the module theory to present quotient $L$-module. So it is clear that $(\frac{M}{S})$, $\oplus$ is a group, where $(m+S) \oplus (n+S) = (m+n) \oplus S$, for every $m, n \in M$.

Proposition 3.12. Let $M$ be an $L$-module, and $S$ be an $L$-submodule of $M$. Then $\frac{M}{S}$ is an $L$-module. Moreover, if $M$ is an $EL$-module, then $\frac{M}{S}$ is an $EL$-module.

Proof: We define the operation $\bullet : L \times \frac{M}{S} \rightarrow \frac{M}{S}$ by $l \bullet (m+S) = l \cdot m + S$, for every $l \in L$ and $m + S \in \frac{M}{S}$. It is routine to see that $\bullet$ is well defined. By $(LM1)$ and $(LM2)$, the proofs of $(L \frac{M}{S}1)$ and $(L \frac{M}{S}2)$ are routine.

$(L \frac{M}{S}3)$ By $(LM3)$, for all pairs $(l, u)$ with $u \neq 1$, we have
\[
(l \mapsto u) \bullet (m + S) = (l \mapsto u) \cdot m + S = (l' \cdot m + u \cdot m) + S = (l' \cdot m + S) \oplus (u \cdot m + S) = l' \bullet (m + S) \oplus u \bullet (m + S),
\]
for every \( m + S \in \frac{M}{S} \). Then \( \frac{M}{S} \) is an \( L \)-module. Moreover, 
\[
(\frac{M}{S} \cdot \frac{4}{4}) \text{ by } (LM4), \text{ for all pairs } (l, u) \text{ with } l \neq 0, \text{ we have }
\]
\[
(l \uparrow u) \cdot (m + S) = (l \uparrow u) \cdot m + S = l \cdot (u \cdot m) + S = l \cdot (u \cdot (m + S)),
\]
for every \( m + S \in \frac{M}{S} \). Therefore, \( \frac{M}{S} \) is an \( EL \)-module. \( \square \)

**Lemma 3.13.** Let \( M \) be an \( EL \)-module, and \( I \) be an ideal of \( L \). Then 
\[
I_L(M) = \{ \Sigma_{i=1}^{n} t_i \cdot m_i : 0 \neq t_i \in I, m_i \in M, n \in \mathbb{N} \}
\]
is an \( L \)-submodule of \( M \).

**Proof:** It is clear that \( I_L(M) \) is a subgroup of \( M \). Now, for every \( l \in L \) and \( \Sigma_{i=1}^{n} t_i \cdot m_i \in I_L(S) \), by \( (LM2) \), we have
\[
l \cdot \Sigma_{i=1}^{n} t_i \cdot m_i = l \cdot (t_1 \cdot m_1) + l \cdot (t_2 \cdot m_2) + \cdots + l \cdot (t_n \cdot m_n)
\]
and so by \( (LM4) \),
\[
l \cdot \Sigma_{i=1}^{n} t_i \cdot m_i = (l \uparrow t_1) \cdot m_1 + (l \uparrow t_2) \cdot m_2 + \cdots + (l \uparrow t_n) \cdot m_n.
\]
Since by \( (I_3) \), \( t_i \cdot m_i \in I \), for every \( 1 \leq i \leq n \), we get \( l \cdot \Sigma_{i=1}^{n} t_i \cdot m_i \in I_L(M) \). Therefore, \( I_L(M) \) is an \( L \)-submodule of \( M \). \( \square \)

**Definition 3.14.** Let \( I \) be a proper ideal of \( L \). Then \( I \) is called a prime ideal of \( L \), if \( l \uparrow u \in I \) implies \( l \in I \) or \( u \in I \), where \( l, u \in L \).

**Example 3.15.** According to Example 3.4 (i), it is easy to see that \( I_1 = \{1, l\} \) and \( I_2 = \{1, u\} \) are prime ideals of \( L \).

**Theorem 3.16.** Let \( M \) be an \( EL \)-module, \( S \) be an \( L \)-submodule of \( M \) and \( P \) be a prime ideal of \( L \). Then 
\[
S_{N,P} = \{ m \in M : c \cdot m \in P_L (M) + S, \exists 0 \neq c \in (L \setminus P) \cup \{1\} \}
\]
is an \( L \)-submodule of \( M \) and \( P_L (M) + S \subseteq S_{N,P} \).

**Proof:** Let \( m, n \in S_{N,P} \). Then there are \( c_1, c_2 \in (L \setminus P) \cup \{1\} \) such that \( 0 \neq c_1, 0 \neq c_2 \) and \( c_1 \cdot m, c_2 \cdot n \in P \cdot M + S \). Consider \( c = c_1 \uparrow c_2 \). It is clear that \( c \in (L \setminus P) \cup \{1\} \). Then by \( (LM4) \), we have
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\[ c \cdot (m - n) = (c_1 \uparrow c_2) \cdot (m - n) = c_1 \cdot (c_2 \cdot (m - n)) \]
\[ = c_1 \cdot (c_2 \cdot m - c_2 \cdot n) = c_1 \cdot (c_2 \cdot m) - c_1 \cdot (c_2 \cdot n) \]

and so by Lemma 3.13, \( c \cdot (m - n) \in P_L(M + S) \). Now, for every \( l \in L \) and \( m \in S_{N,P} \), we show that \( l \cdot m \in S_{N,P} \). Since \( m \in S_{N,P} \), there is \( 0 \neq c \in (L \setminus P) \cup \{1\} \) such that \( c \cdot m \in P_L(M) \). Then by Lemma 3.13 and (LM4),

\[ c \cdot (l \cdot m) = (c \uparrow l) \cdot m = (l \uparrow c) \cdot m = l \cdot (c \cdot m) \in P_L(M). \]

Hence, \( S_{N,P} \) is an \( L \)-submodule of \( M \). Finally, let \( t \cdot m \in P_L(M) \). Then we have \( 1 \cdot (t \cdot m) \in P_L(M) + S \), where \( c = 1 \in (L \setminus P) \cup \{1\} \). Therefore, \( t \cdot m \in S_{N,P} \) and so \( P_L(M) \subseteq S_{N,P} \).

**Theorem 3.17.** Let \( I \) be an ideal of \( L \), and \( M \) be an \( \text{EL} \)-module. Then \( \frac{M}{IL(M)} \) is an \( \text{EL} \cdot \frac{L}{I} \)-module. Moreover, if \( M \) is an \( \text{EL} \)-module, then \( \frac{M}{IL(M)} \) is an \( \text{EL} \cdot \frac{L}{I} \)-module.

**Proof:** The module \( \frac{M}{IL(M)} \) can be defined by Lemma 3.13. Then we define the operation \( \bullet : \frac{L}{I} \times \frac{M}{IL(M)} \rightarrow \frac{M}{IL(M)} \) by \([l] \bullet (m + IL(M)) = l \cdot m + IL(M), \) for every \([l] \in \frac{L}{I}\) and \( m + IL(M) \in \frac{M}{IL(M)}. \) Since

\[ I \cdot \frac{M}{IL(M)} = \{ l \cdot (m + IL(M)) : l \in L, m \in M \} \]
\[ = \{ l \cdot m + IL(M) : l \in L, m \in M \} = IL(M), \]

we have \( I \subseteq \text{Ann}_L(\frac{M}{IL(M)}) \) and so with a proof similar to the proof of Theorem 3.9, \( \bullet \) is well defined.

(1) By (LM1), \([1] \bullet (m + IL(M)) = 1 \cdot m + IL(M) = m + IL(M), \) for every \( m \in M; \)

(2) By (LM2), we have
\[
[l] \cdot ((m + I_L(M)) \oplus (n + I_L(M))) = [l] \cdot (m + n + I_L(M)) = l \cdot (m + n) + I_L(M) = (l \cdot m + I_L(M)) \oplus (l \cdot n + I_L(M)) = [l] \cdot (m + I_L(M)) \oplus [l] \cdot (n + I_L(M)),
\]
for every \([l] \in \frac{L}{I_L(M)}\) and \((m + I_L(M)), (n + I_L(M)) \in \frac{M}{I_L(M)}\):

\[
\frac{L}{I_L(M)} \text{ by } (LM3), \text{ for every } m + I_L(M) \in \frac{M}{I_L(M)} \text{ and for all pairs } ([l], [u]) \text{ with } [u] \neq [1], \text{ we have}
\]

\[
([l] \rightarrow [u]) \cdot (m + I_L(M)) = [l \rightarrow u] \cdot (m + I_L(M)) = (l \rightarrow u) \cdot m + I_L(M) = (l' \cdot m + u \cdot m) + I_L(M) = (l' \cdot m + I_L(M)) \oplus (u \cdot m + I_L(M)) = [l'] \cdot (m + I_L(M)) \oplus [u] \cdot (m + I_L(M));
\]

Hence, \(M\) is an \(\frac{L}{I_L(M)}\) module. Moreover,

\[
\frac{L}{I_L(M)} \text{ by } (LM4), \text{ for every } m + I_L(M) \in \frac{M}{I_L(M)} \text{ and for all pairs } ([l], [u]) \text{ with } [l] \neq [0], \text{ we have}
\]

\[
([l] \uparrow [u]) \cdot (m + I_L(M)) = [l \uparrow u] \cdot (m + I_L(M)) = (l \uparrow u) \cdot m + I_L(M) = l \cdot (u \cdot m) + I_L(M) = [l] \cdot (u \cdot m + I_L(M)) = [l] \cdot ([u] \cdot (m + I_L(M))).
\]

Therefore, \(\frac{M}{I_L(M)}\) is an \(\frac{E}{T}\) module. 

**Definition 3.18.** Let \(M\) be an \(L\)-module and \(S\) be a proper \(L\)-submodule of \(M\). Then \(S\) is called a prime \(L\)-submodule of \(M\), if by \(l \cdot m \in S\), we have \(m \in S\) or \(l \in (S : M) = \{l \in L : l \cdot M \subseteq S\}\).

**Example 3.19.** By Example 3.2(i), \(2\mathbb{Z}\) is a prime \(L\)-submodule of \(\mathbb{Z}\).
Note. Let $M$ be an $L$-module, $I \subseteq L$ and $D \subseteq M$. Then we set $ID = \{i \cdot d : i \in I \text{ and } d \in D\}$, and $I_t = \{\alpha \in L : t \rightarrow \alpha = 1\}$, for every $t \in L$. It is clear that $1, t \in I_t$ and so $I_t \neq \emptyset$.

**Theorem 3.20.** Let $L$ be bounded and $L$-commutative, $M$ be an $L$-module and $S$ be a proper $L$-submodule of $M$. Then $S$ is a prime $L$-submodule of $M$ if and only if $I_tD \subseteq S$ implies $D \subseteq S$ or $I_t \subseteq (S : M)$, for any $L$-submodule $D$ of $M$ and $t \in L$.

**Proof:** $(\Rightarrow)$ Let $S$ be a prime $L$-submodule of $M$ and $I_tD \subseteq S$, where $D$ is an $L$-submodule of $M$ and $t \in L$. We show that $D \subseteq S$ or $I_t \subseteq (S : M)$. Let $I_t \not\subseteq (S : M)$ and $D \not\subseteq S$. Then there are $x \in I_t$ and $d \in D$ such that $x \cdot M \not\subseteq S$ and $d \not\in S$. Since $ID \subseteq S$, we have $x \cdot d \in S$ and so by $d \not\in S$, we get $x \in (S : M)$, which is a contradiction.

$(\Leftarrow)$ Let by $I_tD \subseteq S$, we have $D \subseteq S$ or $I_t \subseteq (S : M)$, for any $L$-submodule $D$ of $M$ and $t \in L$. Suppose $x \cdot m \in S$ and $m \not\in S$, for any $x \in L$ and $m \in M$. For every $\alpha \in I_x$, we have

$$\alpha \cdot m = (1 \rightarrow \alpha) \cdot m = ((x \rightarrow \alpha) \rightarrow \alpha) \cdot m = (x \uparrow \alpha) \cdot m = (\alpha \uparrow x) \cdot m$$

$$= \alpha \cdot (x \cdot m) \in S.$$

Now, consider $D = \langle m \rangle = \{y \cdot m : y \in L\}$. Then

$$I_xD = \{\alpha \cdot (y \cdot m) : \alpha, y \in L\} = \{y \cdot (\alpha \cdot m) : \alpha, y \in L\} \subseteq S$$

and so $I_x \subseteq (S : M)$ or $D \subseteq S$. Since $m \not\in S$, we have $I_x \subseteq (S : M)$ and so $x \in (S : M)$. Therefore, $S$ is a prime $L$-submodule of $M$. \hfill $\Box$

**Proposition 3.21.** For every $x, y \in L$,

(i) $x' \rightarrow (x \rightarrow y) = 1$;

(ii) $(x \rightarrow y) \rightarrow x' = (y \rightarrow x) \rightarrow y'$.

**Proof:** (i) By (L2), we have $x' \rightarrow (x \rightarrow y) = (x \rightarrow 0) \rightarrow (x \rightarrow y) = (0 \rightarrow x) \rightarrow (0 \rightarrow y) = 1 \rightarrow 1 = 1$, for every $x, y \in L$. (ii) By (L2), we have $(x \rightarrow y) \rightarrow x' = (x \rightarrow y) \rightarrow (x \rightarrow 0) = (y \rightarrow x) \rightarrow (y \rightarrow 0) = (y \rightarrow x) \rightarrow y'$, for every $x, y \in L$. \hfill $\Box$
Lemma 3.22. Let $L$ be a bounded $KL$-algebra, $M$ be an $EL$-module and $S$ be a proper $L$-submodule of $M$. Then $P_S = (S : M) \cup \{1\}$ is an ideal of $L$.

Proof: (I1) It is clear that $1 \in P_S$.
(II) Let $x, x \rightarrow y \in P_S$. Because of the nature of the definition of $P_S$, we need to consider three cases:

(1) If $x = 1$, then $y = 1 \rightarrow y = x \rightarrow y \in P_S$.

(2) Let $x \rightarrow y = 1$. Then for $y = 1$, the problem is solved. Thus, let $y \neq 1$. Since $y = 1 \rightarrow y = (x \rightarrow y) \rightarrow y = x \uparrow y$, by (LM4), we have

$$y \cdot m = (x \uparrow y) \cdot m = (y \uparrow x) \cdot m = y \cdot (x \cdot m) \in S,$$
for every $m \in M$.

Thus, $y \in (S : M)$ and so $y \in P_S$.

(3) Let $x \neq 1$ and $x \rightarrow y \neq 1$. Then $x \cdot m, (x \rightarrow y) \cdot m \in S$, for every $m \in M$. It results that $x \cdot m + (x \rightarrow y) \cdot m \in S$, for every $m \in M$. Now, by Proposition 3.21(i) and (LM3), for every $m \in M$, we have

$$m = 1 \cdot m = (x' \rightarrow (x \rightarrow y)) \cdot m = x \cdot m + (x \rightarrow y) \cdot m \in S,$$

which is a contradiction.

Therefore, $P_S = (S : M) \cup \{1\}$ is an ideal of $L$.

Definition 3.23. Let $L$ be bounded and $M$ be an $L$-module. Then $M$ is called a torsion free $L$-module, if $l \cdot m = 0$ implies $l = 0$ or $m = 0$, for every $l \in L$ and $m \in M$.

Example 3.24. By Example 3.2(ii), $M$ is a torsion free $L$-module.

Theorem 3.25. Let $L$ be a bounded $KL$-algebra, $M$ be an $EL$-module and $S$ be a proper $L$-submodule of $M$. Then $S$ is a prime $L$-submodule of $M$ if and only if $P_S = (S : M) \cup \{1\}$ is a prime ideal of $L$ and $\frac{M}{S}$ is a torsion free $\frac{L}{P_S}$-module.

Proof: $(\Rightarrow)$ Let $S$ is a prime $L$-submodule of $M$. By Lemma 3.22, $P_S$ is an ideal of $L$. At first, we show that $P_S$ is a prime ideal of $L$. Let $x \uparrow y \in P_S$, for any $x, y \in P_S$. We consider three cases:
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(1) If \( x = 1 \) or \( y = 1 \), then \( x \in P_S \) or \( y \in P_S \).

(2) If \( x \uparrow y \neq 1 \), \( x \neq 1 \) and \( y \neq 1 \), then by (LM4), we have \( x \cdot (y \cdot m) = (x \uparrow y) \cdot m \in S \), for every \( m \in S \). Hence, \( x \in (S : M) \) or \( y \cdot m \in S \), for every \( m \in M \). It results that \( x \in P_S \) or \( y \in P_S \).

(3) Let \( x \uparrow y = 1 \), \( x \neq 1 \) and \( y \neq 1 \). Then \( (x \rightarrow y) \rightarrow y = (x \uparrow y) = 1 \) and so \( x \rightarrow y \leq y \). Since \( y \leq x \rightarrow y \), we have \( x \rightarrow y = y \) and so by (LM3), \( (x \rightarrow y) \cdot m = x' \cdot m + y \cdot m = y \cdot m \), for every, \( m \in M \).

Now, we define the operation \( \cdot : \frac{L}{P_S} \times \frac{M}{S} \rightarrow \frac{M}{S} \) by \([l] \cdot (m + S) = l \cdot m + S \), for every \([l] \in \frac{L}{P_S} \) and \( m + S \in \frac{M}{S} \). By the similar way to the proof of Theorem 3.17, \( \frac{M}{S} \) is an \( \frac{L}{P_S} \)-module. Finally, let \([l] \cdot (m + S) = S \), for any \([l] \in \frac{L}{P_S} \) and \( m + S \in \frac{M}{S} \). Then \( l \cdot m + S = S \) and so \( l \cdot m \in S \). It results that \( l \in (S : M) \subseteq P_S \) or \( m \in S \) and so \([l] = P_S \) or \( m + S = S \). Therefore, \( \frac{M}{S} \) is a torsion free \( \frac{L}{P_S} \)-module.

(\( \Rightarrow \)) Let \( P_S = (S : M) \cup \{1\} \) be a prime ideal of \( L \) and \( \frac{M}{S} \) be a torsion free \( \frac{L}{P_S} \)-module. If \( l \cdot m \in S \), for any \( l \in L \) and \( m \in S \), then \([l] \cdot (m + S) = l \cdot m + S = S \) and so \([l] = P_S \) or \( m + S = S \). It means that \( l = 1 \rightarrow l \in P_S \). Therefore, \( S \) is a prime \( L \)-submodule of \( M \).

4. Conclusions and future works

In this paper, we have presented the definitions of \( L \)-modules, \( L \)-submodules and prime \( L \)-submodules, and some results about prime \( L \)-submodules. We intend to study \( L \)-modules in specific cases, too. For examples, free \( L \)-modules, projective(injective) \( L \)-modules, and so on. Because \( L \)-algebras
cover a number of algebraic structures (such as BCK-algebras, etc.), the results of this paper can be generalized to those algebraic structures. We hope that we have taken an effective step in this regard.

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References


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